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Relaxation behaviour of dispersive bistability of homogeneously broadened medium

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Abstract. The optical bistable behaviour of homogeneously broadened detuned medium placed inside a Fabry Perot cavity has been investigated. The steady state behaviour of the mean values of collective operators give a bistable output for the transmitted intensity. The bistable behaviour depends on the detuning of the medium. By sufficiently increasing the value of parameter $C = \frac{\gamma_R}{2\gamma_\perp}$, (see text), one can keep the system bistable for larger range of detuning. Linear stability analysis is presented. The negative slope region is unstable. The relaxation behaviour of the detuned medium at the two stable states is investigated. The dissimilarity between the absorptive and dispersive bistability is discussed by phase shift between the transmitied and the incident intensities along the two stable branches.

1. Introduction

Recently much attention has been paid [McCall (1974), Bonifacio and Lugiato (1976), (1978), Agarwal *et al* (1977), (1978)] to the study of a homogeneously broadening absorptive medium placed inside a Fabry Perot cavity. The importance of these studies lies in the fact that optical bistable system gives promise of being used as optical memory element, amplifiers, pulse shapers, limitters, clippers and switches etc. The pure absorptive models discussed in the literature correspond to the ideal situation where the incident coherent pump is exactly in tune with the two level homogenelously broadened medium. The question arises as to what shall happen to the bistable behaviour of the medium if it is not exactly in tune with the incident coherent pump ? We address ourselves to the question in the paper. In section 2 we present our model equations. These equations go over to the Bonifacio-Lugiato (1976) model when dealing with the resonant medium. Our equations differ from the B.L. model equations in the additional terms which depend on the detuning $\delta = \omega_L - \omega_0$, ω_L being the frequency of incident coherent pump and ω_0 the frequency difference of two level atoms comprising the medium. Since the detuned medium shows dispersion we call our model 'dispersive model'.

The steady state solutions of dispersive bistable system are presented. The relationship between the transmitted and the incident intensities are plotted for $\beta = 2$, $c = 10$. It is found (Tewari and Tewari 1979) that the system remains bistable, only for a limited range of detunings. The question of stability of steady states is looked into in section 3. Using the Routh-Hurwitz theorem, the condition for $+ve$ roots of relaxation matrix to occur is determined. This condition is satisfied in the negative slope region of the bistable curve implying that the steady states lying in this part of bistability curve are unstable. The approach to steady states lying on the *cooperative branch* and on *one atom* branch is examined by studying the roots of the relaxation matrix. It is found that approach to 'cooperative' steady states is non-oscillatory but the approach to the 'one atom' steady states is oscillatory. The oscillation being of the order of Rabi frequency. Further, one of the roots of the relaxation matrix changes its sign twice; once at the upper bistability threshold E_M^2 and second time at the lower bistability threshold E_m^2. As β the detuning parameter, is increased to the upper limit β_M, the points where the root changes sign, approach one another till they actually coincide at $\beta = \beta_M$. These changes of the sign of the root show that one of the normal modes of the system becomes soft at the upper and lower bistability thresholds. The approach to Zero of the soft mode with the intensity of coherent pump is discussed.

Finally, we break up, in section 4, the transmitted light into dispersed and 'unabsorbed' parts. We discuss the variation of relative magnitude of the two components of transmitted intensity. We find a sudden change in phase of transmitted light relative to the incident light at the two bistability thresholds. The shift of phase can be observed as a fringe shift in an interference fringe pattern obtained by superposition of incident and transmitted light. Secon 5 records our conclusions.

