Lower bounds for the first eigenvalue of the magnetic Laplacian

Bruno Colbois and Alessandro Savo

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Abstract

We consider a Riemannian cylinder Ω endowed with a closed potential 1-form A and study the magnetic Laplacian Δ_A with magnetic Neumann boundary conditions associated with those data. We establish a sharp lower bound for the first eigenvalue and show that the equality characterizes the situation where the metric is a product. We then look at the case of a planar domain bounded by two closed curves and obtain an explicit lower bound in terms of the geometry of the domain. We finally discuss sharpness of this last estimate.

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1 Introduction

Let (Ω, g) be a compact Riemannian manifold with boundary. Consider the trivial complex line bundle $\Omega \times \mathbf{C}$ over Ω; its space of sections can be identified with $C^\infty(\Omega, \mathbf{C})$, the space of smooth complex valued functions on Ω. Given a smooth real 1-form A on Ω we define a connection ∇^A on $C^\infty(\Omega, \mathbf{C})$ as follows:

$$\nabla^A_X u = \nabla_X u - iA(X)u \tag{1}$$

for all vector fields X on Ω and for all $u \in C^\infty(\Omega, \mathbf{C})$; here ∇ is the Levi-Civita connection assocated to the metric g of Ω. The operator

$$\Delta_A = (\nabla^A)^\star \nabla^A \tag{2}$$

is called the *magnetic Laplacian* associated to the magnetic potential A, and the smooth two form

$$B = dA$$

is the associated *magnetic field.* We will consider Neumann magnetic conditions, that is:

$$\nabla^A_N u = 0 \quad \text{on} \quad \partial\Omega, \tag{3}$$

where N denotes the inner unit normal. Then, it is well-known that Δ_A is self-adjoint, and admits a discrete spectrum

$$0 \leq \lambda_1(\Delta_A) \leq \lambda_2(\Delta_A) \leq \ldots \to \infty.$$

The above is a particular case of a more general situation, where $E \to M$ is a complex line bundle with a hermitian connection ∇^E, and where the magnetic Laplacian is defined as $\Delta_E = (\nabla^E)^\star \nabla^E$.
The spectrum of the magnetic Laplacian is very much studied in analysis (see for example [3] and the references therein) and in relation with physics. For *Dirichlet boundary conditions*, lower estimates of its fundamental tone have been worked out, in particular, when Ω is a planar domain and B is the constant magnetic field; that is, when the function $\star B$ is constant on Ω (see for example a Faber-Krahn type inequality in [9] and the recent[12] and the references therein, also for Neumann boundary condition). The case when the potential A is a closed 1-form is particularly interesting from the physical point of view (Aharonov-Bohm effect), and also from the geometric point of view. For Dirichlet boundary conditions, there is a series of papers for domains with a pole, when the pole approaches the boundary (see [1, 13] and the references therein). Last but not least, there is a Aharonov-Bohm approach to the question of nodal and minimal partitions, see chapter 8 of [4].
For *Neumann boundary conditions*, we refer in particular to the paper [10], where the authors study the multiplicity and the nodal sets corresponding to the ground state λ_1 for non-simply connected planar domains with harmonic potential (see the discussion below).
Let us also mention the recent article [11] (chapter 7) where the authors establish a *Cheeger type inequality* for λ_1; that is, they find a lower bound for $\lambda_1(\Delta_A)$ in terms of the geometry of Ω and the potential A. In the preprint [8], the authors approach the problem via the Bochner method and in [6], the authors look at the problem of finding upper bounds for the spectrum.
Finally, in a more general context (see [2]) the authors establish a lower bound for $\lambda_1(\Delta_A)$ in terms of the *holonomy* of the vector bundle on which Δ_A acts. In both cases, implicitly, the flux of the potential A plays a crucial role.

- From now on we will denote by $\lambda_1(\Omega, A)$ the first eigenvalue of Δ_A on (Ω, g).

1.1 Main lower bound

Our lower bound is partly inspired by the results in [10] for plane domains. First, recall that if c is a closed parametrized curve (a loop), the quantity:

$$\Phi_c^A = \frac{1}{2\pi} \oint_c A$$

is called the *flux* of A across c. (We assume that c is travelled once, and we will not specify the orientation of the loop, so that the flux will only be defined up to sign: this will not affect any of the statements, definitions or results which we will prove in this paper). Let then Ω be a fixed plane domain with one hole, and let Φ^A be the flux of the harmonic potential A across the inner boundary curve. In Theorem 1.1 of [10] it is first remarked that $\lambda_1(\Omega, A)$ is positive if and only if Φ^A is not an integer (but see the precise statement in Section 2.1 below). Then, it is shown that $\lambda_1(\Omega, A)$ is maximal precisely when Φ^A is congruent to $\frac{1}{2}$ modulo integers. The proof relies on a delicate argument involving the nodal line of a first eigenfunction; in particular, the conclusion does not follow from a specific comparison argument, or from an explicit lower bound.

In this paper we give a geometric lower bound of $\lambda_1(\Omega, A)$ when Ω is, more generally, a *Riemannian cylinder*, that is, a domain (Ω, g) diffeomorphic to $[0,1] \times \mathbf{S}^1$ endowed with a Riemannian metric g, and when A is a closed potential 1-form : hence, the magnetic field B associated to A is equal to 0. The lower bound will depend on the geometry of Ω and, in an explicit way, on the flux of the potential A.

Let us write $\partial\Omega = \Sigma_1 \cup \Sigma_2$ where

$$\Sigma_1 = \{0\} \times \mathbf{S}^1, \quad \Sigma_2 = \{1\} \times \mathbf{S}^1.$$

We will need to foliate the cylinder by the (regular) level curves of a smooth function ψ and then we introduce the following family of functions.

$$\mathcal{F}_\Omega = \{\psi : \Omega \to \mathbf{R} : \quad \textit{ψ is constant on each boundary component} \\ \textit{and has no critical points inside Ω.}\}$$

As Ω is a cylinder, we see that $\mathcal{F}_\Omega$ is not empty. If $\psi \in \mathcal{F}_\Omega$, we set:

$$K = K_{\Omega,\psi} = \frac{\sup_\Omega |\nabla\psi|}{\inf_\Omega |\nabla\psi|}.$$

It is clear that, in the definition of the constant K, we can assume that the range of ψ is the interval $[0,1]$, and that $\psi = 0$ on Σ_1 and $\psi = 1$ on Σ_2. Note that the level curves of the function ψ are all smooth, closed and connected; moreover they are all homotopic to each other so that the flux of a closed 1-form A across any of them is the same, and will be denoted by Φ^A.

We say, briefly, that Ω is *K-foliated by the level curves of ψ.* We also denote by $d(\Phi^A, \mathbf{Z})$ the minimal distance between Φ^A and the set of integer $\mathbf{Z}$:

$$d(\Phi^A, \mathbf{Z})^2 = \min\Big\{(\Phi^A - k)^2 : k \in \mathbf{Z}\Big\}.$$

Finally, we say that Ω *is a Riemannian product* if it is isometric to $[0,a] \times \mathbf{S}^1(R)$ for suitable positive constants a, R.

Theorem 1.
a) Let (Ω, g) be a Riemannian cylinder, and let A be a closed 1-form on Ω. Assume that Ω is K-foliated by the level curves of the smooth function $\psi \in \mathcal{F}_\Omega$. Then:

$$\lambda_1(\Omega, A) \geq \frac{4\pi^2}{KL^2} \cdot d(\Phi^A, \mathbf{Z})^2, \tag{4}$$

where L is the maximum length of a level curve of ψ and Φ^A is the flux of A across any of the boundary components of Ω.
b) Equality holds if and only if the cylinder Ω is a Riemannian product.

