TOPICS ON z-IDEALS OF COMMUTATIVE RINGS

by

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Contents

Abstract

The first few chapters of the dissertation will catalogue what is known regarding z-ideals in commutative rings with identity. Some special attention will be paid to z-ideals in function rings to show how the presence of the topological description simplifies z-covers of arbitrary ideals. Conditions in an f-ring that ensure that the sum of z-ideals is a z-ideal will be given.

In the latter part of the dissertation I will generalise a result in higher order z-ideals and introduce a notion of higher order d-ideals.

Keywords: Commutative ring, z-ideal, d-ideal, minimal prime ideal, radical ideal, f-ring, von Neumann regular ring, higher order z-ideal, higher order d-ideal, d-termination.

Declaration

Student number: 457 777 72

Degree: Master of science (Mathematics)

I declare that *Topics on z-ideals of commutative rings* is my own work and that all the sources that I have used or quoted have been indicated and acknowledged by means of complete references.

Batsile Tlharesakgosi.

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Modimo *re a leboga.*

List of notation

We use the following standard notation throughout this dissertation.

$\emptyset$ Set with no elements

$\mathbb{N}$ Set of all positive integers

$\mathbb{Z}$ Set of all integers

$\mathbb{R}$ Set of all real numbers

Other notions are introduced throughout the dissertation, as is appropriate.

Chapter 1

Introduction and preliminaries

1.1 A brief history on z-ideals

The birth of z-ideals came about in the study of the ideal structure of the ring $C(X)$ of real-valued continuous functions on a completely regular Hausdorff space X. This was undertaken by Kohls [24], and is recorded in the book *Rings of Continuous Functions* by Gillman and Jerison [13].

Although Kohls showed that in the ring $C(X)$ z-ideals can be described purely algebraically, it was Mason [31] who initiated the study of z-ideals in commutative rings with identity.

1.2 Synopsis of the dissertation

The dissertation consists of four chapters. It is mainly about the study of z-ideals in commutative rings with identity. Chapter 1 is introductory. It is a chapter in which we fix notation and provide the requisite background needed to read the dissertation.

In Chapter 2 we define z-ideals algebraically, and show the various equivalent ways of defining these ideals. We show that every ideal has a "z-cover", by which we mean a smallest z-ideal containing it. In the classical function rings $C(X)$ the sum of two z-ideals is a z-ideal (see, for instance, [13] and [38]). It has recently been shown by Dube and Ighedo [12] that, in fact, in any function ring $\mathcal{R}L$ of continuous real-valued functions on a frame L, the sum of two z-ideals is a z-ideal. There are rings in which the sum of z-ideals is not a z-ideal, as shown by Henriksen and Smith [16]. We also discuss, mainly in the class of f-rings, sufficient conditions for the sum of z-ideals to be a z-ideal. Most results in this regard were proved by Larson [27].

There are special types of z-ideals that have received a great deal of attention both in rings and Riesz spaces. These are called d-ideals, and have been studied by many mathematicians such as Huijsman and de Pagter [19]. Other authors, such as Azarpanah et al [4] call these z°-ideals. In Chapter 3 we give various properties and characterisations of d-ideals. We also give sufficient conditions when their sums are also d-ideals. Rings in which d-ideals coincide with z-ideals will be characterised.

Chapter 4 will be the last chapter in the dissertation, and will consist of completely new concepts which have hitherto not been considered. These ideals will be generalisations of d-ideals. We will develop as far as possible the theory of these generalisations, which we will call higher order d-ideals. We also give a generalisation of a result of the authors in [11].

1.3 Basic definitions

In this section we shall give definitions of basic terms which will be frequently used throughout this dissertation. These may be found in the book *Advanced modern Algebra* ([37]). All rings considered in this dissertation are commutative with identity.

An *ideal* in a commutative ring A is a subset I of A such that

(i) $0 \in I$,

(ii) if $x \in I$, then $-x \in I$,

(iii) if $x, y \in I$, then $x + y \in I$,

(iv) if $x \in I$ and $a \in A$, then $xa \in I$.

The ring A itself and $\{0\}$, the subset consisting of 0 alone, are always ideals in a commutative ring A. An ideal $I \neq A$ is called a *proper ideal.*

Example 1.3.1. If an ideal I in a commutative ring contains 1, then $I = A$, for I contains $a = a1$ for every $a \in A$. Indeed, if I contains a unit u, then $I = A$, for then I contains $u^{-1}u = 1$.

An ideal I in commutative ring A is called a *prime ideal* if it is a proper ideal, that is, $I \neq A$, and $ab \in I$ implies that $a \in I$ or $b \in I$.

A *principal ideal* of a ring A is an ideal generated by one element. We shall denote the ideal generated by an element a by $\langle a \rangle$. An ideal I in a ring A is a *minimal ideal* if $I \neq \langle 0 \rangle$ and there is no ideal J with $\langle 0 \rangle \subsetneq J \subsetneq I$. A ring need not contain minimal ideals. For example, $\mathbb{Z}$ has no minimal ideals: every nonzero ideal I in $\mathbb{Z}$ has the form $I = \langle n \rangle$ for some nonzero integer n, and $I = \langle n \rangle \supsetneq \langle 2n \rangle \neq \langle 0 \rangle$. The minimal prime ideals of a ring A will be denoted by $\mathrm{Min}(A)$.

An ideal I in a commutative ring A is a *maximal ideal* if I is a proper ideal and there is no proper ideal J with $I \subsetneq J \subsetneq A$. The set of maximal ideals of a ring A will be denoted by $\mathrm{Max}(A)$.

If A and B are rings, a *ring homomorphism* is a function $\phi\colon A \to B$ such that

(i) $\phi(1) = 1$,

(ii) $\phi(x + y) = \phi(x) + \phi(y)$ for all $x, y \in A$,

(iii) $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in A$.

A ring homomorphism that is also a bijection is called an *isomorphism*. Rings A and B are called isomorphic, denoted by $A \cong B$, if there is an isomorphism $\phi\colon A \to B$.

Remark 1.3.2. If $\phi\colon A \to B$ is an isomorphism, then condition (i)$\phi(1)$ in the definition of a ring homomorphism becomes redundant.

A set X is a *partially ordered set* if it has a binary relation $\preceq$ defined on it that satisfies, for all $x, y, z \in X$,

(i) Reflexivity: $x \preceq x$,

(ii) Antisymmetry: if $x \preceq y$ and $y \preceq z$, then $x = y$,

(iii) Transitivity: if $x \preceq y$ and $y \preceq z$, then $x \preceq z$.

A partially ordered set X is a *chain* if, for all, $x, y \in X$, either $x \preceq y$ or $y \preceq x$.

The set of real numbers $\mathbb{R}$ with its usual ordering is a chain.

Recall that an *upper bound* of a nonempty subset Y of a partially ordered set X is an element $x_0 \in X$, not necessarily in Y, with $y \preceq x_0$ for every $y \in Y$.

We state the Zorn's lemma next.