2. The model

We consider a homogeneously broadened medium consisting of $N \gg 1$, two level atoms of frequency difference ω_0. The medium is placed inside a Fabry Perot cavity with rate constant $k = C(1-R)/L$; C being the velocity of light, R the reflectivity of mirrors and L the length of cavity. The medium is constantly excited by the external coherent pump field of angular frequency ω_L, in tune with the cavity The atomic medium however is detuned with respect to the incident coherent pump i.e., $\omega_L \neq \omega_0$ The coherent pump generates an inversion R_3, and a polarization wave having both the in phase and out of phase components, R_1 and R_2 respectively. The R_1, R_2, R_3 form a three component Bloch vector $\boldsymbol{R}$, the dynamics of which is given by the following coupled equations in the rotating wave approximation

$$\dot{R}_1 = -\delta R_2 + 2g(a+\alpha)_I R_3 - R_1/T'_2 \tag{1}$$

$$\dot{R}_2 = \delta R_1 + 2g(a+\alpha)_R R_3 - R_2/T'_2 \tag{2}$$

$$\dot{R}_3 = -2g(a+\alpha)_I R_1 - 2g(a+\alpha)_R R_2 - \frac{(R_3+N)}{T_1} \tag{3}$$

and

$$\dot{a} = \frac{ig}{2}(R_1 - iR_2) - ka. \tag{4}$$

In the above equations α and a are respectively, the suitably normalized, external field amplitude and the internal field amplitude. The internal field is the reaction field generated by the polarization wave of the Bloch vector.

T_1 and T_2' are the longitudinal and transverse relaxation times which take into account the incoherent interactions of two level atomic medium. The subscript I and R imply the imaginary and real part of the field $(a+\alpha)$. g is the coupling constant between the field and the atom.

The in phase component R_1 and the imaginary part of the internal field a in the model equations above arise due to the non-zero value of the detuning $\delta = \omega_L - \omega_0$. It is easily verified that equations 1, 2, 3, 4 go over to the Bonifacio-Lugiato model equations on making $\delta = 0$ and identifying

$$R_1 = 0, \quad R_2 = -S, \quad R_3 = -\Delta \tag{5}$$

and noting also that the ground state of system has been represented by the $-N$ value of the inversion R_3 and not by $+N/2$ as has been done in the Bonifacio-Lugiato model.

We discuss in the following, results obtained from the steady state solutions of the equations (1–4), under the assumption of $K \ll T_1^{-1}, T'_2{}^{-1}$. We obtain the steady state solution for the macroscopic operators given by

$$R_1 = \frac{\pm\sqrt{\beta}NF_R - NF_I}{\sqrt{2}(1+\beta+|F|^2)}, \tag{6}$$

$$R_2 = \frac{\mp\sqrt{\beta}NF_I - NF_R}{\sqrt{2}(1+\beta+|F|^2)}, \tag{7}$$

$$R_3 = -N\frac{(1+\beta)}{(1+\beta+|F|^2)}. \tag{8}$$

Here F and E are proportional to transmitted and incident light fields and are given by

$$F = F_R + iF_I = \sqrt{T_1 T'_2}.2g.(a+\alpha), \tag{9}$$

$$E = \sqrt{T_1 T'_2}.2g\alpha \tag{10}$$

$$\pm\sqrt{\beta} = T'_2\delta. \tag{11}$$

We have also taken $T'_2/T_1 = 2$ in expressions (6–8). The steady state relationship between the incident and transmitted light is

$$E = F\left[1+\frac{2C}{(1+\beta+|F|^2)} \mp i\frac{2C\sqrt{\beta}}{(1+\beta+|F|^2)}\right] \tag{12}$$

where $C = \dfrac{T'_2 g^2 N}{2K}$. An equation similar to equation (12) with $\beta = 0$, $F \equiv x$ and $E \equiv y$ is used to demonstrate the bistable output of the absorptive medium. For, it is found that if $C > 4$, x is a multivalued function of incident field y for $y_m \leqslant y \leqslant y_M$. One has then for a bistability curve a low transmission branch called cooperative branch and a high transmission branch called 'one atom' branch.

