A NOTE ON THE MOTION OF VISCOUS CONDUCTING LIQUID BETWEEN TWO CONCENTRIC ROTATING CYLINDERS IN THE PRESENCE OF A RADIAL MAGNETIC FIELD

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ABSTRACT. The note treats of the problem of a flow of conducting liquid between two rotating non-conducting infinite cylinders. The response of the velocity of the liquid to a radial magnetic field is found to be transient in character.

INTRODUCTION

The problem of flows of conducting liquid (e.g. mercury, liquid sodium metal) contained between two boundaries is considered to be an important problem in magnetohydrodynamics because of its immediate and wide applications in plasma physics and also because of its relevance to astrophysical problems, vide, Jeffreys (1966), Plumpton and Ferraro (1961). The present note is an attempt towards this end and it seeks to investigate the interaction of the motion of conducting liquid with a prescribed radial magnetic field. The liquid is contained between two infinite concentric cylinders when both the cylinders rotate with angular velocities for sometime. It is believed that this particular problem has not yet been solved, although similar efforts have been made by Singh (1963) and Singh (1965). The Laplace transform has been found useful in the solution of the problem.

PROBLEM, EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

Let a conducting liquid be contained between two infinite circular cylinders rotating with angular velocities; and let a, b be the radii of the inner and outer cylinders, ω_1, ω_2 be their angular velocities with which they start. There is an original magnetic field having the induction represented by B_0 in the radial direction. As our problem is to obtain the velocity of the motion, we have to solve the equations representing the electromagnetic field and the hydrodynamic field.

In cylindrical polar co-ordinates (r, θ, z), the components of velocity vector $\vec{v}$ are given by.

$$u_\theta = v = v(r),\ u_r = u_z = 0 \quad p = p(r)$$

These equations now simplify

$$\frac{dp}{dr} = \frac{\rho v^2}{r} \quad \cdots \quad (1)$$

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} - \frac{\sigma}{\rho}\frac{B_0^2}{r^2} = \frac{1}{\nu}\frac{\partial v}{\partial t} \quad \cdots \quad (2($$

where B_0 = radial magnetic field.
ρ = density of the liquid
σ = conductivity of the liquid
v = velocity of the liquid
p = hydrostatic pressure.
ν = kinematic co-efficient of viscosity.

The boundary conditions for the velocity in the present case are

$$v = \omega_1 a\{H(t)-H(t-\tau)\} \quad \text{when} \quad r = a$$
$$v = \omega_2 b\{H(t)-H(t-\tau) \quad \text{when} \quad r = b \qquad (3)$$

where $H(t)$ is the unit step function equal to unity when $t < 0$ and equal to zero when $t < 0$.

SOLUTION OF THE PROBLEM

To solve the problem, let us introduce the Laplace transform $\bar{f}(P)$ of a function $f(t)$ defined by

$$\bar{f}(P) = \int_0^\infty f(t)e^{-Pt}\,dt \ (P > 0)$$

The Laplace transform of equation (2) gives

$$\frac{d^2\bar{v}}{dr^2} + \frac{1}{r}\frac{d\bar{v}}{dr} - \left(\frac{n^2}{r^2} + q^2\right)\bar{v} = 0 \qquad (4)$$

where $\qquad n^2 = 1+\lambda^2, \quad \lambda^2 = \frac{\sigma}{\rho}B_0^2, \quad q = \sqrt{\left(\frac{P}{\nu}\right)}$

The solution of (4) is

$$\bar{v} = AI_n(qr)+Bk_n(qr)$$

where A, B are constants and $I_n(qr)$ and $K_n(qr)$ are modified Bessel's functions.

The boundary conditions now become, when transformed,

$$\frac{\omega_1 a}{P}(1-e^{-P\tau}) \quad \text{on} \quad r = \alpha$$

$$v = \frac{\omega_2 b}{P}(1-e^{-P\tau}) \quad \text{on} \quad r = b \quad \cdots \quad (6)$$

Solving for A and B we get,

$$v = 1-e^{-p\tau}\Big[\omega_1 a\,.\,\frac{I_n(qr)K_n(qb)-K_n(qr)I_n(qb)}{I_n(qa)K_n(qb)-K_n(qa)I_n(qb)}$$

$$-\omega_2 b\,.\,\frac{I_n(qr)\,.\,K_n(qa)-K_n(qr)I_n(qa)}{I_n(qa)K_n(qb)-K_n(qa)I_n(qb)}\Big] \quad \ldots \quad (7)$$

