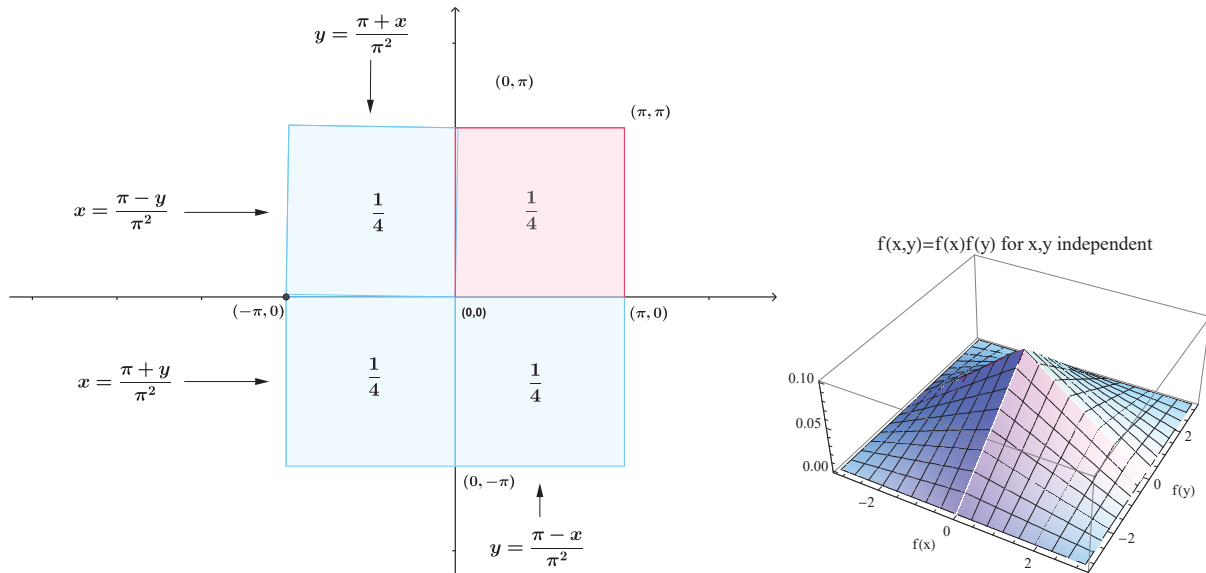


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**Random discrete groups of Möbius transformations:
Probabilities and limit set dimensions.**

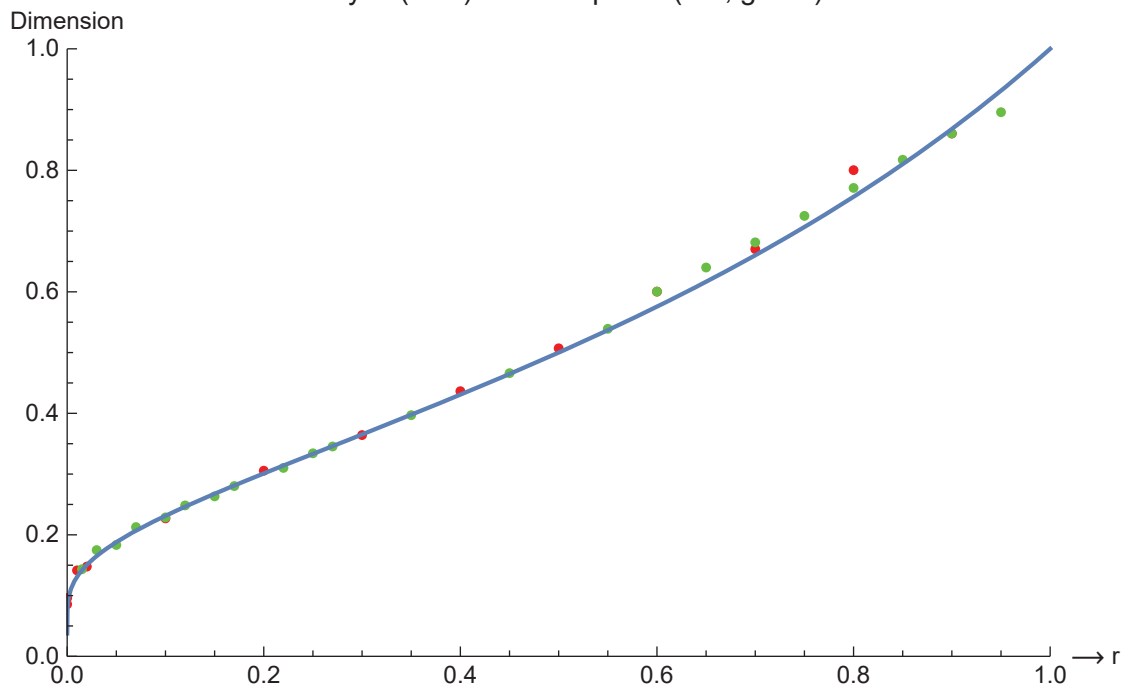
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This thesis represents original work of the author unless otherwise attributed.

