ON THE LORENTZ AND OTHER GROUPS

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ABSTRACT. The object of this paper is to make a comparative study of the Lorentz group, the 4-dimensional rotation group, the Galilei group and the group, which is the other extreme limit of the Lorentz group. This is done by representing the elements of all these groups in terms of a spatial vector and a rotation in space.

INTRODUCTION

It is well known that Galilei group is, in a natural manner, represented by a spatial vector (velocity) and a rotation in space. The Galilei group may be spoken of as the limit of the Lorentz group when the translational velocity in units of that of light tends to zero. It has been shown by the author (Sen Gupta 1966) and Levy-Leblond (1965) that similar to the Galilei group there is a group of space-time transformations which is the other extreme limit of the Lorentz Group, the limit in which the translational velocity tends to infinity. This group is also represented, in a natural manner, by a spatial velocity and a rotation in sapace. The group structures, e.g. the laws of composition of both these groups are easily expressed when they are represented by a spatial vector and a rotation in space. The main object of this short paper is a comparative study of the Lorentz group with respect to these two groups, which are the two extreme limits of Lorentz group and the 4-dimensional rotation group. In order to do this, it is quite convinient to para-moterize the Lorentz group in a mannor similar to the other two groups, i.e. in terms of a spatial vector and a rotation in space. This is accomplished by de-composing the 4×4 matrix corresponding to any Lorentz transformation into a ymmetric matrix and an orthogonal one. The distinctive property of the Lorentz transformation makes the latter orthogonal 4×4 matrix corresponds to a rotation in space with or without time reversal. The former symmetric factor is the characteristic of the Lorentz transformation, in as much as, it is the accelera-ting part (Wigner 1939), i.e. it corresponds to the uniform velocity motion. This is completely determined by a 3-dimensional spatial vector.

This decomposition is carried out in the next section; further, the properties of the characteristic symmetric factor are investigated. In section 3, we discuss the law of composition of the elements of the Lorentz group and compare it with those of the other two limiting groups. In the last section we endeavour to extend our studies to the 4-dimensional orthogonal group.

In the expressions for Lorentz transformations we use the space-time vector $\bar{x} = (\boldsymbol{r}, ct)$. A bar over a matrix indicates its transposed. In the following, we will always use capital Greek letters to represent 4×4 matrices and capital Latin letters to represent 3×3 matrices. In view of the problem we want to discuss it will be advantageous to write any 4×4 matrix with the help of a 3×3 matrix and two-3-dimensional vectors along with a scalar, e.g.

$$\begin{vmatrix} A & \boldsymbol{k} \\ \bar{\boldsymbol{k}}' & s \end{vmatrix}.$$

In particular

$$\epsilon = \begin{vmatrix} E & 0 \\ 0 & -1 \end{vmatrix}, \qquad \ldots \quad (1)$$

where E is the 3×3 unit matrix. It may not be irrelevant to mention that all our discussions are with real numbers. We will use the nottion $\boldsymbol{k}.\bar{\boldsymbol{l}}$ to represent the 3×3 matrix $\{k_i l_j\}$. In this paper we will confine ourselves only to the homogenous space-time transformations.

DECOMPOSITION OF A LORENTZ TRANSFORMATION

Let Λ be the 4×4 matrix corresponding to a Lorentz transformation. Hence,

$$\Lambda\epsilon\bar{\Lambda}\epsilon = 1 \qquad \ldots \quad (2)$$

(i) *Polar decomposition*

It is well known that any real non-singular matrix can be decomposed into a symmetric and an orthogonal factors. So that we can write

$$\Lambda = (\Lambda\bar{\Lambda})^{\frac{1}{2}}\{(\Lambda\bar{\Lambda})^{-\frac{1}{2}}\Lambda\}. \qquad \ldots \quad (3)$$