Lemma 1.3.3. *If X is a nonempty partially ordered set in which every chain has an upper bound in X, then X has a maximal element.*

A set M of elements of a ring A is said to be an *m-system* if and only if it has the following property: If $a, b \in M$, there exists $x \in A$ such that $axb \in M$.

Chapter 2

z-Ideals of commutative rings

2.1 Characterisations of z-ideals

All rings are assumed to be commutative with identity. In consequence, all maximal ideals are prime. We let $\mathrm{Max}(A)$ denote the set of all maximal ideals of a ring A, and we define $\mathfrak{M}(a) = \{M \in \mathrm{Max}(A) \,|\, a \in M\}$ for all $a \in A$. We now define the ideals that will form the main study in this dissertation.

Definition 2.1.1. An ideal I of a ring A is a *z-ideal* if for every $a, b \in A$, $\mathfrak{M}(a) = \mathfrak{M}(b)$ and $a \in I$ imply $b \in I$.

Before we proceed let us give some examples of z-ideals.

Example 2.1.2. (a) Every maximal ideal is a z-ideal. Indeed, if M is a maximal ideal of a ring A, and a, b are elements of A such that $\mathfrak{M}(a) = \mathfrak{M}(b)$ and $a \in M$, then b belongs to every maximal ideal containing a. In particular, $b \in M$, which then shows that M is a z-ideal.

(b) Intersections of z-ideals is a z-ideal. Let $\{I_\lambda \mid \lambda \in \Lambda\}$ be a family of z-ideals of a ring A. We need to show that $\bigcap\{I_\lambda \mid \lambda \in \Lambda\}$ is a z-ideal. Consider $x, y \in A$ such that $x \in \bigcap\{I_\lambda \mid \lambda \in \Lambda\}$ and $\mathfrak{M}(x) = \mathfrak{M}(y)$. Since $x \in \bigcap\{I_\lambda \mid \lambda \in \Lambda\}$, then for any $\beta \in \Lambda$, we have that $x \in I_\beta$. It follows that $y \in I_\beta$ since I_β is a z-ideal. But I_β is chosen arbitrarily. Therefore $y \in \bigcap\{I_\lambda \mid \lambda \in \Lambda\}$ making $\bigcap\{I_\lambda \mid \lambda \in \Lambda\}$ a z-ideal.

Less obvious examples of z-ideals are minimal prime ideals in certain types of rings. A ring is said to be *reduced* if it has no nonzero nilpotent elements. The *Jacobson radical* of a ring A, denoted by $\mathrm{Jac}(A)$, is the intersection of all maximal ideals of A. If A has zero Jacobson radical, then A is reduced because the ideal of nilpotent elements of A is the intersection of all prime ideals of A, and is therefore zero as well, which then says A has no nonzero nilpotent element.

Let us recall the following from [15, Lemma 1.1] that a prime ideal P in a reduced ring is minimal prime if and only if for every $x \in P$ there is an $a \in A \smallsetminus P$ such that $ax = 0$.

Proposition 2.1.3. *Every minimal prime ideal in a ring with zero Jacobson radical is a z-ideal.*

Proof. Let P be a minimal prime ideal in a ring A with zero Jacobson radical. Let $x, y \in A$ be such that $\mathfrak{M}(x) = \mathfrak{M}(y)$ and $x \in P$. Since P is minimal prime, there exists $a \notin P$ such that $ax = 0$. We must show that $y \in P$. We claim that $ay = 0$. If this were false, then there would be a maximal ideal M that does not contain ay, since an element which belongs to every maximal ideal is zero as A has zero Jacobson radical. Since M is a maximal ideal, this would mean that the ideal $\langle M, ay\rangle$ is the whole ring, and so there would be elements $r \in A$ and $m \in M$ such that $ray + m = 1$, which then implies $x = xm$, and hence $x \in M$. But $\mathfrak{M}(x) = \mathfrak{M}(y)$, so we would have $y \in M$, and hence $ay \in M$, leading to a contradiction. Therefore $ay = 0 \in P$, and since P is prime with $a \notin P$, we deduce that $y \in P$. Thus, P is a z-ideal. □

The following results show the various equivalent ways of defining these ideals and they are given without proofs in [31].

Lemma 2.1.4. *Let A be a ring and $a, b \in A$. Then $\mathfrak{M}(a) \supseteq \mathfrak{M}(b)$ if and only if $\mathfrak{M}(a) = \mathfrak{M}(ab)$.*

Proof. ($\Rightarrow$): Assume $\mathfrak{M}(b) \subseteq \mathfrak{M}(a)$. Suppose $M \in \mathfrak{M}(ab)$, so that $ab \in M$. Since M is a maximal ideal, it is prime. It follows that $a \in M$ or $b \in M$.

(1) If $a \in M$, then $M \in \mathfrak{M}(a)$.

(2) If $b \in M$, then $M \in \mathfrak{M}(b)$, and since $\mathfrak{M}(b) \subseteq \mathfrak{M}(a)$ by hypothesis, it follows that $M \in \mathfrak{M}(a)$.

Consequently, $\mathfrak{M}(ab) \subseteq \mathfrak{M}(a)$.

Conversely, let $M \in \mathfrak{M}(a)$. Then $a \in M$, and hence $ab \in M$ since M is an ideal, giving $M \in \mathfrak{M}(ab)$. It follows that $\mathfrak{M}(a) \subseteq \mathfrak{M}(ab)$. Therefore $\mathfrak{M}(a) = \mathfrak{M}(ab)$.

($\Leftarrow$): Assume $\mathfrak{M}(a) = \mathfrak{M}(ab)$. Let $M \in \mathfrak{M}(b)$. We have $ab \in M$ since M is an ideal. This implies that $M \in \mathfrak{M}(ab)$. But by hypothesis $\mathfrak{M}(ab) = \mathfrak{M}(a)$; so $M \in \mathfrak{M}(a)$, showing that $\mathfrak{M}(b) \subseteq \mathfrak{M}(a)$. □

Lemma 2.1.5. *Let A be a ring and $I \subseteq A$ be an ideal. Then I is a z-ideal if and only if for any $a, b \in A$, $\mathfrak{M}(a) \supseteq \mathfrak{M}(b)$ and $b \in I$ imply $a \in I$.*

Proof. ($\Rightarrow$): Assume I is a z-ideal, and consider any $a, b \in A$ with $b \in I$ and $\mathfrak{M}(b) \subseteq \mathfrak{M}(a)$. But by Lemma 2.1.4, $\mathfrak{M}(b) \subseteq \mathfrak{M}(a)$ if and only if $\mathfrak{M}(a) = \mathfrak{M}(ab)$. We then have $ab \in I$, since I is an ideal. By hypothesis I is a z-ideal implying $a \in I$.

