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"Unbiased estimation of autoregressive models for bounded stochastic processes"

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Abstract

The paper investigates the estimation bias of autoregressive models for bounded stochastic processes and the performance of the standard procedures in the literature that aim to correcting the estimation bias. It is shown that, in some cases, the bounded nature of the stochastic processes worsen the estimation bias effect, which suggests the design of bound-specific bias correction methods. The paper focuses on two popular autoregressive estimation bias correction procedures which are extended to cover bounded stochastic processes. Finite sample performance analysis of the new proposal is carried out using Monte Carlo simulations which reveal that accounting for the bounded nature of the stochastic processes leads to improvements in the estimation of autoregressive models. Finally, an illustration is given using the current account balance of some developed countries, whose shocks persistence measures are computed.

JEL Classification: C22, C32, E32, Q43.

Keywords: Bounded stochastic processes, estimation bias, unit root tests, current account balance.

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1 Introduction

Since the seminal paper of Nelson and Plosser (1982), any analysis that considers the use of time series data always begins with the study of the time properties of the variables. This usually implies the use of some unit root tests and the statistical inference that is drawn from their application is relevant for subsequent analyses. For instance, a quite popular practice is to determine the degree of shocks persistence by means of estimating autoregressive models.

The latter analysis provides very interesting insights about the evolution of the variable being studied, including the analysis of the persistence in variables such as the real exchange rates, where some practitioners have studied the number of periods that a shock takes to vanish – see Balli et al. (2014), among others. Similarly, Watson (2014) studies the effect of the Great Recession on inflation persistence. This type of analyses, however, is not straightforward given that we should take into account that the OLS estimator is consistent but biased in finite samples, and this bias must be removed in order to appropriately measure the degree of persistence. There are various proposals in the literature which try to correct this finite sample bias. We can cite here the contributions of Andrews (1993), Andrews and Chen (1994), Kilian (1998), Hansen (1999), Rossi (2005) and Perron and Yabu (2009a), among others, which develop different valid techniques to remove the mentioned bias.

However, some commonly employed variables in this type of works may be affected by the presence of bounds. There are some important macroeconomic variables such as nominal interest rates, unemployment rates, exchange rates and the great ratios, among others, that are bounded by definition, preventing these variables from exhibiting a large variance. This feature generates tension in the statistical inference associated with standard unit root tests and, hence, the estimation of the degree of shocks persistence.

The standard order of integration analysis of time series considers that an I(1) non-stationary stochastic process can vary freely in the limit, that is, they ignore the constraints that impose the existence of bounds. This fact is relevant because the behavior of this type of variables might seem to be stationary when, in fact, they are non-stationary. In this regard, Cavaliere (2005) and Cavaliere and Xu (2014) show that standard unit root tests might reach misleading conclusions if the bounded nature of the time series is not accounted for. Therefore, it is sensible to analyze the influence of these bounds on the determination of the time series properties of the variables.

The goal of this paper is to assess whether the use of bias-corrected autoregressive parameters allows us to obtain statistics such as shock persistence measures, the long-run variance (LRV) estimates and unit root tests statistics for bounded time series with good finite sample performance. To address this issue, this paper investigates the performance of some popular bias correction methods mentioned above when they are applied to bounded stochastic processes. The first stage of the analysis focuses on some of these standard bias correction procedures, showing that, in general, the amount of estimation bias that is corrected is small when the bounded nature of the time series is ignored. This suggests extending the standard bias correction procedures that incorporate the possible effect of the bounds on the estimation of autoregressive models for (possibly persistent) time series.

The paper proceeds as follows. Section 2 describes the model for bounded stochastic processes and investigates the consistence and finite sample bias of the OLS estimation procedure. In addition, we assess the performance of some of the most relevant standard methods proposed in the literature that correct the finite sample bias. Section 3 proposes an extension of two bias correction procedures for bounded stochastic processes. Section 4 analyzes the finite sample performance of the suggested approaches. Section 5 conducts an empirical illustration focusing on the current account balances of a sample of OECD countries. Finally, Section 6 concludes.

2 The model

Let x_t be a stochastic process with a data generating process (DGP) given by:

$$x_t = \mu + y_t \tag{1}$$

$$y_t = \alpha y_{t-1} + u_t, \tag{2}$$

$t = 1, \ldots, T$, where $x_t \in \left[\underline{b}, \overline{b}\right]$ almost surely for all t, $y_0 = O_p(1)$, and $\left[\underline{b}, \overline{b}\right]$ denote the boundaries that affect the time series. The disturbance term u_t is assumed to admit the following decomposition:

$$u_t = \varepsilon_t + \underline{\xi_t} - \overline{\xi_t}, \tag{3}$$

with $\varepsilon_t = C(L)\upsilon_t$, where $C(L) = \sum_{j=0}^{\infty} c_j L^j$ with $\sum_{j=0}^{\infty} j\,|c_j| < \infty$, and υ_t is a martingale difference sequence adapted to the filtration $F_t = \sigma - field\{\upsilon_{t-j}; j \geq 0\}$. The LRV of ε_t is given by $\sigma^2 = \lim_{T\to\infty} E[T^{-1}(\sum_{t=1}^{T}\varepsilon_t)^2]$. The variables $\underline{\xi_t}$ and $\overline{\xi_t}$ are non-negative processes (regulators) such that $\underline{\xi_t} > 0$ if and only if $\alpha y_{t-1} + \varepsilon_t < \underline{b} - \mu$ and $\overline{\xi_t} > 0$ if and only if $\alpha y_{t-1} + \varepsilon_t > \overline{b} - \mu$. The stochastic processes involved in (3) satisfy the Assumptions A and B in Cavaliere and Xu (2014), so that $(\underline{b} - \mu) = \underline{c}\sigma T^{1/2}$ and $\left(\overline{b} - \mu\right) = \overline{c}\sigma T^{1/2}$, with $\underline{c} \leq 0 \leq \overline{c}$, $\underline{c} \neq \overline{c}$. This representation can be particularized to the cases of stochastic processes that are only limited below – i.e., $x_t \in [\underline{b}, \infty]$ – or only limited above – i.e., $x_t \in \left[-\infty, \overline{b}\right]$ – but also covers the case of unbounded processes – i.e., $x_t \in [-\infty, \infty]$.

Estimation of autoregressive models is at the heart of popular practices in empirical economics such as order of integration analysis and the computation of shock persistence measures. However, it is well known that their estimation provides biased estimates in finite samples, although the bias disappears asymptotically. In this regard, it is of interest to study whether dealing with bounded stochastic processes defined by (1) to (3) presents any different features compared to the unbounded situation. The following theorem shows, for a simple model specification, that the estimation of autoregressive models for bounded processes is consistent.

Theorem 1 *Let $\{x_t\}_{t=1}^{T}$ be the stochastic process given by (1) and (2) with $\mu = 0$, $|\alpha| < 1$ and $\varepsilon_t \sim iid\,(0, \sigma^2)$ with $E\left(\varepsilon_t^4\right) < \infty$. Then, as $T \to \infty$ the ordinary least-squares (OLS) estimator:*

$$\hat{\alpha} \xrightarrow{p} \alpha.$$

The proof is given in the appendix. Although the estimation of the autoregressive parameter is consistent, there might be some estimation bias in finite samples. To show the extent of the additional estimation bias effect introduced by bounds, we have conducted a small simulation experiment using a symmetric bounded stochastic process defined by (1) to (3) with α taking values between 0 and 1 in steps of 0.05 size – i.e., $\alpha = \{0, 0.05, 0.1, \ldots, 0.95, 1\}$ – $\varepsilon_t \sim iid\,(0, 1)$, $\bar{c} = \{0.1, 0.3, 0.5, 0.7, 1\}$ and $T = \{50, 100, 200, 500\}$ – 1,000 replications are conducted. Figure 1 reveals that the estimation bias depends not only on T and α, but also on $\bar{c}$. Regardless of T and α, the bias is bigger the narrower the rank of variation defined by the bounds. As expected, the magnitude of the bias reduces as T increases, but even for large T we observe non-negligible bias for small $\bar{c}$. Finally, note that for large values of $\bar{c}$, for which the time series is near-unbounded, the estimation bias tends to increase as α approaches one.

We make use of different standard approaches in the literature that try to correct the estimation bias of autoregressive models. First, we focus on the median-unbiased (MU) estimation procedure for AR(1) models in Andrews (1993), which requires the computation of look-up tables to obtain a correspondence between the value of the OLS estimation of the autoregressive parameter ($\hat{\alpha}$) and the median of the empirical distribution that is obtained assuming that $\alpha = \hat{\alpha}$, which defines $\hat{\alpha}_{MU}$, i.e., the median-unbiased autoregressive estimator of α. Andrews (1993) suggests using $\hat{\alpha}_{MU}$ instead of $\hat{\alpha}$. Other alternatives have been proposed in the literature – see, for instance, Kilian (1998), Hansen (1999), Rossi (2005)) – although initial simulations, not reported here to save space, reveal that they are beaten by the Andrews (1993) MU estimator.

Figures 2 to 5 compare the mean of the estimation bias of OLS and the Andrews MU procedures for $\bar{c} = \{0.1, 0.3, 0.7, 1\}$, $T = \{50, 100, 200, 500\}$ and $\alpha = \{0, 0.05, 0.1, \ldots, 0.95, 1\}$. In general, MU estimation gives more accurate estimates of the autoregressive parameter, regardless of the values of $\bar{c}$, α and T. However, the MU estimation bias is still important for small values of $\bar{c}$, although it clearly reduces as $\bar{c}$ increases. It is worth noticing that these improvements are obtained using an estimation procedure designed for unbounded processes, but we have shown that the amount of bias correction depends on $\overline{c}$. This feature leads us to hypothesize that better results can be expected if bound-specific MU estimates are used, which implies extending the proposal of Andrews (1993) to bounded time series.

3 Bias correction methods for bounded stochastic processes

The estimators mentioned above ignore the bounded nature of x_t, an important feature that should be taken into account in order to improve the estimates of the autoregressive parameters. To address this issue, we have proceeded to modify two bias correction procedures considering that $x_t \in \left[\underline{b}, \overline{b}\right]$.

3.1 The MU estimation procedure of Andrews

Andrews MU estimator requires the computation of look-up tables that establish a correspondence between the OLS and the MU estimate of the autoregressive parameter. This is an intensive computational problem since these look-up tables have to be obtained for different combinations of $[\underline{c}, \overline{c}]$ values. As an example, Table 1 presents the asymptotic look-up table for the AR(1) symmetric bounds case. It provides values of the median of the distribution of $\hat{\alpha}$ for a grid of α and $\overline{c}$ values – a Matlab code is available to compute look-up tables for any bounds.[1] The computation of look-up tables for AR(p) processes can be done following the proposal in Andrews and Chen (1994), although it would have a higher computational cost since it requires the use of bootstrapping.

3.2 Weighted symmetric least-squares estimation procedure

Following Roy and Fuller (2001), Roy, Falk and Fuller (2004) and Perron and Yabu (2009a), we suggest the use of the modified estimator given by:

$$\hat{\alpha}_{TW} = \hat{\alpha}_W + C\left(\hat{\tau}_W\right)\hat{\sigma}_W, \tag{4}$$

where $\hat{\alpha}_W$ denotes the weighted symmetric least-squares (WSLS) estimate of the autoregressive parameter for AR(1) models proposed in Fuller (1996):

$$\hat{\alpha}_W = \frac{\sum_{t=2}^{T} \hat{y}_t \hat{y}_{t-1}}{\sum_{t=2}^{T-1} \hat{y}_t^2 + T^{-1} \sum_{t=1}^{T} \hat{y}_t^2},$$

[1] Note that using $\overline{c} = 1$ approaches the unbounded case covered in Andrews (1993), Table 1.

with $\hat{y}_t$ are the OLS estimated residuals in (1),

$$\hat{\sigma}_W^2 = \frac{\sum_{t=2}^{T} \left(\hat{y}_t - \hat{\alpha}_W \hat{y}_{t-1}\right)^2}{(T-2)\left[\sum_{t=2}^{T-1} \hat{y}_t^2 + T^{-1} \sum_{t=1}^{T} \hat{y}_t^2\right]},$$

and $\hat{\tau}_W = \left(\hat{\alpha}_W - 1\right)/\hat{\sigma}_W$ is the pseudo t-ratio statistic to the null hypothesis that $\alpha = 1$. The modification in (4) requires the definition of $C\left(\hat{\tau}_W\right)$ that, following Roy and Fuller (2001) and Perron and Yabu (2009a), is given by the following discontinuous function:

$$C\left(\hat{\tau}_W\right) = \begin{cases} -\hat{\tau}_W & \text{if } \hat{\tau}_W > \tau_{pct} \\ I_p T^{-1} \hat{\tau}_W - 2\left[\hat{\tau}_W + K\left(\hat{\tau}_W + A\right)\right]^{-1} & \text{if } -A < \hat{\tau}_W \leq \tau_{pct} \\ I_p T^{-1} \hat{\tau}_W - 2\left[\hat{\tau}_W\right]^{-1} & \text{if } -(2T)^{1/2} < \hat{\tau}_W \leq -A \\ 0 & \text{if } \hat{\tau}_W \leq -(2T)^{1/2} \end{cases}, \tag{5}$$

with $K = \left[\left(1 + I_p T^{-1}\right) \tau_{pct} \left(\tau_{pct} + A\right)\right]^{-1} \left[2 - I_p T^{-1} \tau_{pct}^2\right]$, $I_p = \lfloor (p+1)/2 \rfloor$, $\lfloor \cdot \rfloor$ being the integer part, p denotes the order of the autoregressive model – $p = 1$ in this case – and τ_{pct} is a percentile of the limiting distribution of $\hat{\tau}_W$ when $\alpha = 1$. The percentile τ_{pct} is either set at the median (τ_{50}) or at the 85th percentile (τ_{85}) of the distribution of $\hat{\tau}_W$. Finally, the function K depends on the deterministic specification that is used in (1) – i.e., a constant or a linear time trend.[2] The value of the constant A is empirically chosen in Roy and Fuller (2001) after conducting simulation experiments, and they set it at $A = 5$ for unbounded stochastic processes.[3]

Table 2 summarizes selected percentiles of the distribution of $\hat{\tau}_W$ for different values of the (symmetric) bound parameters – the last row shows the percentiles for unbounded stochastic processes. As can be seen, the limiting distribution of $\hat{\tau}_W$ depends on the bounds, with a limiting distribution shifted more to the left the narrower the rank of variation defined by the bounds.

