RADIAL PULSATIONS OF AN INFINITE CYLINDER OF FINITE CONDUCTIVITY AND OF VARIABLE DENSITY IN THE PRESENCE OF A MAGNETIC FIELD

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ABSTRACT. In this paper we have studied the effect of finite conductivity and variable density on the radial oscillations of an infinite cylinder in the presence of a magnetic field. We find that the effect of variable density is to increase the frequency of pulsations and the effect of finite but large conductivity is to damp the mechanical and the magnetic oscillations. Besides damping the pulsations, the effect of finite conductivity is to produce variations in the phase of both mechanical and magnetic oscillations.

INTRODUCTION

The problem of radial pulsations of an infinite cylinder in the presence of magnetic field has been extensively investigated due to its importance in many astrophysical phenomena. Chandrasekhar and Fermi (1953) first considered the problem in the presence of a constant magnetic field assuming the cylinder to be infinitely conducting. Lyttekens (1954) reconsidered the problem in the presence of a magnetic field varying as the square root of the pressure. Chopra and Talwar (1955) studied the problem in the presence of a more general magnetic field of which the field considered by Lytekens (1954) is a particular case. Gurm (1961) discussed the problem with variable density in the presence of a non-uniform magnetic field parallel to the axis of the cylinder. Hans (1966) has considered the problem of radial pulsations in the presence of helical currents assuming the density to be uniform. Recently Srivastava and Kushwaha (1966) have studied the pulsations of an infinite cylinder with variable density in the presence of a heli-magnetic field. However, the magnetic field assumed there gives rise to a volume current which vanishes at the axis but tends to infinity at the surface.

All the above mentioned authors have taken the conductivity to be infinite. The effect of finite conductivity was first taken into account by Bhatnagar and Nagpaul (1957). They have shown that large but finite conductivity does not change the period of pulsation but damps the mechanical and magnetic pulsations and also produces a variation in phase in both of them.

In the present paper we have studied the combined effects of finite conductivity and variable density on the radial pulsations. In part A we have first solved the problem assuming infinite conductivity for a general magnetic field and in Part B we have considered the effect of large but finite conductivity for two particular magnetic field configurations.

PART A

THE CASE OF INFINITE CONDUCTIVITY

2. *Description of Steady State*

We consider the radial pulsations of a self-gravitating infinite cylinder of infinite conductivity in the presence of the following magnetic field configuration :

$$\vec{H}_0^{(i)} = \left(o, Kr, \left\{H_s^2+(H_0^2-H_s^2)\left(1-\frac{r^2}{R^2}\right)\right\}^{\frac{1}{2}}\right) (r \leqslant R), \quad \dots \quad (2.1)$$

$$\vec{H}_0^{(o)} = \left(0, \frac{KR^2}{r}, H_s\right) (r \geqslant R), \quad (2.2)$$

where the superscripts (i) and (o) denote inside and outside of the cylinder whose radius is taken to be R and K, H_s and H_0 are constants. We note that this form of magnetic field gives rise to a volume current, which however, remains finite on the surface of the cylinder unlike in Srivastava and Kustwaha (1966).

We assume that the density of the cylinder varies according to the density law given by

$$\rho_0 = \rho_c\left(1-\frac{r^2}{R^2}\right), r \leqslant R \quad \dots \quad (2.3)$$

where ρ_c is the value of the density at the axis. We assume that outside the cylinder there is vacuum.

With the above magnetic field and density law given by (2.1) and (2.3) respectively, the total equilibrium pressure inside is given by

$$P_0^{(i)} = p_0^{(i)}+p_0^{(i)}{}_{mag}.$$

$$= -\frac{\mu}{8\pi}(H_0^2-H_s^2-2K^2R^2)\left(1-\frac{r^2}{R^2}\right) + \frac{\pi G\rho_c^2R^2}{12}\left(1-\frac{r^2}{R^2}\right)^2\left(5-\frac{2r^2}{R^2}\right)$$

$$+\frac{\mu}{8\pi}\left\{K^2R^2+H_s^2+(H_0^2-H_s^2)\left(1-\frac{r^2}{R^2}\right)\right\}. \quad \dots \quad (2.4)$$