We note that by including the detuning of the medium in equation (12) our transmitted field becomes complex. We shall now onward fix the phase of the incident field E by choosing it to be real. We observe here that the ratio (F/E) would be a measure of the phase shift between the transmitted and incident light. We shall discuss the shift in phase at the end and digress here to the equation relating the transmitted and incident intensities i.e.

$$E^2 = |F|^2\left[1+\frac{4C}{(1+\beta+|F|^2} + \frac{4C(1+\beta)}{(1+\beta+|F|^2)^2}\right] \tag{13}$$

which may also be termed as a relationship between the output and input powers. Equation (13) also predicts a bistable output between $|F|^2$ and E^2.

That is $|F|^2$ is a multivalued function of incident intensity lying between the upper (E_M^2) and lower E_m^2 bistability thresholds, if

$$> 1+3\sqrt{(1+\beta)} \qquad \text{for } \beta \to 0 \tag{14a}$$

$$> 1+\sqrt{\frac{27}{4}\beta} \qquad \text{for } \beta \to \infty. \tag{14b}$$

Considering C to be a constant for an experimental set up one can define a maximum value β_M (given by equation (15)) beyond which the dispersive system is no longer bistable (Tewari and Tewari, 1979)

$$1+\beta_M = \frac{4}{27}\;\frac{(C-1)^3}{C}\,. \tag{15}$$

As has been reported earlier one finds that the cooperative branch transmitted intensity depends upon β in that as $\beta \to \beta_M$ the cooperative branch shifts till it meets continously and monotonically the one atom branch at $E^2 = E_M^2 = E_m^2$ for $\beta = \beta_M$. For larger values of β, $|F|^2$ remains a monotonic function of E^2.

We are interested here in the bistable behaviour of the dispersive medium and therefore would confine ourselves to the region covered by $\beta \leqslant \beta_M$. We give in Figure 1, the $|F|^2$, E^2 bistable output curve for $C = 10$ and $\beta = 2 < \beta_M = 9{\cdot}8$

In the next section we look into the stability of the steady states of such a bistability curve.

3. A stability analysis

Since we are working in the condition $k \ll T_1^{-1}, T'^{-1}_2$ we can adiabatically eliminate the internal field variables from equations (1–3). Linearizing the resulting equations we obtain the following matrix equation for the fluctuations of the macroscopic operators

$$\Delta\dot{R} = M\Delta R, \tag{16}$$

where ΔR is a column matrix given by ΔR_1, ΔR_2, ΔR_3 (the fluctuations in macroscopic operators) and the matrix M is

$$M \quad \begin{bmatrix} -1+\frac{2C}{N}R_3 & \mp\sqrt{\beta} & \frac{2C}{N}R_1 \\ \pm\sqrt{\beta} & -1+\frac{2C}{N}R_3 & \sqrt{2}E+\frac{2C}{N}R_2 \\ -\frac{4C}{N}R_1 & -\sqrt{2}E-\frac{4C}{N}R_2 & -2 \end{bmatrix}. \tag{17}$$

To determine the relaxation behaviour we construct the characteristic equation by setting

$$|\lambda I - M| = 0 \tag{18}$$

where I is the 3×3 unit matrix. The resulting equation (19) is cubic in λ

$$\begin{aligned}
&\lambda^3+4\left(1+\frac{C(1+\beta)}{1+\beta+|F|^2}\right)\lambda^2 \\
&+\left\{\left(1+\frac{2C(1+\beta)}{1+\beta+|F|^2}\right)^2+4\left(1+\frac{2C(1+\beta)}{1+\beta+|F|^2}\right)\right. \\
&\left.+\beta+2\left(|F|^2-\frac{2C|F|^2}{1+\beta+|F|^2}\right)\right\}\lambda \\
&+\left\{2\left(1+\frac{2C(1+\beta)}{1+\beta+|F|^2}\right)^2+2\beta-\frac{4C\beta|F|^2}{1+\beta+|F|^2}\right. \\
&\left.+2\left(1+\frac{2C(1+\beta)}{1+\beta+|F|^2}\right)\left(|F|^2-\frac{2C|F|^2}{(1+\beta+|F|^2)}\right)\right\}
\end{aligned} \tag{19}$$