Let us focus on the median of the distribution as the percentile used in the bias correction. First, note that the use of $A = 5$ for the unbounded stochastic process case does not pose incongruences for the definition of the function in (5), since $-A < \tau_{0.5}$.[4] However, we can see that the median of the distribution $\hat{\tau}_W$ moves away from -1.21 as the rank of variation defined by the bounds decreases, which might produce poor

[2] See Roy and Fuller (2001) for the function that corresponds to the linear time trend. It is worth noticing that Perron and Yabu (2009b) use the same function when testing for multiple shifts in the trend.

[3] Roy and Fuller (2001) also set $A = 5$ for the linear time trends, whereas Perron and Yabu (2009b) specify $A = 10$.

[4] This is also valid for the linear time trend case, for which Roy and Fuller (2001) estimated $\tau_{0.5} = -1.96$ and $A = 5$, as mentioned above. Note that the consideration of slope trend shifts in Perron and Yabu (2009b) lead them to specify $A = 10$ for the one break case – it is well known that the limiting distribution of $\hat{\tau}_W$ shifts to the left as the number of structural breaks increases.

performance of the correction when $\bar{c} < 0.5$.[5] In this regards, an extensive simulation experiment has been conducted to assess the sensitivity of the modified estimator to the specification of the constant $A = \{5, 6, \ldots, 15\}$. Results available upon request indicate that the modified estimator shows good performance when $A = 5$ and $\bar{c} > 0.1$, and only marginal differences are found for the other values of A. Besides, for small values of the bound parameter ($\bar{c} \leq 0.1$), we find that $A = 10$ gives good results.

This method can be easily extended to AR(p) models for which the autoregressive parameter α is estimated from:

$$\hat{y}_t = \alpha \hat{y}_{t-1} + \sum_{j=1}^{k} \psi_j \Delta \hat{y}_{t-j} + \varepsilon_t, \tag{6}$$

with $\hat{y}_t$ being the OLS estimated residuals in (1). In this case, the WSLS estimate of α ($\hat{\alpha}_W$) can be obtained as described in Fuller (1996) and its truncated version (TWSLS) $\hat{\alpha}_{TW}$ is computed as defined in (5).

3.3 Implementation of the estimation procedure

The empirical implementation of the method of bias correction in bounded time series that we propose needs some additional steps. Given a time series with known theoretical limits $\underline{b}$ and $\overline{b}$, we can estimate the bounds as:

$$[\hat{\underline{c}}, \hat{\bar{c}}] = \left[\frac{\underline{b} - \hat{D}_t}{\hat{\sigma} T^{1/2}}, \frac{\overline{b} - \hat{D}_t}{\hat{\sigma} T^{1/2}} \right],$$

which requires an estimation of the deterministic component (D_t) and the long-run variance (σ^2). In our case $D_t = \mu$ and it can be easily estimated by regressing the series to a constant. However, the estimation of the long-run variance deserves further attention because its estimation also suffers from bias estimation problems of the autoregressive parameters. To address this issue we suggest using the following iterative estimation method:

1. Estimate the LRV ignoring the bounds. In this regard, we can use the parametric estimation method proposed in Ng and Perron (2001) and Perron and Qu (2007), which also allows us to select the optimal lag if we have modelled the series as an autoregressive process.

[5] Our guess is based on the fact that Roy and Fuller (2001) define $A = 5$ for the linear time trend case, for which the median of the distribution of $\hat{\tau}_W$ is $\tau_{0.5} = -1.96$. Consequently, we might expect that $A = 5$ is also valid for cases where $\bar{c} \geq 0.5$, although it should be borne in mind that the K function involved in the correction depends on the deterministic specification.

2. Compute an initial educated estimation of the bounds:

$$\left[\hat{\underline{c}}^0, \hat{\bar{c}}^0\right] = \left[\frac{\left(\underline{b} - \hat{D}_t\right)}{\hat{\sigma}_0 T^{1/2}}, \frac{\left(\bar{b} - \hat{D}_t\right)}{\hat{\sigma}_0 T^{1/2}}\right].$$

3. Compute an estimation of α according to one of these procedures:

   (a) For the MU-based procedure, compute the look-up tables corresponding to $[\hat{\underline{c}}^0, \hat{\bar{c}}^0]$ by simulation and obtain $\hat{\alpha}_{MU}$

   (b) For the truncated WSLS-based procedure, compute the percentiles of the $\hat{\tau}_W$ distribution corresponding to $[\hat{\underline{c}}^0, \hat{\bar{c}}^0]$ by simulation and obtain $\hat{\alpha}_{TW}$ as defined in (5).

4. Use $\hat{\alpha}$ from the previous step to estimate the LRV again as follows,

$$y_t - \hat{\alpha} y_{t-1} = \mu + \sum_{j=1}^{k} \psi_j \Delta y_{t-j} + \varepsilon_t$$

$$\hat{\sigma}_1^2 = \frac{\sum_{t=1}^{T} \hat{\varepsilon}_t^2}{(1-\hat{\alpha})^2}.$$

5. Obtain the new bounds as:

$$\left[\hat{\underline{c}}^1, \hat{\bar{c}}^1\right] = \left[\frac{\left(\underline{b} - \hat{D}_t\right)}{\hat{\sigma}_1 T^{1/2}}, \frac{\left(\bar{b} - \hat{D}_t\right)}{\hat{\sigma}_1 T^{1/2}}\right].$$

6. Iterate until $\left|\sum_{t=1}^{T} \hat{\varepsilon}_{t,l}^2 - \sum_{t=1}^{T} \hat{\varepsilon}_{t,l-1}^2\right| < Tol$, where Tol is the desired level of tolerance and l the step of iteration.

It is worth noticing that the implementation of the procedure can be done performing only one iteration (steps 1 to 3) or carrying out multiple iterations (steps 1 to 6). The potential gain of the multiple iterative estimation method is measured through Monte Carlo simulation experiments in the next section.

4 Finite sample performance

In this section we analyze the different bias correction methods discussed above. Simulations are organized according to whether the estimation is conducted without iteration (one-step estimation procedure) or using the iterative scheme defined in the previous section.

4.1 One-step estimation procedure

4.1.1 The AR(1) case

The DGP is given by (1) to (3) with $\mu = 0$ and $\varepsilon_t \sim iid\ N(0,1)$. The symmetric bounds are defined by $[\underline{c}, \overline{c}] = [-\overline{c}, \overline{c}]$, with $\overline{c} = \{0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1,\ 1.5\}$, $\alpha = \{0,\ 0.1,\ 0.2,\ ...,\ 1\}$, $T = 100$ and 1,000 replications are used. We consider three different cases depending on the method that is used to estimate the order of the autoregressive process. First, we focus on the situation in which p is known, as required by the procedure of Andrews (1993). Furthermore, for the WSLS method, we also treat p as an additional unknown parameter which is estimated using both the MAIC information criterion in Ng and Perron (2001) and the BIC information criterion, specifying a maximum of $p_{\max} = \lfloor 12(T/100)^{1/4} \rfloor$ lags.

Tables 3 and 4 present the mean of the empirical distribution of the $\hat{\alpha}_{MU}$ and $\hat{\alpha}_{TW}$ estimators, respectively, for different values of $\overline{c}$ and α, assuming that the order of the autoregressive correction in (6) is known – i.e., $k = p - 1$. The modified $\hat{\alpha}_{TW}$ estimator using either the asymptotic $\tau_{pct} = \tau_{50}$ or $\tau_{pct} = \tau_{85}$ percentiles. Let us first focus on the $\hat{\alpha}_{MU}$ estimator. As can be seen in Table 3, for $T = 50$ and $\overline{c} > 0.1$, the $\hat{\alpha}_{MU}$ estimator tends to under-estimate α, being the distortion more pronounced as α approaches one. Note that the bias correction does not provide good results for $\overline{c} = 0.1$.[6] The performance improves as T increases, showing that the estimation bias is almost fully corrected in most cases – the exception is found for $\alpha > 0.8$ and $\overline{c} = 0.1$, where the estimated value of the autoregressive parameter is below the populational one. It is worth noticing that there is a mild under-estimation distortion when $\alpha = 1$, something that is not found in the I(0) stationary cases.

Table 4 reports two versions of the $\hat{\alpha}_{TW}$ estimator, depending on the τ_{pct} value that is used in the truncation. In general, and for a given set of T, α and $\overline{c}$ values, the estimator $\hat{\alpha}_{TW}$ that is based on τ_{50} – in what follows, $\hat{\alpha}_{TW}^{\tau_{50}}$ – outperforms the one based on τ_{85} – henceforth, $\hat{\alpha}_{TW}^{\tau_{85}}$. In most cases, both estimators $\hat{\alpha}_{TW}^{\tau_{50}}$ and $\hat{\alpha}_{TW}^{\tau_{85}}$ produce the same result but, when they differ, $\hat{\alpha}_{TW}^{\tau_{50}}$ does better than $\hat{\alpha}_{TW}^{\tau_{85}}$. As expected, the bias correction improves as T increases, regardless of α and $\overline{c}$.

When the analysis compares the $\hat{\alpha}_{MU}$ and $\hat{\alpha}_{TW}$ estimators, we notice that for $T = 50$, $\alpha \geq 0.2$ and $\overline{c} < 0.3$, $\hat{\alpha}_{MU}$ outperforms $\hat{\alpha}_{TW}^{\tau_{50}}$. However, the opposite situation is found when $\overline{c} \geq 0.3$, where a minimal dominance of $\hat{\alpha}_{TW}^{\tau_{50}}$ is observed regardless of α – note that differences are small. The picture is qualitatively similar for $T = 200$. Considering these elements, $\hat{\alpha}_{MU}$ offers an overall compromise in terms of bias estimation correction compared to $\hat{\alpha}_{TW}$ because, although $\hat{\alpha}_{TW}^{\tau_{50}}$ outperforms $\hat{\alpha}_{MU}$ in some cases, the difference is small.

[6] It can be stated that $\hat{\alpha} \to 1$ when $\overline{c}$ approaches to zero with $[\underline{c}, \overline{c}] = [-\overline{c}, \overline{c}]$, independently of the value of α, which evidences the difficulty of estimating α in these cases.

So far, the analysis is conditional on the fact that the true order of the autoregressive model is used, something that is not realistic from an empirical point of view. Tables 5 and 6 summarize the performance of $\hat{\alpha}_{TW}$ when k in (6) is selected using the MAIC and BIC information criteria, respectively. In general, the use of MAIC leads $\hat{\alpha}_{TW}$ to over-estimate α compared to the BIC-based results. Consequently, BIC-based estimates is preferred to MAIC-based ones. The comparison of Tables 4 and 6 reveals that the estimation of p improves the performance of $\hat{\alpha}_{TW}^{\tau_{50}}$ in cases in which $\hat{\alpha}_{MU}$ outperformed $\hat{\alpha}_{TW}^{\tau_{50}}$ with known p. Further, if the comparison is established between $\hat{\alpha}_{TW}^{\tau_{50}}\,(BIC)$ and $\hat{\alpha}_{MU}$, we can observe that there are some cases in which $\hat{\alpha}_{MU}$ outperforms $\hat{\alpha}_{TW}^{\tau_{50}}\,(BIC)$ – for instance, $T = 50$ with $\bar{c} < 0.3$, and $T = 200$ with $\bar{c} < 0.3$ and $\alpha < 0.7$ – although there are many cases in which the opposite situation is found – for instance, for $T = 50$, $\bar{c} \geq 0.3$ and $\alpha > 0.3$.

Taking all these elements together, we can conclude that there is no clear predominance of one estimator over the other, but considering that p is not known in practice, the use of $\hat{\alpha}_{TW}^{\tau_{50}}$ with k in (6) selected using BIC provides a good compromise in terms of bias estimation correction for empirical applications. Finally, it is worth mentioning that the computation of $\hat{\alpha}_{TW}^{\tau_{50}}$ is based on the percentile of the limiting distribution of $\hat{\tau}_W$, whereas $\hat{\alpha}_{MU}$ uses look-up tables that have been simulated for the specific empirical size. This feature might explain the superiority of $\hat{\alpha}_{MU}$ over $\hat{\alpha}_{TW}^{\tau_{50}}$ in some cases, so using the percentiles of the finite sample distribution of $\hat{\tau}_W$ might lead to some improvements. Notwithstanding, an important advantage is that the computational cost to obtain $\hat{\alpha}_{TW}^{\tau_{50}}\,(BIC)$ is smaller than getting sample size-specific look-up tables that deliver $\hat{\alpha}_{MU}$.

