Random signal detection in correlated non-Gaussian noise

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RÉSUMÉ

Le problème de la détection d'un signal aléatoire noyé dans un bruit additif non gaussien modélisé par un processus sphériquement invariant est adressé. Une structure asymptotiquement optimale pour la détection d'un signal gaussien est synthétisée. Les performances de cette structure de détection sont obtenues par des simulations de Monte Carlo. De plus, des comparaisons sont effectuées avec le détecteur localement optimal et le détecteur optimal dans un bruit gaussien corrélé.

ABSTRACT

The problem of detecting the presence of a random signal embedded in additive correlated non-Gaussian noise modeled as a spherically invariant random process is addressed. An asymptotically optimum structure for the detection of a Gaussian signal is synthesized. The performance of the detection structure is assessed via Monte Carlo computer simulations. Moreover, a comparison with the locally optimum detector and with the fully optimum detector for correlated Gaussian noise is made.

1 Introduction

The synthesis and the performance analysis of detection algorithms optimized against non-Gaussian noise are generally very difficult. Therefore, to obtain detection structures of easy implementation, some simplifying assumptions about the signal of interest (SOI) and the noise have usually been made. For example, under the weak-signal assumption, locally optimum (LO) (that is, optimum for a vanishingly small signal) detectors have been synthesized, which, in the case of noise samples modeled as independent and identically distributed (IID) random variables (RVs), are slight modifications of the detection structures optimized against Gaussian noise. Moreover, since in weak-signal conditions relatively large sample sizes are needed to obtain a reasonable value of the detection probability, asymptotically optimum (AO) detectors have also been synthesized, whose operating characteristics approach optimality as the sample size increases and the signal amplitude vanishes (see [3] and references therein).

To study the detection of signals embedded in non-Gaussian noise without resorting to the assumption (often violated in practice) of IID noise samples, the noise is modeled here as a spherically invariant random process (SIRP). The rationale for adopting the SIRP model is twofold: (i) it is adequate in many application fields, such as underwater and seismology; (ii) it is a generalization of the Gaussian model and hence retains many of its properties.

As regards the SOI, in this paper it is modeled as a random process. The random signal model is appropriate in situations where the signal propagates through turbulent media or along multiple paths. For example, in underwater sound detection the random dispersion due to turbulence and inhomogeneities in propagation media does not allow one to adopt a deterministic or quasi-deterministic model for the SOI. Moreover, regardless of the channel characteristics, the random signal model is also adequate when the understanding of the signal generating mechanism is insufficient.

Recently, the locally optimum array detector for a random signal embedded in spherically invariant (SI) noise has been synthesized [2]. It has been shown that in weak-signal conditions, the LO detector assures a significant performance improvement over the LO detector designed for the Gaussian noise environment. However, for moderate and high values of the signal-to-noise ratio, the LO detector presents a performance collapse, and then other detection structures need to be considered. In particular, when only one sensor is utilized, the LO performs very poorly and worse than the conventional detector.

In this paper, to propose a detection structure that overcomes the drawbacks of the LO detector when single channel detection is considered, the AO detector for a random signal embedded in additive SI noise is synthesized. At first, an asymptotic sufficient statistic is derived for an arbitrary random signal; then, the AO detector for the case of a Gaussian SOI is synthesized. The obtained detection structure does not depend on the noise univariate probability density function (PDF). However, it requires the knowledge, but for a scale factor, of the noise correlation matrix. The performance of the synthesized detection structure is assessed via Monte Carlo computer simulations, by assuming the generalized Cauchy model for the univariate PDF of the noise. Moreover, a performance comparison with the locally optimum detector synthesized in [2] and the fully optimum detector designed for a Gaussian signal embedded in additive correlated Gaussian noise is made.