$$\equiv \lambda^3+C_1\lambda^2+C_2\lambda+C_3=0. \tag{20}$$

The general solution for a cubic equation can be written down. Such a solution however has a complicated structure and for our purpose we find that a numerical study of the solutions is better suited. However, the stability analysis of steady states can be done by using the Routh-Hurwitz theorem One constructs from the equation (19) an array of numbers viz ,

$$\left(1, C_1, \frac{C_1C_2-C_3}{C_1}, C_3\right) \tag{21}$$

and then looks for the number of times sign of the terms of the array changes as this number is the number of positive roots of the equation (20). In our case, first two terms of the array are always $+ve$, the C_3 term can be expressed in terms of the slope of the bistability curve as

$$C_3 = 2(1+\beta+|F|^2)\frac{dE^2}{d|F|^2}. \tag{22}$$

The third term can also be simplified to

$$2A(A+1)(A+2)+\frac{8C\beta}{A}(A+1)+\frac{C_3}{A}(A+2)-\beta(A+1) \tag{23}$$

with

$$A = 1 + \frac{2C(1+\beta)}{1+\beta+|F|^2} \tag{24}$$

If our system is bistable we note (Figure 1) that C_3 has definite regions of positive and negative value. Along the cooperative and 'one atom' branches slope of $|E|^2$ V$_s$ $|F|^2$ is positive but the intermediate branch has a negative slope region. By a little tedious algebraic calculation it is possible to show that the third term remains positive even in the region where C_3 is most negative, which gives us

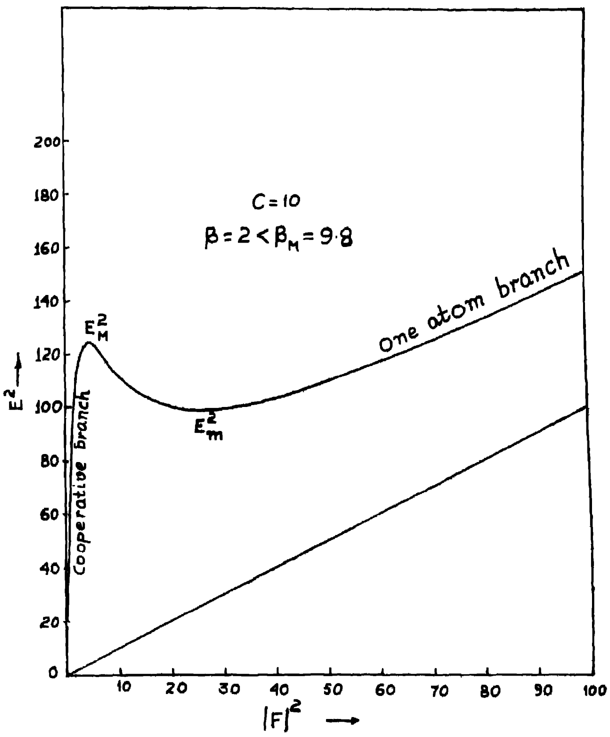

Figure 1. The optical bistability curve for $\beta = 2$, $C = 10$ dispersive case is plotted. For $E_m^2 < E^2 < E_M^2$ one gets three values of transmitted intensity $|F|^2$. As β is increased E_M^2 and F_m^2 are little effected but E_m^2 and F_M^2 increase fast to make $F_M^2 = F_m^2$ and $E_M^2 = E_m^2$ at $\beta = \beta_M$. The system then ceases to be bistable.

only one change of sign between the 3rd and 4th term. Hence only one root of equation (20) is positive in the region where C_3 is negative. Our system is thus unstable in the region with $-ve$ slope.

B. *Relaxation behaviour* :

We discuss here the relaxation of the dispersive medium showing optical bistability by studying the roots of the relaxation matrix*. For simplicity, instead of giving explicity the complicated structure of the roots of equation (20) we discuss essential features of them with the help of Figure 2, wherein we have plotted the three roots in the different steady state regions of optical bistability curve for $c = 10$ and $\beta = 2$.