4.1.2 The AR(p) case

This section generalizes the analysis considering the DGP defined by an AR(2) process:

$$\begin{aligned} x_t &= \mu + y_t \\ y_t &= \alpha y_{t-1} + \psi \Delta y_{t-1} + u_t, \end{aligned}$$

with $\mu = 0$, $\alpha = \{0.8, 0.9, 0.95, 1\}$ and $\psi = 0.5$. The disturbance term u_t is defined in (3) with $\varepsilon_t \sim iid\ N(0,1)$. In this case, we only report simulation results for $\hat{\alpha}_{TW}$ since the estimator of Andrews (1993) was proposed only for the AR(1) case. As mentioned above, it would be possible to compute the generalization of the MU estimator for AR(p) cases suggested in Andrews and Chen (1994), but the high computational cost implied prevents us from doing so.[7] Table 7 presents the mean of the distribution of $\hat{\alpha}_{TW}$ using either $\tau_{pct} = \tau_{50}$ and $\tau_{pct} = \tau_{85}$, when p is known or estimated using the MAIC and BIC

[7] The procedure of Andrews and Chen (1994) would require the use of bootstrap for each replication of the Monte Carlo simulation experiment.

information criteria.

Let us focus on the case where k in (6) is known. First, note that, for a given T, the results for $\hat{\alpha}_{TW}^{\tau 50}$ and $\hat{\alpha}_{TW}^{\tau 85}$ are almost equivalent when $\overline{c} \geq 0.3$, with the mean of the estimators around α when $T = 200$. It is worth noticing that a mild under-estimation is produced when $\alpha = 1$, which suggests that the super-efficient estimator of Perron and Yabu (2009a) could be used in this case. Second, $\hat{\alpha}_{TW}^{\tau 50}$ provides better estimates than $\hat{\alpha}_{TW}^{\tau 85}$ for $\overline{c} = 0.2$ although, in both cases, the estimated values are below α. Interestingly, these estimates collapse around a similar value regardless of α. For instance, when $\overline{c} = 0.2$, the mean of $\hat{\alpha}_{TW}^{\tau 50}$ is estimated around 0.5 for $T = 50$ and 0.74 for $T = 200$, regardless of α. Finally, a strange phenomenon appears for $\overline{c} = 0.1$ because the estimation bias correction worsens as T increases – as before, the mean of the estimators seems to collapse around the same value for all range of α parameters that have been specified.

Simulations based on the MAIC selection of k in (6) show over-estimation of α when $\alpha = 0.8$, but the results are reasonably good for $T = 200$ with $\overline{c} \geq 0.5$. Similar to the previous case, $\hat{\alpha}_{TW}^{\tau 50}$ and $\hat{\alpha}_{TW}^{\tau 85}$ tend to collapse around a given value when $\overline{c} \leq 0.3$, providing useless estimates. For instance, when $\overline{c} = 0.1$, the mean of $\hat{\alpha}_{TW}^{\tau 50}$ is around 0.94 ($T = 50$) and 0.96 ($T = 200$) regardless of α. Better results are obtained when k is chosen using the BIC. In this case, the estimates are located around α for $T = 200$ and $\overline{c} \geq 0.4$, although mild under-estimation is produced when $\alpha = 1$ – this might be improved with the super-efficient estimator of Perron and Yabu (2009a). The estimation procedure does not deliver satisfactory results for $\overline{c} \leq 0.3$. Thus, a similar picture to the known k case is obtained for $\overline{c} = 0.2$ or $\overline{c} = 0.3$, that is, estimates clearly below the true α which collapse around a given value regardless of α. Finally, we also observe the same behavior for $\overline{c} = 0.1$, although the value to which the estimators tend increases with T.

In all, results based on BIC provide a good compromise from the empirical point of view considering that, first, the order of the autoregressive correction is unknown and, second, the performance of the estimation bias correction is similar to the known p case. However, we have noticed that the performance of the estimators is not satisfactory for small values of $\overline{c}$, say $\overline{c} \leq 0.3$.

4.1.3 The ARMA(1,1) case

The third set of DGP that is tried out is given by:

$$\begin{aligned} x_t &= \mu + y_t \\ y_t &= \alpha y_{t-1} + u_t + \theta u_{t-1}, \end{aligned}$$

with $\mu = 0$, $\alpha = \{0.8, 0.9, 0.95, 1\}$ and $\theta = \{-0.8, -0.4, 0.4, 0.8\}$. The disturbance term u_t is assumed to be decomposed as defined in (3) with $\varepsilon_t \sim iid\ N(0,1)$. Given the results above, we base the simulations on the use of the BIC information criterion to select the lag length in (6).

Tables 8 and 9 present the mean of the distribution of $\hat{\alpha}_{TW}^{\tau_{50}}$ and $\hat{\alpha}_{TW}^{\tau_{85}}$, respectively. For $T = 200$, the results reported in both tables are virtually equivalent, regardless of τ_{pct} and $\bar{c}$ while, for $T = 50$, the results are similar when $\bar{c} \geq 0.4$. For $\bar{c} \geq 0.4$, negative values of θ lead us to under-estimate α – tending to zero when there is a common factor in the lag polynomials (i.e., $\alpha = -\theta = 0.8$) – whereas positive values of θ do not have significant effects on the estimation bias correction, which improves as T increases. In general, under-estimation effects are observed when $\bar{c} \leq 0.3$, being more pronounced when $\tau_{pct} = \tau_{85}$ is used. This suggests that $\hat{\alpha}_{TW}^{\tau_{50}}$ should be preferred to $\hat{\alpha}_{TW}^{\tau_{85}}$.

4.2 Iterative estimation procedure

This section focuses on the AR(1) model defined in Section 4.1.1, but estimating α in an iterative fashion using a maximum of 20 iterations. Table 10 reports the simulation results for different values of α and $\bar{c} = \{0.1, 0.2, 0.3, 0.4, 0.5\}$ – results do not change for larger values of $\bar{c}$. For ease of comparison, results based on the non-iterative estimation procedure are also shown.

The MU estimator shows similar performance regardless of the estimation method implementation that is used. There are marginal gains derived from the use of the iterative estimation procedure for $\bar{c} \geq 0.3$, but the performance worsens for $\bar{c} = 0.1$. In general, the $\hat{\alpha}_{TW}^{\tau_{50}}$ $(MAIC)$ non-iterative procedure presents over-estimation distortions, which also prevail for the iterative estimation procedure – in this case, under-estimation distortions are observed for $\bar{c} < 0.3$ for some α. Consequently, the unstable performance of $\hat{\alpha}_{TW}^{\tau_{50}}$ $(MAIC)$ leads us to discourage the use of this estimator in practice. Finally, the non-iterative $\hat{\alpha}_{TW}^{\tau_{50}}$ (BIC) estimator over-estimates α for $\bar{c} = 0.1$ and $\alpha \leq 0.8$, and under-estimates α for $\bar{c} \leq 0.2$ and $\alpha \geq 0.85$. Better performance is shown for the iterative version, although under-estimation is observed for $\bar{c} = 0.1$, and for $\bar{c} = 0.2$ with $\alpha \geq 0.85$.

As can be seen, $\hat{\alpha}_{MU}$ outperforms $\hat{\alpha}_{TW}^{\tau_{50}}$ (BIC), although this superiority benefits from the fact that the former method is based on the correct order of the autoregressive model. As mentioned above, considering that p is unknown in practice, $\hat{\alpha}_{TW}^{\tau_{50}}$ (BIC) represents a good compromise in terms of bias estimation correction for empirical applications.

5 Empirical illustration

The persistence of the current account balance (CAB) disequilibrium is a crucial issue for assessing the long-term solvency and sustainability of the external debt of a country. Different conditions related to the order of integration of the ratio of CAB over GDP ($cab_t = CAB_t/GDP_t * 100$) have been proposed in the literature to test external sustainability and, consequently, the unit root approach has been extensively applied.[8] Most of the existing empirical literature considers that the current account is an unbounded variable, being Herwartz and Xu (2008) one exception that considers the presence of bounds when analysing CAB over GDP sustainability. If this feature is not accounted for when testing the order of integration of the variables or computing their degree of persistence, the conclusions drawn from the unit root test statistics or the autoregressive parameter estimates can be misleading.

The cab_t variable is not theoretically bounded, since flows of goods, services, income and transfers could exceed the total value of GDP. However, economic prudence does not advise maintaining large imbalances, which are a reflection of serious dysfunctions in the internal macroeconomic fundamentals, in the current account balance. Policy makers can introduce the control of CAB into their macro-prudential objectives more or less explicitly. Recently, and due to the aftermath of the Great Recession in Europe and, especially, in the European Monetary Union, a special system for monitoring macroeconomic imbalances was introduced for countries belonging to the euro area. This system called, Macroeconomic Imbalance Procedure (MIP) was established in 2011 and aims to identify and prevent potentially harmful macroeconomic imbalances that could adversely affect economic stability in a particular Member State, the euro area, or the EU as a whole. It controls a total of fourteen indicators, covering the major sources of macroeconomic imbalances and setting indicative thresholds for each of them. Among them, we can find several related to the health of the external foreign sector and, most interestingly for our case, the thresholds for cab_t, which are +6% and -4%, with a dynamic of a 3-year backward moving average. Consequently, we could define a first set of bounds to be $cab_t \in \left[\underline{b}, \overline{b}\right] = [-4, 6]$.

A second set of bounds can also be settled following the strategy in Herwartz and Xu (2008), who consider a potential set of bounds arising from the observed country-specific minimum ($\underline{b} = \min(cab_t)$) and maximum ($\overline{b} = \max(cab_t)$) values of cab_t – which defines $\left[\underline{\hat{c}}, \overline{\hat{c}}\right] = \left[(\underline{b} - \hat{D}_t)/(\hat{\sigma} T^{1/2}), (\overline{b} - \hat{D}_t)/(\hat{\sigma} T^{1/2})\right]$. In addition, and following the spirit in Herwartz and Xu (2008), we increase this initial range up to 300 per cent in absolute value – i.e., $\left[\underline{\hat{c}} - \delta\omega/2, \overline{\hat{c}} + \delta\omega/2\right]$, $\omega = |\overline{\hat{c}} - \underline{\hat{c}}|$ and $\delta = \{0, 0.1, 0.2, 0.3, \ldots, 1, 1.5, 2, 2.5, 3\}$ – so that the robustness of the analysis can be carried out using different sets of bounds. The key issue here is how to select among these values of bounds. The suggestion in Herwartz and Xu (2008) bases on the p-values of the augmented Dickey-Fuller (ADF) unit root test, so that the bounds are selected such that the p-values of the ADF statistic

[8] The economic theory underpinning this empirical literature stems from the inter-temporal approach to the current account, which was initially proposed by Sachs (1981) and Buiter (1981) and later extended by Obstfeld and Rogoff (1995) and by Gourinchas and Rey (2007). This approach considers a country's inter-temporal budget constraint that links the net foreign asset position and the future dynamics of the current account. Recently, Camarero et al. (2015) contribute to the discussion in the context of the European monetary integration process.

with and without bounds equalize – the so-called "break-even" bounds, which are denoted by $[\underline{b}, \overline{b}] = [\underline{b}^*, \overline{b}^*]$. The use of "break-even" bounds warrants a minimum range under which the standard ADF unit root test does not suffer from oversizing.

The data set comprises sixteen developed OECD countries for which annual data is available from 1980 to 2016. The source is the International Monetary Fund, World Economic Outlook Database, October 2016 and the evolution of the series is displayed in Figure 6. This figure includes the horizontal lines defined by the upper and lower thresholds $[\underline{b}, \overline{b}] = [-4, 6]$, although some countries, such as Norway, Sweden, Switzerland and the United Kingdom, are not monitored by the MIP. As it can be seen, only for France and Italy the observed cab_t time series lay inside the MIP range during the period of analysis. It should be born in mind that these values define a target range that would condition policies undertaken by government under MIP surveillance system so that convergence of cab_t towards the target range is pursued. Therefore, although such values can be acting as attractors, it is manifestly evident that for most of cases the time series lay outside the target range. In this case the second strategy that defines the set of bounds based on the observed range of values for each country plays an important role in the analysis.

The model estimated for each country (i) is given by:

$$cab_{i,t} = \mu_i + \rho_i cab_{i,t-1} + \sum_{j=1}^{k_i} \psi_{i,j} \Delta cab_{i,t-j} + \varepsilon_{i,t},$$

$i = 1, \ldots, 16$ and $t = 1980, \ldots, 2016$. Table 11 shows that the degree of persistence (measured by $\hat{\rho}_i$) is relatively low and, on the whole, far from the unit root neighbourhood when standard OLS or bias-corrected OLS estimates are used.[9] Nevertheless, when the bounded nature of time series is considered, most countries show a unit root, emphasizing the insufficiency of the market to promote effective adjustments to offset external disequilibria, and supporting the surveillance measures proposed by the European Commission, as Camarero et al. (2015) highlight. These results are obtained regardless of whether the MIP-based or the min/max-based bounds are used since the bias corrected estimates are almost equivalent in both cases. Finally, when the break-even bounds are specified, bias corrected estimates show slightly smaller values than the ones obtained with the use of the MIP-based and the min/max-based bounds. However, this is somewhat to be expected given the fact that the break-even bounds define a set of bounds that is closer to the unbounded case.

[9] For Andrews method an AR(1) model is estimated. For the WSLS the autoregressive order is selected using BIC, although similar results are obtained when the MAIC is applied.

6 Conclusions

This paper analyzes the behavior of the first order autoregressive estimator when the stochastic process being studied is influenced by the presence of bounds that regulate its evolution. We consider both the standard OLS estimator as well as some of the techniques proposed in the literature in order to correct the finite sample bias of this estimator.

We first show that the presence of bounds clearly distorts the performance of both types of estimators. The more limited the stochastic process – i.e., the narrower the fluctuation bands – the higher the distortion effect. This is especially harmful when the autoregressive parameter takes values close to 1, given that the estimated values tend to take values close to 0. This clearly alters the interpretation of the results, leading practitioners to observe a scarce level of persistence when the variable is, in fact, extremely persistent.