2 Synthesis of the detector

The problem of detecting a random signal in additive noise can be represented by the hypothesis test

$$\begin{aligned} H_0: \quad & \boldsymbol{x} = \boldsymbol{z}, \\ H_1: \quad & \boldsymbol{x} = \boldsymbol{v} + \boldsymbol{z}, \end{aligned} \tag{1}$$

where $\boldsymbol{x}$ and $\boldsymbol{z}$ denote N-dimensional column vectors whose components are samples of the received signal and the noise. The N components of the vector $\boldsymbol{v}$ are samples of the random SOI. The vector $\boldsymbol{z}$ is assumed to be an SI random vector, that is,

$$\boldsymbol{z} = a\, \boldsymbol{n}_g,$$

where a is a nonnegative RV independent of $\boldsymbol{n}_g$, which is a zero-mean Gaussian vector characterized by a covariance matrix $\sigma_g^2 \boldsymbol{K}$, with σ_g^2 the variance of each component of $\boldsymbol{n}_g$. The signal vector $\boldsymbol{v}$ has zero mean and covariance matrix $\boldsymbol{K}_{\boldsymbol{v}}$ and, moreover, is statistically independent of $\boldsymbol{z}$.

The closure property of the SI vectors under deterministic linear transformations assures that the detector synthesized on the basis of the hypothesis test

$$\begin{aligned} H_0: \quad & \boldsymbol{y} = \boldsymbol{w}, \\ H_1: \quad & \boldsymbol{y} = \boldsymbol{s} + \boldsymbol{w}, \end{aligned} \tag{2}$$

retains the optimality properties of the detector synthesized starting from (1). In (2), $\boldsymbol{y} = \boldsymbol{C}^{-1}\boldsymbol{x}$ and $\boldsymbol{s} = \boldsymbol{C}^{-1}\boldsymbol{v}$, where $\boldsymbol{C}$ is an N×N invertible lower triangular matrix obtained by the Cholesky decomposition $\boldsymbol{K} = \boldsymbol{C}\boldsymbol{C}^T$ of the matrix $\boldsymbol{K}$. The vector $\boldsymbol{w} = \boldsymbol{C}^{-1}\boldsymbol{z}$, due to the closure property of the SI vectors, is still an SI vector with the same modulating RV a as $\boldsymbol{z}$, zero mean, and covariance matrix

$$\boldsymbol{K}_{\boldsymbol{w}} = E[a^2]\sigma_g^2 \boldsymbol{I},$$

where $E[\cdot]$ denotes statistical expectation and $\boldsymbol{I}$ is the N×N identity matrix.

The optimum test for H_0 vs H_1, in the sense of maximizing the detection probability for a fixed false-alarm rate (Neyman-Pearson criterion), compares the likelihood ratio

$$\Lambda(\boldsymbol{y}) = \frac{p_{\boldsymbol{y}}(\boldsymbol{y} \mid H_1)}{p_{\boldsymbol{y}}(\boldsymbol{y} \mid H_0)}$$

to some threshold chosen to achieve the desired false-alarm probability.

Taking into account the adopted noise model, the likelihood ratio can be written as

$$\Lambda(\boldsymbol{y}) = \frac{\int_0^{+\infty} p_{\boldsymbol{y}}(\boldsymbol{y} \mid H_1, a) p_a(a) da}{\int_0^{+\infty} p_{\boldsymbol{y}}(\boldsymbol{y} \mid H_0, a) p_a(a) da},$$

where $p_a(a)$ is the PDF of the modulating random variable. Note that the conditional joint PDF $p_{\boldsymbol{y}}(\boldsymbol{y} \mid H_0, a)$ is a product of Gaussian PDFs:

$$p_{\boldsymbol{y}}(\boldsymbol{y} \mid H_0, a) = \prod_{i=1}^{N} \frac{1}{a\sigma_g\sqrt{2\pi}} \exp\left(-\frac{y_i^2}{2\sigma_g^2 a^2}\right)$$

Moreover, the conditional joint PDF $p_{\boldsymbol{y}}(\boldsymbol{y} \mid H_1, a)$ can be written as

$$\begin{aligned} p_{\boldsymbol{y}}(\boldsymbol{y} \mid H_1, a) &= \int_{R^N} p_{\boldsymbol{s}}(\boldsymbol{s}) p_{\boldsymbol{y}}(\boldsymbol{y} \mid H_1, a, \boldsymbol{s}) d\boldsymbol{s} \\ &= \int_{R^N} p_{\boldsymbol{s}}(\boldsymbol{s}) \prod_{i=1}^{N} \frac{1}{a\sigma_g\sqrt{2\pi}} \exp[-\frac{(y_i - s_i)^2}{2\sigma_g^2 a^2}] d\boldsymbol{s} \end{aligned}$$