($\Leftarrow$): Assume that the stated condition holds. To show that I is a z-ideal, consider any a and b in A with $\mathfrak{M}(a) = \mathfrak{M}(b)$ and $b \in I$. We must show that $a \in I$. Since $\mathfrak{M}(a) = \mathfrak{M}(b)$, we have $\mathfrak{M}(a) \supseteq \mathfrak{M}(b)$ and $b \in I$, so by the stated condition $a \in I$. It follows that I is a z-ideal. □

For use in the upcoming lemma, and elsewhere, we introduce the following notation. For any $a \in A$,

$$M(a) = \bigcap \mathfrak{M}(a).$$

Lemma 2.1.6. *Let A be a ring and $I \subseteq A$ be an ideal. Then I is a z-ideal if and only if for every $a \in I$, $M(a) \subseteq I$.*

Proof. ($\Rightarrow$): Assume I is a z-ideal, and let $a \in I$. We must show that $M(a) \subseteq I$. So let $x \in M(a)$. Since $M(a) = \bigcap \mathfrak{M}(a)$, this implies that x belongs to every maximal ideal containing a. Therefore the set of all maximal ideals containing a contains the set of all maximal ideals containing x. That is, $\mathfrak{M}(x) \subseteq \mathfrak{M}(a)$. We therefore have

$$\mathfrak{M}(x) \subseteq \mathfrak{M}(a) \text{ and } a \in I$$

which implies $x \in I$ since I is a z-ideal. Since x is an arbitrary element of $M(a)$, it follows that $M(a) \subseteq I$.

($\Leftarrow$): Assume that $M(a) \subseteq I$ for every $a \in I$. To show that I is a z-ideal, consider any x and y in A with $\mathfrak{M}(x) = \mathfrak{M}(y)$ and $y \in I$. We must show that $x \in I$. But now we have

$$x \in M(x) = \bigcap \mathfrak{M}(x) = \bigcap \mathfrak{M}(y) = M(y),$$

and since $y \in I$, the hypothesis says $M(y) \subseteq I$. So $x \in I$. Therefore I is a z-ideal. □

Next we show that every ideal has a *z-cover*, by which we mean a smallest z-ideal containing it, namely, $I_z = \bigcap\{J \mid J \text{ is a } z\text{-ideal}, J \supseteq I\}$. Before we do that, we will give the definition of the radical of an ideal I of a ring A. The *radical of an ideal* I, denoted by $\sqrt{I}$, is defined by

$$\sqrt{I} = \{a \in A \mid a^n \in I \text{ for some } n \in \mathbb{N}\}.$$

An ideal I is called a *radical ideal* if $I = \sqrt{I}$. An ℓ-ring A has a *root property* if every positive element in A has a square root. Next we give some properties of z-covers and we shall give the proofs which were not given in [32].

Lemma 2.1.7. *Let A be a ring and I, J be ideals of A. Then we have the following.*

(1) $I \subseteq J \implies I_z \subseteq J_z$.

(2) $\left(\sum I_\alpha\right)_z = \left(\sum (I_\alpha)_z\right)_z$.

(3) $(I_z)_z = I_z$.

(4) $I \subseteq \sqrt{I} \subseteq I_z$.

(5) For any positive integer n, $(I^n)_z = I_z$.

(6) If I, J are z-ideals in a commutative ring A with root property then, IJ is a z-ideal iff $IJ = I \cap J$.

(7) If I is a z-ideal, then $I = \sqrt{I} = I_z$.

Proof. (1) Let $I \subseteq J$ and $J_z = \bigcap\{K \mid K \text{ is a } z\text{-ideal}, K \supseteq J\}$. If K is a z-ideal containing J, then $K \supseteq I$. So $J_z \supseteq I$. That is $I \subseteq J_z$. But J_z is a z-ideal and I_z is the smallest z-ideal containing I. Therefore $I_z \subseteq J_z$.

(2) As proved in [30, Proposition 3.1 (a)], both $\left(\sum I_\alpha\right)_z$ and $\left(\sum I_{\alpha z}\right)_z$ are z-ideals containing $\sum I_\alpha$ so by the minimality of the former, it is contained in the latter. Conversely $\left(\sum I_\alpha\right)_z$ is a z-ideal containing each I_α, therefore containing each $(I_\alpha)_z$ and so contains $\left(\left(\sum I_\alpha\right)_z\right)_z$.

(3) If J is a z-ideal, then $J_z = J$. Since I_z is a z-ideal then $I_z = (I_z)_z$.

(4) To show that $I \subseteq \sqrt{I}$, we recall from the definition that

$$\sqrt{I} = \{a \in A \,:\, a^n \in I \text{ for some } n \in \mathbb{N}\}.$$

It is therefore clear from the above that $I \subseteq \sqrt{I}$.

To show that $\sqrt{I} \subseteq I_z$, we let $a \in \sqrt{I}$. So there exist $n \in \mathbb{N}$ such that $a^n \in I$. Consider any z-ideal $J \supseteq I$. Then $a^n \in J$. So a^n is an element of every z-ideal containing I. Since $\mathfrak{M}(a) = \mathfrak{M}(a^n)$, if $a^n \in J$ and J is a z-ideal, then $a \in J$. So

$$a \in \bigcap \{J \mid J \text{ is a } z\text{-ideal containing } I\} = I_z.$$

Therefore $\sqrt{I} \subseteq I_z$.

(5) Since $I^n \subseteq I$, by the first part we have $(I^n)_z \subseteq I_z$. To show the other containment, suppose we define the following sets,

$$\{K \subseteq A \mid K \text{ is a } z\text{-ideal with } K \supseteq I^n\}$$

and

$$\{H \subseteq A \mid H \text{ is a } z\text{-ideal with } H \supseteq I\}.$$

Now since $I^n \subseteq I$, we have that

$$\{K \subseteq A \mid K \text{ is a } z\text{-ideal with } K \supseteq I^n\} \subseteq \{H \subseteq A \mid H \text{ is a } z\text{-ideal with } H \supseteq I\}.$$

But $\bigcap\{H \subseteq A \mid H \text{ is a } z\text{-ideal with } H \supseteq I\} \subseteq \bigcap\{K \subseteq A \mid K \text{ is a } z\text{-ideal with } K \supseteq I^n\}$ since a collection with more sets has a smaller intersection. Now

$$\bigcap\{H \subseteq A \mid H \text{ is a } z\text{-ideal with } H \supseteq I\} = I_z,$$

also

$$\bigcap\{K \subseteq A \mid K \text{ is a } z\text{-ideal with } K \supseteq I^n\} = (I^n)_z.$$

Therefore $I_z \subseteq (I^n)_z$.

(6) ($\Rightarrow$): Let I and J be z-ideals in a ring A with root property. Suppose IJ is a z-ideal. Then $\sum a_i b_i \in IJ$ for $a_i \in I$ and $b_i \in J$ $\forall i \in I$. So since I and J are ideals, we have $\sum a_i b_i \in I$ and $\sum a_i b_i \in J$, and hence $\sum a_i b_i \in I \cap J$. Therefore $IJ \subseteq I \cap J$.