In order to remove this effect, we have proposed some modifications of the methods proposed by Andrews (1993) and Perron and Yabu (2009a) that account for the bounded nature of the time series. Simulation experiments have evidenced that these extensions are quite helpful in order to appropriately determine shock persistence for bounded stochastic processes. Notwithstanding, estimation bias persists in those cases for which the rank variation defined by the bounds is very narrow.

Finally, we have applied these new methods to the analysis of the current account balance of a sample of developed countries. Our results show that the use of the proposed methods improve our knowledge about the stochastic properties of the variables under study, allowing us to carry out more adequate shock persistence analysis.

A Mathematical appendix

Lemma 1 *Let* $\{y_t\}_{t=1}^{T}$ *be the stochastic process given by (2)-(3) with* $\mu = 0$*. Then:*
a) $E\left(y_t y_{t-1}\right) = E\left(\varepsilon_t^* \varepsilon_{t-1}^*\right) + O_p(T^{1/2})$.
b) $E\left(y_t y_{t+k} y_{t+k+l} y_{t+k+l+m}\right) = E\left(\varepsilon_t^* \varepsilon_{t+k}^* \varepsilon_{t+k+l}^* \varepsilon_{t+k+l+m}^*\right) + O_p(T^{1/2})$.

$$\text{with } \varepsilon_t^* = \sum_{i=1}^{t} \alpha^i \varepsilon_{t-i} \text{ and } r_t^* = \sum_{i=1}^{t} \alpha^i (\xi_{t-i} - \bar{\xi}_{t-i})$$

Proof. For statement (a), we have:

$$\begin{aligned} E\left(y_t y_{t-1}\right) &= E\left[\left(\varepsilon_t^* + r_t^*\right)\left(\varepsilon_{t-1}^* + r_{t-1}^*\right)\right] \\ &= E\left[\varepsilon_t^* \varepsilon_{t-1}^*\right] + E\left[\varepsilon_t^* r_{t-1}^*\right] + E\left[r_t^* \varepsilon_{t-1}^*\right] + E\left[r_t^* r_{t-1}^*\right] \\ &= E\left[\varepsilon_t^* \varepsilon_{t-1}^*\right] + R. \end{aligned}$$

Then, we should prove that R is an $O_p(T^{1/2})$. To do it, it suffices to note that:

$$\begin{aligned} \left| T^{-1} \sum_{t=1}^{T} \left(\varepsilon_t^* r_{t-1}^* + r_t^* \varepsilon_{t-1}^* + r_t^* r_{t-1}^*\right) \right| &\leq T^{-1} \sum_{t=1}^{T} \left| \left(\varepsilon_t^* r_{t-1}^* + r_t^* \varepsilon_{t-1}^* + r_t^* r_{t-1}^*\right) \right| \leq \\ &\leq T^{-1} \left\{ \left[\max\left(\left|\varepsilon_t^*\right|\right) + \max\left(\left|r_t^*\right|\right)\right] \sum_{t=1}^{T} \left|r_{t-1}^*\right| \right\} \\ &+ T^{-1} \max\left(\left|\varepsilon_{t-1}^*\right|\right) \sum_{t=1}^{T} \left|r_t^*\right| = o_p\left(1\right), \end{aligned}$$

given that $\max\left(\left|\varepsilon_t^*\right|\right)$, $\max\left(\left|r_t^*\right|\right)$ are $o_p\left(T^{1/2}\right)$ and $\sum_1^T \left|r_t^*\right|$ is $O_p\left(T^{1/2}\right)$, according to the results of Cavaliere and Xu (2014).
For statement (b), we write:

$$\begin{aligned} E\left(y_t y_{t+k} y_{t+k+l} y_{t+k+l+m}\right) &= E\left[\left(\varepsilon_t^* + r_t^*\right)\left(\varepsilon_{t+k}^* + r_{t+k}^*\right)\right. \\ &\quad \left.\left(\varepsilon_{t+k+l}^* + r_{t+k+l}^*\right)\left(\varepsilon_{t+k+l+m}^* + r_{t+k+l+m}^*\right)\right] \\ &= E\left(\varepsilon_t^* \varepsilon_{t+k}^* \varepsilon_{t+k+l}^* \varepsilon_{t+k+l+m}^*\right) + M, \end{aligned}$$

with M containing the covariates between ε_t^* and r_t^*, which can be generarically defined as $E\left[(\varepsilon_t^*)^s (r_t^*)^{4-s}\right]$, $s = 1, 2, 3$, and $E\left[(r_t^*)^4\right]$. We have to prove that all these elements are

$o_p\left(T^{-2}\right)$. To that end, we should note that:

$$\left|T^{-1/2}\sum_{t=1}^{T}\varepsilon_t^*\varepsilon_{t-i}^*\varepsilon_{t-j}^*r_t^*\right| \leq T^{-3/2}\sum_{t=1}^{T}\left|\max(|\varepsilon_t^*|)^3 r_t^*\right| \leq T^{-1/2}\max\left[(|\varepsilon_t^*|)^3\right]\sum_{t=1}^{T}r_t^* = O_p\left(1\right),$$

$$\begin{aligned}\left|T^{-1/2}\sum_{t=1}^{T}\varepsilon_t^*\varepsilon_{t-i}^*r_t^*r_{t-i}^*\right| &\leq T^{-1/2}\sum_{t=1}^{T}\left|\max(\varepsilon_t^*)^2\max(r_t^*)r_t^*\right| \\ &\leq T^{-1/2}\max\left[(\varepsilon_t^*)^2\right]\max\left[(|r_t^*|)\right]\sum_{t=1}^{T}r_t^* = O_p\left(1\right),\end{aligned}$$

$$\begin{aligned}\left|T^{-1/2}\sum_{t=1}^{T}\varepsilon_t^*r_t^*r_{t-i}^*r_{t-j}^*\right| &\leq T^{-1/2}\sum_{t=1}^{T}\left|\max(\varepsilon_t^*)\max\left[(|r_t^*|)^2\right]r_t^*\right| \\ &\leq T^{-1/2}\max\left[\varepsilon_t^*\right]\max\left[(|r_t^*|)^2\right]\sum_{t=1}^{T}r_t^* = O_p\left(1\right).\end{aligned}$$

■

Lemma 2 *Let* $\{y_t\}_{t=1}^{T}$ *be the stochastic process given by (2)-(3). Further, let us define* $N = (T-1)^{-1}\sum_{t=1}^{T-1}y_t y_{t-1}$ *and* $D = T^{-1}\sum_{t=1}^{T}y_t^2$*. Then:*
a) $E\left(N\right) = \frac{\alpha}{1-\alpha^2} + o_p\left(T^{-1}\right)$.
b) $E\left(D\right) = \frac{1}{1-\alpha^2} + o_p\left(T^{-1}\right)$.
c) $Var(D) = \frac{2\left(1+\alpha^2\right)}{T\left(1-\alpha^2\right)^3} + O_p\left(T^{-3/2}\right)$.
d) $Cov(N,D) = \frac{4\alpha}{T\left(1-\alpha^2\right)^3} + O_p\left(T^{-3/2}\right)$.

Proof. Statements (a) and (b) derive from results in Lemma 1. To prove (c) we should note that:

$$\begin{aligned}E\left(D^2\right) &= \frac{1}{T^2}E\left[\left(\sum_{t=1}^{T}y_t^2\right)^2\right] = \frac{1}{T^2}E\left[\left(\sum_{t=1}^{T}\left(\varepsilon_t^*+r_t^*\right)^2\right)^2\right] \\ &= \frac{1}{T^2}E\left[\left(\sum_{t=1}^{T}\left(\varepsilon_t^*\right)^2\right)^2 + O_p(T^{1/2})\right] \\ &= \frac{1}{T^2}\left[\frac{3T}{\left(1-\alpha^2\right)^2} + 2\sum_{i=1}^{T-1}\left(T-i\right)\frac{1+2\alpha^{2i}}{\left(1-\alpha^2\right)^2} + O_p(T^{1/2})\right] \\ &= \frac{1}{\left(1-\alpha^2\right)^2} + \frac{2\left(1+\alpha^2\right)}{T\left(1-\alpha^2\right)^3} + O_p(T^{-3//2}),\end{aligned}$$

and, subsequently, we have that:

$$Var\left(D\right) = \frac{2\left(1+\alpha^2\right)}{T\left(1-\alpha^2\right)^3} + O_p(T^{-3/2}).$$

Similarly, in order to prove (d), we should note that:

$$\begin{aligned}
E\left(ND\right) &= \frac{1}{T\left(T-1\right)} E\left[\left(\sum_{t=2}^{T} y_{t-1}y_t\right)\left(\sum_{t=1}^{T} y_t^2\right)\right] \\
&= \frac{1}{T\left(T-1\right)}\left[2(T-1)E\left(y_t^3 y_{t-1}\right) + 2\sum_{j=3}^{T}(T-1-j)E(y_t y_{t-1} y_{t-1-j}^2)\right] \\
&= \frac{1}{T\left(T-1\right)}\left[2(T-1)E\left[(\varepsilon_t^* + r_t^*)^3(\varepsilon_{t-1}^* + r_{t-1}^*)\right]\right. \\
&\quad \left. +2\sum_{j=3}^{T}(T-1-j)E((\varepsilon_t^* + r_t^*)(\varepsilon_{t-1}^* + r_{t-1}^*)(\varepsilon_{t-1-j}^* + r_{t-1-j}^*)^2\right].
\end{aligned}$$

Rearranging terms,

$$\begin{aligned}
E\left(ND\right) &= \frac{1}{T\left(T-1\right)}\left[2(T-1)\left\{E\left[(\varepsilon_t^*)^3\varepsilon_{t-1}^*\right] + O_p(T^{1/2})\right\}\right. \\
&\quad \left. +2\sum_{j=3}^{T}(T-1-j)\left\{E\left[(\varepsilon_t^*\varepsilon_{t-1}^*(\varepsilon_{t-1-j}^*)^2\right] + O_p(T^{1/2})\right\}\right] \\
&= \frac{\alpha}{\left(1-\alpha^2\right)^2} + \frac{4\alpha}{T\left(1-\alpha^2\right)^3} + O_p(T^{-3/2}).
\end{aligned}$$

Then,

$$Cov\left(N,D\right) = \frac{4\alpha}{T\left(1-\alpha^2\right)^3} + O_p(T^{-3/2}).$$

■

A.1 Proof of Theorem 1

Let us consider that the variable y_t is generated by (2) and (3) with $|\alpha| < 1$, $\mu = 0$ and $\varepsilon_t \sim iid\left(0,1\right)$. Following Mariott and Pope (1954), let us define the first order autorregresive parameter as $r = N/D$, where

$$N = \frac{1}{T-1}\sum_{t=1}^{T-1} y_t y_{t-1}, \tag{7}$$

$$D = \frac{1}{T}\sum_{t=1}^{T} y_t^2. \tag{8}$$

Then, the expected value of r is given by:

$$E(r) = \frac{\nu}{\delta}\left\{1 - \frac{Cov(N,D)}{\nu\delta} + \frac{Var(D)}{\delta^2}\right\}, \tag{9}$$

where $E(N) = \nu$ and $E(D) = \delta$. Using the results of Lemmas 1 and 2 we have that:

$$E(r) = E(\hat{\alpha}) = \alpha - \frac{2\alpha}{T},$$

so that

$$\hat{\alpha} \xrightarrow{p} \alpha.$$

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Table 1: Andrews MU estimates for symmetric bounded stochastic processes

	α_{MU}				
α	$\bar{c} = 0.1$	$\bar{c} = 0.3$	$\bar{c} = 0.5$	$\bar{c} = 0.7$	$\bar{c} = 1$
0	-0.01	-0.01	-0.01	-0.00	-0.00
0.1	0.07	0.09	0.09	0.10	0.10
0.2	0.15	0.19	0.19	0.19	0.19
0.3	0.23	0.29	0.29	0.29	0.29
0.4	0.29	0.39	0.39	0.40	0.39
0.5	0.36	0.49	0.49	0.49	0.49
0.6	0.41	0.58	0.59	0.58	0.59
0.7	0.45	0.69	0.68	0.69	0.69
0.8	0.48	0.79	0.79	0.78	0.79
0.9	0.51	0.87	0.89	0.89	0.88
1	0.53	0.92	0.96	0.96	0.97

Table 2: Percentiles of the limiting distribution of $\hat{\tau}_W$ for different (symmetric) bounds

$(\underline{c}, \bar{c})$	1%	2.5%	5%	7%	7.5%	10%	15%	50%	85%
$(-0.1, 0.1)$	-9.16	-9.01	-8.88	-8.82	-8.80	-8.74	-8.64	-8.25	-7.89
$(-0.2, 0.2)$	-5.39	-5.18	-5.02	-4.94	-4.93	-4.86	-4.76	-4.38	-4.07
$(-0.3, 0.3)$	-4.58	-4.21	-3.94	-3.82	-3.79	-3.70	-3.56	-3.11	-2.80
$(-0.4, 0.4)$	-4.17	-3.85	-3.58	-3.44	-3.41	-3.28	-3.09	-2.52	-2.17
$(-0.5, 0.5)$	-3.75	-3.49	-3.27	-3.15	-3.13	-3.02	-2.85	-2.22	-1.79
$(-0.6, 0.6)$	-3.37	-3.14	-2.95	-2.86	-2.84	-2.74	-2.60	-2.04	-1.56
$(-0.7, 0.7)$	-3.15	-2.89	-2.70	-2.61	-2.59	-2.50	-2.38	-1.89	-1.42
$(-0.8, 0.8)$	-3.11	-2.81	-2.56	-2.45	-2.43	-2.33	-2.20	-1.74	-1.32
$(-0.9, 0.9)$	-3.10	-2.79	-2.54	-2.40	-2.38	-2.26	-2.08	-1.59	-1.20
$(-1.0, 1.0)$	-3.14	-2.81	-2.55	-2.40	-2.38	-2.25	-2.06	-1.46	-1.08
$(-1.5, 1.5)$	-3.13	-2.81	-2.53	-2.39	-2.36	-2.23	-2.03	-1.20	-0.49
$(-\infty, \infty)$	-3.12	-2.80	-2.53	-2.39	-2.37	-2.24	-2.04	-1.21	-0.24