In the pure absorptive case one obtains,

a) three real roots along the *cooperative* branch and the unstable branch,

b) one real and a pair of complex conjugate roots along the *one atom* branch.

For the dispersive medium, one real root and a pair of complex conjugate roots along all the branches of the bistability curve are obtained. In Figure 2 the behaviour of the pair of complex conjugate roots is shown by plotting real (λ_R) and imaginary (λ_I) parts for various incident intensities. The pure real root (λ_2 as it is customarily called) is also plotted. In our diagram curves linked with arrows indicating to the right imply that they correspond to *cooperative* branch and those curves with arrows pointing to the left correspond to *one atom* branch. Only positive root is shown along the unstable branch and has no arrows associated with it.

(a) *Cooperative branch* :

Along this branch because of the finite value of λ_I we may expect oscillation of the fluctuations of macroscopic operators with frequency λ_I. Noting, however, the fact that $|\lambda_R| \gg |\lambda_I|$, we obtain very quick damping of such oscillations The cooperative damping is very large for small incident intensities where λ_I is also small, $\lambda_I \simeq \sqrt{\beta}$. As E^2 increases and approaches the upper bistability threshold E_M^2, $|\lambda_I|$ increases and λ_R decreases, however $|\lambda_R| \gg |\lambda_I|$ throughout. At $\beta = \beta_M$, the λ_R and λ_I become equal and meet at $E^2 = E_M^2 = E_m^2$.

The λ_2 root along the cooperative branch shows an interesting behaviour. Noting that $|\lambda_2| \ll |\lambda_R|$ the dynamics of the fluctuation will be dominated by the λ_2 root. As E^2 approaches E_M^2, $|\lambda_2|$ decreases fast and ultimately becomes

* Our discussion of relaxation behaviour is semiclassical and assumes regression theorem. On the basis of regression theorem it is reasonable to assume that oscillatory character of approach to steady state is necessary to have oscillations of the correlation fluctuations in the stationery state. See e.g. Agarwal *et al* (1978) and Bonifacio *et al* (1976).

zero at $E^2 = E_M^2$. The dynamics dominated by λ_2 thus slow down; reminiscence of phase transition showing critical slowing down. This is also interpreted as one of the 'normal modes' of the system becoming long lived. Such a mode is referred to as soft mode.