3. *Linearized Equations*

The linearized equations governing the variations in the physical quantities, in Lagrangian description of the motion, are

$$\frac{\delta\rho}{\rho_0} = -\frac{1}{r_0}\frac{\partial}{\partial r_0}(r_0\,\xi) \qquad \ldots \quad (3.1)$$

$$\delta p = \frac{\gamma p_0}{\rho_0}\,\delta\rho \text{ (assuming adiabatic pulsations)}$$

$$= -\frac{\gamma p_0}{r_0}\frac{\partial}{\partial r_0}(r_0\xi) \qquad \ldots \quad (3.2)$$

$$\rho_0\frac{\partial^2\xi}{\partial t^2} = \frac{\partial}{\partial r_0}\left[\frac{\gamma p_0}{r_0}\frac{\partial}{\partial r_0}(r_0\xi)\right] + \frac{4Gm(r_0)\rho_0}{r_0^2}\xi$$

$$+\frac{\mu}{4\pi}\left[\text{curl}\,\vec{H}_0\times\delta\vec{H}+\text{curl}\,\delta\vec{H}\times\vec{H}_0\right]_{rad} \qquad \ldots \quad (3.3)$$

where ξ is the Eulerian displacement; δp, $\delta\rho$ and $\delta\vec{H}$ are the variations in pressure, density and magnetic field respectively; r_0 is the equilibrium value of the Lagrangian coordinate r; $m(r_0)$ is mass of the cylinder of radius r_0 and unit thickness and γ is the ratio of the specific heats.

The change in the magnetic field following the motion of the fluid is given by

$$\delta\vec{H} = \text{curl}\,(\vec{\xi}\times\vec{H}_0)+(\vec{\xi}\cdot\nabla)\vec{H}_0 \qquad \ldots \quad (3.4)$$

while $m(r_0)$ is given by

$$m(r_0) = \pi\rho_c r_0^2\left(1-\frac{r_0^2}{2R^2}\right) \qquad \ldots \quad (3.5)$$

From (3.4) we get

$$\delta\vec{H} = \left(0,\ -H_\theta\frac{\partial\xi}{\partial r_0},\ -\frac{H_z}{r_0}\frac{\partial}{\partial r_0}(r_0\xi)\right)$$

where H_θ and H_z are given by (2.1).

We assume that the physical quantities vary with time like $\exp(i\omega t)$ so that the frequency of the pulsations is given by ω.

Then, on using (3.6) in (3.3), we get

$$\frac{d}{dr_0}\left[\left(\frac{\gamma p_0}{r_0}+\frac{\mu H_z^2}{4\pi r_0}\right)\frac{d}{dr_0}(r_0\xi)+\frac{\mu H_\theta^2}{4\pi}\frac{d\xi}{dr_0}\right]+\frac{\mu H_\theta^2}{2\pi r_0}\frac{d\xi}{dr_0}$$

$$=-\rho_0\left(\omega^2+\frac{4Gm(r_0)}{r_0^2}\right)\xi, \qquad \ldots (3.7)$$

where ξ has now the meaning of an amplitude.

Denoting by ψ the relative displacement ξ/r_0 and transforming the independent variable to x, where $x = r_0/R$, *eqn.* (3.7) reduces after substituting the values of $p_0, \vec{H}_0, \rho_0$ and $m(r_0)$, to

$$x(A+Bx^2+Cx^4+Dx^6)\frac{d^2\psi}{dx^2}+(E+Fx^2+Gx^4+Hx^6)\frac{d\psi}{dx}$$

$$+x(I+Jx^2+Lx^4)\psi = 0, \qquad (3.8)$$

where

$$A = \frac{5}{12}-\tfrac{1}{2}(V_0^2-V_B^2-2V_S^2)+\frac{1}{\gamma}V_0^2$$

$$B = -1-\left(\frac{1}{\gamma}-\tfrac{1}{2}\right)(V_0^2-V_B^2)-\left(1-\frac{1}{\gamma}\right)V^2_S$$

$$\begin{aligned}
C &= 3/4\\
D &= -1/6\\
E &= 3A\\
F &= 5B+V_0^2/\gamma\\
G &= 21/4\\
H &= -3/2\\
I &= 1/\gamma[\omega^2/\pi G\rho_c+4]+4B\\
J &= -1/\gamma[\omega^2/\pi G\rho_c+4]-2/\gamma+6\\
L &= 2/\gamma-2
\end{aligned} \qquad \ldots (3.9)$$

$V_0^2 = \mu H_0^2/4\pi\rho_c/\pi G\rho_c R^2$; $V^2_B = \mu H^2_S/4\pi\rho_c/\pi G\rho_c R^2$;

$V_S^2 = \mu K^2R^2/4\pi\rho_c/\pi G\rho_c R^2$.

The above equation is to be solved under the following boundary conditions :

$$\xi = 0 \quad \text{when} \quad r_0 = 0$$

and

$$\delta P^{(i)} = \delta P^{(0)} \quad \text{at} \quad r_0 = R$$

which reduce to

$$\psi \text{ is finite at } x = 0$$

and

$$(V^2_S + V^2_B)\psi' + (V^2_S + 2V^2_B)\psi = 0 \text{ at } x = 1, \tag{3.10}$$

where dash denotes differentiation with respect to x.

In passing we mention that the differential equation governing ξ, the amplitude of pulsation, given in Srivastava and Kushwaha (1966) differs from our equation (3.7) in as much as that instead of the term $\frac{\mu H_\theta^2}{2\pi r_0} \frac{d\xi}{dr_0}$ in (3.7) there occurs term $\frac{H_\theta^2}{4\pi r_0^2} \frac{d}{dr_0}(r_0\xi)$. Further we note that the boundary condition given there, namely, $\frac{d\psi}{dx} = \psi\ (\psi = \xi/R)$ seems to be incorrect as can be seen by putting $V_B = 0$ in our boundary condition.