Table 3: Bias corrected estimator using Andrews median-unbiased estimator

T	$\alpha\backslash\overline{c}$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.5
50	0	0.39	0.08	0.07	0.06	0.07	0.06	0.06	0.06	0.06	0.06	0.06
	0.1	0.41	0.15	0.13	0.13	0.13	0.13	0.13	0.13	0.13	0.13	0.13
	0.2	0.43	0.23	0.21	0.21	0.21	0.21	0.21	0.21	0.21	0.21	0.21
	0.3	0.46	0.32	0.31	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30
	0.4	0.49	0.43	0.41	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.40
	0.5	0.51	0.54	0.51	0.50	0.49	0.49	0.50	0.50	0.49	0.49	0.49
	0.6	0.53	0.63	0.61	0.60	0.59	0.59	0.59	0.59	0.59	0.59	0.59
	0.7	0.55	0.70	0.72	0.71	0.69	0.69	0.69	0.69	0.69	0.69	0.69
	0.8	0.58	0.76	0.80	0.81	0.80	0.80	0.79	0.79	0.79	0.79	0.79
	0.85	0.58	0.79	0.82	0.84	0.85	0.85	0.84	0.84	0.84	0.84	0.84
	0.9	0.59	0.81	0.84	0.86	0.87	0.89	0.89	0.89	0.89	0.89	0.88
	0.95	0.60	0.82	0.86	0.87	0.88	0.90	0.91	0.92	0.92	0.92	0.92
	1	0.62	0.83	0.89	0.91	0.91	0.92	0.92	0.93	0.94	0.94	0.96
200	0	0.04	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03
	0.1	0.11	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
	0.2	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20
	0.3	0.30	0.29	0.30	0.30	0.30	0.30	0.30	0.30	0.29	0.30	0.30
	0.4	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.39	0.40	0.40
	0.5	0.51	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.49	0.50	0.50
	0.6	0.62	0.59	0.60	0.60	0.60	0.60	0.60	0.60	0.60	0.60	0.59
	0.7	0.73	0.69	0.69	0.69	0.69	0.69	0.69	0.69	0.69	0.69	0.69
	0.8	0.81	0.80	0.79	0.79	0.79	0.79	0.79	0.79	0.79	0.79	0.79
	0.85	0.83	0.87	0.85	0.85	0.84	0.84	0.84	0.84	0.84	0.84	0.84
	0.9	0.86	0.92	0.91	0.90	0.90	0.90	0.90	0.89	0.90	0.90	0.90
	0.95	0.88	0.94	0.95	0.95	0.95	0.95	0.95	0.95	0.95	0.95	0.95
	1	0.90	0.96	0.97	0.98	0.98	0.98	0.98	0.98	0.99	0.99	0.99

Table 4: Mean of the distribution of $\hat{\alpha}_{TW}$. The AR(1) case with k known

		τ_{50}											τ_{85}										
T	$\alpha\backslash\bar{c}$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.5	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.5
50	0	0.22	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.03	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
	0.1	0.24	0.09	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.04	0.09	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.11
	0.2	0.25	0.17	0.20	0.21	0.21	0.21	0.21	0.21	0.21	0.21	0.21	0.06	0.17	0.20	0.21	0.21	0.21	0.21	0.21	0.21	0.21	0.21
	0.3	0.30	0.24	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.07	0.24	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30
	0.4	0.34	0.33	0.39	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.08	0.31	0.39	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.40
	0.5	0.37	0.42	0.48	0.49	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.11	0.37	0.47	0.49	0.50	0.50	0.50	0.49	0.49	0.49	0.49
	0.6	0.39	0.50	0.57	0.59	0.59	0.59	0.59	0.59	0.59	0.59	0.59	0.12	0.43	0.56	0.58	0.59	0.59	0.59	0.59	0.59	0.59	0.58
	0.7	0.41	0.57	0.66	0.68	0.68	0.68	0.68	0.68	0.68	0.68	0.68	0.14	0.48	0.63	0.67	0.68	0.68	0.68	0.68	0.68	0.68	0.67
	0.8	0.44	0.63	0.74	0.76	0.77	0.77	0.77	0.77	0.77	0.77	0.77	0.16	0.52	0.69	0.75	0.76	0.77	0.77	0.77	0.77	0.76	0.76
	0.85	0.45	0.66	0.76	0.80	0.82	0.82	0.82	0.82	0.81	0.81	0.81	0.17	0.54	0.71	0.77	0.80	0.81	0.81	0.81	0.81	0.81	0.80
	0.9	0.45	0.69	0.78	0.83	0.85	0.86	0.86	0.86	0.86	0.86	0.85	0.17	0.55	0.72	0.79	0.83	0.84	0.85	0.85	0.85	0.85	0.84
	0.95	0.47	0.70	0.80	0.83	0.86	0.89	0.90	0.90	0.91	0.90	0.90	0.19	0.57	0.73	0.80	0.83	0.86	0.88	0.89	0.89	0.89	0.88
	1	0.49	0.72	0.84	0.88	0.90	0.91	0.92	0.93	0.94	0.94	0.95	0.18	0.58	0.76	0.83	0.86	0.88	0.90	0.91	0.92	0.93	0.93
200	0	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00
	0.1	0.08	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.08	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
	0.2	0.15	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.15	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20
	0.3	0.23	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.23	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30
	0.4	0.30	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.30	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.40
	0.5	0.37	0.49	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.37	0.49	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50
	0.6	0.43	0.59	0.60	0.60	0.60	0.60	0.60	0.60	0.60	0.60	0.60	0.42	0.59	0.60	0.60	0.60	0.60	0.60	0.60	0.60	0.60	0.60
	0.7	0.51	0.68	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.47	0.68	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.70
	0.8	0.61	0.77	0.79	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.51	0.77	0.79	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
	0.85	0.65	0.81	0.84	0.84	0.84	0.84	0.84	0.84	0.84	0.84	0.84	0.53	0.80	0.84	0.84	0.84	0.84	0.84	0.84	0.84	0.84	0.84
	0.9	0.68	0.86	0.88	0.89	0.89	0.89	0.89	0.89	0.89	0.89	0.89	0.55	0.83	0.88	0.89	0.89	0.89	0.89	0.89	0.89	0.89	0.89
	0.95	0.71	0.89	0.93	0.94	0.94	0.94	0.94	0.94	0.94	0.94	0.94	0.57	0.85	0.91	0.93	0.94	0.94	0.94	0.94	0.94	0.94	0.94
	1	0.74	0.92	0.95	0.97	0.97	0.98	0.98	0.98	0.99	0.99	0.99	0.59	0.87	0.93	0.95	0.96	0.97	0.97	0.98	0.98	0.98	0.98

Table 5: Mean of the distribution of $\hat{\alpha}_{TW}$. The AR(1) case with k estimated using MAIC

		τ_{50}											τ_{85}										
T	$\alpha\backslash\bar{c}$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.5	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.5
50	0	0.94	0.88	0.72	0.51	0.39	0.34	0.30	0.26	0.24	0.23	0.20	0.92	0.85	0.53	0.33	0.25	0.22	0.21	0.20	0.18	0.17	0.12
	0.1	0.94	0.88	0.73	0.56	0.45	0.41	0.38	0.35	0.33	0.31	0.29	0.92	0.84	0.57	0.40	0.33	0.31	0.29	0.28	0.27	0.26	0.22
	0.2	0.94	0.87	0.75	0.61	0.52	0.48	0.45	0.43	0.41	0.40	0.37	0.91	0.84	0.62	0.47	0.41	0.39	0.38	0.37	0.36	0.35	0.31
	0.3	0.94	0.88	0.77	0.66	0.59	0.55	0.52	0.50	0.49	0.48	0.45	0.91	0.85	0.66	0.54	0.49	0.47	0.46	0.45	0.44	0.43	0.40
	0.4	0.94	0.90	0.79	0.70	0.64	0.62	0.59	0.57	0.56	0.55	0.53	0.92	0.86	0.70	0.61	0.56	0.55	0.54	0.53	0.52	0.52	0.49
	0.5	0.95	0.90	0.81	0.73	0.69	0.67	0.65	0.64	0.63	0.62	0.61	0.93	0.87	0.73	0.66	0.63	0.62	0.61	0.60	0.60	0.59	0.57
	0.6	0.94	0.92	0.83	0.76	0.74	0.72	0.71	0.70	0.69	0.69	0.67	0.92	0.88	0.76	0.71	0.69	0.68	0.68	0.67	0.67	0.66	0.64
	0.7	0.95	0.94	0.87	0.81	0.78	0.77	0.76	0.76	0.75	0.75	0.74	0.93	0.89	0.81	0.77	0.75	0.74	0.74	0.74	0.73	0.73	0.71
	0.8	0.94	0.95	0.90	0.87	0.85	0.84	0.83	0.82	0.82	0.81	0.81	0.92	0.90	0.85	0.83	0.82	0.81	0.81	0.81	0.80	0.80	0.79
	0.85	0.95	0.95	0.92	0.89	0.88	0.87	0.86	0.86	0.85	0.85	0.84	0.92	0.91	0.85	0.85	0.85	0.84	0.84	0.84	0.84	0.83	0.82
	0.9	0.95	0.94	0.92	0.91	0.90	0.90	0.90	0.89	0.89	0.88	0.87	0.92	0.89	0.86	0.86	0.87	0.87	0.87	0.87	0.87	0.87	0.86
	0.95	0.96	0.95	0.92	0.91	0.91	0.92	0.92	0.93	0.92	0.92	0.91	0.92	0.89	0.86	0.86	0.88	0.89	0.90	0.90	0.90	0.90	0.89
	1	0.95	0.94	0.93	0.93	0.93	0.93	0.94	0.95	0.95	0.95	0.96	0.92	0.89	0.87	0.88	0.89	0.90	0.91	0.92	0.93	0.94	0.94
200	0	0.97	0.71	0.18	0.12	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.97	0.50	0.13	0.11	0.11	0.11	0.11	0.11	0.11	0.10	0.09
	0.1	0.97	0.74	0.27	0.21	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.97	0.57	0.22	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.18
	0.2	0.97	0.78	0.36	0.30	0.29	0.29	0.29	0.29	0.29	0.29	0.29	0.97	0.62	0.31	0.29	0.29	0.29	0.29	0.29	0.29	0.29	0.28
	0.3	0.97	0.81	0.44	0.39	0.39	0.38	0.38	0.38	0.38	0.38	0.38	0.97	0.67	0.40	0.39	0.38	0.38	0.38	0.38	0.38	0.38	0.37
	0.4	0.98	0.84	0.54	0.48	0.48	0.48	0.48	0.48	0.48	0.48	0.47	0.98	0.73	0.49	0.48	0.48	0.48	0.47	0.47	0.47	0.47	0.46
	0.5	0.99	0.87	0.62	0.57	0.57	0.57	0.57	0.57	0.57	0.56	0.56	0.99	0.78	0.58	0.57	0.57	0.57	0.56	0.56	0.56	0.56	0.56
	0.6	0.99	0.89	0.71	0.66	0.66	0.66	0.66	0.65	0.65	0.65	0.65	0.99	0.83	0.67	0.66	0.66	0.65	0.65	0.65	0.65	0.65	0.65
	0.7	1.00	0.91	0.79	0.75	0.75	0.74	0.74	0.74	0.74	0.74	0.74	0.99	0.87	0.76	0.74	0.74	0.74	0.74	0.74	0.74	0.74	0.74
	0.8	1.00	0.94	0.86	0.84	0.83	0.83	0.83	0.83	0.83	0.83	0.82	1.00	0.91	0.84	0.83	0.83	0.83	0.83	0.82	0.82	0.82	0.82
	0.85	1.00	0.96	0.89	0.87	0.87	0.87	0.87	0.87	0.87	0.86	0.86	0.99	0.93	0.88	0.87	0.87	0.87	0.86	0.86	0.86	0.86	0.86
	0.9	0.99	0.98	0.93	0.91	0.91	0.91	0.90	0.90	0.90	0.90	0.90	0.99	0.96	0.91	0.91	0.90	0.90	0.90	0.90	0.90	0.90	0.90
	0.95	0.99	0.98	0.96	0.95	0.95	0.95	0.95	0.95	0.94	0.94	0.94	0.98	0.96	0.94	0.94	0.94	0.94	0.94	0.94	0.94	0.94	0.94
	1	0.99	0.98	0.98	0.97	0.98	0.98	0.98	0.99	0.99	0.99	0.99	0.98	0.95	0.96	0.96	0.97	0.97	0.98	0.98	0.98	0.98	0.98

Table 6: Mean of the distribution of $\hat{\alpha}_{TW}$. The AR(1) case with k estimated using BIC