- It is clear that we can also state the lower bound as follows:

$$\lambda_1(\Omega, A) \geq \frac{4\pi^2}{\tilde{K}_\Omega} \cdot d(\Phi^A, \mathbf{Z})^2,$$

where $\tilde{K}_\Omega$ is an invariant depending only on Ω:

$$\tilde{K}_\Omega = \inf_{\psi \in \mathcal{F}_\Omega} K_{\Omega,\psi} L_\psi^2 \quad \text{and} \quad L_\psi = \sup_{r \in \text{range}(\psi)} |\psi^{-1}(r)|.$$

It is is not always easy to estimate K. In Section 2.4 we will show how to estimate K in terms of the metric tensor. Note that $K \geq 1$; we will see that in many interesting situations (for example, for revolution cylinders, or for smooth embedded tubes around a closed curve) one has in fact $K = 1$.

1.2 Doubly connected planar domains

We now estimate the constant K above when Ω is an annular region in the plane, bounded by the inner curve Σ_1 and the outer curve Σ_2.

- We assume that the inner curve Σ_1 is convex.

From each point $x \in \Sigma_1$, consider the ray $\gamma_x(t) = x + tN_x$, where N_x is the exterior normal to Σ_1 at x and $t \geq 0$. Let $Q(x)$ be the first intersection of $\gamma_x(t)$ with Σ_2, and let

$$r(x) = d(x, Q(x)).$$

We say that Ω is *starlike with respect to* Σ_1 if the map $x \to Q(x)$ is a bijection between Σ_1 and Σ_2; equivalently, if given any point $y \in \Sigma_2$, the geodesic segment which minimizes distance from y to Σ_1 is entirely contained in Ω.
For $x \in \Sigma_1$, we denote by θ_x the angle between γ_x' and the outer normal to Σ_2 at the point $Q(x)$, and we let

$$m \doteq \min_{x \in \Sigma_1} \cos\theta_x.$$

Note that as Ω is starlike w.r.t. Σ_1, one has $\theta_x \in [0, \frac{\pi}{2}]$ and then $m \geq 0$.

- To have a positive lower bound, we will assume that $m > 0$ (that is, Ω is *strictly* starlike w.r.t. Σ_1).

We also define

$$\begin{cases} \beta = \min\{r(x) : x \in \Sigma_1\} \\ B = \max\{r(x) : x \in \Sigma_1\} \end{cases} \tag{5}$$

Note that β and B are, respectively, the minimum and maximum thickness of the annulus; obviously B has nothing to do with the magnetic field (which in our case is zero because the magnetic potential is closed).
We then have the following result.

Theorem 2. *Let Ω be an annulus in $\mathbf{R}^2$, which is strictly-starlike with respect to its inner (convex) boundary component Σ_1. Assume that A is a closed potential having flux Φ^A around Σ_1. Then:*

$$\lambda_1(\Omega, A) \geq \frac{4\pi^2}{L^2} \frac{\beta m}{B} d(\Phi^A, \mathbf{Z})^2$$

where β and B are as in (18)*, and L is the length of the outer boundary component. If Σ_2 is also convex, then $m \geq \beta/B$ and the lower bound takes the form:*

$$\lambda_1(\Omega, A) \geq \frac{4\pi^2}{L^2} \frac{\beta^2}{B^2} d(\Phi^A, \mathbf{Z})^2.$$

In section 4, we will explain why we need to control $\frac{\beta}{B}$, L, and why we need to impose the starlike condition. If $\beta = B$ and Σ_2 is the circle of length L we get the estimate

$$\lambda_1(\Omega, A) \geq \frac{4\pi^2}{L^2} d(\Phi^A, \mathbf{Z})^2$$

which is the first eigenvalue of the magnetic Laplacian on the circle with potential A (see section 5.1). If Σ_2 and Σ_1 are two concentric circles of respective lengths L and $L_\epsilon \to L$, the domain is a thin annulus with $\lambda_1 \to \frac{4\pi^2}{L^2} d(\Phi^A, \mathbf{Z})^2$ which shows that our estimate is sharp.

Our aim is to use these estimates on cylinders as a basis stone in order to study the same type of questions on compact surfaces of higher genus.

2 Proof of the main theorem

2.1 Preliminary facts and notation

First, we recall the variational definition of the spectrum. Let Ω be a compact manifold with boundary and Δ_A the magnetic Laplacian with Neumann boundary conditions. One verifies that

$$\int_\Omega (\Delta_A u)\bar{u} = \int_\Omega |\nabla^A u|^2,$$

and the associated quadratic form is then

$$Q_A(u) = \int_\Omega |\nabla^A u|^2.$$

The usual variational characterization gives:

$$\lambda_1(\Omega, A) = \min\left\{\frac{Q_A(u)}{\|u\|^2} : \ u \in C^1(\Omega, \mathbb{C})/\{0\}\right\} \tag{6}$$

The following proposition (which is well-known) expresses the *gauge invariance* of the spectrum of the magnetic Laplacian.

Proposition 3. a) *The spectrum of Δ_A is equal to the spectrum of $\Delta_{A+d\phi}$ for all smooth real valued functions ϕ; in particular, when A is exact, the spectrum of Δ_A reduces to that of the classical Laplace-Beltrami operator acting on functions (with Neumann boundary conditions if $\partial\Omega$ is not empty).*
b) *If A is a closed 1-form, then A is gauge equivalent to a unique (harmonic) 1-form $\tilde{A}$ satisfying*

$$\begin{cases} d\tilde{A} = \delta\tilde{A} = 0 \quad \textit{on} \quad \Omega \\ \tilde{A}(N) = 0 \quad \textit{on} \quad \partial\Omega \end{cases}$$

The form $\tilde{A}$ is often called the Coulomb gauge *of A. Note that $\tilde{A}$ is the harmonic representative of A for the absolute boundary conditions.*

Proof. a) This comes from the fact that $\Delta_A e^{-i\phi} = e^{-i\phi}\Delta_{A+d\phi}$ hence Δ_A and $\Delta_{A+d\phi}$ are unitarily equivalent.
b) Consider a solution ϕ of the problem:

$$\begin{cases} \Delta\phi = \delta A \quad \text{on} \quad \Omega, \\ \dfrac{\partial\phi}{\partial N} = A(N) \quad \text{on} \quad \partial\Omega. \end{cases}$$

Then one checks that $\tilde{A} = A - d\phi$ is a Coulomb gauge of A. As ϕ is unique up to an additive constant, $d\phi$, hence $\tilde{A}$, is unique. □

We now focus on the first eigenvalue. Clearly, if $A = 0$, then $\lambda_1(\Omega, A) = 0$ simply because Δ_A reduces to the usual Laplacian, which has first eigenvalue equal to zero and first eigenspace spanned by the constant functions. If A is exact, then Δ_A is unitarily equivalent to Δ, hence, again, $\lambda_1(\Omega, A) = 0$. In fact one checks easily from the definition of the connection that, if $A = d\phi$ for some real-valued function ϕ then $\nabla^A e^{i\phi} = 0$, which means that $u = e^{i\phi}$ is ∇^A-parallel hence Δ_A-harmonic. On the other hand, if the magnetic field $B = dA$ is non-zero then $\lambda_1(\Omega, A) > 0$.

It then remains to examine the case when A is closed but not exact. The situation was clarified in [14] for closed manifolds and in [10] for Neumann boundary conditions.

Theorem 4. *The following statements are equivalent:*

a) $\lambda_1(\Omega, A) = 0$*;*

b) $dA = 0$ *and* $\Phi^A_c \in \mathbf{Z}$ *for any closed curve* c *in* Ω*.*

Thus, the first eigenvalue vanishes if and only if A is a closed form whose flux around every closed curve is an integer; equivalently, if A has non-integral flux around at least one closed loop, then $\lambda_1(\Omega, A) > 0$.