where $p_{\boldsymbol{y}}(\boldsymbol{y} \mid H_1, a, \boldsymbol{s})$ is the conditional joint PDF of the N components of $\boldsymbol{y}$, which is just the N-dimensional Gaussian PDF $p_{\boldsymbol{w}}(\boldsymbol{y} - \boldsymbol{s} \mid a)$ of the noise vector $\boldsymbol{w}$ conditioned to a, and $p_{\boldsymbol{s}}(\boldsymbol{s})$ is the joint PDF of the N components of the signal vector $\boldsymbol{s}$.

Owing to the implementation difficulties of the fully optimum detector, in this paper, an asymptotically optimum structure of easier implementation is proposed. Such a detector can be synthesized following an approach similar to that considered in [1] to obtain a detection structure for weak radar signals with unknown parameters embedded in SI noise.

At first, the likelihood ratio can be written as

$$\Lambda(\boldsymbol{y}) = \frac{\int_0^{\infty} a^{-N} e^{-\frac{q}{2a^2}} W(\boldsymbol{y}, a)\, p_a(a)\, da}{\int_0^{\infty} a^{-N} e^{-\frac{q}{2a^2}} p_a(a) da}, \tag{3}$$

where

$$q \triangleq \sum_{i=1}^{N} \frac{y_i^2}{\sigma_g^2} = \frac{1}{\sigma_g^2} \|\boldsymbol{y}\|^2$$

and

$$W(\boldsymbol{y}, a) \triangleq \int_{R^N} e^{-\frac{1}{2\sigma_g^2 a^2}\|\boldsymbol{s}\|^2 + \frac{1}{\sigma_g^2 a^2}\boldsymbol{y}^T\boldsymbol{s}} p_{\boldsymbol{s}}(\boldsymbol{s}) d\boldsymbol{s},$$

where T denotes matrix transposition. Then, by substituting $\lambda = \frac{q}{Na^2}$ in both the integrals in (3), it results that

$$\Lambda(\boldsymbol{y}) = \frac{\int_0^{\infty} \Psi_N(\lambda)\, \lambda^{-\frac{1}{2}}\, W\left(\boldsymbol{y}, \sqrt{\frac{q}{N\lambda}}\right) p_a\left(\sqrt{\frac{q}{N\lambda}}\right) d\lambda}{\int_0^{\infty} \Psi_N(\lambda)\, \lambda^{-\frac{1}{2}}\, p_a\left(\sqrt{\frac{q}{N\lambda}}\right) d\lambda},$$

where the function

$$\Psi_N(\lambda) = \frac{(N/2)^{N/2}}{\Gamma(N/2)} \lambda^{N/2-1} e^{-\frac{N}{2}\lambda}$$

has been introduced. Finally, since

$$\lim_{N\to\infty} \Psi_N(\lambda) = \delta(\lambda - 1),$$

it results that the likelihood ratio can be asymptotically expressed as

$$\Lambda(\boldsymbol{y}) = W\left(\boldsymbol{y}, \sqrt{\frac{q}{N}}\right)$$

$$= \int_{R^N} e^{-\frac{N}{2\sigma_g^2 q} \|\boldsymbol{s}\|^2 + \frac{N}{\sigma_g^2 q} \boldsymbol{y}^T \boldsymbol{s}} p_{\boldsymbol{s}}(\boldsymbol{s}) d\boldsymbol{s}.$$

Since the obtained AO detection structure depends on the joint PDF of the N components of the signal vector, to go further one needs to specify such a PDF. In the sequel, the SOI is modeled as a zero-mean Gaussian process, so that $\boldsymbol{s}$ is a Gaussian vector with zero mean and covariance matrix $\boldsymbol{K}_{\boldsymbol{s}}$. Thus, after simple manipulations, the AO detector for a Gaussian signal embedded in SI noise can be expressed as

$$T^{AO}(\boldsymbol{y}) = \boldsymbol{y}^T \left[\boldsymbol{I} + \left(\frac{\boldsymbol{K}_{\boldsymbol{s}}}{\frac{1}{N} \|\boldsymbol{y}\|^2} \right)^{-1} \right]^{-1} \boldsymbol{y} \frac{1}{\frac{1}{N} \|\boldsymbol{y}\|^2}$$

$$- \ln \left\{ \det \left[\boldsymbol{I} + \frac{\boldsymbol{K}_{\boldsymbol{s}}}{\frac{1}{N} \|\boldsymbol{y}\|^2} \right] \right\},$$

where det[·] denotes the determinant of the matrix in the brackets.

