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CONSEQUENCES OF GENERALISED KIRCHHOFF'S LAWS AND PROOF OF THEVENIN AND NORTON THEOREMS.

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ABSTRACT. Some properties of the conductivity matrix of the generalised Kirchhoff's laws for continuous media have been deduced. The diagonal elements of this matrix are shown to be all positive and non-diagonal elements all negative. For the usual network of resistors the non-diagonal elements are identified with the branch conductances with negative sign. The two distinct approaches for studying the amplification of vacuum tubes are reconciled. The two well-known theorems in network theory are deduced from the generalised Kirchhoff's laws.

INTRODUCTION

The generalised Kirchhoff's laws for a continuous conducting medium of specific conductivity κ and having m electrodes embedded in it can be expressed in the following matrix equations (Mitra and Roy 1966)

$$J = CV - VC$$

$$J \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \vec{i}$$

where (i) J represents the cross-current matrix, (ii) V the voltage matrix, a diagonal matrix, with its elements $v_1, v_2, \ldots v_m$, the potentials on the m electrodes, (iii) C is the conductivity matrix, which is symmetric and depends on the geometry of the system, (iv) $\vec{i} \equiv (i_1, i_2, \ldots i_m)$ is the current (column) vector whose elements are the total current flowing into the electrodes from outside sources. When the continuous medium degenerates into a network of line conductors, the elements of the conductivity matrix can be easily identified with the reciprocal of the resistances of the elements of the network (but with reversed sign*) by means of the macroscopic Ohm's law.

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*In our earlier paper Indian J. Phys, (1966), 50, 38, 2nd para the relation should read C_{13} $- 1/R_{13}$ instead of $C_{13} = 1/R_{13}$.

In this paper we shall first deduce some properties of the elements of the conductivity matrix C and then give a proof of the well-known theorems of Thevenin and Norton in network theory.

1. *Theorem* : The diagonal elements c_{ii} of the conductivity matrix C are all positive and (ii) the nondiagonal elements c_{ij} are all negative.

The first part is easy to prove from the quadratic form for viz,

$$\int\int\int |\operatorname{grad}\phi|^2 d\tau = -\sum_{l=1}^{m} \int\int_{S_l} \phi \frac{\partial\phi_{\pm}}{\partial n} dS$$

$$= \frac{4\pi}{\kappa} \sum_{l=1}^{m} v_l i_l$$

where $\dfrac{\partial}{\partial n}$ denotes derivative along the outward drawn normal as usual.

Since, $$\vec{i} = C$$

$$\sum_{l=1}^{m} v_l i_l = \vec{v}' \, C \, \vec{v} = \frac{\kappa}{4\pi} \iiint |\operatorname{grad}\phi|^2 d\tau \geqslant 0$$

Thus the quadratic form $\vec{v}' \; C \; \vec{v}$ is positive semi-definite, and therefore, $c_{ii} \geqslant 0$.

The second part is not so obvious and the proof is somewhat elaborate. To obtain the first column of C it is to be multiplied by the unit vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Physically it means that the first column of C represents the total currents flowing into the electrodes when the first electroded is kept at unit potential and the rest at zero potential, that is, $v_1 = 1, v_2 = v_3 = \,.\,.\, = v_m = 0$. By Earnshaw's theorem the potential function ϕ cannot have any maximum or minimum in the region outside the surfaces $S_0, S_1, S_2, \ldots, S_m$. Since the potential function is a continuous function in this region, which is a closed region enclosed by S_0, it must attain its upper and lower bounds within it or on the boundaries of the region. Since Earnshaw's theorem excludes the possibility of the upper and lower bounds occurring within the region, therefore it must be on the boundary of the region. The largest value of the potential is 1 and occurs on the first surface and the lowest is 0 on the other surfaces. In the immediate neighbourhood of the first surface S_1, the potential ϕ must be less than the potential on S_1. Thus $\dfrac{\partial\phi_+}{\partial n}$ must be throughout negative on S_1 and positive on $S_2, S_3, \ldots, S_m$; because an equipotential surface must exist in the neighbourhood of S_1 and cannot have common points with S_1 which excludes the possibility of $\frac{\partial\phi_+}{\partial n} = 0$ at any point on S_1.