2.2 Proof of the lower bound

From now on we assume that Ω is a Riemannian cylinder. Fix a first eigenfunction u associated to $\lambda_1(\Omega, A)$ and fix a level curve

$$\Sigma_r = \{\psi = r\}, \quad \text{where } r \in [0, 1].$$

As ψ has no critical points, Σ_r is isometric to $\mathbf{S}^1(\frac{L_r}{2\pi})$, where L_r is the length of Σ_r. The restriction of A to Σ_r is a closed 1-form denoted by $\tilde{A}$; we use the restriction of u to Σ_r as a test-function for the first eigenvalue $\lambda_1(\Sigma_r, \tilde{A})$ and obtain:

$$\lambda_1(\Sigma_r, \tilde{A}) \int_{\Sigma_r} |u|^2 \leq \int_{\Sigma_r} |\nabla^{\tilde{A}} u|^2. \tag{7}$$

By the estimate on the eigenvalues of a circle done in Section 2.3.3 below we see :

$$\lambda_1(\Sigma_r, \tilde{A}) = \frac{4\pi^2}{L_r^2} d(\Phi^{\tilde{A}}, \mathbf{Z})^2,$$

where $\Phi^{\tilde{A}}$ is the flux of $\tilde{A}$ across Σ_r. Now note that $\Phi^{\tilde{A}} = \Phi^A$, because $\tilde{A}$ is the restriction of A to Σ_r; moreover $L_r \leq L$ by the definition of L. Therefore:

$$\lambda_1(\Sigma_r, \tilde{A}) \geq \frac{4\pi^2}{L^2} d(\Phi^A, \mathbf{Z})^2 \tag{8}$$

for all r. Let X be a unit vector tangent to Σ_r. Then:

$$\begin{aligned}\nabla^{\tilde A}_X u &= \nabla_X u - i\tilde A(X)u\\ &= \nabla_X u - iA(X)u\\ &= \nabla^A_X u.\end{aligned}$$

The consequence is that:

$$|\nabla^{\tilde A}u|^2 = |\nabla^{\tilde A}_X u|^2 = |\nabla^A_X u|^2 \le |\nabla^A u|^2. \tag{9}$$

• *Note that equality holds in* (9) *iff* $\nabla^A_N u = 0$ *where* N *is a unit vector normal to the level curve* Σ_r *(we could take* $N = \nabla\psi/|\nabla\psi|$*).*

For any fixed level curve $\Sigma_r = \{\psi = r\}$ we then have, taking into account (7), (8) and (9):

$$\frac{4\pi^2}{L^2} d(\Phi^A, \mathbf{Z})^2 \int_{\psi=r} |u|^2 \le \int_{\psi=r} |\nabla^A u|^2. \tag{10}$$

Assume that $B_1 \le |\nabla\psi| \le B_2$ for positive constants B_1, B_2. Then the above inequality implies:

$$\frac{4\pi^2}{L^2} d(\Phi^A, \mathbf{Z})^2 \cdot B_1 \int_{\psi=r} \frac{|u|^2}{|\nabla\psi|} \le B_2 \int_{\psi=r} \frac{|\nabla^A u|^2}{|\nabla\psi|}. \tag{11}$$

• *Note that if equality holds in* (10) *and* (11) *then necessarily* $B_1 = B_2$ *and then* $\nabla\psi$ *must be constant.*

We now integrate both sides from $r = 0$ to $r = 1$ and use the coarea formula. Conclude that

$$\frac{4\pi^2}{L^2} d(\Phi^A, \mathbf{Z})^2 \cdot B_1 \int_\Omega |u|^2 \le B_2 \int_\Omega |\nabla^A u|^2.$$

As u is a first eigenfunction, one has:

$$\int_\Omega |\nabla^A u|^2 = \lambda_1(\Omega, A) \int_\Omega |u|^2.$$

Recalling that $K = \frac{B_2}{B_1}$ we finally obtain the estimate (4).

2.3 Proof of the equality case

If the cylinder Ω is a Riemannian product then it is obvious that we can take $K = 1$ and then we have equality by Proposition 8 below. Now assume that we do have equality: we have to show that Ω is a Riemannian product. Going back to the proof, we must have the following facts.

F1. *All level curves of ψ have the same length L.*

F2. *By the remark after (11), $|\nabla\psi|$ must be constant and, by renormalization, we can assume that it is everywhere equal to* 1. Then, $\psi : \Omega \to [0, a]$ for some $a > 0$ and we set

$$N \doteq \nabla\psi.$$

F3. *The eigenfunction u on Ω restricts to an eigenfunction of the magnetic Laplacian of each level set $\Sigma_r = \{\psi = r\}$, with potential given by the restriction of A to Σ_r.*

F4. *One has $\nabla^A_N u = 0$ identically on Ω.*

2.3.1 First step: description of the metric

Lemma 5. *Ω is isometric to the product $[0, a] \times \mathbf{S}^1(\frac{L}{2\pi})$ with metric*

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \theta^2(r,t) \end{pmatrix}, \quad (r,t) \in [0,a] \times [0,L] \tag{12}$$

where $\theta(r,t)$ is positive and periodic of period L in the variable t. Moreover $\theta(0,t) = 1$ for all t.

Proof. We first show that the integral curves of N are geodesics; for this it is enough to show that $\nabla_N N = 0$ on Ω. Let $e_1(x)$ be a vector tangent to the level curve of ψ passing through x. Then, we obtain a smooth vector field e_1 which, together with N, forms a global orthonormal frame. Now

$$\langle \nabla_N N, N \rangle = \frac{1}{2} N \langle N, N \rangle = 0.$$

On the other hand, as the Hessian is a symmetric tensor:

$$\langle \nabla_N N, e_1 \rangle = \nabla^2\psi(N, e_1) = \nabla^2\psi(e_1, N) = \langle \nabla_{e_1} N, N \rangle = \frac{1}{2} e_1 \langle N, N \rangle = 0.$$

Hence $\nabla_N N = 0$ as asserted. As each integral curve of $N = \nabla\psi$ is a geodesic meeting Σ_1 orthogonally, we see that ψ is actually the distance function to Σ_1. We introduce coordinates on Ω as follows. For a fixed point $p \in \Omega$ consider the unique integral curve γ of N passing through p and let $x \in \Sigma_1$ be the intersection of γ with Σ_1 (note that x is the foot of the unique geodesic which minimizes the distance from p to Σ_1). Let r be the distance of p to Σ_1. We then have a map $\Omega \to [0, a] \times \Sigma_1$ which sends p to (r, x). Its inverse is the map $F : [0, a] \times \Sigma_1 \to \Omega$ defined by

$$F(r, x) = \exp_x(rN).$$

Note that F is a diffeomeorphism; we call the pair (r, x) the *normal coordinates* based on Σ_1. We introduce the arc-length t on Σ_1 (with origin in any assigned point of Σ_1) and recall that L is length of Σ_1 (which is also the length of Σ_2) by **F1**). Let us compute the metric g in normal coordinates. Since $N = \frac{\partial}{\partial r}$ one sees that $g_{11} = 1$ everywhere; for any fixed $r = r_0$ we have that $F(r_0, \cdot)$ maps Σ_1 diffeomorphically onto the level set $\{\psi = r_0\}$ so that $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial t}$ will be mapped onto orthogonal vectors, and indeed $g_{12} = 0$. Setting $\theta(r,t)^2 = \langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \rangle$ one sees that the metric takes the form (12). Finally note that $\theta(0, t) = 1$ for all t, because $F(0, \cdot)$ is the identity. □

2.3.2 Second step : Gauge invariance

Lemma 6. *Let Ω be any Riemannian cylinder and $A = f(r,t)\,dr + h(r,t)\,dt$ a closed 1-form on Ω. Then, there exists a smooth function ϕ on Ω such that*

$$A + d\phi = H(t)\,dt$$

for a smooth function $H(t)$ depending only on t. Hence, by gauge invariance, we can assume from the start that $A = H(t)\,dt$.

Proof. Consider the function $\phi(r,t) = -\int_0^r f(x,t)\,dx$. Then:

$$A + d\phi = \tilde{h}(r,t)\,dt$$

for some smooth function $\tilde{h}(r,t)$. As A is closed, also $A + d\phi$ is closed, which implies that $\frac{\partial \tilde{h}}{\partial r} = 0$, that is, $\tilde{h}(t,r)$ does not depend on r; if we set $H(t) \doteq \tilde{h}(t, 0)$ we get the assertion. □

- We point out the following consequence. If $u = u(r,t)$ is an eigenfunction, we know from **F4** above that $\nabla_N^A u = 0$, where $N = \frac{\partial}{\partial r}$. As $\nabla_N^A u = \frac{\partial u}{\partial r} - iA(\frac{\partial}{\partial r})u$ and $A = H(t)\,dt$ we obtain $A(\frac{\partial}{\partial r}) = 0$ hence $\frac{\partial u}{\partial r} = 0$ at all points of Ω. This implies that

$$u = u(t) \tag{13}$$

depends only on t.