For comparison purposes, let us report the LO detector statistic for the problem at hand, which was derived in [2]:

$$T^{LO}(\boldsymbol{y}) = \boldsymbol{y}^T \boldsymbol{K}_{\boldsymbol{s}} \boldsymbol{y} \frac{\left[G^2(q) - 2\dot{G}(q)\right]}{\sigma_g^2}$$

$$-tr\left\{\boldsymbol{K}_{\boldsymbol{s}}\right\} G(q), \tag{4}$$

where $tr\{\cdot\}$ denotes the trace of the matrix in the brackets, $\dot{G}(q)$ is the derivative of $G(q)$ with respect to the quadratic form q and the nonlinearity with memory

$$G(q) \triangleq \frac{\int_0^{+\infty} a^{-N-2} \exp(-\frac{q}{2a^2}) p_a(a) da}{\int_0^{+\infty} a^{-N} \exp(-\frac{q}{2a^2}) p_a(a) da} \tag{5}$$

has been introduced.

Note that, the AO detection structure does not depend on the PDF of the modulating RV and, consequently, on the univariate PDF of the noise. Moreover, the knowledge, but for a scale factor, of the noise correlation matrix is required. On the other hand, the synthesis of the LO detector requires the knowledge of the univariate PDF and the correlation matrix of the noise. As regard the a priori knowledge on the signal to be detected, the synthesis of the AO detector requires the knowledge of the joint PDF of the N components of the signal vector, whereas the implementation of the LO detector requires only the knowledge of the covariance matrix of the signal vector.

Finally, let us report the well known optimum (for any value of N) detection statistic for a Gaussian signal embedded in Gaussian noise:

$$T^{OG}(\boldsymbol{y}) = \boldsymbol{y}^T \left[\boldsymbol{I} + \left(\frac{\boldsymbol{K}_{\boldsymbol{s}}}{\sigma_g^2} \right)^{-1} \right]^{-1} \boldsymbol{y}.$$

3 Performance assessment

The present section is aimed at assessing the performance of the AO detector synthesized in the previous section. Moreover, the performance is compared with that of the LO detector for a random signal embedded in SI noise and with that of the OG detector, that is, the fully optimum detector for correlated Gaussian noise.

Since the analytical evaluation of the conditional PDFs of the decision variable under both hypotheses is an intractable problem, the performance (detection probability $P(d)$ for a fixed false-alarm rate $P(fa)$) has been carried out by Monte Carlo simulations.

In the simulations, the generalized Cauchy model has been considered for the univariate PDF of the noise RVs:

$$p_z(x) = \frac{(2\sigma_g^2 \nu)^\nu \ \Gamma(\nu + \frac{1}{2})}{\sqrt{\pi}\ \Gamma(\nu)(2\sigma_g^2 \nu + x^2)^{\nu + \frac{1}{2}}}, \tag{6}$$

where $\Gamma(\cdot)$ is the gamma function and ν is a positive shape parameter. Note that as ν approaches infinity, the resulting PDF approaches the Gaussian density. Moreover, the variance of z is given by $\sigma_z^2 = \sigma_g^2 \nu/(\nu - 1)$ for $\nu > 1$ and is not finite for any value of ν belonging to the interval (0, 1], which has not been considered in the simulation experiments. This model is quite flexible encompassing a wide class of PDFs and, moreover, adequately models the univariate statistics of certain data arising from noise sources. For example, it can suitably model the extremely-low frequency atmospheric noise caused by lightning strokes.