$(\Lambda\bar{\Lambda})^{\frac{1}{2}}$ is symmetric and $(\Lambda\bar{\Lambda})^{-\frac{1}{2}}\Lambda$ is orthogonal. In order to determine uniquely the square root $(\Lambda\bar{\Lambda})^{\frac{1}{2}}$, we first note that $\Lambda\bar{\Lambda}$ is symmetric, hence it can be diagonalised by a real orthogonal matrix Δ (say); so that

$$\Lambda\bar{\Lambda} = \Delta\Lambda_d\bar{\Delta}, \qquad \ldots \quad (4)$$

where Λ_d is a diagonal matrix whose elements are the eigen-values of $\Lambda\bar{\Lambda}$. They are real positive. Further, $\Lambda\bar{\Lambda}$ being a Lorentz transformations the eigen values of $\Lambda\bar{\Lambda}$ are e^{θ_1}, $e^{-\theta_1}$, e^{θ_2} and $e^{-\theta_2}$ where θ's are real. (It will be shown later that for $\Lambda\bar{\Lambda}$ at least one of the θ's is zero. Thus

$$\Lambda_d = (e^{\theta_1},\ e^{-\theta_1},\ e^{\theta_2},\ e^{-\theta_2}). \quad .. \quad (5)$$

Let

$$\Lambda_d^{\frac{1}{2}} = (e^{\theta_1/2},\ e^{-\theta_1/2},\ e^{\theta_2/2},\ e^{-\theta_2/2}) \quad .. \quad (6)$$

and

$$\Lambda_d^{-\frac{1}{2}} = (e^{-\theta_1/2},\ e^{\theta_1/2},\ e^{-\theta_2/2},\ e^{\theta_2/2}). \quad .. \quad (7)$$

We define (Gantmacher 1959)

$$(\Lambda\bar{\Lambda})^{\frac{1}{2}} = \Delta\Lambda_d^{\frac{1}{2}}\bar{\Delta} \quad .. \quad (8)$$

and

$$(\Lambda\bar{\Lambda})^{-\frac{1}{2}} = \Delta\Lambda_d^{-\frac{1}{2}}\bar{\Delta}. \quad .. \quad (9)$$

With this definition of $(\Lambda\bar{\Lambda})^{\frac{1}{2}}$ one can easily verify that it is also a Lorentz transformation.

Next, it can be easily established that any 4×4 orthogonal matrix Γ, which is also a Lorentz transformation, is only a space-rotation with or without reversal of time. This is because of the fact that Γ commutes with ϵ. Hence any Λ is the product of a symmetric Lorentz transforation Σ and a rotation in space,

$$\Lambda = \Sigma\Gamma. \quad .. \quad (10)$$

In order to avoid complications we confine our attention to the proper Lorentz transformation, i.e. only those transformations which are continuously connected to unity. So that

$$\Gamma = \begin{vmatrix} A & 0 \\ 0 & 1 \end{vmatrix} \quad \Gamma(A) \quad .. \quad (11)$$

and det. $\Sigma = 1$ and det. $A = 1$.

(ii) *Symmetric Lorentz Transformations*

Now we will try to find the most general form of a symmetric Lorentz transformation Σ. Let

$$\Sigma = \begin{vmatrix} S & \boldsymbol{k} \\ \bar{\boldsymbol{k}} & s \end{vmatrix} \quad .. \quad (12)$$

Since Σ satisfies eq. (2),

$$S\,S - \boldsymbol{k}.\bar{\boldsymbol{k}} = E \quad .. \quad (13)$$

$$S\boldsymbol{k} - s\boldsymbol{k} = 0 \quad .. \quad (14)$$

and

$$s^2 - k^2 = 1;\ \{k \equiv +(\bar{\boldsymbol{k}}.\boldsymbol{k})^{\frac{1}{2}}\}. \quad .. \quad (15)$$