Conversely let $f \in I \cap J$. We can write $f = (f^{\frac{1}{3}})^3$. So since I and J are z-ideals and $\mathfrak{M}(f^{\frac{1}{3}}) = \mathfrak{M}(f)$ then $f^{\frac{1}{3}} \in I$ and $f^{\frac{1}{3}} \in J$. Therefore $f^{\frac{1}{3}} \cdot f^{\frac{1}{3}} = (f^{\frac{1}{3}})^2 \in I$ and $(f^{\frac{1}{3}})^2 \cdot f^{\frac{1}{3}} = f \in IJ$ giving $I \cap J \subseteq IJ$. Hence $IJ = I \cap J$.

($\Leftarrow$): Suppose $IJ = I \cap J$. Since I and J are z-ideals and the intersection of z-ideals are z-ideals, $I \cap J = IJ$ is a z-ideal.

(7) If I is a z-ideal then $I = I_z$. By (4) $I \subseteq \sqrt{I} \subseteq I_z = I$. Therefore $I = \sqrt{I}$. $\square$

Remark 2.1.8. We deduce from item (7) of the foregoing lemma that every z-ideal is a radical ideal. Consequently, every z-ideal is an intersection of prime ideals. It is however not the case that in every ring prime ideals are z-ideals. See [13, 14B.4] for an example where prime ideals are not z-ideals.

2.2 Miscellaneous results on z-ideals

In this section we recall from [31] some miscellaneous results on z-ideals of commutative rings. We examine rings with zero Jacobson radical to investigate the z-ideal structure. We record results showing the relationship between z-ideals and minimal prime ideals. We begin with the following theorem.

Theorem 2.2.1. *If P is minimal in the class of prime ideals containing a z-ideal I, then P is a z-ideal.*

Proof. Let P be a prime ideal containing I, and suppose that P is not a z-ideal. There exist $a \notin P$ and $b \in P$ such that $\mathfrak{M}(a) = \mathfrak{M}(b)$. Consider the set

$$S = (A \smallsetminus P) \cup \{cb^n \mid n \in \mathbb{N} \text{ and } c \notin P\}.$$

Since P is prime, S is closed under multiplication. Furthermore, S does not meet I; for if $cb^n \in I$, then ca belongs to the z-ideal I, and hence to the prime ideal P, whence $c \in P$. By [13, Theorem 0.16], there is a prime ideal containing I and disjoint from S, and hence contained properly in P. So P is not minimal. $\square$

Next we shall give characterisations of regular ring in terms of z-ideals. In order to present the next results we remind the reader that a ring A is *von Neumann regular* if for every $a \in A$, there exists an element $x \in A$ such that $axa = a$. We shall also recall that a principal ideal of a ring A is an ideal generated by one element. The intersections of maximal ideals are the most obvious z-ideals and they will be called *strong z-ideals*. Also any principal z-ideal $\langle x \rangle$ is strong, for if $y \in \bigcap\{M \mid M \supseteq \langle x\rangle\}$, then $\mathfrak{M}(y) \supseteq \mathfrak{M}(\langle x\rangle) = \mathfrak{M}(x)$ so $y \in \langle x\rangle$. We start with following lemma that is going to be useful in the proof of the next theorem. Recall that if I and J are ideals of a ring A, then IJ is the ideal whose elements are finite linear combinations of the $\sum i_\alpha j_\alpha$, where each $i_\alpha \in I$ and each $j_\alpha \in J$.

Lemma 2.2.2. *If $\langle a\rangle$ is a principal ideal, then $\langle a\rangle^2 = \langle a^2\rangle$.*

Proof. Let $x \in \langle a\rangle^2$. Then there are finitely many elements $r_1, \ldots, r_n$ and $s_1, \ldots, s_n$ in A such that

$$\begin{aligned} x &= r_1 a s_1 a + \cdots + r_n a s_n a \\ &= (r_1 s_1)a^2 + \cdots + (r_n s_n)a^2 \\ &= (r_1 s_1 + \cdots + r_n s_n)a^2 \end{aligned}$$

which shows that $x \in \langle a^2\rangle$. Therefore $\langle a\rangle^2 \subseteq \langle a^2\rangle$.

On the other hand, if $y \in \langle a^2\rangle$, then $y = ra^2 = (ra)a$ for some $r \in A$, whence $y \in \langle a\rangle^2$, showing that $\langle a^2\rangle \subseteq \langle a\rangle^2$. □

Theorem 2.2.3. *The following are equivalent for a ring A.*

(1) Every ideal is a strong z-ideal.

(2) Every ideal is a z-ideal.

(3) Every principal ideal is a z-ideal.

(4) A is a von Neumann regular ring.

Proof. (1) ⇒ (2): Suppose that every ideal is a strong z-ideal. Then every ideal is an intersection of maximal ideals which implies that every ideal is a z-ideal.

(2) ⇒ (3): Suppose that every ideal is a z-ideal. Then every principal ideal is a z-ideal.

(3) ⇒ (4): Now the hypothesis says the principal ideal $\langle a\rangle$ is a z-ideal. By Lemma 2.1.7, this implies $\langle a^2\rangle = (\langle a^2\rangle)_z = (\langle a\rangle)_z = \langle a\rangle$. Since $a \in \langle a\rangle$, this implies $a \in \langle a^2\rangle$, hence there exists $b \in A$ such that $a = a^2 b$. So A is von Neumann regular.

(4) ⇒ (1): Every ideal in a von Neumann regular ring is the intersection of the maximal ideals containing it. Hence every ideal is a strong z-ideal. □

Next we recall that the *annihilator* of $I \subseteq A$ is the ideal

$$\mathrm{Ann}(I) = \{a \in A \,|\, ax = 0 \text{ for every } x \in I\}.$$

If I and J are ideals of a ring A, their *ideal quotient* $(J : I)$ is the set

$$(J \,:\, I) = \{a \in A \,|\, aI \subseteq J\}.$$

The next result shows how z-ideals are related to their ideal quotients.

Proposition 2.2.4. *If J is a z-ideal, so is $(J \,:\, I)$ for any I.*

Proof. If $\mathfrak{M}(a) = \mathfrak{M}(b)$ where $bI \subseteq J$, then $\mathfrak{M}(ai) = \mathfrak{M}(bi)$ for all $i \in I$. Since $bi \in J$ and J is a z-ideal, then $ai \in J$ for all $i \in I$, that is $a \in (J \,:\, I)$. □

For use in the following theorem, we recall that a z-ideal I in a ring A is a *maximal z-ideal* if I is a proper z-ideal and there is no proper z-ideal J with $I \subsetneqq J \subsetneqq A$. The *inductive set* is non-empty partially ordered set in which every element has a successor.

Theorem 2.2.5. *If P is a prime ideal, then either P is a z-ideal or the maximal z-ideals contained in P are prime z-ideals.*

Proof. Put $\mathcal{S} = \{I \subseteq A \,|\, I \text{ is a } z\text{-ideal and } I \subseteq P\}$. Then $\{0\} \in \mathcal{S}$ and $\mathcal{S}$ is inductive, so by Zorn's lemma, $\mathcal{S}$ has a maximal element. Let I be one. Then $I = P$ iff P is a prime z-ideal. If $I \subsetneqq P$ then there exists a prime ideal Q minimal with respect to containing I and contained in P; $Q \neq P$ since Q will be a z-ideal (Theorem 2.2.1). Moreover, either $Q = I$ in which case I is prime, or $I \subsetneqq Q$, which contradicts the maximality of I. □

For use in the following proposition, we recall that a *minimal z-ideal* is a nonzero z-ideal which contains no z-ideal except $\{0\}$. Even though minimal prime ideals exist, minimal z-ideals may not. Example of such rings is a field, because it has only one z-ideal which is a maximal ideal $\{0\}$.