2.3.3 Third step : spectrum of circles and Riemannian products

In this section, we give an expression for the eigenfunctions of the magnetic Laplacian on a circle with a Riemannian metric g and a closed potential A. Of course, we know that any metric g on a circle is always isometric to the canonical metric $g_{\rm can} = dt^2$, where t is arc-length. But our problem in this proof is to reconstruct the global metric of the cylinder and to show that it is a product, and we cannot suppose a priori that the restricted metric of each level set of ψ is the canonical metric. The same is true for the restricted potential: we know that it is Gauge equivalent to a potential of the type $a\,dt$ for a scalar a, but we cannot suppose a priori that it is of that form.
We refer to Appendix 5.1 for the complete proof of the following fact.

Proposition 7. *Let (M, g) be the circle of length L endowed with the metric $g = \theta(t)^2\,dt^2$ where $t \in [0, L]$ and $\theta(t)$ is a positive function, periodic of period L. Let $A = H(t)\,dt$. Then, the eigenvalues of the magnetic Laplacian with potential A are:*

$$\lambda_k(M, A) = \frac{4\pi^2}{L^2}(k - \Phi^A)^2, \quad k \in \mathbf{Z}$$

with associated eigenfunctions

$$u_k(t) = e^{i\phi(t)} e^{\frac{2\pi i(k-\Phi^A)}{L} s(t)}, \quad k \in \mathbf{Z}.$$

where $\phi(t) = \int_0^t H(\tau)\,d\tau$ and $s(t) = \int_0^t \theta(\tau)\,d\tau$.
In particular, if the metric is the canonical one, that is, $g = dt^2$, and the potential 1-form is harmonic, so that $A = \frac{2\pi\Phi^A}{L}dt$, then the eigenfunctions are simply :

$$u_k(t) = e^{\frac{2\pi i k}{L}t}, \quad k \in \mathbf{Z}.$$

We remark that if the flux Φ^A is not congruent to $1/2$ modulo integers, then the eigenvalues are all simple. If the flux is congruent to $1/2$ modulo integers, then there are two consecutive integers $k, k+1$ such that $\lambda_k = \lambda_{k+1}$. Consequently, the lowest eigenvalue has multiplicity two, and the first eigenspace is spanned by

$$e^{i\phi(t)} e^{\frac{\pi i}{L} s(t)}, \; e^{i\phi(t)} e^{-\frac{\pi i}{L} s(t)}.$$

The following proposition is an easy consequence (for a proof, see also Appendix 5.1).

Proposition 8. *Consider the Riemannian product $\Omega = [0, a] \times \mathbf{S}^1(\frac{L}{2\pi})$, and let A be a closed $1-$form on Ω. Then, the spectrum of Δ_A is given by*

$$\frac{\pi^2 h^2}{a^2} + \frac{4\pi^2}{L^2}(k - \Phi^A)^2, \quad h, k \in \mathbf{Z}, h \geq 0.$$

In particular,

$$\lambda_1(\Omega, A) = \frac{4\pi^2}{L^2} d(\Phi^A, \mathbf{Z})^2.$$

2.3.4 Fourth step : a calculus lemma

In this section, we state a technical lemma which will allow us to conclude. The proof is conceptually simple, but perhaps tricky at some points; then, we decided to put it in Appendix 5.2.

Lemma 9. *Let $s : [0, a] \times [0, L] \to \mathbf{R}$ be a smooth, non-negative function such that*

$$s(0, t) = t, \quad s(r, 0) = 0, \quad s(r, L) = L \quad \text{and} \quad \frac{\partial s}{\partial t}(r, t) \doteq \theta(r, t) > 0.$$

Assume that there exist smooth functions $p(r), q(r)$ with $p(r)^2 + q(r)^2 > 0$ such that

$$p(r)\cos(\frac{\pi}{L}s(r, t)) + q(r)\sin(\frac{\pi}{L}s(r, t)) = F(t)$$

where $F(t)$ depends only on t. Then p and q are constant and $\frac{\partial s}{\partial r} = 0$ so that

$$s(r, t) = t$$

for all (r, t).

2.3.5 End of proof of the equality case

Assume that equality holds. Then, if u is an eigenfunction, we know that $u = u(t)$ by the discussion in (13) and u restricts to an eigenfunction on each level circle Σ_r for the potential $A = H(t)\,dt$ above (see Fact 3 at the beginning of Section 2.3 and the second step above).

We assume that Φ^A is congruent to $\frac{1}{2}$ modulo integers. This is the most difficult case; in the other cases the proof is a particular case of this, it is simpler and we omit it.

Recall that each level set Σ_r is a circle of length L for all r, with metric $g = \theta(r, t)^2\,dt$. As the flux of A is congruent to $\frac{1}{2}$ modulo integers, we see that there exist complex-valued functions $w_1(r), w_2(r)$ such that

$$u(t) = e^{i\phi(t)}\Big(w_1(r)e^{\frac{\pi i}{L}s(r,t)} + w_2(r)e^{-\frac{\pi i}{L}s(r,t)}\Big),$$

which, setting $f(t) = e^{-i\phi(t)}u(t)$, we can re-write

$$f(t) = w_1(r)e^{\frac{\pi i}{L}s(r,t)} + w_2(r)e^{-\frac{\pi i}{L}s(r,t)}. \tag{14}$$

Recall that here $\phi(t) = \int_0^t H(\tau)\,d\tau$ and

$$s(r, t) = \int_0^t \theta(r, \tau)\,d\tau.$$

We take the real part on both sides of (14) and obtain smooth real-valued functions $F(t), p(r), q(r)$ such that

$$F(t) = p(r)\cos(\frac{\pi}{L}s(r,t)) + q(r)\sin(\frac{\pi}{L}s(r,t)).$$

Since $\theta(0,t) = 1$ for all t, we see

$$s(0,t) = t.$$

Clearly $s(r,0) = 0$; finally, $s(r,L) = \int_0^L \theta(r,\tau)\, d\tau = L$, being the length of the level circle Σ_r. Thus, we can apply Lemma 9 and conclude that $s(r,t) = t$ for all t, that is,

$$\theta(r,t) = 1$$

for all (r,t) and the metric is a Riemannian product.
It might happen that $p(r) = q(r) \equiv 0$. But then the real part of $f(t)$ is zero and we can work in an analogous way with the imaginary part of $f(t)$, which cannot vanish unless $u \equiv 0$.

2.4 General estimate of $K_{\Omega,\psi}$

We can estimate $K_{\Omega,\psi}$ for a Riemannian cylinder $\Omega = [0,a] \times \mathbf{S}^1$ if we know the explicit expression of the metric in the normal coordinates (r,t), where $t \in [0, 2\pi]$ is arc-length :

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}.$$

If g^{ij} is the inverse matrix of g_{ij}, and if $\psi = \psi(r,t)$ one has:

$$|\nabla\psi|^2 = g^{11}\Big(\frac{\partial\psi}{\partial r}\Big)^2 + 2g^{12}\frac{\partial\psi}{\partial r}\frac{\partial\psi}{\partial t} + g^{22}\Big(\frac{\partial\psi}{\partial t}\Big)^2.$$

The function $\psi(r,t) = r$ belongs to $\mathcal{F}_\Omega$ and one has: $|\nabla\psi|^2 = g^{11}$, which immediately implies that we can take

$$K_{\Omega,\psi} \leq \frac{\sup_\Omega g^{11}}{\inf_\Omega g^{11}}.$$

Note in particular that if Ω is rotationally invariant, so that the metric can be put in the form:

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \alpha(r)^2 \end{pmatrix},$$

for some function $\alpha(r)$, then $K_{\Omega,\psi} = 1$. The estimate becomes

$$\lambda_1(\Omega, A) \geq \frac{4\pi^2}{L^2} \cdot d(\Phi^A, \mathbf{Z})^2, \tag{15}$$

where L is the maximum length of a level curve $r = \text{const}$.

