Application of Semi-Definite Relaxation to Multiuser Detection in a CDMA context

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Résumé – De nombreuses problématiques de traitement du signal se ramènent à la résolution d'un problème d'optimisation combinatoire. Récemment, la Relaxation Semi-Definie (SDR) s'est révélée être une approche prometteuse en la matière, permettant une relaxation réaliste de problèmes NP-complets. Dans cet article, nous présentons un algorithme efficace pour résoudre SDR avec une complexité réduite. L'objet principale est d'étudier des méthodes de programmation non linéaires qui reposent sur un changement de variable consistant à remplacer la variable symétrique définie positive X intervenant dans SDR par une variable rectangulaire V à travers la décomposition $X = V^T V$. Des résultats récents sur les rangs de matrices de corrélations extrémales permettent de conduire à un algorithme de faible complexité avec une perte négligeable en matière de performances. Des résultats très encourageants ont été obtenus pour résoudre des problèmes d'optimisation combinatoire de grande dimension, tel que celui qui intervient dans la détection multi utilisateur en mode CDMA.

Abstract –Many signal processing applications boil down to solve combinatorial optimization problems. Recently, Semi-Definite Relaxation (SDR) has been shown to be a very promising approach to combinatorial optimization, where SDR serves as tractable convex relaxation of NP hard problems. In this paper, we present an efficient algorithm for solving SDR with a low complexity. The main focus of this paper is on non-linear programming algorithms based on a change of variables that replaces the symmetrical, positive Semi-Definite variable X in SDR with a rectangular variable V according to $X = V^T V$. Some recent results on the rank of extreme correlation matrices permit to derive a low-complexity algorithm with almost no performance loss. Very encouraging results are obtained to solve large scale combinatorial optimization programs, as the one arising in multi-user detection for CDMA systems.

1. Introduction

It is well known that the maximum likelihood (ML) detector for a multiuser signal in a CDMA context leads a large-dimensional combinatorial optimization problem. The typical dimension of this problem prohibits the use of classical global combinatorial techniques, such as the branch-and-bound technique (whose complexity is exponential). This has triggered a very active field of research, whose main goal is to design sub-optimal algorithms with polynomial complexities but close-to-optimal performance (see [1] and references therein). Recently, several authors have considered a relaxation method, the Semi-Definite Relaxation SDR, which consists in replacing the original combinatorial optimization problem by linear programming over the cone of semi-definite matrices, cf. [2]. These authors have all considered interior point methods to solve the resulting convex problem. In this paper, we investigate a computationally simpler sub-optimal solution, based on a change of variable consisting in parameterizing the cone of semi-definite positive matrices as the product of rectangular matrices. The advantages of this approach over interior point methods have been evidenced in a recent work in [3].

This paper is organized as follows. In section 2, the relaxation of the ML criterion to derive SDR is presented and in section 3 a sub-optimal solution of this non-linear optimization problem is considered. The interest of the proposed method is illustrated by extensive numerical Monte-Carlo simulations in section 4.

2. The Maximum-Likelihood criterion and its relaxation

The key equation of many digital communication problems can be written as $\mathbf{Y} = \mathbf{A}\mathbf{b} + \varepsilon$ where $\mathbf{Y}$ is an observed vector, $\mathbf{A}$ is a known matrix, $\mathbf{b}$ is a vector of transmitted symbols to be estimated and ε is a zero mean additive white Gaussian noise. For ease of presentation, it is assumed that all these quantities are real-valued and that the symbol alphabet is binary; adaptation to the complex case is straightforward.

This problem is of course prototypal of many signal processing applications including multi-user detection of CDMA signals. In this case, $\mathbf{Y}$ corresponds to the received signal after chip matched filtering and sampling (see [1]), and the columns of $\mathbf{A}$ are the convolution of the codes with the channel impulse responses. Assuming BPSK modulation, the Maximum-Likelihood detection scheme consists in finding:

$$\arg\min_{\mathbf{b}} \|\mathbf{Y} - \mathbf{Ab}\|^2 = \arg\min_{\mathbf{X}(\mathbf{b})} \mathrm{Tr}(\mathbf{QX})$$

Where:

$$\mathbf{Q} = \begin{bmatrix} \|\mathbf{Y}\|^2 & -\mathbf{Y}^T\mathbf{A} \\ -\mathbf{A}^T\mathbf{Y} & \mathbf{A}^T\mathbf{A} \end{bmatrix}, \qquad \mathbf{X}(\mathbf{b}) = \begin{bmatrix} 1 \\ \mathbf{b} \end{bmatrix}\begin{bmatrix} 1 & \mathbf{b}^T \end{bmatrix}$$