Proposition 2.2.6. *If A has minimal nonzero ideals, they are strong minimal z-ideals.*

Proof. A minimal ideal has the form $\langle e \rangle$ where $e^2 = e$ [21, Lemma 2] . But $\langle e \rangle = \mathrm{Ann}((1-e)A)$ is a z-ideal and since it is principal, it is strong.

Suppose A has minimal z-ideals (though not necessarily minimal ideals). If I is one such then the z-ideal $\mathrm{Ann}(1-x) \subseteq I$ for all $x \in I$, so either

(a) $\mathrm{Ann}(1-x) = \{0\}$ for all x or

(b) $\mathrm{Ann}(1-x) = I$ for some x.

There exist rings in which (a) cannot occur. If we write

$$m(I) = \{x \mid x = ax \text{ for some } a \in I\} = \bigcup_{x \in I} \mathrm{Ann}(1-x)$$

then by [22] if $\mathfrak{M}(I) = \mathfrak{M}(m(I))$, condition (a) will not hold. Again if A is *weakly regular* (every nonzero ideal contains an idempotent $\neq \{0\}$), every ideal has an element e such that $\mathrm{Ann}(1-e) \neq \{0\}$ and (a) does not hold. In such rings, $x \in \mathrm{Ann}(1-x) \implies I = \langle x \rangle$, where $x^2 = x$ so these z-ideals are again strong and minimal. □

Next we show that z-ideals behave nicely under contractions. Before we do that let us recall that a ring is said to be reduced if it has no nonzero nilpotent elements. Let $\phi \colon A \to B$ be a ring homomorphism, where A and B are reduced rings. By the *contraction* of an ideal J of B we mean the ideal $J^c = \phi^{-1}(J)$.

Lemma 2.2.7. *Every proper z-deal of B contracts to a proper z-ideal of A if and only if every maximal ideal of B contracts to a proper z-ideal of A.*

Proof. ($\Rightarrow$): Suppose that every z-ideal of B contracts to a z-ideal of A and let $\phi^{-1}(I)$ be a z-ideal in A. To show that the maximal ideal I of B contracts to a z-ideal $\phi^{-1}(I)$ of A, consider x and y in B with $\mathfrak{M}_B(\phi(x)) = \mathfrak{M}_B(\phi(y))$ and $\phi(x) \in I$. We must show that $\phi(y) \in I$. Since $\phi^{-1}(I)$ is a z-ideal in A, the hypothesis says I is a z-ideal in B. So $\phi(y) \in I$.

($\Leftarrow$): Suppose that every maximal ideal of B contracts to a proper z-ideal of A and let J be a proper z-ideal in B. To show that $\phi^{-1}(J)$ is a z-ideal, consider x and y in A with $\mathfrak{M}_A(x) = \mathfrak{M}_A(y)$ and $y \in \phi^{-1}(J)$. Let M be a maximal ideal of B containing $\phi(y)$. Then $y \in \phi^{-1}(M)$ and since $\mathfrak{M}(x) = \mathfrak{M}(y)$ and $\phi^{-1}(M)$ is a z-ideal by hypothesis, hence $x \in \phi^{-1}(M)$ which implies $\phi(x) \in M$. Thus we have shown that $\mathfrak{M}_B(\phi(x)) = \mathfrak{M}_B(\phi(y))$ and $\phi(x)$ is in the z-ideal J. So $\phi(y) \in J$ implies $y \in \phi^{-1}(J)$. Therefore $\phi^{-1}(J)$ is a z-ideal. □

In [25, page 27], Lambek gives a characterisation that if A is a commutative ring with identity, then an ideal Q of A is a maximal ideal if and only if for every x not in Q, there exists r in A such that $1 - rx \in Q$. We will make use of this result in the proof of the following lemma which we need to deduce that surjective ring homomorphism contract z-ideals to z-ideals.

Lemma 2.2.8. *If $\phi\colon A \to B$ is a surjective ring homomorphism, then every maximal ideal of B contracts to a maximal ideal of A.*

Proof. Let $M \in \mathrm{Max}(B)$. We must show that $\phi^{-1}[M] \in \mathrm{Max}(A)$. Suppose that $x \notin \phi^{-1}[M]$, then we have $\phi(x) \notin M$. Since $M \in \mathrm{Max}(B)$ and $\phi(x) \notin M$, there exists $u \in B$ such that

$$1_B - u\phi(x) \in M.$$

Since ϕ is surjective and $u \in B$, there exists $r \in A$ such that $\phi(r) = u$. It follows that

$$\phi(1_A) - \phi(r)\phi(x) = \phi(1_A - rx) \in M.$$

Since $\phi(1_A - rx) \in M$, $x \notin \phi^{-1}[M]$ and $r \in A$, then $1_A - rx \in \phi^{-1}[M]$. Therefore every maximal ideal of B contracts to maximal ideal of A. □

Corollary 2.2.9. *If $\phi\colon A \to B$ is a surjective ring homomorphism, then every z-ideal of B contracts to a z-ideal of A.*

2.3 Sums of z-ideals

In this section we will discuss, mainly in the class of f-rings, sufficient conditions for the sum of z-ideals to be a z-ideal. Most results in this regard were proved by Larson [27]. Recall that all rings are assumed to be commutative with identity.

A *lattice-ordered ring* is a ring A with a lattice structure such that, for all $a, b, c \in A$,

$$(a \wedge b) + c = (a + c) \wedge (b + c)$$

or, equivalently

$$(a \vee b) + c = (a + c) \vee (b + c),$$

and

$$0 \leq ab \text{ whenever } 0 \leq a \text{ and } 0 \leq b.$$

An f-*ring* is a lattice-ordered ring A in which the identity

$$(a \wedge b)c = (ac) \wedge (bc)$$

holds for all $a, b \in A$ and $c \geq 0$ in A.

We present some of the background facts concerning f-rings which are used in showing that the sum of z-ideals is a z-ideal. Given an f-ring A and $x \in A$, we let

$$A^+ = \{a \in A \,:\, a \geq 0\}$$

and

$$x^+ = x \vee 0, \quad x^- = (-x) \vee 0 \quad \text{and} \quad |x| = x \vee (-x).$$

A ring ideal I of an f-ring A is an ℓ-*ideal* if $|x| \leq |y|$ and $y \in I$ implies $x \in I$. An ℓ-ideal I of an f-ring A is *square dominated* if

$$I = \{a \in A \,:\, |a| \leq x^2 \text{ for some } x \in A \text{ such that } x^2 \in I\}.$$

We give a lemma, recorded in [26, Lemma 2.1], that will be used later and that also gives a characterisation of commutative reduced f-rings in which minimal prime ℓ-ideals are square dominated.