Example 10. Yet more generally, one can fix a smooth closed curve γ on a Riemannian surface M and consider the tube of radius R around γ:

$$\Omega = \{x \in M : d(x, \gamma) \leq R\}.$$

It is well-known that if R is sufficiently small (less than the injectivity radius of the normal exponential map) then Ω is a cylinder with smooth boundary which can be foliated by the level sets of ψ, the distance function to γ. Clearly $|\nabla\psi| = 1$ and (15) holds as well.

A concrete example where we could estimate the width R is the case of a compact surface M of genus ≥ 2 and curvature $-a^2 \leq K \leq -b^2$, $a \geq b > 0$. Let γ be a simple closed geodesic. Then, using the Gauss-Bonnet theorem, one can show that R is bounded below by an explicit positive constant $R = R(\gamma, a)$, hence the R-neighborhood of γ is diffeomorphic to the product $S^1 \times (-1, 1)$ (see for example [5]). If we take Ω as the Riemannian cylinder of width $R(\gamma, a)$ having one boundary component equal to γ then we can foliate Ω with the level sets of the distance function to γ and so $K = 1$ and (15) holds, with L given by the length of the other boundary component.

3 Proof of Theorem 2: plane annuli

Let Ω be an annulus in $\mathbf{R}^2$, which is starlike with respect to its inner convex boundary component Σ_1. Assume that A is a closed potential having flux Φ^A around Σ_1. Recall that we have to show:

$$\lambda_1(\Omega, A) \geq \frac{4\pi^2}{L^2}\frac{\beta m}{B} d(\Phi^A, \mathbf{Z})^2 \tag{16}$$

where β, B and m will be recalled below and L is the length of the outer boundary component. If we assume that Σ_2 is also convex, then we show that $m \geq \beta/B$ and the lower bound takes the form:

$$\lambda_1(\Omega, A) \geq \frac{4\pi^2}{L^2}\frac{\beta^2}{B^2} d(\Phi^A, \mathbf{Z})^2. \tag{17}$$

Before giving the proof let us recall notation. For $x \in \Sigma_1$, the ray γ_x is the geodesic segment $\gamma_x(t) = x + tN_x$, where N_x is the exterior normal to Σ_1 at x and $t \geq 0$. The ray γ_x meets Σ_2 at a first point $Q(x)$, and we let $r(x) = d(x, Q(x))$. For $x \in \Sigma_1$, we denote by θ_x the angle between the ray γ_x' and the outer normal to Σ_2 at the point $Q(x)$, and we let

$$m \doteq \min_{x \in \Sigma_1} \cos\theta_x.$$

We assume that Ω is strictly starlike, that is, $m > 0$; in particular $Q(x)$ is unique. Recall also that:

$$\beta = \min_{x \in \Sigma_1} r(x), \quad B = \max_{x \in \Sigma_1} r(x). \tag{18}$$

We construct a suitable smooth function ψ and estimate the constant $K = K_{\Omega,\psi}$ with respect to the geometry of Ω. The starlike assumption implies that each point in Ω belongs to a unique ray γ_x. Then we can define a function $\psi : \Omega \to [0, 1]$ as follows:

$$\psi = \begin{cases} 0 \quad \text{on} \quad \Sigma_1 \\ 1 \quad \text{on} \quad \Sigma_2 \\ \text{linear on each ray from } \Sigma_1 \text{ to } \Sigma_2. \end{cases}$$

Estimates (16) and (17) now follow from Theorem 1 together with the following Proposition.

Proposition 11. a) *At all points of Ω one has: $\frac{1}{B} \leq |\nabla\psi| \leq \frac{1}{\beta m}$. Therefore:*

$$K_{\Omega,\psi} = \frac{\sup_\Omega|\nabla\psi|}{\inf_\Omega|\nabla\psi|} \leq \frac{B}{\beta m}.$$

b) *One has*

$$\sup_{r\in[0,1]} |\psi^{-1}(r)| = L = |\Sigma_2|.$$

c) *If Σ_2 is also convex, then $m \geq \beta/B$ hence we can take $K = \beta^2/B^2$.*

The proof of the Proposition 11 depends on the following steps.

Step 1. *On the ray γ_x joining x to $Q(x)$, consider the point $Q_t(x)$ at distance t from x, and let $\theta_x(t)$ be the angle between γ_x' and $\nabla\psi(Q_t(x))$. Then the function*

$$h(t) = \cos(\theta_x(t))$$

is non-increasing in t. As $\theta_x(r(x)) = \theta_x$ we have in particular:

$$\cos(\theta_x(t)) \geq \cos(\theta_x) \geq m$$

for all $t \in [0, r(x)]$ and $x \in \Sigma_1$.

Step 2. *The function $r \to |\psi^{-1}(r)|$ is non-decreasing in r.*

Step 3. *If Σ_2 is also convex we have $m \geq \beta/B$.*

We will prove Steps 1-3 below.

Proof of Proposition 11. a) At any point of Ω, let $\nabla^R\psi$ denote the radial part of $\nabla\psi$, which is the gradient of the restriction of ψ to the ray passing through the given point. As such restriction is a linear function, one sees that

$$\frac{1}{B} \leq |\nabla^R\psi| \leq \frac{1}{\beta}.$$

Since $|\nabla\psi| \geq |\nabla^R\psi|$ one gets immediately

$$|\nabla\psi| \geq \frac{1}{B}.$$

Note that $\theta_x(t)$, as defined above, is precisely the angle between $\nabla\psi$ and $\nabla^R\psi$, so that, using Step 1,

$$|\nabla^R\psi| = |\nabla\psi| \cos\theta_x(t) \geq m|\nabla\psi|$$

hence:

$$|\nabla\psi| \leq \frac{1}{m}|\nabla^R\psi| \leq \frac{1}{\beta m}.$$

as asserted. It is clear that b) and c) are immediate consequences of Steps 2-3.

Proof of Step 1. We use a suitable parametrization of Ω. Let l be the length of Σ_1 and consider a parametrization $\gamma : [0,l] \to \Sigma_1$ by arc-length s with origin at a given point in Σ_1. Let $N(s)$ be the outer normal vector to Σ_1 at the point $\gamma(s)$. Consider the set:

$$\tilde{\Omega} = \{(t,s) \in [0,\infty) \times [0,l) : t \leq \rho(s)\}$$

where we have set $\rho(s) = r(\gamma(s))$. The starlike property implies that the map $\Phi : \tilde{\Omega} \to \Omega$ defined by

$$\Phi(t,s) = \gamma(s) + tN(s)$$

is a diffeomorphism. Let us compute the Euclidean metric tensor in the coordinates (t,s). Write $\gamma'(s) = T(s)$ for the unit tangent vector to γ and observe that $N'(s) = k(s)T(s)$, where $k(s)$ is the curvature of Σ_1 which is everywhere non-negative because Σ_1 is convex. Then:

$$\begin{cases} d\Phi(\dfrac{\partial}{\partial t}) = N(s) \\ d\Phi(\dfrac{\partial}{\partial s}) = (1 + tk(s))T(s) \end{cases}$$

If we set $\Theta(t,s) = 1 + tk(s)$ the metric tensor is:

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \Theta^2 \end{pmatrix}$$

and an orthonormal basis is then (e_1, e_2), where

$$e_1 = \frac{\partial}{\partial t}, \quad e_2 = \frac{1}{\Theta}\frac{\partial}{\partial s}.$$

In these coordinates, our function ψ is written:

$$\psi(t,s)=\frac{t}{\rho(s)}.$$

Now

$$\begin{cases}\langle\nabla\psi,e_1\rangle=\dfrac{\partial\psi}{\partial t}=\dfrac{1}{\rho(s)}\\ \langle\nabla\psi,e_2\rangle=\dfrac{1}{\Theta}\dfrac{\partial\psi}{\partial s}=-\dfrac{t\rho'(s)}{\Theta(t,s)\rho(s)^2}\end{cases}.$$