The above equations may be easily solved for s and the symmetric matrix S in terms of $\boldsymbol{k}$, as a given vector. The solution is

$$S(\boldsymbol{k}) = E - \frac{1-s_k}{k^2}\ \boldsymbol{k}\cdot\bar{\boldsymbol{k}} \quad .. \quad (16)$$

$$s_k = +\sqrt{1+k^2}. \quad .. \quad (17)$$

Thus Σ is uniquely determined by a vector $\boldsymbol{k}$; $k \neq 0$. We will use the notation $\Sigma(\boldsymbol{k})$ to represent such a Σ. This result can also be obtained directly by assuming the factorization (Sen Gupta 1965). It is of interest to note that the eigen vectors of $S(\boldsymbol{k})$ are $\boldsymbol{l}_1$, $\boldsymbol{l}_2$ and $\boldsymbol{k}$. with eigen values 1, 1 and s_k; $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ are two mutually orthogonal vectors both orthogonal to $\boldsymbol{k}$. The eigen vectors of $\Sigma(\boldsymbol{k})$ are

$$(\boldsymbol{l}_1,\ 0),\ (\boldsymbol{l}_2,\ 0),\ (\boldsymbol{k},\ k),\ (\boldsymbol{k},\ -k)$$

with respective eigen values

$$1,\ 1,\ \ (s_k+k),\ \ (s_k-k).$$

$\Sigma(\boldsymbol{k})$ corresponds physically to a pure Lorentz transformation with the velocity along the direction $\boldsymbol{k}$ and magnitude ck.

(iii) *Composition of $\Sigma(\boldsymbol{k})$'s and uniqueness of the decomposition*

By direct multiplication one can obtain

$$\Sigma(\boldsymbol{k})\Sigma(\boldsymbol{l}) = \Sigma(\boldsymbol{p}(\boldsymbol{k},\boldsymbol{l}))\Gamma(A(\boldsymbol{k},\boldsymbol{l})), \quad .. \quad (18)$$

where

$$\boldsymbol{p}(\boldsymbol{k},\boldsymbol{l}) = S(\boldsymbol{k})\boldsymbol{l} + s_l\boldsymbol{k} \quad .. \quad (19)$$

and

$$A(\boldsymbol{k},\boldsymbol{l}) = S^{-1}(\boldsymbol{p})\{S(\boldsymbol{k})\ S(\boldsymbol{l}) + \boldsymbol{k}\cdot\bar{\boldsymbol{l}}\}. \quad .. \quad (20)$$

Thus $\Sigma(\boldsymbol{k})$'s forms a sub-group only with $\boldsymbol{k}$'s along the same direction, which is well known and

$$\Sigma^{-1}(\boldsymbol{k}) = \Sigma(-\boldsymbol{k}). \quad .. \quad (21)$$

In order to show the uniqueness of the decomposition, let us assume two distinct decompositon of the same Λ,

$$\Lambda = \Sigma(\boldsymbol{k}_1)\,\Gamma(A_1) = \Sigma(\boldsymbol{k}_2)\Gamma(A_2);$$

so that
$$\Gamma(A_2)\Gamma(\bar{A}_1) = \Sigma(-\boldsymbol{k}_2)\Sigma(\boldsymbol{k}_1)$$
$$= \Sigma(\boldsymbol{p}(-\boldsymbol{k}_2, \boldsymbol{k}_1)\Gamma(A(-\boldsymbol{k}_2, \boldsymbol{k}_1)).$$

This can only happen when $\boldsymbol{p} = 0$, i.e. $\boldsymbol{k}_1 = \boldsymbol{k}_2$ which will imply further $A_1 = A_2$. It should be mentioned we would have obtained similar results by writing

$$\Lambda = \Gamma(A')\Sigma(\boldsymbol{k}').$$

Evidently $\boldsymbol{k}' \neq \boldsymbol{k}$ but they are related by

$$A'\boldsymbol{k}' = \boldsymbol{k}. \quad .. \quad (22)$$

On the other hand it is of interest to note that

$$A' = A. \quad .. \quad (23)$$

These results follow from the fact

$$\Gamma(B)\Sigma(\boldsymbol{k})\Gamma(\bar{B}) = \Sigma(B\boldsymbol{k}). \quad .. \quad (24)$$

The relation between this decomposition and usual one (Wigner 1939) is given by

$$\Lambda = \Sigma(\boldsymbol{k})\,\Gamma(A) = \Gamma(A')\,\Sigma\,(\boldsymbol{k}_3)\,\Gamma(A'\,A), \quad .. \quad (25)$$

where
$$A'\boldsymbol{k}_3 = \boldsymbol{k}$$

and $\boldsymbol{k}_3$ is the vector along third spatial axis with magnitude k.