Lemma 2.3.1. *Let A be a commutative reduced f-ring.*

(1) A radical ℓ-ideal I is square dominated if every prime ℓ-ideal minimal with respect to containing I is square dominated.

(2) Every minimal prime ℓ-ideal of A is square dominated if and only if for every $a \in A^+$, the ℓ-ideal $\mathrm{Ann}(a) = \{b \in A \colon ab = 0\}$ *is square dominated.*

Proof. (1) Let $b \in I^+$. Let $M = \{c_1^2 \cdots c_n^2 \colon n \in \mathbb{N};\ b \leq c_i^2\}$. Then M is an m-system. Suppose that $M \cap I = \emptyset$. Then there is a prime ℓ-ideal P such that $I \subseteq P$ and $M \cap P = \emptyset$. Let $P_1 \subseteq P$ be a prime ℓ-ideal minimal with respect to containing I. By hypothesis, P_1 is square dominated, so there is a $p \in A$ such that $b \leq p^2$ and $p^2 \in P_1$. But then $p^2 \in M \cap P$, contrary to assumption. So $M \cap I \neq \emptyset$. Let $c_1^2 \cdots c_n^2 \in M \cap I$, where $b \leq c_i^2$ for each i. Then $b \leq c_1^2 \wedge \cdots \wedge c_n^2 = (|c_1| \wedge \cdots \wedge |c_n|)^2$. Also, $0 \leq (|c_1| \wedge \cdots \wedge |c_n|)^{2n} \leq c_1^2 \cdots c_n^2$. This implies $(|c_1| \wedge \cdots \wedge |c_n|)^{2n} \in I$, and because I is radical, $(|c_1| \wedge \cdots \wedge |c_n|)^2 \in I$.

(2) ($\Rightarrow$): Suppose that every minimal prime ℓ-ideal is square dominated. Let $a \in A^+$. Suppose P is a prime ℓ-ideal minimal with respect to containing $\mathrm{Ann}(a)$. Then

$$M = \{b\colon b \in A \smallsetminus P\} \cup \{a^n\colon n \in \mathbb{N}\} \cup \{ba^n\colon b \in A \smallsetminus P,\ n \in \mathbb{N}\}$$

is an m-system such that $M \cap \mathrm{Ann}(a) = \emptyset$. So there is a prime ℓ-ideal P_1 satisfying

$$\mathrm{Ann}(a) \subseteq P_1 \subseteq P.$$

But our choice of P implies $P_1 = P$ and $a \notin P$.

Now if P_2 is a minimal prime ℓ-ideal contained in P, $a \notin P_2$ implies $\mathrm{Ann}(a) \subseteq P_2$. Hence $P_2 = P$, and P is in fact a minimal prime ℓ-ideal which is square dominated. So every prime ℓ-ideal minimal with respect to containing $\mathrm{Ann}(a)$ is square dominated, and part (1) implies $\mathrm{Ann}(a)$ is square dominated.

($\Leftarrow$): Let P be a minimal prime ℓ-ideal, and $f \in P$. By [26, 1.2], there is a $g \notin P$ such that $fg = 0$. By hypothesis, $\mathrm{Ann}(g)$ is square dominated. So there is $f_1 \in \mathrm{Ann}(g)$ such that $f \leq f_1^2$ and $f_1^2 \in P$. □

There is a large class of f-rings, specifically those f-rings in which minimal prime ℓ-ideals are square dominated, in which the sum of any two radical ℓ-ideals is a radical ℓ-ideal. We are going to use the following two results recorded in [27] to show that in an f-ring in which minimal prime ℓ-ideals are square dominated, if the sum of any two minimal prime ℓ-ideals is a z-ideal, then the sum of any two z-ideals is a z-ideal.

Theorem 2.3.2. *Let A be an f-ring. In A, the sum of a radical ℓ-ideal and a square dominated radical ℓ-ideal is a radical ℓ-ideal.*

Proof. Suppose that I and J are radical ℓ-ideals and that J is square dominated. Let $a^2 \in I + J$ with $a \geq 0$. Then $a^2 \leq i + j$ for some $i \in I^+, j \in J^+$. Since J is square dominated, $j \leq j_1^2$ for some $j_1 \in A^+$ with $j_1^2 \in J$. So $a^2 \leq i + j_1^2$. Let $x = a - (a \wedge j_1)$ and $y = a \wedge j_1$. Since J is a radical ℓ-ideal, $j_1 \in J$ and $y \in J$. Now for any positive elements a, j_1 of any totally ordered ring, $a \wedge j_1 = a$ or $a \wedge j_1 = j_1$. In the first case $(a - (a \wedge j_1))^2 = 0$, and in the second case $(a - (a \wedge j_1))^2 = (a - j_1)^2 = a^2 - aj_1 - j_1 a + j_1^2 \leq a^2 - 2j_1^2 + j_1^2 = a^2 - j_1^2$. Therefore in any totally ordered ring, $(a - (a \wedge j_1))^2 \leq 0 \vee (a^2 - j_1^2)$. This implies that in the f-ring A,

$x^2 = (a - (a \wedge j_1))^2 \leq 0 \vee (a^2 - j_1^2) \leq i$. Thus, $x^2 \in I$ and hence $x \in I$. We have $a = x + y$ with $x \in I$ and $y \in J$. Therefore $a \in I + J$. □

Lemma 2.3.3. *Let A be an f-ring in which minimal prime ℓ-ideals are square dominated. In A, the sum of any two prime ℓ-ideals is prime.*

Proof. Let I and J be prime ℓ-ideals of A. Let I_1 and J_1 be minimal prime ℓ-ideals contained in I and J respectively. We will show that $I + J$ is an intersection of prime ℓ-ideals. To do so, we let $z \in A$ be such that $z \notin I + J$ and we will show that there is a prime ℓ-ideal containing $I + J$ but not z. The ℓ-ideal $I_1 + J_1$ is prime, and the prime ℓ-ideals containing it form a chain. By the maximal principle, there is a prime ℓ-ideal Q containing $I_1 + J_1$ which is maximal with respect to not containing z. By the previous theorem, $I + J_1$ is a radical ℓ-ideal. It also contains a prime ℓ-ideal and is therefore prime. Similarly, $I_1 + J$ is prime. Thus $I \subseteq I + J_1 \subseteq Q$ and $J \subseteq I_1 + J \subseteq Q$. This implies that $I + J \subseteq Q$ and $z \notin Q$. Therefore $I + J$ is an intersection of prime ℓ-ideals. So it is a radical ℓ-ideal. It also contains a prime ℓ-ideal and is therefore prime. □

In [14, Theorem 4.7], Gillman and Kohls show that in $C(X)$, the f-ring of all real-valued continuous functions defined on the topological space X, an ℓ-ideal is an intersection of ℓ-ideals, each of which contains a prime ℓ-ideal. Following [27], their proof easily generalises the proof that in an f-ring, an ℓ-ideal which contains all nilpotent elements of the f-ring is an intersection of ℓ-ideals, each of which contains a prime ℓ-ideal. We will make use of this result in the proof of the following theorem.