Thus the current density $\lambda_1 = -\kappa \frac{\partial\phi_+}{\partial n} > 0$ and $\lambda_2, \lambda_3, .., \lambda_m$ are all < 0,

$$c_{11} = \iint_{S_1} \lambda_1 dS > 0$$

$$c_{21} = \iint_{S_2} \lambda_2 dS < 0$$

$$c_{31} = \iint_{S_3} \lambda_3 dS < 0$$

....

....

For other electrodes this property can be similarly established. Hence the theorem is proved.

This proves also that the current density matrix $\Lambda(s)$ has positive elements on the diagonal and negative elements in the rest, excepting the 0th diagonal element, which may be of arbitrary sign.

The elements of C are thus analogous to the co-efficients of electrostatic capacity. The only difference is that the sum of the elements of a column (or row) of C is 0.

As has been shown in our earlier work, the cross-elements for a network of resistors, is the same as the conductance of the resistor $\frac{-1}{R_{ij}}$ connecting the ith and jth node, but with a negative sign. Since c_{ij} have been proved to be negative, the identification with the conductance is compltely free from any ambiguity, because the resistances must be positive quantities.

Application to Thermionic Tubes: This identification nicely reconciles the two distinct approaches for studying the theory of amplification in thermionic valves, viz, in terms of the electrostatic capacities between the electrodes or by means of the transconductances. We have shown just now that these are identical in magnitude but opposite in sign. But, there is one ***essential difference***. The elements of the conductivity matrix are not exactly identical with the usual co-efficients of capacity between the electrodes, unless the medium is ***infinitely extended*** (Mitra and Roy 1966). For a clearer understanding, let us suppose that the conductivity of the medium κ be reduced to 0 in the limit, keeping the potentials undisturbed. The problem then reduces to a purely electrostatic problem of finding the potential function ϕ with the same boundary conditions (i) $\phi = v_1, v_2, \ldots v_m$ on the electrodes and (ii) $\frac{\partial\phi}{\partial n} = 0$ on the outer boundary S_0 of the medium. This problem is identical to the current flow problem discussed in the earlier work. Here a capacity matrix say, Q takes the place of the conductivity matrix C and these are identical excepting for the constant factor κ, which is present as a constant multiple in each element of C i.e., $C = \kappa Q$ (note : this Q should not be confused with the Q-matrix mentioned in the earlier work). The elements of Q are the co-efficients of electrostatic capacity for this particular boundary value problem with $\frac{\partial\phi}{\partial n} = 0$ on a surface S_0 enclosing the conductors $S_1, S_2, \ldots S_0$. Only when this enclosing surface is extended to infinity, that the elements of Q will become identical with the usual co-efficients of electrostatic capacities. The diagonal elements of Q will be larger than the co-efficients of self-capacity. In fact, the effect of the enclosure is neglected when the theory of the amplification of thermonic tubes are studied from the electrostatic view-point (Spangenberg, 1948,).

3. *The Form* (5) *Kirchhoff's Laws for a Network* : For a network containing resistors, the generalised Kirchhoff's laws have to be modified a little, as in this case the elements of the conductance matrix are to be identified with the branch conductances with negative sign. Denoting by g_{ij} the conductance of the resistor connecting the ith and jth node (that is the reciprocal of the resistance R_{ij}), the Kirchhoff's laws become

$$J = VG - GV \qquad \ldots (1)$$

$$J\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix} = \vec{i} \qquad \ldots (2)$$

where G is the conductivity matrix having its cross elements, $g_{12}, g_{13}, \ldots$ all positive and the diagonal elements, g_{ii} all negative and are equal to the row sum (or column sum) of the ith row, with reversed, sign, that is

$$g_{ii} = -(g_{i1} + g_{i2} + \cdots + g_{im}) \qquad \ldots (3)$$

When two nodes (say the ith and jth) are not connected by any resistor, $g_{ij} = 0$. Since,

$$G\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix} = 0 \qquad \ldots (4)$$

$$\vec{i} = J \begin{bmatrix} 1 \\ \cdot \\ \cdot \\ 1 \end{bmatrix} = (VG-GV) \begin{bmatrix} 1 \\ \cdot \\ \cdot \\ 1 \end{bmatrix} = -G\vec{v} \qquad .. \quad (5)$$