4. *Integration of* (3.8)

On substituting

$$\psi = \sum_{n=0}^{\infty} a_n x^{n+\nu}, \quad a_0 \neq 0$$

in (3.8) and equating the coefficients of various powers of x to zero, we find that ν satisfies the indicial equation

$$\nu(\nu+2) = 0$$

which gives $\nu = 0$ or $\nu = -2$. In view of the b.c. that ψ must remain finite at $x = 0$, the solution corresponding to $\nu = -2$ is inadmissable. Hence, we have

$$\psi = \sum_{n=0}^{\infty} a_n x^n. \quad \dots \tag{4.1}$$

From the other equations we find that

$$a_{2r+1} = 0 \text{ for all } r$$

and that the coefficients a_{2r} are given by the recurrence relation

$$4r(r+1)Aa_{2r} + \{I + (2r-2)F + (2r-2)(2r-3)B\}a_{2r-2}$$
$$+\{J + (2r-4)G + (2r-4)(2r-3)C\}a_{2r-4}$$
$$+\{L + (2r-6)H + (2r-6)(2r-7)D\}a_{2r-6} = 0. \quad \dots \tag{4.2}$$

The series (4.1) is convergent for $0 \leqslant x \leqslant 1$. The b.C (3.10) gives

$$a_0 + \sum_{r=1} \left(1+2r \frac{V_S^2+V_B^2}{V_S^2+2V_B^2} \right) a_{2r} = 0 \qquad \ldots \quad (4.3)$$

The equation (4.3) together with the recurrence relation (4.2) determines the frequencies when V_0^2, V_B^2 and V_S^2 are given.

We have calculated the frequencies in the following manner. After fixing the values of V_0^2, V_B^2 and V_S^2, we first obtain an approximation to the correct value of $W = 1/\gamma\,(\omega^2/\pi G\rho_c+4)$ from (4.3) by taking a first few terms. Using this value of W as a trial value, we integrate the equation (3.8) numerically from 0 to 1 using Milne's method. This gives us the values of ψ' and ψ at $x = 1$, which in general will not satisfy (3.10). Let the error be λ_1, say. We then choose another value of W and again integrate (3.8) as above. Let the error now be λ_2. Guided by the magnitudes of λ_1 and λ_2 we proceed in this manner till we get two values of W for which the corresponding λ's are of opposite signs. We then interpolate between these two values of W to get approximately the value of W for which the boundary condition (3.10) is satisfied. This value was further sharpened by further interpolation and numerical integration.

The starting values of ψ and ψ' for the numerical integration were obtained from the series (4.1) while the corresponding values of ψ'' were calculated from the equation (3.8) itself.

In table 1 we have given the frequencies for various values of V^2_0, V^2_B and V^2_S. We have calculated only the first modes in each case.

Table 1

$\gamma = 5/3$; $a_0 = 1$

V^2_0	V^2_B	V^2_S	$W = 1/\gamma\left(\frac{\omega^2}{\pi G\rho_c}+4\right)$
0.2	0.1	0.1	6.7356
2.0	1.5	1.0	26.8207
0.417	0	0	7.1229
1.667	0	0	16.5845
3.75	0	0	32.00
15.0	0	0	115.5933
0	0	1.0	13.12
0	0	9.0	86.3312

From table 1 we find that as V_0^2, V_B^2 and V_S^2 increase, W increases and thus the frequency increases.

To find the effect of variable density, we compare our results with those given in Bhatnagar and Nagpaul (1957). We first note that, when $H_s = H_0$, the parameters f and A given there are related to the corresponding parameters in our case through

$$V_0^2 = \frac{\gamma f}{4} \text{ and } W = \frac{A}{2} + \frac{2}{\gamma}$$

where we have replaced ρ by the mean density $\rho_c/2$. A comparison now shows that the frequencies in our cases are larger than the corresponding cases in Bhatnagar and Nagpaul (1957). Thus we conclude that the effect of variable density is to increase the frequency of pulsation.

PART B

5. In this part we consider the radial pulsations of a self-gravitating cylinder whose conductivity is now taken to be finite. We discuss the oscillations in the presence of two types of magnetic fields. First we consider the case of a uniform magnetic field parallel to the axis and then we consider a purely azimuthal field. In both cases we have taken the density of the cylinder to vary according to the law given by (2.3).

6. *Case (i) : Uniform axial magnetic field*

The initial magnetic field in this case is given by $\vec{H_0} = (0, 0, H_0)$ both inside and outside the cylinder. This is a particular case of the general magnetic field given by (2.1) when $H_s = H_0$ and $K = 0$.