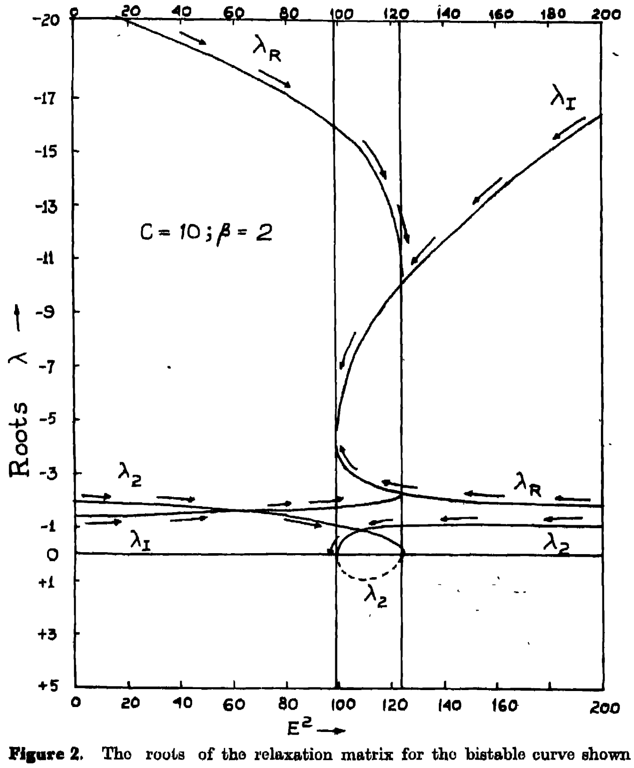

Figure 2. The roots of the relaxation matrix for the bistable curve shown in Figure 1 i.e., $\beta = 2$, $C = 10$ are plotted here. The pair of complex conjugate roots is shown in terms of λ_R and λ_I the real and imaginary parts. λ_2 the pure real root is also plotted. The arrows pointing to the right correspond to co-operative branch and those pointing to the left correspond to one atom branch. λ_R and λ_I are not plotted for unstable branch but λ_2 is plotted in dotted line. The two vertical lines correspond to E_M^2 and E_m^2. The closed loop in between these two lines gives the region of positive λ_2 depicting the unstable branch. At $\beta = \beta_M$; $\lambda_R(E_m^2) = \lambda_R(E_m^2) = \lambda_I(E_M^2) = \lambda_I(E_m^2)$. For $\beta < \beta_M$, $\lambda_R(E_M^2)| > |\lambda_R(E_m^2)$ |&| $\lambda_I(E_M^2)| < \lambda_I(E_m^2)$.

b) *Unstable branch* :

The positive real root along the unstable branch is plotted. We note that it is the root λ_2 that changes sign, at E_M^2 as one shifts from cooperative to the unstable branch. Because of the +ve nature of the root the fluctuations blow up and the system is unstable As we approach the lower bistability thresholds along the unstable branch the root again approaches zero and changes sign at $E^2 = E_m^2$. For small values of $\beta\,(\beta < \beta_M)$ we have well defined regions for λ_2 to be positive As $\beta \rightarrow \beta_M$ the region of positive values of λ_2 decreases, so also do the positive values of λ_2. At $\beta = \beta_M$ the positive value region culminates into a point with $\lambda_2 = 0$ at $E^2 = E_M^2 = E_m^2$.

c) *One atom branch* :

On this branch we note $|\lambda_2| \lesssim |\lambda_R|$ and $|\lambda_I| \gg |\lambda_R|$ Consequently, the oscillation of the fluctuation of the macroscopic operators survive for some time and dominate the dynamics. The damping of the oscillation is governed by λ_R. For larger values of E^2, λ_R decreases to satisfy $\lambda_R/\lambda_2 = 1{\cdot}5$. Thus λ_R and λ_2 are of same order of magnitude along the one atom branch. However as one approaches $E^2 = E_m^2$, the lower bistability threshold along the one atom branch, $|\lambda_I|$ decreases, $|\lambda_R|$ increases and $|\lambda_2|$ decreases to zero at $E^2 = E_m^2$. Consequently the dynamics of the medium near E_m^2 is controlled by λ_2 and thus once again the medium shows critical slowing down etc. etc. As $\beta \rightarrow \beta_M$, $|\lambda_R|$ and $|\lambda_I|$ approach one another near E_m^2 and for $\beta = \beta_M$ they actually meet at $E^2 = E_m^2$. The discontinuity in the plots for λ_R and λ_I which results essentially due to the fact that we avoided plotting the values of unstable branch, vanishes at $\beta = \beta_M$ and $E^2 = E_M^2 = E_m^2$. The λ_R and λ_I plots therefore become two continuous curves which cross at $E^2 = E_M^2 = E_m^2$

Finally the soft mode root λ_2 approaches zero near upper bistability thresholds as const.$(E_M^2 - E_m^2)^{\frac{1}{2}}$ and near the lower bistability thresholds it approaches to zero as Const.$(E^2 - E_m^2)^{\frac{1}{2}}$. The region of validity of the approximate result near lower bistability is small and that near upper bistability is larger.

In the next section we discuss the phase shift we had pointed to in section 2, equation (12).