It is obvious that both the Loop and Nodal laws of Kirchhoff are implicit in the two equations (1) and (2) and equation (5) is the result of combining the two laws. The conductivity matrix G is symmetric and singular, the diagonal elements being the negative sum of the conductivities of the elements of the corresponding row or column (that is, the negative sum of all the conductivities of the branches connecting the particular node). *A node, say, the kth which is connected to a source, we call it 'live', is distinguished from the 'floating' nodes by having the current flowing into it from outside, viz,* i_k by $i_k \neq 0$.

Proof of Thevenin Theorem : Now we proceed to prove the Thevenin Theorem from equation (5). Let us suppose that the first $k-1$ nodes are connected to sources that is, $i_1, i_2, .., i_{k-1}$ are $\neq 0$. The remaining nodes from k onwards are floating, viz, $i_k = i_{k+1} = .. = i_m = 0$. The equation (5) takes the form in this case,

$$\begin{array}{rcl}
-i_1 & = & g_{11}v_1+g_{12}v_2+\cdot\cdot+g_{1m}v_m \\
-i_2 & = & g_{21}v_1+g_{22}v_2+\cdot\cdot+g_{2m}v_m \\
\cdot\cdot & \cdot\cdot & \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \\
\cdot\cdot & \cdot\cdot & \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \\
-i_{k-1} & = & g_{k-1,\,1}v_1+g_{k-1,\,2}v_2+\cdot\cdot\cdot+g_{k-1,\,m}v_m \\
0 & = & g_{k,\,1}v_1+g_{k,2}v_2+\cdot\cdot+g_{k,\,m}v_m \\
0 & = & g_{k+1,\,1}v_1+g_{k+1,\,2}v_2+\cdot\cdot+g_{k+1,\,m}v_m \\
\cdot\cdot & \cdot\cdot & \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \\
\cdot\cdot & \cdot\cdot & \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \\
0 & = & g_{m,1}v_1+g_{m,2}v_2+\cdot\cdot+g_{m,m}v_m
\end{array} \qquad .. \quad (6)$$

The voltages on the 'floating' nodes k, $k+1$, .. m are linearly dependent on the voltages on the 'live' nodes $v_1, v_2, .., v_{k-1}$ and can be determined in terms of these from the kth to the mth equations. Let the matrix G be partitioned at the $(k-1)$th row and $(k-1)$th column

$$G \equiv \begin{pmatrix} G_1 & D_1 \\ D_1' & G_2 \end{pmatrix} \qquad .. \quad (7)$$

where the partitioned matrices G_1 and G_2 are square matrices, G_1 having $k-1$ rows and G_2 having $m-k+1$ rows; D_1 is a rectangular matrix of $k-1$ rows and $m-k+1$ columns and D_1' is its transpose because $G' = G$.

Thus,

$$\begin{bmatrix} v_k \\ v_{k+1} \\ \cdot \\ \cdot \\ v_m \end{bmatrix} = -G_2^{-1}D_1' \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ v_{k-1} \end{bmatrix} \qquad .. \quad (8)$$

The branch current j_{rs} between two 'floating' nodes, say, the rth and sth is $g_{rs}(v_r-v_s)$. The Thevenin theorem gives a method of determining this by solving a simpler circuit problem.

But computation of the reciprocal matrix G_2^{-1} is necessary for determinng only two elements of the vector, a labour not worth doing. Some artifice can be found for simplying the computation. Let Γ_2 be the matrix formed from G_2 by removing the (rs)th element g_{rs}. Thus,

$$G_2-\Gamma_2 = g_{rs}(E_{rr}-E_{rs}-E_{sr}+E_{ss}) \qquad .. \quad (9)$$

where E_{rr} is the matrix having 1 in the diagonal position r and E_{rs} is the matrix having 1 in the (rs)th position. The rth diagonal element of Γ_2 is different from that of G_2, being the sum of the elements of the rth row from which g_{rs} has been removed.