Theorem 2.3.4. *Let A be an f-ring in which minimal prime ℓ-ideals are square dominated. In A, the sum of any two radical ℓ-ideals is a radical ℓ-ideal.*

Proof. Let I and J be radical ℓ-ideals. We will show that $I + J$ is an intersection of prime ℓ-ideals. To do so, we let $z \in A$ such that $z \notin I + J$ and we show that there is a prime ℓ-ideal containing $I + J$ but not z. By Gillman and Kohl's result mentioned above, there is an ℓ-ideal Q containing $I + J$ and containing a prime ℓ-ideal but not containing z. Let P be a minimal prime ℓ-ideal contained in Q. By Theorem 2.3.2, $P + I$ is a radical ℓ-ideal. Also, it contains a prime ℓ-ideal and so is prime. Similarly, $P + J$ is prime. Then by Lemma 2.3.3, $(P + I) + (P + J)$ is prime. Since $(P + I) + (P + J) \subseteq Q$, $z \notin (P + I) + (P + J)$ and $I + J \subseteq (P + I) + (P + J)$. □

For use in the next example recall that a principal ideal of a ring A is an ideal generated by one element. The ideal generated by one element a is denoted by $\langle a\rangle$. Larson in [26, Lemma 2.1] proves that every mininal prime ℓ-ideal of a commutative reduced f-ring A is square dominated if and only if for every $a \in A^+$, the ℓ-ideal $\mathrm{Ann}(a)$ is square dominated. We will use this result of Larson to show that the converse of Theorem 2.3.4 does not hold, as we show next.

Example 2.3.5. Let $C([0,1])$ be the ring of real-valued continuous functions on the closed interval $[0,1]$. In $C([0,1])$, denote by i the function $i(x) = x$, and by e the function $e(x) = 1$. Let $A = \{f \in C([0,1]) : f = ae + g \text{ where } a \in \mathbb{R}, g \in \langle i\rangle\}$ with coordinate operations. Then A is a commutative reduced f-ring.

Next we show that the sum of two radical ℓ-ideals of A is a radical ℓ-ideal. So suppose I and J are radical ℓ-ideals. If I or J contains an element $f = ae + g$ such that $a \neq 0$, then it can be shown that I or J is square dominated. Then by Theorem 2.3.2, $I + J$ is a radical ℓ-ideal. So we may now suppose that both $I, J \subseteq \langle i\rangle$. If $f^2 \in I + J$, then there is $i_1 \in I^+$, $j_1 \in J^+$ such that $f^2 = i_1 + j_1$. Also $f \in \langle i\rangle$ which implies $|f| \leq ni$ and $f^2 \leq n^2 i^2$ for some $n \in \mathbb{N}$. So $i_1 \leq n^2 i^2$ and $j_1 \leq n^2 j^2$. Therefore $\sqrt{i_1} \leq ni$ and $\sqrt{i_1} \in A$. Since I is a radical ℓ-ideal, $\sqrt{i_1} \in I$. Similarly $\sqrt{j_1} \in J$. So $f \leq \sqrt{i_1} + \sqrt{j_1}$ implies $f \in I + J$. Thus $I + J$ is a radical ℓ-ideal.

Next we show that not every minimal prime ℓ-ideal of A is square dominated. Let f be a function such that $0 \leq f \leq i$ with $f(x) = 0$ for all $x \in [1/4, 1]$, $f(x) = 0$ for all $x \in [1/(4n+2), 1/4n]$, and $f(1/(4n+3)) = 1/(4n+3)$ for all $n \in \mathbb{N}$. Also, let g be a function such that $0 \leq g \leq i$ with $g(x) = 0$ for all $x \in [1/4, 1]$, $g(1/(4n+1)) = 1/(4n+1)$, and $g(x) = 0$ for all

$$x \in [1/(4n+4), 1/(4n+2)]$$

for all $n \in \mathbb{N}$. Then $g \in \mathrm{Ann}(f)$, and there is no element $h \in A$ which satisfies $g \leq h^2$ and $h^2 \in \mathrm{Ann}(f)$. So $\mathrm{Ann}(f)$ is not square dominated, and by Larson's results mentioned above, it implies that not every minimal prime ℓ-ideal of A is square dominated.

Next we turn our attention to the sum of two z-ideals which are ℓ-ideals. An f-ring A has *bounded inversion* if every $a \in A$ with $a \geq 1$ is invertible. In a commutative ring with identity every maximal ideal is a z-ideal, and if every maximal ideal is an ℓ-ideal (or equivalently if for all $x \geq 1, x^{-1}$ exists), then a z-ideal is always an ℓ-ideal [27].

Mason established two results concerning z-ideals and Larson in [27] slightly modified them to get the following two theorems. We are going to use them later when we show that the sum of any two z-ideals which are ℓ-ideals of a ring A is a z-ideal.

Theorem 2.3.6. *If A is an f-ring and $P \subseteq A$ is minimal in the class of prime ℓ-ideals containing a z-ideal I which is an ℓ-ideal, then P is also a z-ideal.*

Proof. We show that if Q is a prime ℓ-ideal containing I which is not a z-ideal, it is not minimal. Let Q be a prime ℓ-ideal containing an ℓ-ideal I, and suppose that Q is not a z-ideal. There exist $a \notin Q$ and $b \in Q$ such that $\mathfrak{M}(a) = \mathfrak{M}(b)$.

Consider the set

$$S = (A \setminus Q) \cup \{cb^n \mid n \in \mathbb{N} \text{ and } c \notin Q\}.$$

Since Q is prime, S is closed under multiplication. Furthermore, S does not meet I; for, if $cb^n \in I$, then ca belongs to the z-ideal I, and hence in the prime ℓ-ideal Q, whence $c \in Q$. By [13, Theorem 0.16], there is a prime ℓ-ideal containing I and disjoint from S and hence contained properly in Q. So Q is not minimal. □

An f-ring A is *1-convex* if for any $u, v \in A$ such that $0 \leq u \leq v$, there is a $w \in A$ such that $u = vw$. For the proof of the upcoming theorem we need to recall the following lemma from [10, Lemma 3.6].