The total equilibrium pressure in this case is

$$P_0^{(i)} = \frac{\pi G \rho_c^2 R^2}{12} \left(1 - \frac{r^2}{R^2}\right)^2 \left(5 - \frac{2r^2}{R^2}\right) + \frac{\mu H_0^2}{8\pi}. \quad \dots (6.1)$$

Linearized Equations

The linearized equations governing the variations in the physical quantities are

$$\delta p = \frac{\gamma p_0}{\rho_0} \delta\rho \quad \text{(assuming adiabatic pulsations)} \quad \dots (6.2)$$

$$\frac{\delta\rho}{\rho_0} = -\frac{1}{r_0} \frac{\partial}{\partial r_0} (r_0 \zeta) \quad \dots (6.3)$$

$$\rho_0 \frac{\partial^2 \xi}{\partial t^2} = \frac{\partial}{\partial r_0}\left[\frac{\gamma p_0}{r_0}(r_0 \xi)\right] + \frac{4Gm(r_0)\rho_0}{r_0^2}\xi$$

$$+\frac{\mu}{4\pi}[\text{curl } \vec{H}_0 \times \vec{\delta H} + \text{curl } \vec{\delta H} \times \vec{H}_0]_{rad} \quad \ldots (6.4)$$

$$\frac{\partial}{\partial t}\delta H_z = -\frac{H_0}{r_0}\frac{\partial}{\partial r_0}\left(r_0 \frac{\partial \xi}{\partial t}\right) + \frac{1}{4\pi\mu\sigma}\frac{1}{r_0}\frac{\partial}{\partial r_0}\left(r_0 \frac{\partial}{\partial r_0}\delta H_z\right) \quad \ldots (6.5)$$

We assume, as before, that all the physical quantities vary with time like $\exp(i\omega t)$ and introduce the following non-dimensional uantities :

$$x = \frac{r_0}{R}\ ; \ \psi = \xi/x; \ \bar{p}_0 = \frac{p_0}{\pi G \rho_c^2 R^2}\ ; \ \delta\bar{H}_z = \frac{\delta H_z}{H_0},$$

$$V_0^2 = \frac{\mu H_0^2}{4\pi\rho_c} / \pi G \rho_c R^2; \ \frac{S}{\sigma} = \frac{1}{\sigma}\cdot\frac{1}{4\pi\mu\omega R^2} \quad \ldots (6.6)$$

Equations (6.4) and (6.5) then become

$$x(1-x^2)\left(W - \frac{2}{\gamma}x^2\right)\psi = -\frac{d}{dx}\left\{\frac{\bar{p}_0}{x}(x^2\psi)\right\} + \frac{V_0^2}{\gamma}\frac{d}{dx}\delta H_z \quad \ldots (6.7)$$

$$\delta\bar{H}_z = -\frac{1}{x}\frac{d}{dx}(x^2\psi) - \frac{iS}{\sigma}\frac{1}{x}\frac{d}{dx}\left(x\frac{d}{dx}\delta H_z\right) \quad \ldots (6.8)$$

where

$$W = \frac{1}{\gamma}\left(\frac{\omega^2}{\pi G \rho_c} + 4\right).$$

We shall assume that σ is large but finite so that the squares and higher powers of $1/\sigma$ can be neglected. Expanding the physical quantities involved in terms of

$$\tau = \frac{S_0}{\sigma}, \text{ where } S_0 = \frac{1}{4\pi\mu R^2 \omega_0}$$

we have

$$S = S_0 + \tau S_1 \qquad \psi = \psi_0 + \tau\psi_1 \quad \ldots (6.9$$

$$W = W_0 + \tau W_1 \qquad \delta\bar{H}_z = \overline{\delta H}_{z0} + \tau\overline{\delta H}_{z1}$$

$$\omega = \omega_0 + \tau\omega_1$$

so that

$$S_1 = -\frac{\omega_1}{\omega_0} S_0, \quad W_0 = \frac{1}{\gamma}\left(\frac{\omega_0^2}{\pi G\rho_c}+4\right), \quad W_1 = \frac{2\omega_0\omega_1}{\gamma\pi G\rho_c}.$$

Substituting (6.9) in (6.7) and (6.8) and separating the various order terms, we have, after dropping the bars

$$x(1-x^2)\left(W_0-\frac{2}{\gamma}x^2\right)\psi_0 = -\frac{d}{dx}\left[\frac{p_0}{x}\frac{d}{dx}(x^2\psi_0)\right]+\frac{V_0^2}{\gamma}\frac{d}{dx}\delta H_{z0} \quad \ldots \quad (6.10)$$

$$x(1-x^2)\left\{\left(W_0-\frac{2}{\gamma}x^2\right)\psi_1+W_1\psi_0\right\}$$

$$= -\frac{d}{dx}\left\{\frac{p_0}{x}\frac{d}{dx}(x^2\psi_1)\right\}+\frac{V^2_0}{\gamma}\frac{d}{dx}\delta H_{z1} \quad \ldots \quad (6.11)$$

$$\delta H_{z0} = -\frac{1}{x}\frac{d}{dx}(x^2\psi_0) \quad \ldots \quad (6.12)$$

$$\delta H_{z1} = -\frac{1}{x}\frac{d}{dx}(x^2\psi_1)-i\frac{1}{x}\frac{d}{dx}\left(x\frac{d}{dx}\delta H_{z0}.\right) \quad \ldots \quad (6.13)$$