		τ_{50}											τ_{85}										
T	$\alpha\backslash\bar{c}$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.5	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.5
50	0	0.41	0.13	0.06	0.05	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.24	0.10	0.05	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.03
	0.1	0.42	0.18	0.14	0.14	0.13	0.13	0.13	0.13	0.13	0.13	0.13	0.23	0.15	0.13	0.13	0.13	0.13	0.13	0.13	0.13	0.13	0.12
	0.2	0.41	0.25	0.23	0.23	0.22	0.22	0.22	0.22	0.22	0.22	0.22	0.23	0.23	0.22	0.22	0.22	0.22	0.22	0.22	0.22	0.22	0.22
	0.3	0.46	0.31	0.31	0.32	0.31	0.31	0.31	0.31	0.31	0.31	0.31	0.25	0.29	0.31	0.31	0.31	0.31	0.31	0.31	0.31	0.31	0.31
	0.4	0.48	0.38	0.41	0.41	0.41	0.41	0.41	0.41	0.41	0.41	0.41	0.26	0.35	0.40	0.41	0.41	0.41	0.41	0.41	0.41	0.41	0.40
	0.5	0.50	0.47	0.49	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.27	0.41	0.49	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.49
	0.6	0.51	0.55	0.58	0.59	0.59	0.59	0.59	0.59	0.59	0.59	0.59	0.28	0.47	0.57	0.59	0.59	0.59	0.59	0.59	0.59	0.59	0.58
	0.7	0.53	0.62	0.67	0.68	0.68	0.68	0.68	0.68	0.68	0.68	0.68	0.30	0.53	0.64	0.67	0.68	0.68	0.68	0.68	0.68	0.68	0.67
	0.8	0.56	0.66	0.75	0.77	0.77	0.77	0.77	0.77	0.77	0.77	0.76	0.31	0.56	0.70	0.75	0.76	0.77	0.76	0.76	0.76	0.76	0.75
	0.85	0.57	0.69	0.77	0.81	0.82	0.82	0.82	0.81	0.81	0.81	0.81	0.33	0.57	0.72	0.78	0.80	0.81	0.81	0.81	0.81	0.80	0.80
	0.9	0.58	0.70	0.79	0.83	0.85	0.86	0.86	0.86	0.86	0.85	0.85	0.32	0.58	0.73	0.79	0.83	0.84	0.85	0.85	0.85	0.85	0.84
	0.95	0.57	0.71	0.81	0.83	0.86	0.88	0.90	0.90	0.90	0.90	0.90	0.33	0.59	0.74	0.80	0.84	0.86	0.88	0.89	0.89	0.89	0.88
	1	0.59	0.72	0.83	0.88	0.90	0.91	0.92	0.93	0.94	0.94	0.95	0.32	0.60	0.77	0.83	0.86	0.88	0.90	0.91	0.92	0.92	0.93
200	0	0.57	0.07	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.52	0.06	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
	0.1	0.58	0.15	0.14	0.14	0.14	0.14	0.14	0.14	0.14	0.14	0.14	0.53	0.14	0.14	0.14	0.14	0.14	0.14	0.14	0.14	0.14	0.14
	0.2	0.58	0.24	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.53	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23
	0.3	0.59	0.33	0.32	0.32	0.32	0.32	0.32	0.32	0.32	0.32	0.32	0.54	0.32	0.32	0.32	0.32	0.32	0.32	0.32	0.32	0.32	0.32
	0.4	0.62	0.42	0.42	0.42	0.42	0.42	0.42	0.42	0.42	0.42	0.42	0.57	0.41	0.42	0.42	0.42	0.42	0.42	0.42	0.42	0.42	0.42
	0.5	0.65	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.61	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51
	0.6	0.69	0.60	0.60	0.60	0.60	0.60	0.60	0.60	0.60	0.60	0.60	0.66	0.60	0.60	0.60	0.60	0.60	0.60	0.60	0.60	0.60	0.60
	0.7	0.73	0.69	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.69	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.70
	0.8	0.78	0.77	0.79	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.71	0.77	0.79	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
	0.85	0.78	0.82	0.84	0.84	0.84	0.84	0.84	0.84	0.84	0.84	0.84	0.71	0.81	0.84	0.84	0.84	0.84	0.84	0.84	0.84	0.84	0.84
	0.9	0.79	0.87	0.88	0.89	0.89	0.89	0.89	0.89	0.89	0.89	0.89	0.69	0.84	0.88	0.89	0.89	0.89	0.89	0.89	0.89	0.89	0.89
	0.95	0.80	0.89	0.93	0.94	0.94	0.94	0.94	0.94	0.94	0.94	0.94	0.69	0.86	0.91	0.93	0.94	0.94	0.94	0.94	0.94	0.94	0.94
	1	0.82	0.92	0.95	0.97	0.97	0.98	0.98	0.98	0.99	0.99	0.99	0.70	0.87	0.93	0.95	0.96	0.97	0.97	0.98	0.98	0.98	0.98

Table 7: Mean of the distribution of $\hat{\alpha}_{TW}$. The AR(2) case

k	τ_{pct}	T	$\alpha\backslash\bar{c}$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.5
known	τ_{50}	50	0.8	0.94	0.50	0.61	0.71	0.76	0.77	0.78	0.78	0.78	0.78	0.78
			0.9	0.95	0.50	0.62	0.74	0.80	0.84	0.86	0.87	0.87	0.87	0.88
			0.95	0.93	0.51	0.62	0.74	0.81	0.85	0.87	0.89	0.90	0.91	0.92
			1	0.92	0.49	0.60	0.73	0.81	0.86	0.88	0.90	0.91	0.92	0.95
		200	0.8	0.45	0.71	0.78	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
			0.9	0.45	0.74	0.85	0.88	0.89	0.90	0.90	0.90	0.90	0.90	0.90
			0.95	0.45	0.74	0.87	0.91	0.93	0.94	0.94	0.94	0.95	0.95	0.95
			1	0.44	0.74	0.87	0.92	0.95	0.96	0.97	0.97	0.98	0.98	0.99
	τ_{85}	50	0.8	0.73	0.38	0.60	0.71	0.75	0.77	0.78	0.78	0.78	0.78	0.78
			0.9	0.70	0.40	0.60	0.73	0.80	0.83	0.85	0.86	0.87	0.87	0.87
			0.95	0.70	0.40	0.60	0.73	0.80	0.84	0.86	0.88	0.89	0.90	0.91
			1	0.70	0.39	0.59	0.72	0.80	0.85	0.88	0.90	0.91	0.92	0.94
		200	0.8	0.37	0.71	0.78	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
			0.9	0.38	0.74	0.85	0.88	0.89	0.90	0.90	0.90	0.90	0.90	0.90
			0.95	0.38	0.74	0.87	0.91	0.93	0.94	0.94	0.94	0.94	0.95	0.94
			1	0.38	0.74	0.87	0.92	0.95	0.96	0.97	0.97	0.98	0.98	0.99
MAIC	τ_{50}	50	0.8	0.94	0.88	0.84	0.86	0.88	0.89	0.88	0.88	0.88	0.87	0.87
			0.9	0.96	0.88	0.86	0.89	0.91	0.92	0.92	0.92	0.91	0.91	0.91
			0.95	0.94	0.87	0.86	0.89	0.92	0.93	0.93	0.93	0.94	0.94	0.94
			1	0.94	0.88	0.86	0.89	0.92	0.94	0.94	0.94	0.94	0.94	0.96
		200	0.8	0.96	0.89	0.86	0.84	0.84	0.84	0.84	0.84	0.84	0.84	0.84
			0.9	0.96	0.93	0.91	0.90	0.91	0.91	0.91	0.91	0.91	0.91	0.91
			0.95	0.96	0.93	0.93	0.93	0.94	0.95	0.95	0.95	0.95	0.95	0.95
			1	0.96	0.93	0.91	0.94	0.96	0.97	0.97	0.98	0.98	0.98	0.99
	τ_{85}	50	0.8	0.92	0.84	0.79	0.82	0.85	0.86	0.87	0.87	0.87	0.86	0.85
			0.9	0.94	0.84	0.80	0.85	0.88	0.90	0.90	0.90	0.90	0.90	0.90
			0.95	0.91	0.83	0.80	0.85	0.88	0.90	0.91	0.92	0.92	0.93	0.93
			1	0.92	0.83	0.80	0.85	0.89	0.91	0.92	0.93	0.93	0.93	0.95
		200	0.8	0.96	0.87	0.84	0.84	0.84	0.84	0.84	0.84	0.84	0.84	0.84
			0.9	0.95	0.90	0.89	0.90	0.91	0.91	0.91	0.91	0.91	0.91	0.91
			0.95	0.95	0.90	0.91	0.93	0.94	0.95	0.95	0.95	0.95	0.95	0.95
			1	0.95	0.90	0.90	0.93	0.95	0.96	0.97	0.98	0.98	0.98	0.99
BIC	τ_{50}	50	0.8	0.49	0.51	0.68	0.75	0.78	0.78	0.79	0.79	0.79	0.79	0.79
			0.9	0.50	0.52	0.70	0.78	0.83	0.86	0.87	0.87	0.88	0.88	0.88
			0.95	0.51	0.54	0.70	0.78	0.83	0.86	0.89	0.90	0.91	0.92	0.92
			1	0.49	0.53	0.68	0.77	0.83	0.87	0.90	0.91	0.92	0.93	0.95
		200	0.8	0.59	0.72	0.78	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
			0.9	0.60	0.75	0.85	0.88	0.89	0.90	0.90	0.90	0.90	0.90	0.90
			0.95	0.61	0.75	0.87	0.91	0.93	0.94	0.94	0.94	0.95	0.95	0.95
			1	0.61	0.75	0.87	0.92	0.95	0.96	0.97	0.97	0.98	0.98	0.99
	τ_{85}	50	0.8	0.28	0.45	0.64	0.74	0.77	0.78	0.78	0.79	0.79	0.79	0.78
			0.9	0.28	0.45	0.66	0.76	0.82	0.84	0.86	0.87	0.87	0.87	0.87
			0.95	0.26	0.46	0.66	0.76	0.81	0.85	0.88	0.89	0.90	0.91	0.91
			1	0.26	0.45	0.64	0.75	0.82	0.86	0.89	0.90	0.91	0.92	0.94
		200	0.8	0.57	0.72	0.78	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
			0.9	0.57	0.75	0.85	0.88	0.89	0.90	0.90	0.90	0.90	0.90	0.90
			0.95	0.58	0.75	0.87	0.91	0.93	0.94	0.94	0.94	0.95	0.95	0.94
			1	0.58	0.75	0.87	0.92	0.95	0.96	0.97	0.97	0.98	0.98	0.99

Table 8: Mean of the distribution of $\hat{\alpha}_{TW}^{50}$. The ARMA(1,1) case with BIC

T	α	θ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.5
50	0.8	-0.8	0.37	-0.05	-0.03	-0.03	-0.02	-0.02	-0.02	-0.02	-0.02	-0.02	-0.02
		-0.4	0.49	0.53	0.49	0.49	0.46	0.49	0.47	0.47	0.47	0.46	0.46
		0	0.57	0.68	0.74	0.76	0.75	0.74	0.74	0.73	0.73	0.72	0.72
		0.4	0.58	0.76	0.81	0.81	0.84	0.82	0.82	0.82	0.82	0.82	0.82
		0.8	0.42	0.57	0.68	0.74	0.81	0.81	0.81	0.80	0.81	0.80	0.80
	0.9	-0.8	0.39	0.07	0.15	0.11	0.12	0.12	0.12	0.12	0.12	0.12	0.12
		-0.4	0.58	0.58	0.56	0.60	0.58	0.58	0.58	0.58	0.58	0.58	0.57
		0	0.62	0.62	0.78	0.81	0.81	0.81	0.81	0.81	0.80	0.80	0.80
		0.4	0.46	0.67	0.78	0.86	0.88	0.88	0.87	0.87	0.87	0.86	0.86
		0.8	0.35	0.40	0.68	0.77	0.84	0.85	0.87	0.86	0.87	0.86	0.86
	0.95	-0.8	0.09	0.26	0.26	0.21	0.27	0.29	0.29	0.27	0.27	0.27	0.27
		-0.4	0.50	0.51	0.62	0.63	0.64	0.64	0.64	0.64	0.64	0.63	0.63
		0	0.62	0.63	0.77	0.84	0.83	0.83	0.84	0.84	0.84	0.84	0.83
		0.4	0.40	0.63	0.72	0.83	0.86	0.89	0.90	0.89	0.89	0.88	0.88
		0.8	0.42	0.52	0.68	0.75	0.82	0.84	0.85	0.88	0.88	0.88	0.88
	1	-0.8	0.23	0.19	0.19	0.37	0.38	0.38	0.38	0.36	0.36	0.36	0.36
		-0.4	0.35	0.56	0.70	0.74	0.75	0.75	0.74	0.76	0.75	0.74	0.74
		0	0.70	0.73	0.75	0.87	0.87	0.87	0.88	0.89	0.89	0.89	0.89
		0.4	0.39	0.69	0.79	0.84	0.86	0.90	0.93	0.92	0.93	0.93	0.93
		0.8	0.42	0.50	0.68	0.74	0.82	0.85	0.87	0.91	0.91	0.93	0.92
200	0.8	-0.8	0.05	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04
		-0.4	0.59	0.65	0.66	0.64	0.64	0.64	0.64	0.64	0.64	0.64	0.64
		0	0.70	0.82	0.81	0.82	0.82	0.82	0.82	0.82	0.82	0.82	0.82
		0.4	0.53	0.83	0.86	0.85	0.86	0.86	0.86	0.86	0.86	0.86	0.86
		0.8	0.34	0.75	0.85	0.87	0.87	0.87	0.87	0.87	0.87	0.87	0.87
	0.9	-0.8	0.23	0.28	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25
		-0.4	0.74	0.84	0.80	0.80	0.80	0.79	0.79	0.79	0.79	0.79	0.79
		0	0.64	0.87	0.89	0.90	0.90	0.90	0.90	0.90	0.90	0.90	0.90
		0.4	0.59	0.83	0.89	0.91	0.92	0.92	0.92	0.92	0.92	0.92	0.92
		0.8	0.31	0.70	0.86	0.90	0.92	0.93	0.93	0.93	0.93	0.93	0.93
	0.95	-0.8	0.35	0.54	0.51	0.49	0.49	0.49	0.49	0.49	0.49	0.49	0.49
		-0.4	0.74	0.93	0.89	0.88	0.88	0.88	0.88	0.88	0.87	0.87	0.87
		0	0.69	0.90	0.93	0.94	0.94	0.94	0.94	0.94	0.94	0.94	0.94
		0.4	0.49	0.80	0.89	0.93	0.94	0.95	0.95	0.95	0.95	0.95	0.95
		0.8	0.30	0.71	0.85	0.91	0.93	0.95	0.96	0.96	0.96	0.96	0.96
	1	-0.8	0.85	0.94	0.85	0.83	0.83	0.82	0.81	0.81	0.81	0.81	0.81
		-0.4	0.75	0.98	0.98	0.98	0.98	0.97	0.97	0.97	0.97	0.97	0.97
		0	0.74	0.93	0.96	0.96	0.98	0.98	0.98	0.99	0.99	0.99	0.99
		0.4	0.51	0.82	0.90	0.93	0.95	0.97	0.98	0.98	0.98	0.98	0.98
		0.8	0.33	0.70	0.87	0.92	0.94	0.95	0.97	0.98	0.98	0.98	0.99