It follows that

$$|\nabla\psi|^2=\frac{1}{\rho^2}+\frac{t^2\rho'^2}{\Theta^2\rho^4}=\frac{\Theta^2\rho^2+t^2\rho'^2}{\Theta^2\rho^4}.$$

Recall the radial gradient, which is the orthogonal projection of $\nabla\psi$ on the ray, whose direction is given by e_1. If we fix $x\in\Sigma_1$, we have

$$\theta_x(t)=\text{angle between }\nabla\psi\text{ and }e_1$$

and we have to study the function

$$h(t)=\cos\theta_x(t)=\frac{\langle\nabla\psi,e_1\rangle}{|\nabla\psi|}=\frac{1}{\rho(s)|\nabla\psi|}$$

for a fixed s. From the above expression of $|\nabla\psi|$ and a suitable manipulation we see

$$h(t)^2=\frac{\Theta^2}{\Theta^2+t^2g^2}$$

where $g=\rho'(s)/\rho(s)$. Now

$$\frac{d}{dt}\frac{\Theta^2}{\Theta^2+t^2g^2}=\frac{2t\Theta g^2}{(\Theta^2+t^2g^2)^2}\left(t\frac{\partial\Theta}{\partial t}-\Theta\right)$$

As $\Theta(t,s)=1+tk(s)$ one sees that $t\dfrac{\partial\Theta}{\partial t}-\Theta=-1$ hence

$$\frac{d}{dt}h(t)^2=-\frac{2t\Theta g^2}{(\Theta^2+t^2g^2)^2}\leq 0$$

Hence $h(t)^2$ is non-increasing and, as $h(t)$ is positive, it is itself non-increasing. □

Proof of Step 2. In the coordinates (t,s) the curve $\psi^{-1}(r)$ is parametrized by $\alpha:[0,l]\to\tilde{\Omega}$ as follows:

$$\alpha(u)=(r\rho(u),u)\quad u\in[0,l].$$

Then:

$$|\psi^{-1}(r)| = \int_0^l \sqrt{g(\alpha'(u), \alpha'(u))}\, du$$
$$= \int_0^l \sqrt{r^2\rho'(u)^2 + (1 + rk(u)\rho(u))^2}\, du$$

Convexity of Σ_1 implies that $k(u) \geq 0$ for all u; differentiating under the integral sign with respect to r one sees that indeed $\frac{d}{dr}|\psi^{-1}(r)| \geq 0$ for all $r \in [0, 1]$.

Proof of Step 3. Let T_x be the tangent line to Σ_2 at $Q(x)$ and $H(x)$ the point of T_x closest to x. As Σ_2 is convex, $H(x)$ is not an interior point of Ω, hence

$$d(x, H(x)) \geq \beta.$$

The triangle formed by $x, Q(x)$ and $H(x)$ is rectangle in $H(x)$, then we have:

$$r(x)\cos\theta_x = d(x, H(x)).$$

As $r(x) \leq B$ we conclude:

$$B\cos\theta_x \geq \beta,$$

which gives the assertion.

4 Sharpness of the lower bound

4.1 An upper bound

In this short paragraph, we give a simple way to get an upper bound when the potential A is *closed*. Then, we will use this in different kinds of examples, in order to show that the assumptions of Theorem 2 are sharp. The geometric idea is the following: if we have a region $D \subset \Omega$ such that the first absolute cohomology group $H^1(D)$ is 0, then we can estimate from above the spectrum of Δ_A in Ω in terms of the spectrum of the usual Laplacian on D. The reason is that the potential A is 0 on D up to a gauge transformation; then, on D, Δ_A becomes the usual Laplacian and any eigenfunction of the Laplacian on D may be extended by 0 on Ω and thus used as a test function for the magnetic Laplacian on the whole of Ω.

Let us give the details. Let D be a closed subset of Ω such that, for some (small) $\delta > 0$ one has $H^1(D^\delta, \mathbf{R}) = 0$, where $D^\delta = \{p \in \Omega : \text{dist}(p, D) < \delta\}$. This happens when D^δ has a retraction onto D. We write

$$\partial D = (\partial D \cap \partial\Omega) \cup (\partial D \cap \Omega) = \partial^{\text{ext}} D \cup \partial^{\text{int}} D$$

and we denote by $(\nu_j(D))_{j=1}^{\infty}$ the spectrum of the Laplacian acting on functions, with the Neumann boundary condition on $\partial^{\mathrm{ext}}D$ (if non empty) and the Dirichlet boundary condition on $\partial^{\mathrm{int}}D$.

Proposition 12. *Let Ω be a compact manifold with smooth boundary and A a closed potential on Ω. Assume that $D \subset \Omega$ is a compact subdomain such that $H^1(D, \boldsymbol{R}) = H^1(D^\delta, \boldsymbol{R}) = 0$ for some $\delta > 0$. Then we have*

$$\lambda_k(\Omega, A) \leq \nu_k(D)$$

for each $k \geq 1$.

Proof. We recall that for any function ϕ on Ω, the operator Δ_A and $\Delta_{A+d\phi}$ are unitarily equivalent and have the same spectrum. As A is closed and, by assumption, $H^1(D^\delta, \mathbf{R}) = 0$, A is exact on D^δ and there exists a function $\tilde{\phi}$ on D^δ such that $A + d\tilde{\phi} = 0$ on D^δ. We consider the restriction of $\tilde{\phi}$ to D and extend it differentiably on Ω by using a partition of unity (χ_1, χ_2) subordinated to $(D^\delta, \Omega/D)$. Then, setting

$$\phi \doteq \chi_1 \tilde{\phi}$$

we see that ϕ is a smooth function on Ω which is equal to $\tilde{\phi}$ on D so that, on D, one has $A + d\phi = 0$. We consider the new potential $\tilde{A} = A + d\phi$ and observe that $\tilde{A} = 0$ on D. Now consider an eigenfunction f for the mixed problem on D (Neumann boundary conditions on $\partial^{\mathrm{ext}}D$ and Dirichlet boundary conditions on $\partial^{\mathrm{int}}D$), and extend it by 0 on $\Omega \setminus D$. As $\tilde{A} = 0$ on D, we see that

$$|\nabla^{\tilde{A}} f|^2 = |\nabla f|^2,$$

and we get a test function having the same Rayleigh quotient as that of f. Thanks to the usual min-max characterization of the spectrum, we obtain, for all k:

$$\lambda_k(\Omega, A) = \lambda_k(\Omega, \tilde{A}) \leq \nu_k(D).$$

4.2 Sharpness

We will use Proposition 12 to show the sharpness of the hypothesis in Theorem 2. Let us first show that we need to control the ratio $\frac{BL}{\beta}$.

Example 13. In the first situation, we give an example where the ratio $\frac{BL}{\beta} \to \infty$ and the distance β between the two components of the boundary is uniformly bounded from below. We want to show that $\lambda_1 \to 0$. We consider an annulus Ω composed of two concentric balls of radius 1 and $R+1$ and same center, with $R \to \infty$. We have $B = \beta = R$ and $L \to \infty$.

From the assumptions we get the existence of a point $x \in \Omega$ such that the ball $B(x, \frac{R}{2})$ of center x and radius $\frac{R}{2}$ is contained in Ω. Proposition 12 implies that $\lambda_1(\Omega, A)$ is bounded from above by the first eigenvalue of the Dirichlet problem for the Laplacian of the ball, which is proportional to $\frac{1}{R^2}$ and tends to zero because $R \to \infty$.

Example 14. Next, we construct an example to show that if the distance β tends to 0 and B and L are uniformly bounded from below and from above, then again $\lambda_1 \to 0$. We again use Proposition 12. Fix the rectangles :

$$R_2 = [-4, 4] \times [0, 4], \quad R_{1,\epsilon} = [-3, 3] \times [\epsilon, 2]$$

and consider the region Ω_ϵ given by the closure of $R_2 \setminus R_{1,\epsilon}$. Note that Ω_ϵ is a planar annulus whose boundary components are convex and get closer and closer as $\epsilon \to 0$.