THE LAWS OF COMPOSITION OF LORENTZ AND OTHER GROUPS

Let us consider two Lorentz transformations

$$\mathcal{L}(\boldsymbol{k}_1, A_1) = \Sigma(\boldsymbol{k}_1)\,\Gamma(A_1)$$

and
$$\mathcal{L}(\boldsymbol{k}_2, A_2) = \Sigma(\boldsymbol{k}_2)\,\Gamma(A_2).$$

Their product is given by

$$\mathcal{L}(\boldsymbol{k}_1, A_1)\,\mathcal{L}(\boldsymbol{k}_2, A_2) = \Sigma(\boldsymbol{k}_1)\,\Sigma(A_1\boldsymbol{k}_2)\,\Gamma(A_1\,A_2)$$
$$= \Sigma(\boldsymbol{p}(\boldsymbol{k}_1, A_1\boldsymbol{k}_2))\Gamma(A(\boldsymbol{k}_1, A_1\,\boldsymbol{k}_2)A_1A_2)$$
$$= \mathcal{L}(\boldsymbol{p}(\boldsymbol{k}_1, A_1\boldsymbol{k}_2), A(\boldsymbol{k}_1, A_1\boldsymbol{k}_2)A_1A_2). \quad .. \quad (26)$$

The unit element is $\Lambda(0, E)$ and the inverse

$$\mathcal{L}^{-1}(\boldsymbol{k}, A) = \mathcal{L}(-\bar{A}\boldsymbol{k}, \bar{A}). \quad .. \quad (27)$$

The elements of the Galilei group can be also represented by a spatial vector $\boldsymbol{k}$ and a rotation in space. The law of combination of the elements when expressed in this form is given by

$$\mathcal{G}(\boldsymbol{k}_1, A_1)\ \mathcal{G}(\boldsymbol{k}_2, A_2) = \mathcal{G}(A_1\ \boldsymbol{k}_2+\boldsymbol{k}_1, \boldsymbol{A}_1A_2). \quad .. \quad (28)$$

The unit element is $\mathcal{G}(0,\ E)$ and the inverse

$$\mathcal{G}^{-1}(\boldsymbol{k}, A) = \mathcal{G}(-\bar{A}\boldsymbol{k}, \bar{A}). \quad .. \quad (29)$$

Similar expression for the law of composition of the elements $\mathcal{U}$ of the group which is the other extreem limit of the Lorents group is given by (Sen Gupta 1966)

$$\mathcal{U}(\boldsymbol{k}_1, A_1)\ \mathcal{U}(\boldsymbol{k}_2\ A_2) = \mathcal{U}(\bar{A}_2\boldsymbol{k}_1+\boldsymbol{k}_2, A_1A_2). \quad .. \quad (30)$$

The unit element is $\mathcal{U}$ $(0, E)$ and the inverse

$$\mathcal{U}^{-1}(\boldsymbol{k}, A) = \mathcal{U}(-A\boldsymbol{k}, \bar{A}). \quad .. \quad (31)$$

The basic difference in the structure of the Lorentz group becomes transparent in the law of compositions as expressed above.

From eqs. (26), (28) and (30) if follows that

$$\mathcal{L}(0, A_1)\ \mathcal{L}(0, A_2) = \mathcal{L}(0, A_1\ A_2) \quad .. \quad (32)$$

$$\mathcal{G}(0, A_1)\ \mathcal{G}(0, A_2) = \mathcal{G}(0, A_1\ A_2) \quad .. \quad (33)$$

and

$$\mathcal{U}(0, A_1)\ \mathcal{U}(0, A_2) = \mathcal{U}(0, A_1\ A_2). \quad .. \quad (34)$$

The elements $\mathcal{L}(0, \mathrm{A})$ form a sub-group of the Lorentz group. Similarly $\boldsymbol{G}(0, \mathrm{A})$ and $\mathcal{U}(0, A)$ are respectively sub-group of the Galilei group and the other limit group. This sub-group is nothing but the 3-dimensional rotation group. In none of these cases this is an invariant sub-group.