Eliminating δH_{z0} between (6.10) and (6.12), we get

$$(Ax+Bx^3+Cx^5+Dx^7)\psi''_0+(E+Fx^2+Gx^4+Hx^6)\psi'_0$$

$$+(Ix+Jx^3+Lx^5)\psi_0 = 0 \quad \ldots \quad (6.14)$$

where

$$A = \frac{V_0^2}{\gamma}+\frac{5}{12}; \quad B = -1; \quad C = \frac{3}{4}; \quad D = -\frac{1}{6}; \quad E = 3A$$

$$F = 5B; \quad G = \frac{21}{4}; \quad H = -3/2; \quad I = W_0-4 \quad \ldots \quad (6.15)$$

$$J = -W_0+6-\frac{2}{\gamma}; \quad L = \frac{2}{\gamma}-2.$$

Now equation (6.13) can be rewritten with the help of (6.12) as

$$\delta H_{z1} = -\left(x\frac{d\psi_1}{dx}+2\psi_1\right)+\frac{i}{x}\left(x^2\frac{d^3\psi_0}{dx^3}+5x\frac{d^2\psi_0}{dx^2}+\frac{3d\psi_0}{dx}\right). \quad \ldots \quad (6.16)$$

Eliminating δH_{z1} from (6.11) using (6.16), we find

$$(Ax+Bx^3+Cx^5+Dx^7)\psi''_1+(E+Fx^2+Gx^4+Hx^6)\psi'_1$$
$$+(Ix+Jx^3+Lx^5)\psi_1 = -x(1-x^2)W_1\psi_0$$
$$+i\,\frac{V_0^2}{\gamma}\left\{x\psi_0^{iv}+6\psi_0^{iii}+\frac{3}{x}\psi_0^{ii}-\frac{3}{x^2}\psi_0^{i}\right\}. \qquad (6.17)$$

The equations (6.14) and (6.17) are to be solved under the following boundary conditions respectively :

i) a) $\psi_0(0)$ is finite

b) $\psi'(1)+2\psi_0(1) = 0$ ... (6.18)

ii) a) $\psi_1(0)$ is finite ... (6.19)

b) $\psi_1'(1)+2\psi_1(1) = i(\psi_0'''(1)+5\psi_0''(1)+3\psi'(1))$.

From part A, the solution for (6.14) is

$$\psi_0 = \sum_{n=0}^{\infty} a_n x^n \quad \text{where}$$

$a_{2r+1} = 0$ for all r, and a_{2r}'s are given by the recurrence relation (4.2) in which the constants A, B, C, ... are now given by (6.15).

The presence of i in equation (6.17) leads to conclude that ψ_1 and W_1 are complex. Accordingly we set

$$\psi_1 = \zeta+i\eta$$
$$W_1 = \alpha+i\beta.$$

Substituting in (6.17) and separating the real and imaginary parts, we have

$$(Ax+Bx^3+Cx^5+Dx^7)\zeta''+(E+Fx^2+Gx^4+Hx^6)\zeta'$$
$$+(Ix+Jx^3+Lx^5)\zeta = -x(1-x^2)\alpha\psi_0 \qquad \ldots (6.20)$$

and

$$(Ax+Bx^3+Cx^5+Dx^7)\eta''+(E+Fx^2+Gx^6+Hx^6)\eta'$$
$$+(Ix+Jx^3+Lx^5)\eta = -x(1-x^2)\beta\psi_0$$
$$+\frac{V_0^2}{\gamma}\left\{x\psi_0^{iv}+6\psi_0^{iii}+\frac{3}{x}\psi_0''-\frac{3}{x^2}\psi'_0\right\} \qquad \ldots (6.21)$$

The boundary conditions (6.18) now yield

a) ζ and η should be finite at $x = 0$

b) $\zeta'(1)+2\zeta(1) = 0$ and ... (6.22)

$$\eta'(1)+2\eta(1) = \psi'''_0(1)+5\psi'_0(1)+3\psi'_0(1).$$

We shall first solve equation (6.20). In view of the boundary conditions on ζ at $x = 0$, we set

$$\zeta = \sum_{n=0}^{\infty} b_n x^n. \qquad \dots \quad (6.23)$$

Substituting in (6.20) and equating the coefficients of the various powers of x to zero, we get

$$b_{2r+1} = 0 \quad \text{for all } r$$

and

$$\begin{aligned} 4r(r+1)Ab_{2r}&+\{I+(2r-2)F+(2r-2)+(2r-2)(2r-3)B\}b_{2r-2} \\ &+\{J+(2r-4)G+(2r-4)(2r-5)C\}b_{2r-4} \\ &+\{L+(2r-6)H+(2_e-6)(2r-7)D\}b_{2r-6} \\ &+L(a_{2r-2}-a_{2r-4}) = 0. \end{aligned} \qquad (6.24)$$