Now,

$$G_2^{-1}-\Gamma_2^{-1} = G_2^{-1}(\Gamma_2-G_2)\Gamma_2^{-1} = \Gamma_2^{-1}(\Gamma_2-G_2)G_2^{-1}$$

So,

$$\begin{aligned} G_2^{-1}-\Gamma_2^{-1} &= g_{rs}G_2^{-1}(E_{rr}-E_{rs}-E_{sr}+E_{ss})\Gamma_2^{-1} \\ &= g_{rs}\Gamma_2^{-1}(E_{rr}-E_{rs}-E_{sr}+E_{ss})G_2^{-1} \qquad .. \quad (10) \end{aligned}$$

Further

$$\begin{bmatrix} v_k \\ v_{k+1} \\ \cdot \\ \cdot \\ v_m \end{bmatrix} = -G_2^{-1}D_1' \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ v_{k-1} \end{bmatrix} \qquad .. \quad (11)$$

$$\begin{bmatrix} u_k \\ u_{k+1} \\ \cdot \\ \cdot \\ u \end{bmatrix} = -\Gamma_2^{-1}D_1' \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ v_{k-1} \end{bmatrix} \qquad .. \quad (11')$$

where u_k, u_{k+1}, .. u_m are the new floating voltages when the element g_{rs} has been removed from the network.

Thus,

$$\begin{bmatrix} v_k - u_k \\ v_{k+1} - u_{k+1} \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ v_m - u_m \end{bmatrix} = -(G_2^{-1} - \Gamma_2^{-1})D_1' \begin{bmatrix} v_1 \\ v_2 \\ : \\ \cdot \\ v_{k-1} \end{bmatrix}$$

$$= -g_{rs}\Gamma_2^{-1}(E_{rr} - E_{rs} - E_{sr} + E_{ss})G_2^{-1}D_1' \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ v_{k-1} \end{bmatrix} \quad .. \quad (12)$$

$$= g_{rs}\Gamma_2^{-1}(E_{rr} - E_{rs} - E_{sr} + E_{ss}) \begin{bmatrix} v_k \\ v_{k+1} \\ : \\ v_m \end{bmatrix}$$

If we can know all the elements of the rth and sth rows, then the original floating voltages $v_k, v_{k+1}, \ldots, v_m$ can be written in terms of the new floating voltages $u_k, u_{k+1}, \ldots u_m$. But for Thevenin's Theorem we require to calculate only $v_r - v_s$. Let γ'_{pq} denote the elements of Γ_2^{-1}. As Γ_2^{-1} must be symmetric since Γ_2 is symmetric $\gamma'_{rs} = \gamma'_{sr}$. Multiplying both sides of the equation (12) by the row vector $(e_r - e_s)'$ that is, the row vector having 1 in the rth position and -1 in the sth position we get,

$$v_r - u_r - (v_s - u_s) = g_{rs}(\gamma'_{r1} - \gamma'_{s1}, \gamma'_{r2} - \gamma'_{s2}, \ldots\ldots)(E_{rr} - E_{rs} - E_{sr} + E_{ss})$$

$$\times \begin{bmatrix} v_k \\ v_{k+1} \\ : \\ v_m \end{bmatrix}$$

$$= g_{rs}(0, 0, \ldots \gamma'_{rr} - \gamma'_{sr} - \gamma'_{rs} + \gamma'_{ss}, 0, 0, \ldots$$

$$-\gamma'_{rr} + \gamma'_{sr} + \gamma'_{rs} - \gamma'_{ss}, 0, 0, \ldots 0) \begin{bmatrix} v_k \\ v_{k+1} \\ : \\ v_n \end{bmatrix}$$

$$= g_{rs}(\gamma'_{rr} + \gamma'_{ss} - 2\gamma'_{rs})(v_r - v_s)$$

Therefore,

$$v_r - v_s = \frac{u_r - u_s}{1 + g_{rs}(2\gamma'_{rs} - \gamma'_{rr} - \gamma'_{ss})} \quad .. \quad (13)$$