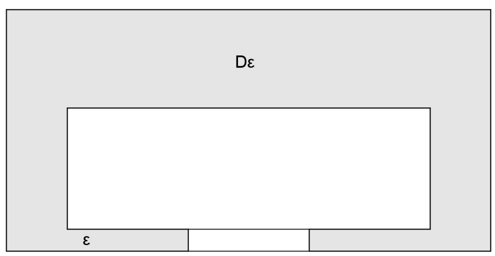

Figure 1: $\lambda_1 \to 0$ as $\epsilon \to 0$

We show that, for any closed potential A one has:

$$\lim_{\epsilon \to 0} \lambda_1(\Omega_\epsilon, A) = 0. \tag{19}$$

Consider the simply connected region $D_\epsilon \subset \Omega_\epsilon$ given by the complement of the rectangle $[-1, 1] \times [0, \epsilon]$. Now D_ϵ has trivial 1-cohomology; by Proposition 12, to show (19) it is enough to show that

$$\lim_{\epsilon \to 0} \nu_1(D_\epsilon) = 0. \tag{20}$$

By the min-max principle :

$$\nu_1(D_\epsilon) = \inf \left\{ \frac{\int_{D_\epsilon} |\nabla f|^2}{\int_{D_\epsilon} f^2} : f = 0 \text{ on } \partial D_\epsilon^{\text{int}} \right\}$$

where

$$\partial D_\epsilon^{\text{int}} = \{(x, y) \in \Omega_\epsilon : x = \pm 1, y \in [0, \epsilon]\}.$$

Define the test-function $f : D_\epsilon \to \mathbf{R}$ as follows.

$$f = \begin{cases} 1 \quad \text{on the complement of } [-2,2]\times[0,\epsilon] \\ x-1 \quad \text{on } [1,2]\times[0,\epsilon] \\ -x-1 \quad \text{on } [-2,-1]\times[0,\epsilon] \end{cases}$$

One checks easily that, for all ϵ:

$$\int_{D_\epsilon} |\nabla f|^2 = 2\epsilon, \quad \int_{D_\epsilon} f^2 \geq \text{const} > 0$$

Then (20) follows immediately by observing that the Rayleigh quotient of f tends to 0 as $\epsilon \to 0$

Example 15. In the example we constructed previously the two boundary components approach each other along a common set of positive measure (precisely, a segment of total length 6). In the next example we sketch a construction showing that, in fact, this is not necessary.

So, let us fix the outside curve Σ_2 and choose a family of inner convex curves Σ_1 such that B is bounded below (say, $B \geq 1$) and $\beta \to 0$ (no other assumption is made). Then, we want to show that $\lambda_1(\Omega, A) \to 0$.

Fix points $x \in \Sigma_2$, $y \in \Sigma_1$ such that $d(x,y) = \beta$. We take $b = 2\beta$ and introduce the balls of center x and radius b and $\sqrt{b}$, denoted by $B(x,b)$ and $B(x,\sqrt{b})$, respectively. Then the set $D = \Omega \setminus (B(x,b) \cap \Omega)$ is simply connected so that, by Proposition 12:

$$\lambda_1(\Omega, A) \leq \nu_1(D)$$

and it remains to show that $\nu_1(D) \to 0$ as $b \to 0$.

Introduce the function $F(r)$ (r being the distance to x):

$$F(r) = \begin{cases} 1 \quad \text{on the complement of } B(x,\sqrt{b}) \\ 0 \quad \text{on } B(x,b) \\ \dfrac{-2}{\ln b}(\ln r - \ln b) \quad \text{on } B(x,\sqrt{b}) - B(x,b) \end{cases}$$

and let f be the restriction of F to D. As $f = 0$ on $\partial^{\text{int}} D = \partial B(x,b) \cap \Omega$, we see that f is a test function for the eigenvalue $\nu_1(D)$. A straightforward calculation shows that, as $b \to 0$, we have

$$\int_D |\nabla f|^2 \to 0;$$

on the other hand, as $B \geq 1$, the volume of D is uniformly bounded from below, which implies that

$$\int_D f^2 \geq C > 0.$$

We conclude that the Rayleigh quotient of f tends to 0 as $b \to 0$, which shows the assertion.

Example 16. The following example shows that we need to impose some condition on the outer curve in order to get a positive lower bound as in Theorem 2.

It is an easy and classical fact that, in order to create a small eigenvalue for the Neumann problem, it is sufficient to deform a domain locally, near a boundary point, as indicated by the mushroom-shaped region shown in the figure below. Up to a gauge transformation, we can suppose that the potential A is locally 0 in a neigborhood of the mushroom, and we have to estimate the first eigenvalue of the Laplacian with Dirichlet boundary condition at the basis of the mushroom (which is a segment of length ϵ) and Neumann boundary condition on the remaining part of its boundary, as required by Proposition 12.

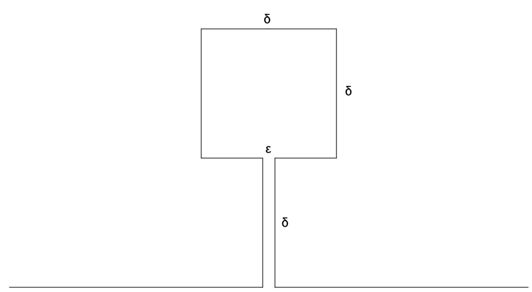

Figure 2: A local deformation implying $\lambda_1 \to 0$

The only point is to take the value of the parameter ϵ much smaller than δ as $\delta \to 0$. Take for example $\epsilon = \delta^4$ and consider a function u taking value 1 in the square of size δ and passing linearly from 1 to 0 outside the rectangle of sizes ϵ, δ. The norm of the gradient of u is 0 on the square of size δ and $\frac{1}{\delta}$ in the rectangle of size δ, ϵ.
Then the Rayleigh quotient is

$$R(u) \leq \frac{\frac{1}{\delta^2}\delta\epsilon}{\delta^2} = \frac{\epsilon}{\delta^3}$$

which tends to 0 as $\delta \to 0$.

Moreover, we can make such local deformation keeping the curvature of the boundary uniformly bounded in absolute value (see Example 2 in [7]).

5 Appendix

5.1 Spectrum of circles and Riemannian products

We first prove Proposition 7.
Let then (M, g) be the circle of length L with metric $g = \theta(t)^2 dt^2$, where $t \in [0, L]$ and $\theta(t)$ is periodic of period L. Given the 1-form $A = H(t)dt$ we first want to find the harmonic 1-form ω which is cohomologous to A; that is, we look for a smooth function ϕ so that $\omega = A + d\phi$ is harmonic. Now a unit tangent vector field to the circle is

$$e_1 = \frac{1}{\theta}\frac{d}{dt}.$$

Write $\omega = G(t)\, dt$. Then

$$\delta\omega = -\frac{1}{\theta}\Big(\frac{G}{\theta}\Big)'.$$

As any 1-form on the circle is closed, we see that ω is harmonic iff $G(t) = c\theta(t)$ for a constant c. We look for ϕ and $c \in \mathbf{R}$ so that

$$\phi' = -H + c\theta.$$

As ϕ must be periodic of period L, we must have $\int_0^L \phi' = 0$. As the volume of M is L, we also have $\int_0^L \theta = L$. This forces

$$c = \frac{1}{L}\int_0^L H(t)\, dt.$$

On the other hand, as the curve $\gamma(t) = t$ parametrizes M with velocity $\frac{d}{dt}$, one sees that the flux of A across M is given by

$$\Phi^A = \frac{1}{2\pi}\int_0^L H(t)\, dt.$$

Therefore $c = \frac{2\pi}{L}\Phi^A$ and a primitive could be

$$\phi(t) = -\int_0^t H + c\int_0^t \theta.$$

Conclusion:

- *The form $A = H(t)dt$ is cohomologous to the harmonic form $\omega = c\theta\, dt$ with $c = \frac{2\pi}{L}\Phi^A$.*

We first compute the eigenvalues. By gauge invariance, we can use the potential ω. In that case

$$\Delta_\omega = -\nabla^\omega_{e_1}\nabla^\omega_{e_1}.$$

Now

$$\nabla^{\omega}_{e_1} u = \frac{u'}{\theta} - icu$$

hence

$$\nabla^{\omega}_{e_1}\nabla^{\omega}_{e_1} u = \frac{1}{\theta}\Big(\frac{u'}{\theta} - icu\Big)' - ic\Big(\frac{u'}{\theta} - icu\Big).$$