Using the recurrence relation (4.2), we obtain from (6.24)

$$b_{2r} = \beta_{2r}b_0+\alpha\mu_{2r}a_0 \qquad (6.25)$$

where β_{2r} are given by

$$\begin{aligned} 4r(r+1)A\beta_{2r}&+\{I+(2r-2)F+(2r-2)(2r-3)B\}\beta_{2r-2} \\ &+\{J+(2r-4)G+(2r-4)(2r-5)C\}\beta_{2r-4} \\ &+\{L+(2r-6)H+(2r-6)(2r-7)D\}\beta_{2r-6} = 0 \end{aligned} \qquad (6.26)$$

while μ_{2r} are given by

$$\begin{aligned} 4r(r+1)A\mu_{2r}&+\{I+(2r-2)F+(2r-2)(2r-3)B\}\mu_{2r-2} \\ &+\{J+(2r-4)G+(2r-4)(2r-5)C\}\mu_{2r-4} \\ &+\{L+(2r-6)H+(2r-6)(2r-7)D\}\mu_{2r-6} \\ &+\beta_{2r-2}-\beta_{2r-4} = 0. \end{aligned} \qquad (6.27)$$

Thus ζ can be written as

$$\zeta = \frac{v_0}{a_0}\,\psi_0 + \alpha\nu(x) \qquad \ldots \quad (6.28)$$

where

$$\nu(x) = a_0 \sum_{n=0} \mu_{2n} x^{2n}.$$

Applying the boundary condition (6.22) we see that, in view of (6.10)

$$\alpha = 0$$

which implies that $W_1 = i\beta$. Hence we conclude, as in Bhatnagar and Nagpaul (1957) that to our approximation the period of oscillation is unaffected by the assumption of finite conductivity.

We shall now solve equation (6.21) so that the parameter β may be determined.

$$\text{Let} \quad \eta = \sum_{n=0}^{\infty} C_n x^n.$$

Substituting in (6.21) and proceeding as before, we get

$$C_{2r+1} = 0 \quad \text{for all } r$$

and

$$\begin{aligned} &4r(r+1)AC_{2r} + \{I + (2r-2)F + (2r-2)(2r-3)B\}C_{2r-2} \\ &+ \{J + (2r-4)G + (2r-4)(2r-5)C\}C_{2r-4} \\ &+ \{L + (2r-6)H + (2r-6)(2r-7)D\}C_{2r-6} \\ &= 16\,\frac{V_0^2}{\gamma}\,r(r+1)^2(r+2)a_{2r+2} - \beta(a_{2r-2} - a_{2r-4}), \qquad (6.29) \end{aligned}$$

from which we obtain, after using (4.2),

$$C_{2r} = C_0\beta_{2r} + a_0\lambda_{2r} + \beta a_0\mu_{2r} \qquad \ldots \quad (6.30)$$

where β_{2r} and μ_{2r} are given by (6.26) and (6.27) respectively while λ_{2r} is given by

$$\begin{aligned} &4r(r+1)A\lambda_{2r} + \{I + (2r-2)F + (2r-2)(2r-3)B\}\lambda_{2r-2} \\ &+ \{J + (2r-4)G + (2r-4)(2r-5)C\}\lambda_{2r-4} \\ &+ \{L + (2r-6)H + (2r-6)(2r-7)D\}\lambda_{2r-6} \\ &= 16\,\frac{V_0^2}{\gamma}\,r(r+1)^2(r+2)\beta_{2r+2} \qquad \ldots \quad (6.31) \end{aligned}$$

Thus

$$\eta = \frac{c_0}{a_0}\psi_0 + a_0 \sum_{n=0}^{\infty} \lambda_{2n} x^{2n} + \beta a_0 \sum_{n=0}^{\infty} \mu_{2n} x^{2n}. \qquad \ldots (6.32)$$

Applying the boundary condition (6.22), we obtain, after taking into consideration (6.18),

$$\beta = \frac{4\sum_{n=0}^{\infty} n^2(n+1)\beta_{2n} - \sum_{n=0}^{\infty}(n+1)\lambda_{2n}}{\sum_{n=0}^{\infty}(n+1)\mu_{2n}} \qquad (6.33)$$