4. Phase shift

In contrast to the pure absorptive case of bistability, where transmitted field is represented by real value of x, we are forced to write the dispersive case transmitted field as complex number. We discuss in this section the relative magnitude of the real and imaginary parts of the transmitted light We can obtain easily

$$F_R = \frac{|F|^2}{E}\left(1 + \frac{2C}{1+\beta+|F|^2}\right) \tag{25}$$

$$F_I = \frac{|F|^2}{E}\left(\frac{\pm 2C\sqrt{\beta}}{1+\beta+|F|^2}\right) \tag{26}$$

We note immediately that at $\beta = 0$, $F_I = 0$ and $F_R = x$ with $y \equiv E$ in pure absorptive model. Since x refers to transmitted light in that model we call F_R to be 'unabsorbed' part of the transmitted light. Further we note that in the dispersive model F_I is non zero due to finite detuning, also sign of F_I is the sign of $(T'_2\delta)$. Consequently we call F_I to be the dispersed part of the transmitted light. On comparing the dispersed and unabsorbed parts of transmitted light we get

$$\frac{F_I}{F_R} = \frac{\pm 2C\sqrt{\beta}}{1+\beta+|F|^2+2C}. \tag{27}$$

Confining ourselves to the bistability case we note that $|F|^2$ remains small along the cooperative branch for small β and changes suddenly to one atom branch values for $E^2 > E_M^2$. Consequently a discontinuity is encountered in F_I/F_R at $E^2 = E_M^2$ on cooperative branch and at $E^2 = E_M^2$ on one atom branch.

Since F_I/F_R is $\tan\phi$ with ϕ as the phase shift between the transmitted and incident light, we observe a discontinuous behaviour in ϕ as well. This would imply a sudden fringe shift in the interference fringe pattern which can be obtained by the superposition of incident and transmitted light. The behaviour of ϕ for $\beta = 2$ is plotted in Figure 3. As $\beta \to \beta_M$ the cooperative branch lifts up by almost a constant amount i.e., remains almost parallel to small β curve

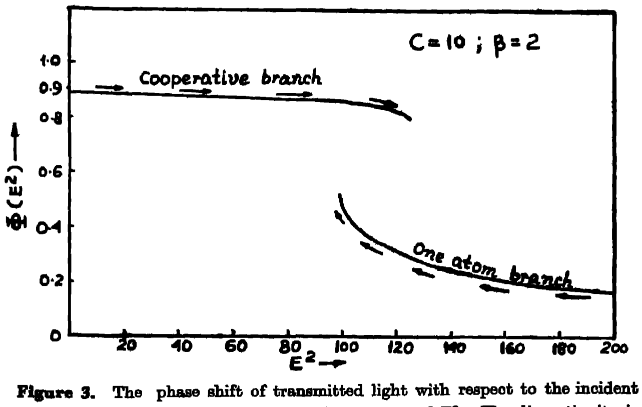

Figure 3. The phase shift of transmitted light with respect to the incident coherent pump light is plotted for various values of E^2. The discontinuity is demonstrated for the example chosen. As β is increased $\phi(E_M^2)$ and $\phi(E_m^2)$ approach one another so as to make $\phi(E_M^2) = \phi(E_m^2)$ at $\beta = \beta_M$. The above graph suggests that for dispersive bistability sudden appearance of dynamical stark shift is accompanied by a sudden change in the phase of transmitted light. It may be measured by a simple experiment of interferometry.

but the 'one atom' ϕ increases as $\beta \to \beta_M$ so as to meet the ϕ of cooperative branch and at $\beta = \beta_M$ the discontinuity vanishes.

5. Conclusion

We have discussed the possibilities of using detuned homogeneously broadened medium as a bistable device. It is found that in principle a detuned medium can remain bistable only for a limited range of detunings. The relaxation behaviour of the detuned medium is examined. In view of the regression theorem and the results of relaxation matrix one would find that along the cooperative branch spectrum of fluctuations would show a single peak structure and along the one atom branch it would show the three peak structure (dynamical stark shift) i.e. a central peak with two side bands roughly placed at $\pm\lambda_I$ It is shown that in addition to the sudden appearance of dynamical stark shift there must appear a sudden shift in the fringe pattern obtained by superposition of incident and transmitted light as the system jumps from cooperative branch to the one atom branch.

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