Determination of dimension vs isometric circle radius
analytic (blue) and computed (red, green)



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It was my good fortune to have a great mathematician for my supervisor, but he made me work for the privilege. With his professional scepticism he made me fight for every claim, standard challenges were "I don't believe it", or maybe "It's either well known or it's wrong, I don't know which". Thanks Gaven.

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Lynette O'Brien, BBS (Hons), MSc (Mathematics)

Well, who could have a more loving, patient and supportive mathematical wife?

ABSTRACT

This thesis addresses two areas related to the quantification of discrete groups. We study "random" groups of Möbius transformations and in particular random two-generator groups; that is, groups where the generators are selected randomly. Our intention is to estimate the likelihood that such groups are discrete and to calculate the expectation of their associated geometric and topological parameters. Computational results of the author [55] that indicate a low probability of a random group being discrete are extended and we also assess the expected Hausdorff dimension of the limit set of a discrete group. In both areas of research analytic determinations are correlated with computational results. Our results depend on the precise notion of randomness and we introduce geometrically natural probability measures on the groups of all Möbius transformations of the circle and the Riemann sphere.

Contents

Acknowledgments	iii
Abstract	iv
1 Introduction	1
1.1 Möbius transformations and hyperbolic geometry	1
1.2 Random groups	1
1.3 Discrete groups	2
1.4 Limit sets	4
1.5 Dimension	5
1.6 Computation	6
1.7 Chapter order	7
2 Foundations	8
2.1 Möbius transformations	8
2.1.1 The cross ratio	8
2.1.2 Conjugation	9
2.1.3 Classification and fixed points	9
2.1.4 Isometric circles	10
2.1.5 The axis of a transformation	13
2.1.6 Discrete groups	13
2.2 Random variables and probability distributions	13
2.2.1 Random variables	14
2.2.2 Kolmogorov's σ -fields	14
2.2.3 Experimental definition	15
2.2.4 Random events	16
3 Random Möbius groups and the Fuchsian space	17
3.1 Distributions on the space of matrices \mathcal{F}	20
3.2 Isometric circles	23
3.3 Distributions on \mathbb{S} and the group \mathfrak{C}	24
3.3.1 Circular uniform distribution	24
3.3.2 The group \mathfrak{C}	25
3.3.3 Arcs and points	26
3.3.4 Matrix entry vectors	27
3.4 Traces, disjointedness and discreteness	32

3.4.1	The parameter $\beta(f)$	35
3.4.2	The parameter $\gamma([f, g])$	37
3.4.3	Jørgensen's inequality	39
3.4.4	Fixed points	40
3.4.5	Translation lengths	43
3.5	Random arcs on a circle	44
3.6	Random arcs to Möbius groups	45
3.7	The topology of the quotient space	47
3.7.1	Commutators and cross ratios	47
3.7.2	Cross ratio of fixed points	49
3.8	Discreteness	51
3.8.1	The Klein combination theorem and isometric circles	52
3.8.2	Intersections of two isometric circles of elements of \mathcal{F}	52
3.8.3	Intersections of the four isometric circles of two elements of \mathcal{F}	54
3.8.4	\mathcal{F} is discrete with $\mathbb{P} \geq \frac{1}{20}$	56
4	Probability and random variables	58
4.1	Isometric circle intersections	58
4.2	Domains of support for random variables	59
4.2.1	Modular domains	60
4.3	Functional transformations of random variables	61
4.3.1	Multi-variable transformations	61
4.3.2	Change of variables	62
4.3.3	Mellin convolutions	62
4.3.4	Unary functions	63
4.4	Elements of a random variable algebra	63
4.4.1	Products and quotients of independent random variables	63
4.4.2	Linear combinations of independent random variables	64
4.5	Linear combinations via characteristic functions	65
4.5.1	A closed form for the p.d.f. of a sum of independent random variables	65
4.5.2	Probabilities for linear combinations	68
4.6	Some distributions of trigonometric functions via the change of variables formula	68
4.7	Some distributions via characteristic functions	70
5	Computational determinations	73
5.1	Algorithmic considerations	73
5.2	\mathfrak{F}_4 σ -field probabilities	74
5.2.1	Detailed analysis	74
6	Limit sets of Möbius transformations	80
6.1	Random Fuchsian groups	80
6.2	Iterated function systems	83
6.2.1	Isometric circles	84
6.2.2	Similarity dimension	84
6.3	A calibration group	85

6.4	Covering set computational determinations	87
6.4.1	Algorithms	87
6.4.2	Some results	88
	Bibliography	88