On putting $r = 0$ in (6.30) we find that

$$c_0 = c_0\beta_0 + a_0\lambda_0 + \beta a_0\mu_0$$

so that

$$\beta_0 = 1, \quad \lambda_0 = \mu_0 = 0.$$

7. From the relation

$$\omega = \omega_0 + \tau\omega_1$$

$$= \omega_0 + \frac{i\gamma\pi G\rho_c}{2\omega_0}\tau\beta,$$

we find that the contribution to the frequency due to the finiteness of the conductivity is purely imaginary. Thus there is no change in the period of oscillation but the mechanical and magnetic pulsations are damped. The damping time is given by

$$t_0 = 8\pi\mu R^2\sigma \left[\frac{1}{\beta}\left(W_0 - \frac{4}{\gamma}\right)\right]. \qquad \ldots (7.1)$$

Besides damping the pulsations, the effect of finite conductivity is also to produce variations in the phase of both mechanical and magnetic pulsations. The variation in phase of ψ, for example is given by

$$\tan\chi = \frac{\tau\eta}{\psi_0} \qquad \ldots (7.2)$$

while the variation of phase in magnetic field is given by

$$\tan\chi_H = \frac{\tau\left\{-\frac{1}{x}\left(x^2\frac{d^3\psi_0}{dx^3} + 5x\frac{d^2\psi_0}{dx^2} + 3\frac{d\psi_0}{dx}\right) + \left(x\frac{d\eta}{dx} + 2\eta\right)\right\}}{\left(x\frac{d\psi_0}{dx} + 2\psi_0\right)}.$$

8. *Case (ii)* : In this case we have taken the initial magnetic fields as

$$\left.\begin{aligned} \vec{H}_0^{(i)} &= (0, Kr, 0) \qquad & r \leqslant R \\ \vec{H}_0^{(0)} &= \left\{ 0, \frac{KR^2}{r}, 0 \right) \qquad & r \geqslant R \end{aligned}\right\} \qquad \ldots \quad (8.1)$$

which is a particular case of (2.1) and (2.2) with $H_s = H_0 = 0$.

With the density law given by (2.3), the total equilibrium pressure is

$$P_0^{(i)} = \frac{\mu K^2 R^2}{4\pi}\left(1-\frac{r^2}{R^2}\right)+\frac{\pi G\rho_c^2 R^2}{12}\left(1-\frac{r}{R^2}\right)\left(5-\frac{2r^2}{R^2}\right)$$

$$+\frac{\mu K^2 R^2}{8\pi}\left(\frac{r}{R}\right)^2. \qquad \ldots \quad (8.2)$$

9. *Linearized Equations* :

The linearized equations relevant to this case are

$$x(1-x^2)\left(W-\frac{2}{\gamma}x^2\right)\psi = -\frac{d}{dx}\left\{\frac{p_0}{x}\frac{d}{dx}(x^2\psi)\right\}$$

$$+\frac{V_s^2}{\gamma}\left\{\frac{d}{dx}(x\delta H_\theta)+2\delta H_\theta\right\} \qquad \ldots \quad (9.1)$$

and

$$\delta H_\theta = -x\frac{d}{dx}(x\psi)-\frac{iS}{\sigma}\left\{\frac{d}{dx}\left[\frac{1}{x}\frac{d}{dx}x\delta H_\theta\right]-2\frac{d^2}{dx^2}(x\psi)\right\}. \qquad \ldots \quad (9.2)$$

Proceeding exactly as in case (i) by setting

$$\psi = \psi_0+\tau\psi_1, \text{ etc}$$

the equations determining the zeroth and first order quantities are

$$x(1-x^2)\left(W_0-\frac{2}{\gamma}x^2\right)\psi_0 = -\frac{d}{dx}\left\{\frac{p_0}{x}\frac{d}{dx}(x^2\psi_0)\right\}+$$

$$+\frac{V_s^2}{\gamma}\left\{\frac{d}{dx}(x\delta H_{\theta 0})+2\delta H_{\theta 0}\right\} \qquad \ldots \quad (9.3)$$

$$\delta H_{\theta 0} = -x\frac{a}{dx}(x\psi_0) \qquad \ldots \quad (9.4)$$

and

$$x(1-x^2)\left\{\left(W_0-\frac{2}{\gamma}x^2\right)\psi_1+W_1\psi_0\right\}=-\frac{d}{dx}\left\{\frac{p_0}{x}\frac{d}{dx}(x^2\psi_1)\right\}$$

$$+\frac{V_s^2}{\gamma}\left\{\frac{d}{dx}(x\delta H_{\theta 1})+2\delta H_{\theta 1}\right\}, \quad \ldots \quad (9.5)$$

$$\delta H_{\theta 1}=-x\frac{d}{dx}(x\psi_1)-i\left\{\frac{d}{dx}\left(\frac{1}{x}\frac{d}{dx}x\overline{\delta H_{\theta 0}}-2\frac{d^2}{dx^2}(x\psi_0)\right]. \quad \ldots \quad (9.6)$$