The principle of SDR is to embed the combinatorial optimization problem into a much simpler convex optimization problem, as explained below. When $\mathbf{b}$ runs through the vectors with coordinate in $\{-1, 1\}$, $\mathbf{X}(\mathbf{b})$ runs through the set of matrices:

$$D_{N+1} = \left\{ \mathbf{X} \in S_{N+1}^+, \mathbf{X}_{i,i} = 1 \,\forall i = 1,...,N+1, \mathrm{rank}(\mathbf{X}) = 1 \right\}$$

where S_{N+1}^+ is the cone of positive symmetrical matrices of dimension (N+1). Our initial problem is then equivalent to minimize the scalar product $\mathrm{Tr}(\mathbf{QX})$ with respect to the matrix $\mathbf{X}$ running through the elements of D_{N+1}. If we ignore the only non-convex *rank one* constraint, we are left with a convex optimization problem, referred to as SDR, which consists in minimizing the same criterion over the set:

$$C_{N+1} = \left\{ \mathbf{X} \in S_{N+1}^+, \mathbf{X}_{i,i} = 1 \,\forall i = 1,...,N+1 \right\}$$

This optimization problem can be approximately solved at any required precision in polynomial time using interior point methods, as suggested in [4].

The solution $\mathbf{X}^*$ of SDR is not necessarily of rank one. To approximate the optimal sequence, we then need to derive a rank one approximation of $\mathbf{X}^*$. A simple solution consists in using a randomization procedure, described below. Let $\mathbf{X}^* = (\mathbf{V}^*)^T \mathbf{V}^*$ be the Cholesky decomposition of the optimal matrix, and K_{rnd} be a parameter chosen arbitrarily. The randomization procedure can be described as follows:

1. For $i = 1,...,K_{rnd}$, generate a random vector $\mathbf{w}_i$ uniformly distributed on the unit sphere of R^{N+1} and set $\mathbf{z}_i = \mathrm{sign}(\mathbf{V}^T\mathbf{w}_i)$

2. Once these K_{rnd} vectors are generated, set $\mathbf{z} = \arg\min_{\mathbf{z}_i}(\mathrm{Tr}(\mathbf{Qz}_i\mathbf{z}_i^T))$.

3. Denoting $\mathbf{z} = \begin{bmatrix} v_0 & \mathbf{v}_1^T \end{bmatrix}^T$ where $v_0 \geq 0$ is the first coordinate of $\mathbf{z}$, set $\mathbf{b}^* = \mathrm{sign}(\mathbf{v}_1)$.

This technique finds it theoretical roots in graph theory (for more details, see [5]). The most commonly used technique to solve the problem SDR is the interior point method [6]. This method is however computationally intensive and memory demanding for high-dimensional data so that it is not adapted to solve in real-time the typically very large problems that arise in the digital communications context (in particular, in the presence of Inter-symbol Interference, see simulations section). In the rest of the article, we then consider a numerically efficient alternative to the interior point method.

3. A Sub-optimal optimization method

3.1 Low rank factorization

To obtain a new formulation of SDR, we introduce, after [3], the change of variable $\mathbf{X} = \mathbf{V}^T\mathbf{V}$ where $\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_{N+1} \end{bmatrix}$ is a real (N+1) by (N+1) matrix, which is taken here upper triangular with positive diagonal elements. In terms of the new variable $\mathbf{V}$, the resulting non-linear program

$$\left\{ \min \mathrm{Tr}(\mathbf{QV}^T\mathbf{V}), \mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_{N+1} \end{bmatrix}, \\ \mathbf{v}_i \in \mathrm{R}^{N+1}, \|\mathbf{v}_i\| = 1 \,\forall i = 1,\cdots,N+1 \right\} \qquad (2)$$

is easily seen to be equivalent to SDR. Note however that the objective function and the constraints are no longer linear, but instead quadratic and the resulting program is non convex. We can simplify this new problem using a result shown by Barvinok [7] and Pataki [8] stating that there exists an optimal solution $\mathbf{X}^*$ of SDR with rank r^* satisfying the inequality $r^*(r^*+1)/2 \leq N+1$. In the sequel, we denote $r_0 = \max\{r \in \mathrm{N}, r(r+1)/2 \leq N+1\}$. We can thus use the above result and solve a program similar to (2):

$$\left\{ \min \mathrm{Tr}(\mathbf{QV}^T\mathbf{V}), \mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_{N+1} \end{bmatrix}, \\ \mathbf{v}_i \in \mathrm{R}^r, \|\mathbf{v}_i\| = 1 \,\forall i = 1,\cdots,N+1 \right\} \qquad (P_r)$$

Since there exists an optimal solution of SDR of rank $r^* \leq r_0$, the non-linear program P_r is then equivalent to SDR whenever $r \leq r_0$. However, as illustrated in the simulation section, it makes sense to consider the problem P_r with values of r much smaller than r_0. While there is no theoretical guarantee that there exists a solution of the original SDR problem with rank r, it is still possible to determine a solution $\mathbf{V}^*$ of the problem P_r, and to use this solution to estimate the symbol vector $\mathbf{b}$.