By inversion theorem,

$$v = \frac{\omega_1 a}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty}\frac{I_n(q'r)K_n(q'b)-K_n(q'r)I_n(q'b)}{I_n(q'a)K_n(q'b)-K_n(q'b)I_n(q'b)}\cdot\frac{e^{\lambda t}}{\lambda}d\lambda$$

$$-\frac{\omega_2 b}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty}\frac{a_n(q'r)K_n(q'a)-K_n(q'r)I_n(q'a)}{I_n(q'a)K_n(q'b)-K_n(q'a)I_n(q'b)}\cdot\frac{e^{\lambda t}}{\lambda}g\lambda$$

for $t<\tau$ ... (8)

and

$$v = \frac{\omega_1 a}{2\pi i}\int_{\nu i-\infty}^{\nu+i\infty}\{e^{\lambda t}-e^{\lambda(t-\tau)}\}\frac{\{I_n(q'r)K_n(q'r)-K_n(q'r)I_n(q'b)}{I_n(q'a)K_n(q'b)-K_n(q'a)I_n(q'b)}\cdot\frac{d\lambda}{\lambda}$$

$$-\frac{\omega_2 b}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty}\{e^{\lambda t}-e^{\lambda(t-\tau)}\}\cdot\frac{I_n(q'r)K_n(q'a)-K_n(q'r)I_n(q'a)}{I_n(q'a)K_n(q'b)-K_n(q'a)I_n(q'b)}\cdot\frac{d\lambda}{\lambda}$$

for $t>\tau$ ... (9)

where $q' = \sqrt{\left(\frac{\lambda}{\nu}\right)}$

Following the method given by Carslaw and Jaeger (1950), we have the solution of (8) and (9) as

$$v = \frac{\omega_1 a^2}{r}\,\frac{b^2-r^2}{b^2-a^2}+\pi\omega_1 a\Big[\sum_{s=1}^{\infty}\frac{J_n(b\alpha_s)Y_n(r\alpha_s)-Y_n(b\alpha_s)J_n(r\alpha_s)}{J_n^2(b\alpha_s)-J_n^2(a\alpha_s)}\times$$

$$J_n(a\alpha_s)J_n(b\alpha_s)\,.\,e^{-\alpha_s^2 t}\Big]$$

$$+\frac{\omega_2 b^2}{r}\cdot\frac{r^2-a^2}{b^1-a^2}-\pi\omega_2 b\Bigg[\sum_{s=1}^{\infty}\frac{J_n(\alpha a_s)Y_n(r\alpha_s)-Y_n(a\alpha_s)J_n(r\alpha_s)}{J_n^2(b\alpha_s)-J_n^2(a\alpha_s)}\times$$

$$\Big[J_n(a\alpha_s)J_n(b\alpha_s)e-v\alpha_s^2 t\Big]$$

for $$t < \tau \qquad \ldots (10)$$

and

$$v = \pi\omega_1 a\Bigg[\sum_{s=1}^{\infty}\{e-v\alpha_s^2 t-e-v\alpha_s^2(t-\tau)\}$$

$$\times\frac{J_n(a\alpha_s)Y_n(r\alpha_s)-Y_n(b\alpha_s)J_n(r\alpha_s)}{J_n^2(b\alpha_s)-J_n^2(a\alpha_s)}\cdot J_n(a\alpha_s)J_n(b\alpha_s)\Bigg]$$

$$-\pi\omega_2 b\Bigg[\sum_{s=1}^{\infty}\{e-v\alpha_s^2 t-e-v\alpha_s^2(t-\tau)\}$$

$$\times\frac{J_n(a\alpha_s)Y_n(r\alpha_s)-Y_n(a\alpha_s)J_n(r\alpha_s)}{J_n^2(b\alpha\ )-J_n^2(a\alpha\ )}\cdot J_n(a\alpha_s)\cdot J_n(b\alpha_s)\Bigg]$$

for $$t > \tau \qquad \ldots (11)$$

Thus the velocities have been obtained and evidently they are transient in character.

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REFERENCES

Carslaw, H. S. and Jaeger, J. C., 1950, *Operational methods in Applied Mathematics*, Dover Publications Inc. New York.

Ferraro, V. C. A. and Plumpton, C., 1961, *Magneto-fluid mechanics*, Oxford Univ. Press.

Jeffreys, A., 1966, *Magneto-Hydrodynamics*, Oliver & Boyd Ltd.

Singh, D., 1963, *App. Sci. Res.*, **10B.**, 413-416.

Singh, G. S., 1965, *Jr. Sc. Engg. Research*, **9**, 284-289.