Eliminating $\delta H_{\theta 0}$ from (9.3) using (9.4), we get

$$(Ax+Bx^3+Cx^5+Dx^7)\psi_0''+(E+Fx^2+Gx^4+Hx^6)\psi'_0$$

$$+(Ix+Jx^3+Lx^5)\psi_0=0 \quad \ldots \quad (9.7)$$

where

$$A=V_s^2+\frac{5}{12} \qquad E=3A \qquad I=W_0+4B$$

$$B=\frac{V_s^2}{\gamma}-(1+V_s^2) \qquad F=5B+\frac{V_s^2}{\gamma} \qquad J=-W_0+6-\frac{2}{\gamma} \quad \ldots \quad (9.8)$$

$$C=\frac{3}{4} \qquad G=\frac{21}{4} \qquad L=\frac{2}{\gamma}-2$$

$$D=-\frac{1}{6} \qquad H=-\frac{3}{2}$$

Similarly eliminating $\delta H\theta_1$ from (9.5) using (9.6) we obtain

$$(Ax+Bx^3+Cx^5+Dx^7)\psi''_1+(E+Fx^2+Gx^4+Hx^6)\psi'_1$$

$$+(Ix+Jx^3+Lx^5)\psi_1=-x(1-x^2)W_1\psi_0$$

$$+i\frac{V_s^2}{\gamma}\left\{x^3\psi_0^{iv}+13x^2\psi_0^{iii}+42x\psi_0^{ii}+30\psi_0\right\}. \quad \ldots \quad (9.9)$$

Equations (9.7) and (9.9) are to be solved under the following boundary conditions :

ψ_0 and ψ_1 should remain finite at $x=0$

and $\psi_0'+\psi_0=0$ at $x=1$, ... (9.10)

$\psi'_1(1)+\psi_1(1)=i(\psi_0'''(1)+8\psi_0''(1)+10\psi'_0(1))$

From part A, the solution of equation (9.7) satisfying the boundary condition is given by

$$\psi_0 = \sum_{n=0}^{\infty} a_{2n} x^{2n} \qquad \ldots \quad (9.11)$$

where the coefficients a_2 are related by the recurrence relation (4.2)

To solve (9.9) we set, as before

$$\psi_1 = \zeta + i\eta$$
$$W_1 = \alpha + i\beta.$$

Proceeding as in case (i) we find that

$$\alpha = 0$$

and that β is given by

$$\beta = \frac{4\sum\limits_{n=0}^{\infty} n(n+1)(2n+1)\beta_{2n} - \sum\limits_{n=0}^{\infty}(2n+1)\lambda_{2n}}{\sum\limits_{n=0}^{\infty}(2n+1)\lambda_{2n}} \qquad (9.12)$$

where β_{2r} and μ_{2r} are given by (6.26) and (6.27) respectively, while λ_{2r} are given by the relation

$$4(r+1)A\lambda_{2r} + \{I + (2r-2)F + (2r-2)(2r-3)B\}\lambda_{2r-2}$$
$$+\{J + (2r-4)G + (2r-4)(2r-5)C\}\lambda_{2r-4}$$
$$+\{L + (2r-6)H + (2r-6)(2r-7)D\}\lambda_{2r-6}$$
$$= \frac{8V_s}{\gamma} r(r+1)(r+2)(2r+1)\beta_{2r}. \qquad (9.13)$$

Thus in the case of a purely azimuthal magnetic field also there is no change in the period of oscillation due to the finiteness of conductivity but the mechanical and magnetic pulsations are damped thus indicating the generality of the results obtained here and in Bhatnagar and Nagpaul (1957). The damping is given as before, by

$$t_0 = 8\pi\mu R^2\sigma \left[\frac{1}{\beta}\left(W_0 - \frac{4}{\gamma}\right)\right], \qquad \ldots \quad (9.14)$$

where, of course, β is now given by (9.12).

In table 2 we have given the values of β and the damping time $t_0/8\pi\mu R^2\sigma$ for some of the values of $V_0{}^2$ and $V_s{}^2$ discussed in part A.

Table 2

V_0^2	V_s^2	W_0	β	$\frac{t_0}{8\pi\mu R^2\sigma}=\frac{1}{\beta}\left(W_0-\frac{4}{\gamma}\right)$
1.667	—	16.5845	80.7124	.176
3.75	—	32.00	185.8800	.157
—	1.0	13.12	1.2867	8.3314
--	9.0	86 3312	146.2818	0.5738

From table 2 we find that as the strength of the magnetic field increases the damping time decreases. We also find that by comparing the first two values of damping times in table 2 with the corresponding values in Bhatnagar and Nagpaul (1957) that due to variable density the damping time decreases.

In passing we mention that the present problem may be also regarded as the investigation of the stability of the cylindrical stratified system with Helical magnetic field under radial disturbances. That is, the disturbances in the direction in which there is inhomogeneity in density and magnetic field. The role of finite conductivity is to increase the stability by damping the disturbances.

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