3.2 The Gauss-Seidel method

We describe below a computationally simple algorithm for obtaining a local optimizer of the non-linear program P_r. Contrary to "classical" problems in the optimization

literature, we concentrate on a non-linear Gauss-Seidel technique whose key features are its ability (i) to handle non-convex equality constraints and (ii) to exploit the sparsity in the problem data (in large CDMA problems, matrix $\mathbf{A}^T\mathbf{A}$ is most often sparse). We stress that the precision here is not an important issue of the problem. We must keep the memory requirements minimal (preserving *e.g.* the inherent sparsity of the data) and also, the number of computations by symbol should be lower than a constant (which prohibits line searching). The Gauss-Seidel technique consists in updating successively the different columns of the matrix $\mathbf{V}$ while letting the others constant. We denote

$$\Phi(\mathbf{v}_1,\cdots,\mathbf{v}_{N+1}) = \mathrm{Tr}(\mathbf{Q}\mathbf{V}^T\mathbf{V}) = \sum_{i,j}Q_{ij}\langle\mathbf{v}_i,\mathbf{v}_j\rangle$$

Let $\mathbf{v}_i^{(n)}$ be the current value of the ith column of the matrix $\mathbf{V}$ at iteration n. Denote also $\mathbf{V}^{(n)} = \begin{bmatrix}\mathbf{v}_1^{(n)} & \cdots & \mathbf{v}_{N+1}^{(n)}\end{bmatrix}$ the current value of the matrix $\mathbf{V}$. To update this matrix, we pass through all its columns and we update them in turn by solving the reduced constrained optimization program:

$$\mathbf{v}_i^{(n+1)} = \arg\min_{\mathbf{v}_i\in S^r}\Phi\left(\mathbf{v}_1^{(n+1)},\cdots,\mathbf{v}_{i-1}^{(n+1)},\mathbf{v}_i,\mathbf{v}_{i+1}^{(n)},\cdots,\mathbf{v}_{N+1}^{(n)}\right)$$

S^r being the unit-sphere of $\mathbf{R}^r$. We notice that these optimization sub-problems can be solved in closed form:

$$\mathbf{v}_i^{(n+1)} = -\frac{\mathbf{s}_i^{(n)}}{\left\|\mathbf{s}_i^{(n)}\right\|},\ \text{where}\ \mathbf{s}_i^{(n)} = \sum_{j<i}Q_{ij}\mathbf{v}_j^{(n+1)} + \sum_{j>i}Q_{ij}\mathbf{v}_j^{(n)}$$

Obviously, we have to update several times the whole sequence of columns before reaching convergence.

An advantage of this method compared to other more sophisticated solutions is its ease of implementation and, because there is no need to compute and store gradients, the amount of computer memory required is low. Moreover, it does not involve line searches, and it is thus possible to determine the numerical complexity of one iteration of the algorithm. The update of one column of $\mathbf{V}$ approximately amounts in $(2N+3)r$ operations, so one iteration amounts in $2(N+3)(N+1) \approx 2N^2r$ operations. We will perform in practice a fixed number of iterations.

3.2 Initialization of the Gauss-Seidel algorithm

In this section a method for computing a starting point $\mathbf{V}^{(0)}$ is presented. The idea is to choose the vector $\mathbf{v}_{N+1}^{(0)}$ as a random vector uniformly distributed on the unit sphere of $\mathbf{R}^r$, and then to compute successively the vectors $\mathbf{v}_N^{(0)},\cdots,\mathbf{v}_1^{(0)}$ by setting:

$$\mathbf{v}_i^{(0)} = \arg\min_{\mathbf{v}_i\in S^r}\left\langle\mathbf{v}_i,\sum_{j=i+1}^{N+1}Q_{ij}\mathbf{v}_j^{(0)}\right\rangle$$

This results in:

$$\mathbf{v}_i^{(0)} = -\frac{\sum_{j>i}Q_{ij}\mathbf{v}_j^{(0)}}{\left\|\sum_{j>i}Q_{ij}\mathbf{v}_j^{(0)}\right\|}$$

4. Numerical results: Performance and complexity

4.1 Performance

In this section, the performance of the algorithm described above is discussed for a CDMA system with long-spreading sequences, derived from the specifications of the Universal Mobile Telecommunications System (UMTS), see [9], [10]. A model for multipaths Rayleigh fading channels, defined by the European Telecommunications Standard Institute (ETSI) and called "Vehicular B" is used. The channel impulse response of length W=60 is supposed to be known at the receiver. We considered slots of 320 chips here for the purpose of simulations. All users emit with the same power. The Bit Error Rate (BER, defined as the average BER over the different users, expressed in %) is computed by means of Monte-Carlo simulations. The SNR is defined here as the energy per transmitted S chips divided by the Gaussian noise spectral level. For each value of the SNR, 50 000 slots are generated.