Again it follows from eq. (28) that

$$\mathcal{G}(\boldsymbol{k}_1, E)\ \mathcal{G}(\boldsymbol{k}_2, E) = \mathcal{G}(\boldsymbol{k}_1+\boldsymbol{k}_2, E) = \mathcal{G}(\boldsymbol{k}_2, E)\mathcal{G}\ (\boldsymbol{k}_1, E), \quad .. \quad (35)$$

i.e., the elements $\mathrm{G}(\boldsymbol{k}, E)$ are a commuting sub-group of the Galilei group. Similarly also for the other group as it follows for eq. (30) that

$$\mathcal{U}(\boldsymbol{k}_1, E)\ \mathcal{U}(\boldsymbol{k}_2, E) = \mathcal{U}(\boldsymbol{k}_1+\boldsymbol{k}_2, E) = \mathcal{U}(\boldsymbol{k}_2\ E)\mathcal{U}(\boldsymbol{k}_1, E). \quad .. \quad (36)$$

But it is no longer true in case of the Lorentz group because of the factor $A(\boldsymbol{k}_1, A_1\ \boldsymbol{k}_2)$ in the right hand side of eq. (26).

It has already been noted that $\mathcal{G}(\boldsymbol{k}, E)$ and $\mathcal{U}(\boldsymbol{k}, E)$ are commuting sub-groups, in fact they are nothing but the Abelian Group of 3-dimensional vector space. Further-more since

$$\mathcal{G}^{-1}(\boldsymbol{k}', A')\mathcal{G}(\boldsymbol{k}, E)\ \mathcal{G}(\boldsymbol{k}', A') = \mathcal{G}(\bar{A}'\boldsymbol{k}, E) \quad (35)$$

and

$$\mathcal{U}^{-1}(\boldsymbol{k}', A')\mathcal{U}(\boldsymbol{k}, E)\, \mathcal{U}(\boldsymbol{k}', A') = \mathcal{U}(\bar{A}'\ \boldsymbol{k}, E), \quad .. \quad (36)$$

they are also invariant sub-groups of the respective groups. In the language of group theory both these groups $\mathcal{G}(\boldsymbol{k}, A)$ and $\mathcal{U}(\boldsymbol{k}, A)$ are group extensions of the Abelian group of 3-dimensional vector space by the operator group of 3-dimensional rotational group. As a matter of fact, they constitute simple illustrative examples of inequivalent extensions as it clear from eqs. (35) and (36).

4-DIMENSIONAL ROTATION GROUP

Finally we try to make a comparative study of the Lorentz group and the 4-dimensional rotation group. Evidently it will be much easy if we represent the 4-dimensional rotation group in a similar manner, i.e. with the help of a 3-dimensional vector and a 3-dimensional rotation group. The element Ω of the 4×4 matrices which represents the 4-dimensional rotation group are orthogonal; hence, the question of their polar decomposition in the form of eq. (3) does not arise. But one can still speak of a decomposition of Ω in the form

$$\Omega = \Theta\Gamma(A) \quad .. \quad (37)$$

where Θ is a symmetric matrix and Γ is an orthogonal one of the form given by eq. (11). Clearly Θ is also an orthogonal matrix,

$$\Theta = \overline{\Theta} \quad \text{and} \quad \Theta\,\overline{\Theta} = \Theta^2 = 1. \quad .. \quad (38)$$

In order to express the involution and symmetric matrix Θ, we proceed as before and write

$$\Theta = \begin{vmatrix} S', & \boldsymbol{k} \\ \bar{\boldsymbol{k}}, & s' \end{vmatrix}. \quad .. \quad (39)$$