After some calculation, the eigenfunction equation $\Delta_\omega u = \lambda u$ takes the form:

$$-u'' + \frac{\theta'}{\theta}u' + 2ic\theta u' + c^2\theta^2 u = \lambda\theta^2 u.$$

Recall the arc-length function $s(t) = \int_0^t \theta(\tau)\,d\tau$. We make the change of variables:

$$u(t) = v(s(t)), \quad \text{that is} \quad v = u \circ s^{-1}.$$

Then:

$$\begin{cases} u' = v'(s)\theta \\ u'' = v''(s)\theta^2 + v'(s)\theta' \end{cases}$$

and the equation becomes:

$$-v'' + 2icv' + c^2 v = \lambda v$$

with solutions :

$$v_k(s) = e^{\frac{2\pi ik}{L}s}, \quad \lambda = \frac{4\pi^2}{L^2}(k - \Phi^A)^2, \quad k \in \mathbf{Z}.$$

Now Gauge invariance says that

$$\Delta_{A+d\phi} = e^{i\phi}\Delta_A e^{-i\phi};$$

and v_k is an eigenfunction of $\Delta_{A+d\phi}$ iff $e^{-i\phi}v_k$ is an eigenfunction of Δ_A. Hence, the eigenfunctions of Δ_A (where $A = H(t)\,dt$) are

$$u_k = e^{-i\phi}v_k,$$

where $\phi(t) = -\int_0^t H + c\,s(t)$ and $c = \frac{2\pi}{L}\Phi^A$. Explicitly:

$$u_k(t) = e^{i\int_0^t H} e^{\frac{2\pi i(k-\Phi^A)s(t)}{L}} \tag{21}$$

as asserted in Proposition 7.

Let us now verify the last statement. If the metric is $g = dt^2$ then $\theta(t) = 1$ and $s(t) = t$. If A is a harmonic 1-form then it has the expression $A = \frac{2\pi\Phi^A}{L}dt$. Taking into account (21) we indeed verify that $u_k(t) = e^{\frac{2\pi ik}{L}t}$.

- We now prove Proposition 8.

Here we assume that Ω is a Riemannian product $[0,a]\times \mathbf{S}^1(\frac{L}{2\pi})$ with coordinates (r,t) and the canonical metric on the circle. We fix a closed potential A on Ω. By gauge invariance we can assume that A is a Coulomb gauge, and by what we said above we have easily

$$A = \frac{2\pi\Phi^A}{L}\,dt.$$

Then A restrict to zero on $[0,a]$; as $A(N)=0$ on $\partial\Omega$ the magnetic Neumann conditions reduce simply to $\dfrac{\partial u}{\partial N}=0$. At this point we apply a standard argument of separation of variables; if $\phi(r)$ is an eigenfunction of the usual Neumann Laplacian on $[0,a]$, and $v(t)$ is an eigenfunction of Δ_A on $\mathbf{S}^1(\frac{L}{2\pi})$, we see that the product $u(r,t)=\phi(r)v(t)$ is indeed an eigenfunction of Δ_A on Ω. As the set of eigenfunctions we obtain that way is a complete orthonormal system in $L^2(\Omega)$, we see that each eigenvalue of the product is the sum of an eigenvalue in the Neumann spectrum of $[0,a]$ and an eigenvalue in the magnetic spectrum of the circle, as computed before. We omit further details.

5.2 Proof of Lemma 9

For simplicity of notation, we give the proof when $a=L=1$. This will not affect generality. Then, assume that $s:[0,1]\times[0,1]\to\mathbf{R}$ is smooth, non-negative and satisfies

$$s(0,t)=t, \quad s(r,0)=0, \quad s(r,1)=1 \quad \text{and} \quad \frac{\partial s}{\partial t}(r,t) \doteq \theta(r,t)>0.$$

Assume the identity

$$F(t)=p(r)\cos(\pi s(r,t))+q(r)\sin(\pi s(r,t)) \tag{22}$$

for real-valued functions $F(t),p(r),q(r)$, such that $p(r)^2+q(r)^2>0$. Then we must show:

$$\frac{\partial s}{\partial r}=0 \tag{23}$$

everywhere.

Differentiate (22) with respect to t and get:

$$F'(t)=-\pi p(r)\theta(r,t)\sin(\pi s)+\pi q(r)\theta(r,t)\cos(\pi s) \tag{24}$$

and we have the following matrix identity

$$\begin{pmatrix}\cos(\pi s) & \sin(\pi s)\\ -\pi\theta\sin(\pi s) & \pi\theta\cos(\pi s)\end{pmatrix}\begin{pmatrix}p\\ q\end{pmatrix}=\begin{pmatrix}F\\ F'\end{pmatrix}.$$

We then see:

$$p(r) = F(t)\cos(\pi s) - \frac{F'(t)}{\pi\theta}\sin(\pi s).$$

Set $t = 0$ so that $s = 0$ and $p(r) = F(0) \doteq p$ is constant; the previous identity becomes

$$p = F(t)\cos(\pi s) - \frac{F'(t)}{\pi\theta}\sin(\pi s). \tag{25}$$

Observe that:

$$\begin{cases} F'(0) = \pi q(r)\theta(r,0) \\ F'(1) = -\pi q(r)\theta(r,1) \end{cases} \tag{26}$$

- Assume $F'(0) = 0$. Then, as $\theta(t,r)$ is positive one must have $q(r) = 0$ for all r, hence $p \neq 0$ and $F(t) = p\cos(\pi s)$, from which, differentiating with respect to r, one gets easily $\dfrac{\partial s}{\partial r} = 0$ and we are finished.
- We now assume that $F'(0) \neq 0$: then we see from (26) that q is not identically zero and the smooth function $F' : [0,1] \to \mathbf{R}$ changes sign. This implies that
- *there exists* $t_0 \in (0,1)$ *such that* $F'(t_0) = 0$.

Now (25) evaluated at $t = t_0$ gives:

$$p = F(t_0)\cos(\pi s(r,t_0))$$

for all r. Differentiate w.r.t. r and get, for all $r \in [0,1]$:

$$0 = \sin(\pi s(r,t_0))\frac{\partial s}{\partial r}(r,t_0).$$

Since $s(r,t)$ is increasing in t, we have

$$0 < s(r,t_0) < s(r,1) = 1.$$

Hence $\sin(\pi s(r,t_0)) > 0$ and we get

$$\frac{\partial s}{\partial r}(r,t_0) = 0.$$

(22) writes:

$$F(t) = p\cos(\pi s) + q(r)\sin(\pi s),$$

and then, differentiating w.r.t. r:

$$0 = -p\pi\sin(\pi s)\frac{\partial s}{\partial r} + q'(r)\sin(\pi s) + \pi q(r)\cos(\pi s)\frac{\partial s}{\partial r}.$$

Evaluating at $t = t_0$ we obtain $0 = q'(r)\sin(\pi s(r, t_0))$ which implies

$$q'(r) = 0$$

hence $q(r) = q$, a constant. We conclude that

$$F(t) = p\cos(\pi s) + q\sin(\pi s)$$

for constants p, q. We differentiate the above w.r.to r and get:

$$0 = \Big(-\pi p\sin(\pi s) + \pi q\cos(\pi s)\Big)\frac{\partial s}{\partial r}$$

for all $(r, t) \in [0, 1] \times [0, 1]$. Now, the expression inside parenthesis is non-zero a.e. on the square. Then one must have $\dfrac{\partial s}{\partial r} = 0$ everywhere and the final assertion follows.

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Bruno Colbois
Université de Neuchâtel, Institut de Mathématiques
Rue Emile Argand 11
CH-2000, Neuchâtel, Suisse
bruno.colbois@unine.ch

Alessandro Savo
Dipartimento SBAI, Sezione di Matematica
Sapienza Università di Roma, Via Antonio Scarpa 16
00161 Roma, Italy
alessandro.savo@uniroma1.it