We noticed that the convergence of the Gauss-Seidel SDR procedure is quick in this context, because the matrix $\mathbf{A}^T\mathbf{A}$ is large but sparse. All the results reported in this section have been obtained using five iterations and a rank equal to r=2. The average BER (in %) is displayed in figure 1 for U=32 users with spreading factor S=32 chips and in figure 2 for U=4 users with spreading factor S=4. It is shown in these figures that the Gauss Seidel SDR technique significantly outperforms both the RAKE receiver and the MMSE detector, and even the EM with hard decision M-step (see [11]).

The use of the Gauss-Seidel algorithm is justified for small spreading factors because (i) in this context, it significantly outperforms the EM with hard decision M-step, and (ii) since there is ISI in this configuration, we cannot estimate independently the different bits of a given user as it would be the case for high spreading factors (S=64 for example).

4.2 Complexity

In the CDMA model described here, the matrix $\mathbf{A}$ is banded with band S+W-1, so that the correlation matrix $\mathbf{A}^T\mathbf{A}$ is also banded with band $\beta = \lceil 1 + (W-1)/Q\rceil U$ where $\lceil x\rceil$ is the smallest integer greater than x. Assume

that each user transmits a bloc of N_d bits (with the notations introduced above, $N = N_d U$). We then need approximately $N_d U \beta^2 + 7 N_d U \beta + 3 N_d U$ flops to solve the linear system corresponding to the MMSE (if we use a band Cholesky procedure). On the other hand, if we use the Gauss-Seidel SDR algorithm and the rank-two relaxation with N_i iterations, we need approximately $(4\beta + 1)N_i N_d U$ flops. The EM algorithm requires $2 N_d U \beta$ flops. In our simulations we took r=2 and $N_i = 2$ so that the MMSE is approximately 3 times as complex as SDR which twice as complex as the EM algorithm.

5. **Conclusion**

In this article, we have explored a semi-definite relaxation of the Maximum Likelihood detector. To avoid computationally intensive interior point methods, a parameterization of semi-definite positive matrices $\mathbf{X}$ as a product of rectangular matrices is proposed, $\mathbf{X} = \mathbf{V}^T \mathbf{V}$. A simple Gauss-Seidel algorithm is used to obtain a (low precision) solution. This technique, which is much simpler to implement than the other methods and which does not rely on line-searches, is a simple and efficient alternative to interior point methods especially at low spreading factor, with much lower complexity. The proposed algorithm outperforms the classical linear MMSE and the EM with hard M-step.

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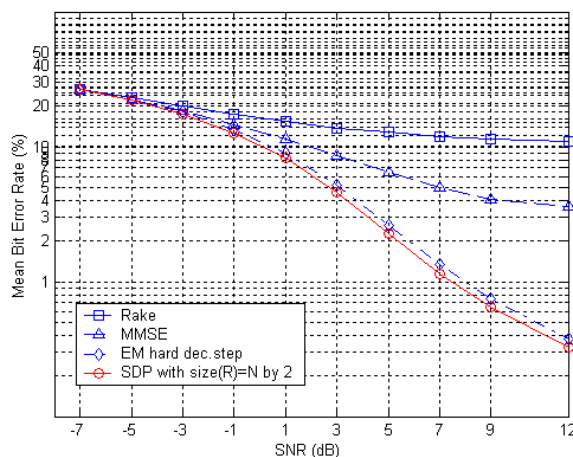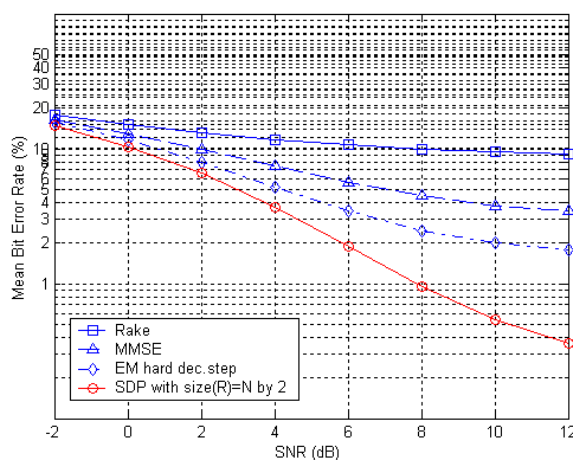
FIG. 1: Bit Error Rate of the different algorithms, S=U=32

FIG. 2: Bit Error Rate of the different algorithms, S=U=4