So that

$$S'S' + \boldsymbol{k}.\bar{\boldsymbol{k}} = E \quad .. \quad (40)$$

$$S'\boldsymbol{k} + s'\boldsymbol{k} = 0 \quad .. \quad (41)$$

and

$$k^2 + s^2 = 1. \quad .. \quad (42)$$

Hence

$$S'(\boldsymbol{k}) = E - \frac{1+s'_k}{k^2}\ \boldsymbol{k}.\bar{\boldsymbol{k}} \quad .. \quad (43)$$

and

$$s_k' = +\sqrt{1-k^2}. \quad .. \quad (44)$$

Thus $\Theta(k)$ is uniquely determined by a spatial vector $\boldsymbol{k}$; $k \neq 0$. It should be noted that det. $\Theta = -1$. The eigen-vectors of $\Theta(\boldsymbol{k})$ are

$$(\boldsymbol{l}_1, 0),\ (\boldsymbol{l}_2, 0),\ (\boldsymbol{k}, s_k'+1),\ (\boldsymbol{k}, s_k'-1)$$

with the respective eigen values,

$$1,\ 1,\ 1,\ -1.$$

As before one can obtain by direct multiplication

$$\Theta(\boldsymbol{k})\Theta(\boldsymbol{l}) = \Theta(\boldsymbol{p}'(\boldsymbol{k}, \boldsymbol{l}))\Gamma(A'(\boldsymbol{k}, \boldsymbol{l}), \quad .. \quad (45)$$

where

$$\boldsymbol{p}'(\boldsymbol{k}, \boldsymbol{l}) = S'(\boldsymbol{k})\boldsymbol{l}+s_l'\boldsymbol{k} \quad .. \quad (46)$$

and

$$A'(\boldsymbol{k}, \boldsymbol{l}) = S'^{-1}(\boldsymbol{p}')\{S'(\boldsymbol{k})S'(\boldsymbol{l})+\boldsymbol{k}.\bar{\boldsymbol{l}}\}. \quad (47)$$

The uniqueness of the decomposition may be proved as in the case of the Lorentz group. Thus any element of the 4-dimensional orthogonal group $\mathcal{O}$ can be represented by a spatial vector k and 3-dimensional rotation group, so that

$$\mathcal{O}(\boldsymbol{k}, A) = \Theta(\boldsymbol{k})\Gamma(A). \quad .. \quad (48)$$

The law of composition in this representation follows from eq. (45)

$$\mathcal{O}(\boldsymbol{k}_1, A_1)\ \mathcal{O}(\boldsymbol{k}_2, A_2) = \Theta\ (\boldsymbol{k}_1)\Gamma(A_1)\Theta(\boldsymbol{k}_2)\Gamma(A_2)$$
$$= \mathcal{O}\ (\boldsymbol{p}'(k_1, A_1\ \boldsymbol{k}_2))\ A(\boldsymbol{k}_1. A_1\boldsymbol{k}_2)A_1A_2), \quad .. \quad (49)$$

In deriving this use has been made of

$$\Gamma(A)\Theta(\boldsymbol{k})\ \Gamma(\bar{A}) = \Theta(A\ \boldsymbol{k}). \quad .. \quad (50)$$

The unit element is $\mathcal{O}(0, E)$ and the inverse

$$\mathcal{O}^{-1}(\boldsymbol{k}, A) = \mathcal{O}(-\bar{A}\boldsymbol{k}, \bar{A}). \quad .. \quad (51)$$

In this case also,

$$\mathcal{O}(0, A_1)\ \mathcal{O}(0, A_2) = \mathcal{O}(0, A_1\, A_2); \quad .. \quad (52)$$

hence, the elements $\mathcal{O}$ $(0, A)$ forms a sub-group, the group of 3-dimensional rotation. As represented by the expression (48) along with the law of composition (49), the difference between the 4-dimensional rotation group and the Lorentz group is only in the expressions for $S'(\boldsymbol{k})$ and s_k' eqs. (43) and (44). Incidentally this introduces the well-known basic difference between them as in this case the reality condition for $s_k' = \sqrt{1-k^2}$ restricts $0 \leqslant k \leqslant 1$, which makes the 4-dimensional rotation group compact; but in the case of the Lorentz group, k is not restricted which makes it non-compact.

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