Table 9: Mean of the distribution of $\hat{\alpha}_{TW}^{85}$. The ARMA(1,1) case with BIC

T	α	θ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.5
50	0.8	-0.8	0.03	-0.05	-0.03	-0.03	-0.02	-0.02	-0.02	-0.02	-0.02	-0.02	-0.02
		-0.4	0.07	0.45	0.49	0.46	0.46	0.46	0.46	0.46	0.46	0.46	0.45
		0	0.14	0.60	0.72	0.74	0.74	0.73	0.73	0.72	0.72	0.72	0.71
		0.4	0.31	0.57	0.75	0.78	0.81	0.81	0.82	0.82	0.81	0.81	0.80
		0.8	0.08	0.44	0.66	0.72	0.79	0.80	0.80	0.80	0.80	0.80	0.79
	0.9	-0.8	0.22	0.07	0.11	0.11	0.12	0.12	0.12	0.12	0.12	0.12	0.12
		-0.4	0.16	0.50	0.55	0.59	0.58	0.57	0.57	0.57	0.57	0.57	0.56
		0	0.34	0.58	0.75	0.78	0.79	0.80	0.80	0.80	0.80	0.80	0.79
		0.4	0.30	0.59	0.74	0.81	0.84	0.85	0.86	0.86	0.86	0.86	0.85
		0.8	-0.01	0.40	0.64	0.74	0.82	0.83	0.85	0.85	0.86	0.86	0.85
	0.95	-0.8	0.00	0.26	0.26	0.20	0.24	0.27	0.27	0.27	0.27	0.27	0.26
		-0.4	0.33	0.44	0.62	0.63	0.64	0.64	0.63	0.63	0.63	0.63	0.62
		0	0.45	0.59	0.73	0.81	0.82	0.82	0.83	0.83	0.83	0.83	0.82
		0.4	0.22	0.59	0.71	0.79	0.83	0.86	0.88	0.88	0.88	0.88	0.87
		0.8	0.09	0.45	0.66	0.72	0.79	0.82	0.84	0.86	0.87	0.87	0.87
	1	-0.8	0.05	0.19	0.19	0.33	0.36	0.36	0.36	0.36	0.35	0.35	0.34
		-0.4	0.27	0.44	0.61	0.73	0.72	0.73	0.72	0.73	0.74	0.74	0.73
		0	0.45	0.69	0.67	0.84	0.84	0.86	0.87	0.87	0.87	0.87	0.87
		0.4	0.23	0.58	0.73	0.80	0.83	0.89	0.90	0.90	0.91	0.92	0.92
		0.8	0.09	0.43	0.66	0.73	0.80	0.83	0.86	0.90	0.91	0.92	0.91
200	0.8	-0.8	0.05	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04
		-0.4	0.59	0.65	0.64	0.64	0.64	0.64	0.64	0.64	0.64	0.64	0.64
		0	0.57	0.80	0.81	0.82	0.82	0.82	0.82	0.82	0.82	0.82	0.82
		0.4	0.44	0.80	0.85	0.85	0.86	0.86	0.86	0.86	0.86	0.86	0.86
		0.8	0.34	0.75	0.84	0.87	0.87	0.87	0.87	0.87	0.87	0.87	0.87
	0.9	-0.8	0.23	0.24	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25
		-0.4	0.70	0.82	0.80	0.79	0.79	0.79	0.79	0.79	0.79	0.79	0.79
		0	0.55	0.86	0.89	0.90	0.90	0.90	0.90	0.90	0.90	0.90	0.90
		0.4	0.46	0.81	0.89	0.91	0.92	0.91	0.92	0.92	0.92	0.91	0.91
		0.8	0.31	0.70	0.86	0.90	0.92	0.93	0.93	0.93	0.93	0.93	0.93
	0.95	-0.8	0.35	0.50	0.49	0.49	0.49	0.49	0.49	0.49	0.49	0.49	0.49
		-0.4	0.74	0.89	0.88	0.88	0.88	0.87	0.87	0.87	0.87	0.87	0.87
		0	0.51	0.84	0.91	0.94	0.94	0.94	0.94	0.94	0.94	0.94	0.94
		0.4	0.45	0.78	0.89	0.93	0.94	0.95	0.95	0.95	0.95	0.95	0.95
		0.8	0.30	0.71	0.85	0.91	0.93	0.95	0.96	0.96	0.96	0.96	0.96
	1	-0.8	0.85	0.85	0.83	0.83	0.81	0.81	0.81	0.81	0.81	0.81	0.80
		-0.4	0.66	0.98	0.95	0.97	0.96	0.97	0.96	0.96	0.97	0.97	0.96
		0	0.61	0.87	0.92	0.95	0.97	0.98	0.98	0.98	0.98	0.98	0.98
		0.4	0.46	0.78	0.90	0.93	0.95	0.96	0.97	0.98	0.98	0.98	0.98
		0.8	0.33	0.70	0.87	0.92	0.94	0.95	0.96	0.97	0.98	0.98	0.99

Table 10: Mean of the distribution of $\hat{\alpha}_{TW}^{50}$ based on the iterative estimation procedure. The AR(1) case, $T = 200$

		Non-iterative					Iterative				
Method	$\alpha \backslash \bar{c}$	0.1	0.2	0.3	0.4	0.5	0.1	0.2	0.3	0.4	0.5
Andrews	0	0.03	0.02	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03
	0.1	0.10	0.10	0.10	0.10	0.10	0.09	0.10	0.10	0.10	0.10
	0.2	0.19	0.19	0.19	0.19	0.19	0.16	0.20	0.20	0.20	0.20
	0.3	0.29	0.29	0.29	0.29	0.29	0.24	0.29	0.30	0.30	0.30
	0.4	0.39	0.39	0.39	0.39	0.39	0.32	0.40	0.40	0.40	0.40
	0.5	0.49	0.49	0.49	0.49	0.49	0.40	0.50	0.50	0.50	0.50
	0.6	0.60	0.59	0.59	0.59	0.59	0.46	0.59	0.60	0.60	0.60
	0.7	0.71	0.69	0.69	0.69	0.69	0.50	0.69	0.69	0.69	0.69
	0.8	0.78	0.79	0.79	0.79	0.79	0.53	0.78	0.79	0.79	0.79
	0.85	0.81	0.84	0.84	0.84	0.84	0.54	0.83	0.85	0.85	0.84
	0.9	0.84	0.88	0.89	0.89	0.89	0.56	0.87	0.90	0.90	0.90
	0.95	0.86	0.91	0.93	0.94	0.94	0.56	0.88	0.94	0.95	0.95
	1	0.88	0.94	0.95	0.96	0.97	0.57	0.90	0.96	0.98	0.98
TW-MAIC	0	0.97	0.71	0.18	0.12	0.11	0.17	0.17	0.17	0.17	0.17
	0.1	0.97	0.74	0.27	0.21	0.20	0.24	0.26	0.26	0.26	0.26
	0.2	0.97	0.78	0.36	0.30	0.29	0.31	0.34	0.34	0.34	0.34
	0.3	0.97	0.81	0.44	0.39	0.39	0.37	0.42	0.42	0.42	0.42
	0.4	0.98	0.84	0.54	0.48	0.48	0.44	0.50	0.50	0.50	0.50
	0.5	0.99	0.87	0.62	0.57	0.57	0.50	0.58	0.58	0.58	0.58
	0.6	0.99	0.89	0.71	0.66	0.66	0.56	0.66	0.66	0.66	0.66
	0.7	1.00	0.91	0.79	0.75	0.75	0.60	0.74	0.74	0.74	0.74
	0.8	1.00	0.94	0.86	0.84	0.83	0.63	0.81	0.82	0.82	0.82
	0.85	1.00	0.96	0.89	0.87	0.87	0.63	0.84	0.86	0.86	0.86
	0.9	0.99	0.98	0.93	0.91	0.91	0.63	0.87	0.90	0.90	0.90
	0.95	0.99	0.98	0.96	0.95	0.95	0.62	0.88	0.93	0.94	0.95
	1	0.99	0.98	0.98	0.97	0.98	0.61	0.88	0.95	0.97	0.97
TW-BIC	0	0.57	0.07	0.05	0.05	0.05	-0.01	-0.01	-0.01	-0.01	-0.01
	0.1	0.58	0.15	0.14	0.14	0.14	0.07	0.09	0.09	0.09	0.09
	0.2	0.58	0.24	0.23	0.23	0.23	0.15	0.19	0.19	0.19	0.19
	0.3	0.59	0.33	0.32	0.32	0.32	0.23	0.29	0.30	0.30	0.30
	0.4	0.62	0.42	0.42	0.42	0.42	0.30	0.39	0.40	0.40	0.40
	0.5	0.65	0.51	0.51	0.51	0.51	0.37	0.49	0.50	0.50	0.50
	0.6	0.69	0.60	0.60	0.60	0.60	0.43	0.59	0.60	0.60	0.60
	0.7	0.73	0.69	0.70	0.70	0.70	0.47	0.68	0.69	0.70	0.70
	0.8	0.78	0.77	0.79	0.80	0.80	0.50	0.77	0.79	0.79	0.79
	0.85	0.78	0.82	0.84	0.84	0.84	0.51	0.80	0.84	0.84	0.84
	0.9	0.79	0.87	0.88	0.89	0.89	0.52	0.83	0.88	0.89	0.89
	0.95	0.80	0.89	0.93	0.94	0.94	0.53	0.84	0.92	0.94	0.94
	1	0.82	0.92	0.95	0.97	0.97	0.53	0.85	0.94	0.96	0.97

Table 11: Persistence of Current Account Balance over GDP ratio

	Bounds ignored					Bounds considered								
	$[\underline{b},\overline{b}]=[-\infty,\infty]$					$[\underline{b},\overline{b}]=[-4,6]$			$[\underline{b},\overline{b}]=[\underline{b}^{\min},\overline{b}^{\max}]$			$[\underline{b},\overline{b}]=[\underline{b}^{*},\overline{b}^{*}]$		
	$\hat{\alpha}_{OLS}$	$\hat{\alpha}_{MU}$	$\hat{\alpha}_{TW}^{\tau 50}$	$\hat{\alpha}_{TW}^{\tau 85}$	k	$\hat{\alpha}_{MU}$	$\hat{\alpha}_{TW}^{\tau 50}$	$\hat{\alpha}_{TW}^{\tau 85}$	$\hat{\alpha}_{MU}$	$\hat{\alpha}_{TW}^{\tau 50}$	$\hat{\alpha}_{TW}^{\tau 85}$	$\hat{\alpha}_{MU}$	$\hat{\alpha}_{TW}^{\tau 50}$	$\hat{\alpha}_{TW}^{\tau 85}$
Austria	0.78	0.81	1	0.92	1	0.91	1	1	0.92	1	1	0.88	1	0.95
Belgium	0.82	0.93	1	0.93	1	1	1	1	0.96	1	1	0.92	1	0.95
Finland	0.90	0.94	0.94	0.91	1	1	1	0.94	1	1	0.94	1	0.94	0.92
France	0.88	0.91	0.92	0.89	1	1	1	0.91	1	1	0.91	1	0.92	0.89
Germany	0.95	1	1	1	1	1	1	1	1	1	1	1	1	1
Greece	0.91	0.98	0.93	0.90	1	1	1	0.93	1	1	0.93	1	0.93	0.91
Ireland	0.88	0.81	1	1	1	1	1	1	1	1	1	1	1	1
Italy	0.79	0.75	0.89	0.85	1	0.93	1	1	1	1	1	0.88	0.89	0.87
Luxembourg	0.68	0.97	0.82	0.76	1	1	1	0.83	1	1	0.83	0.87	0.82	0.77
Netherlands	0.83	1	1	0.93	1	1	1	1	1	1	1	0.94	1	0.94
Norway	0.83	0.97	0.87	0.84	1	1	1	0.88	1	1	0.88	0.93	0.87	0.85
Portugal	0.82	0.90	0.86	0.83	1	1	0.86	0.86	1	0.86	0.86	0.93	0.86	0.83
Spain	0.85	0.94	0.88	0.86	2	1	0.88	0.88	1	0.88	0.88	1	0.88	0.86
Sweden	0.92	1	1	0.98	1	1	1	1	1	1	1	1	1	0.99
Switzerland	0.69	1	0.82	0.79	1	1	1	0.83	0.81	1	0.83	0.77	0.83	0.81
United Kingdom	0.91	1	1	0.97	1	1	1	1	1	1	1	1	1	0.97

Note: $[\underline{b},\overline{b}]=[\underline{b}^{\min},\overline{b}^{\max}]$ denote the bounds defined by the minimum and maximum of the observed values of the $cad_{i,t}$ time series. $[\underline{b},\overline{b}]=[\underline{b}^{*},\overline{b}^{*}]$ denote the "break-even" bounds.