The PDF of the nonnegative modulating RV a, which generates the SI noise vector $\boldsymbol{z}$ with univariate PDF given by (6), has the following expression [4]:

$$p_a(a) = \frac{2\nu^\nu}{a^{2\nu+1}\Gamma(\nu)} \exp(-\nu/a^2) \qquad (a \geqslant 0)$$

Therefore, the nonlinearities with memory $G(q)$ and $\dot{G}(q)$ of the LO detector (see (4) and (5)) are expressed by

$$G(q) = \frac{N + 2\nu}{2\nu + q},$$

$$\dot{G}(q) = -\frac{N + 2\nu}{(2\nu + q)^2}.$$

As regards the correlation of the noise samples, in the simulations a noise correlation matrix $\boldsymbol{K}_{\boldsymbol{z}}$ whose (im)th element is

$$K_z(i - m) = \frac{\nu}{\nu - 1} \sigma_g^2 \rho^{|i-m|}, \qquad |\rho| < 1$$

has been considered.

The components of the SOI vector $\boldsymbol{v}$ have been modeled as statistically independent Gaussian RVs with zero means and unit variances and, hence, $\boldsymbol{K}_{\boldsymbol{s}} = \boldsymbol{C}^{-1}(\boldsymbol{C}^{-1})^T$.

Under the previous assumptions on noise and SOI, the signal-to-noise ratio at the input of the whitening filter $\boldsymbol{C}^{-1}$, say SNR, is given by

$$SNR = \frac{\nu - 1}{\nu \sigma_g^2}.$$

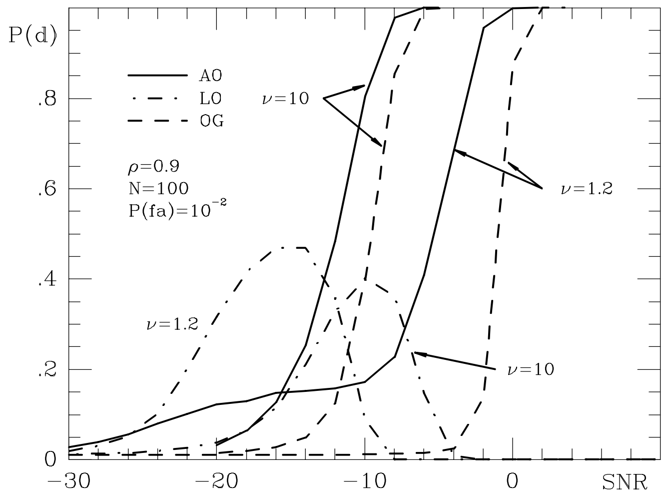

Fig.1

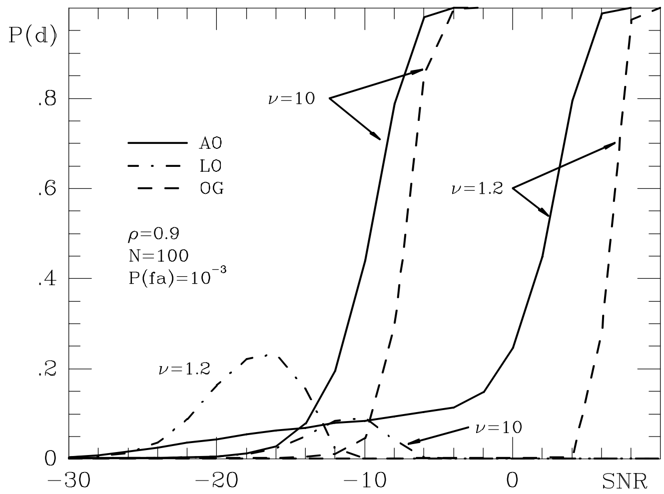

Fig.2

Figures 1 and 2 present the detection probability $P(d)$ of all considered detectors as a function of SNR for two different values of the false-alarm rate ($P(fa) = 10^{-2}$ in Fig.1 and $P(fa) = 10^{-3}$ in Fig.2). In both figures, N=100 samples, a correlation coefficient $\rho = 0.9$ and two different values ($\nu = 1.2$ and $\nu = 10$) of the noise shape parameter have been considered. The number of simulation runs per datum has been assumed to be $100/P(fa)$. The results show that the AO detector outperforms the OG detector on the whole range of values of SNR. Moreover, although the LO detector outperforms both AO and OG detectors in very weak-signal conditions, in such a SNR range the performance is largely unsatisfactory. Thus, the AO detector overcomes the performance limits exhibited by the LO detector and outperforms the OG detector.

References

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