Or,

$$j_{rs} = g_{rs}(v_r - v_s) = \frac{u_r - u_s}{\frac{1}{g_{rs}} + 2\gamma'_{rs} - \gamma_{rr}' - \gamma'_{ss}} \qquad \ldots \quad (14)$$

Now the denominator in the right hand side of the equation (14) represents a resistance which is the sum of the resistance $1/g_{rs}$ and $(2\gamma'_{rs} - \gamma'_{rr} - \gamma'_{ss})$, that is to say, the two resistances $1/g_{rs}$ and $(2\gamma'_{rs} - \gamma'_{rr} - \gamma'_{ss})$ are connected in series, of which the former represents the resistance of the (rs)th branch. Suppose the nodes r and s are connected to voltage sources w_r and w_s and all other voltage sources are grounded. Let l_r and l_s represent the total currents flowing through these nodes $(l_s = -l_r)$, then from equations (6) and (7)

$$\begin{bmatrix} w_k \\ w_{k+1} \\ \vdots \\ w_r \\ \vdots \\ w_s \\ \vdots \\ w_m \end{bmatrix} = \Gamma_2^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ l_r \\ \vdots \\ l_s \\ \vdots \\ 0 \end{bmatrix} - \Gamma_9^{-1} D'_1 \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{bmatrix} - l_r \Gamma_2^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ -1 \\ \vdots \\ 0 \end{bmatrix}$$

Multiplying by $(e_r - e_s)'$ both sides we get,

$$w_r - w_s = l_r(2\gamma'_{rs} - \gamma'_{rr} - \gamma'_{ss})$$

Thus $2\gamma'_{rs} - \gamma'_{rr} - \gamma'_{ss}$ represents the equivalent resistance of the network Γ_2 between the rth and sth nodes, which is the same as the equivalent resistance of the circuit with the source voltages reduced to 0. So

$$\frac{u_r - u_s}{\frac{1}{g_{rs}} + 2\gamma'_{rs} - \gamma'_{rr} - \gamma'_{ss}}$$

can be interpreted as the total currents flowing into the rth and sth nodes in the network Γ_2, when they are connected with a voltage source $u_r - u_s$ in series with the resistance $1/g_{rs}$. This is the well-known Thevenin's Theorem.

Norton Theorem : The Norton's Theorem comes as a simple consequence of equation (14) when defferently interpreted.

$$j_r = g_{rs}(v_{rs} - v_s) = \frac{u_r - u_s}{\frac{1}{g_{rs}} + 2\gamma'_{rs} - \gamma'_{rr} - \gamma'_{ss}}$$

When the series resistance $1/g_{rs}$ is put equal to 0, then the total current flowing into the net work i_N is

$$i_N = \frac{u_r - u_s}{2\gamma'_{rs} - \gamma'_{rr} - \gamma'_{ss}}$$

Let us now imagine that the rth and sth nodes are connected to a current source of strength i_N and the branch element g_{rs} restored to its original position. Then the total current i_N is split into two parallel portions, one through the element g_{rs} and the other through the network Γ_2. If j_N be the total current flowing into Γ_2, and l_N be the current across g_{rs} then

$$i_N = j_N + l_N$$

As these are parallel, the voltage condition gives

$$j_N(2\gamma'_{rs} - \gamma'_{rr} - \gamma'_{ss}) = \frac{l_N}{g_{rs}}$$

Thus,

$$l_N = \frac{i_N}{1 + \dfrac{1}{g_{rs}(2\gamma'_{rs} - \gamma'_{rr} - \gamma'_{ss})}}$$

$$= \frac{u_r - u_s}{\dfrac{1}{g_{rs}} + 2\gamma'_{rs} - \gamma'_{sl} - \gamma'_{ss}}$$

$- j_{rs}$, by equation (14)

This is Norton's Theorem.

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