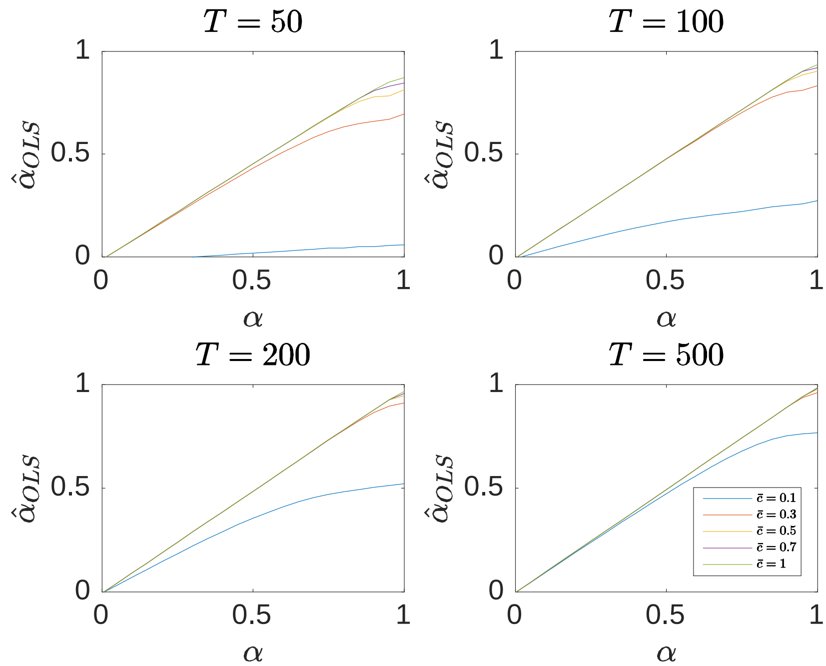

Figure 1: Mean of the OLS α estimate for different (symmetric) bounded time series

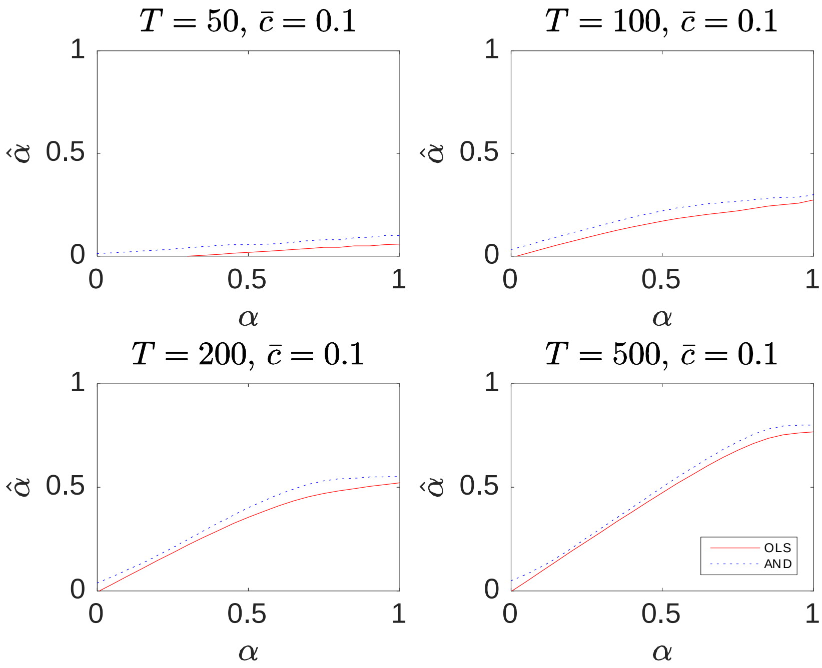

Figure 2: OLS and Andrews MU estimates for symmetric bounded processes with $\bar{c} = 0.1$

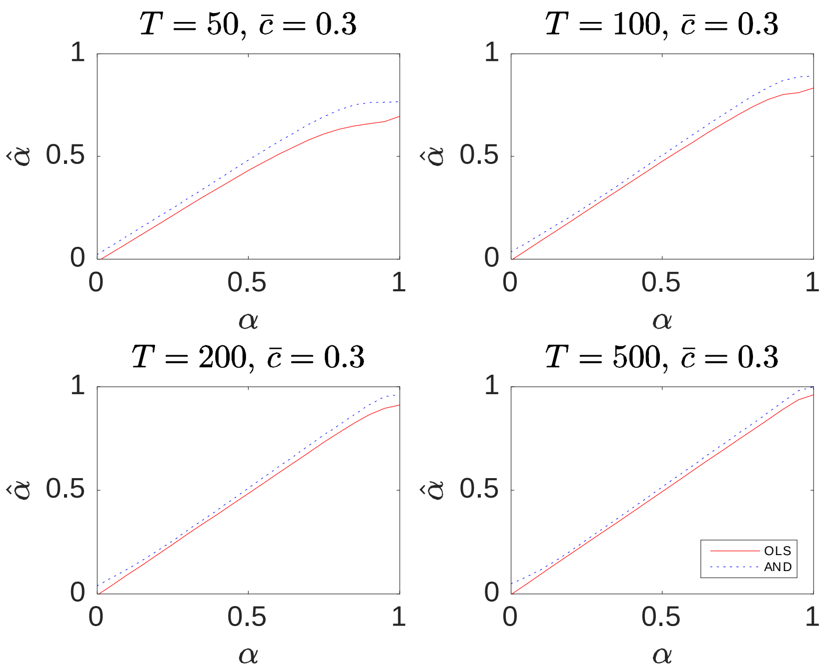

Figure 3: OLS and Andrews MU estimates for symmetric bounded processes with $\bar{c} = 0.3$

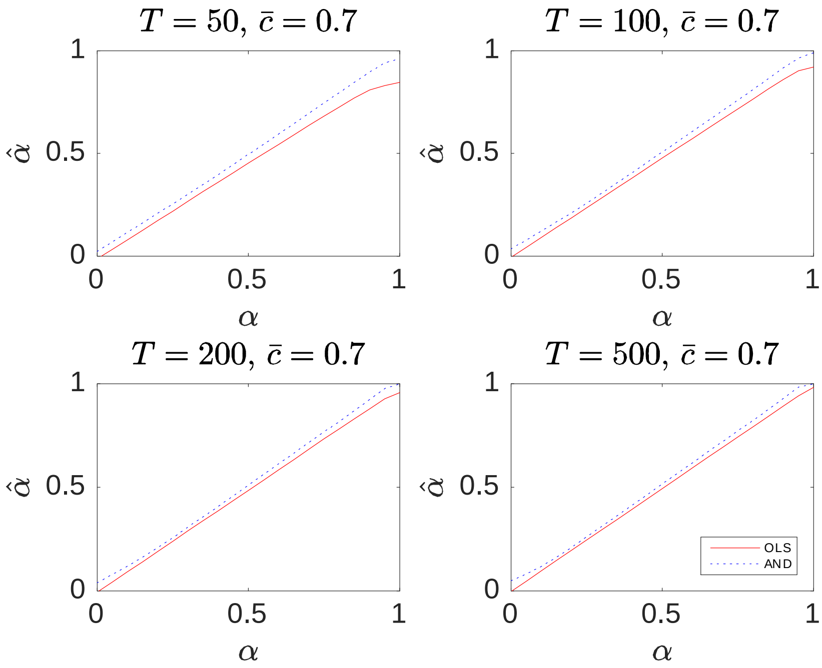

Figure 4: OLS and Andrews MU estimates for symmetric bounded processes with $\bar{c} = 0.7$

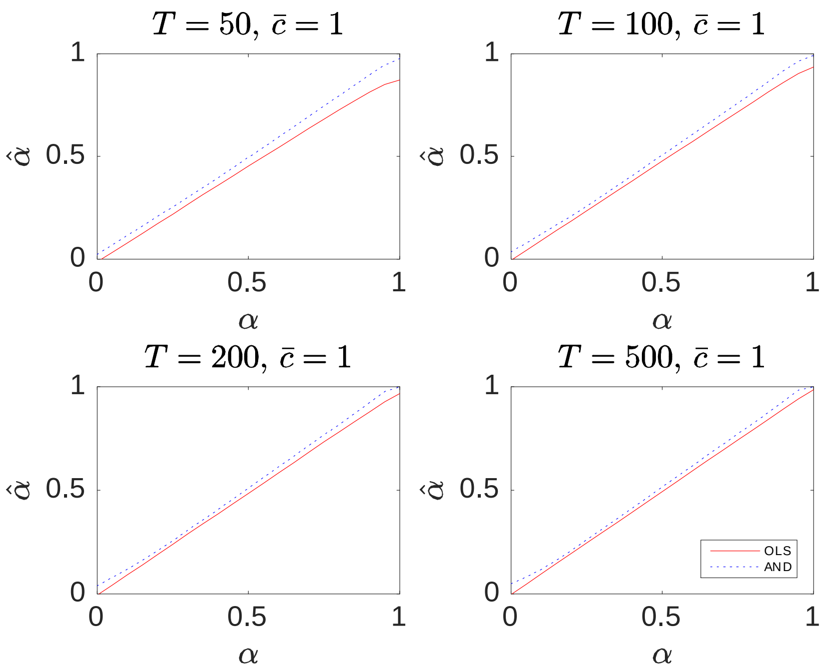

Figure 5: OLS and Andrews MU estimates for symmetric bounded processes with $\bar{c} = 1$

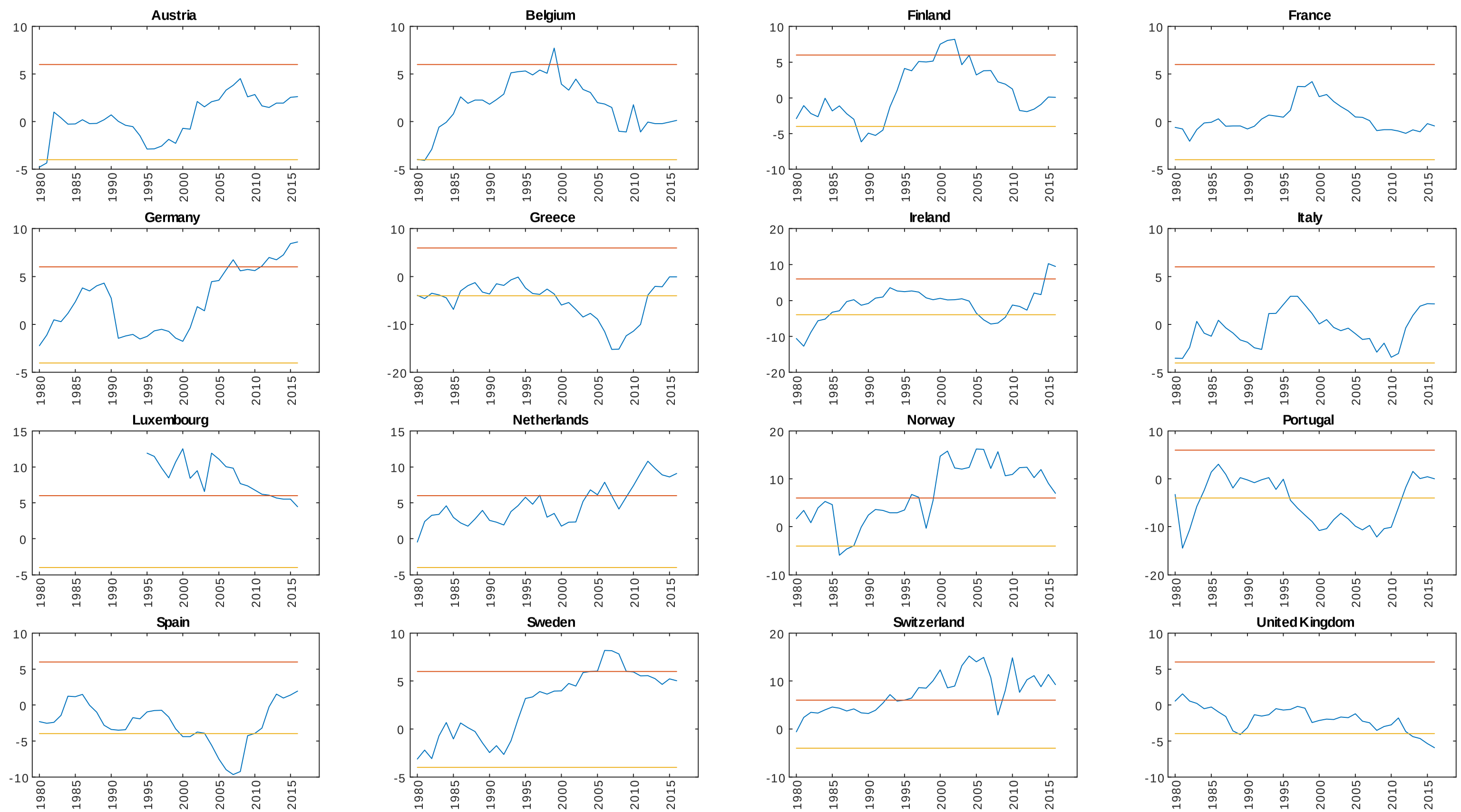

Figure 6: Current Account Balance over GDP ratio for selected countries

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