Lemma 2.3.7. *Let A be an f-ring, I be a prime ideal, and P, Q be convex prime ideals each containing I. Then P and Q are in a chain.*

Proof. If $P = Q$, there is nothing to prove. So we may assume there exists $p \in P$ such that $p \notin Q$. Then, by primeness, $p^2 \notin Q$. Let $q \in Q$ and put $a = p^2 - q^2$. By properties of f-rings, $(a - |a|)(a + |a|) = 0 \in I$. Since I is prime, it contains one of these factors. We show that $a + |a| \notin I$. If not, then $p^2 - q^2 + |a| \in Q$, which implies $p^2 + |a| \in Q$. Since $0 \leq p^2 \leq p^2 + |a|$, this implies $p^2 \in Q$, and hence, $p \in Q$, which is false. So we must have $a - |a| \in I \subseteq P$, that is, $p^2 - q^2 - |a| \in P$, whence we deduce $q^2 + |a| \in P$. As before, this implies $q \in P$, and therefore, $Q \subseteq P$. □

Theorem 2.3.8. *Let A be an f-ring with bounded inversion. Suppose that the sum of any two minimal prime ℓ-ideals of A is a prime z-ideal. Then the sum of any two prime ℓ-ideals not in a chain is a z-ideal. .*

Proof. Suppose P and Q are prime ℓ-ideals of A. Let R_1 and R_2 be minimal prime ℓ-ideals such that $R_1 \subseteq P$ and $R_2 \subseteq Q$. By hypothesis, $R_1 + R_2$ is a prime z-ideal. Since $R_1 \subseteq P$ and $R_2 \subseteq Q$, it follows that $R_1 + R_2 \subseteq P + Q$.

On the other hand, $R_1 + R_2$ is a prime ℓ-ideal containing R_1 and R_2 so is in a chain with both P and Q by Lemma 2.3.7. Since P and Q are not in a chain, both P and Q must be contained in $R_1 + R_2$ whence $R_1 + R_2 = P + Q$ is a prime z-ideal. □

The following result is recorded in [27].

Theorem 2.3.9. *Let A be an f-ring in which minimal prime ℓ-ideals are square dominated. Suppose that the sum of any two minimal prime ℓ-ideals of A is a z-ideal. Then the sum of any two z-ideals which are ℓ-ideals of A is a z-ideal.*

Proof. Suppose I and J are z-ideals which are ℓ-ideals. Then I and J are radical ℓ-ideals since every z-ideal is a radical ℓ-ideal, and by Theorem 2.3.4, $I + J$ is a radical ℓ-ideal. We will show that $I + J$ is the intersection of z-ideals. To do so, we let $z \in A$ such that $z \notin I + J$, and we will show there is a z-ideal containing $I + J$ but not z. Since $I + J$ is a radical ℓ-ideal, it is the intersection of prime ℓ-ideals. So there is a prime ℓ-ideal P containing $I + J$ but not z. Let $P_1, P_2 \subseteq P$ be prime ℓ-ideals minimal with respect to containing I and J respectively. By Theorem 2.3.6, P_1, P_2 are prime z-ideals. It follows from Theorem 2.3.8 that $P_1 + P_2$ is a z-ideal. Also, $I + J \subseteq P_1 + P_2$ and $z \notin (P_1 + P_2)$ since $P_1 + P_2 \subseteq P$. □

For any element a of an f-ring A, $\mathrm{Ann}(a)$ is a z-ideal. Larson in [26, 1.2] gives a characterisation that a prime ℓ-ideal P of a commutative reduced f-ring is minimal if and only if $a \in P$ implies there is a $b \notin P$ such that $ab = 0$. Recall that $A^+ = \{a \in A : a \geq 0\}$. Then we obtain the following results which are recorded in [27].

Corollary 2.3.10. *Let A be a reduced f-ring in which minimal prime ℓ-ideals are square dominated. Suppose that for every $a, b \in A^+$, $\mathrm{Ann}(a) + \mathrm{Ann}(b)$ is a z-ideal. Then the sum of any two z-ideals which are ℓ-ideals of A is a z-ideal.*

Proof. We only need to show that the sum of any two minimal prime ℓ-ideals is a z-ideal. Let P and Q be minimal prime ℓ-ideals. Suppose a, b are in the same set of maximal ideals and $b \in P + Q$. Then $b = p + q$ for some $p \in P$ and $q \in Q$. Also, there is $p_1, q_1 \in A^+$ such that

$p_1 \notin P$, $q_1 \notin Q$, and $pp_1 = 0$, $qq_1 = 0$. So $b = p + q \in \text{Ann}(p_1) + \text{Ann}(q_1)$. By hypothesis, $\text{Ann}(p_1) + \text{Ann}(q_1)$ is a z-ideal. So $a \in \text{Ann}(p_1) + \text{Ann}(q_1) \subseteq P + Q$. □

Following [18], an f-ring A is called *normal* if

$$A = \text{Ann}(a^+) + \text{Ann}(a^-)$$

for all $a \in A$, or equivalently if

$$a \wedge b = 0 \text{ implies } A = \text{Ann}(a) + \text{Ann}(b).$$

In [26, Lemma 2.5] it is shown that in a commutative reduced normal f-ring with identity element, every minimal prime ℓ-ideal is square dominated. We will make use of that result in the proof of the next corollary.

Corollary 2.3.11. *Let A be a reduced normal f-ring. In A, the sum of any two z-ideals which are ℓ-ideals is a z-ideal.*

Proof. In view of the fact that minimal prime ℓ-ideals of A are square dominated and in light of Theorem 2.3.9, we only need to show that the sum of any two minimal prime ℓ-ideals is a z-ideal. So let P and Q be minimal prime ℓ-ideals. We will show that if $P \neq Q$, then $P + Q = A$. If $P \neq Q$, then there is an element $p \in P \setminus Q$. Since P is a minimal prime ℓ-ideal, there is an element $q \notin P$ such that $pq = 0$. Then $p \wedge q = 0$, and $\text{Ann}(p) + \text{Ann}(q) = A$. But $\text{Ann}(p) \subseteq Q$, $\text{Ann}(q) \subseteq P$. So $A = P + Q$. □

Chapter 3

d-Ideals of commutative rings

There are special types of z-ideals that have received a great deal of attention both in rings and Riesz spaces. These are called d-ideals, and have been studied by many mathematicians such as Huijsman and de Pagter [19]. In [2], [3], [5] and [6], d-ideals have been studied under the name z°-ideal. In this chapter all rings are reduced. We give various properties and characterisations of d-ideals. We will also give sufficient conditions when their sums are also d-ideals. Rings in which d-ideals coincide with z-ideals will be characterised.

3.1 Characterisations of d-ideals

We recall that the annihilator of a singleton $\{a\}$ of a ring A is the ideal

$$\mathrm{Ann}(a) = \{x \in A : ax = 0\}$$

and the *double annihilator* is the ideal

$$\mathrm{Ann}^2(a) = \{x \in A : xy = 0 \text{ for all } y \in \mathrm{Ann}(a)\}.$$

Furthermore $\mathrm{Ann}^3(a) = \mathrm{Ann}(a)$.

We start by defining the ideals that will form the main study in this section. The results in this section are recorded in [2].

Definition 3.1.1. An ideal I of a ring A is a *d-ideal* if for any $a, b \in A$, $\mathrm{Ann}(a) = \mathrm{Ann}(b)$ and

$a \in I$ imply $b \in I$. Equivalently, I is a d-ideal if and only if for any $a, b \in A$, $\text{Ann}^2(a) = \text{Ann}^2(b)$ and $a \in I$ imply $b \in I$.

Examples of d-ideals which are recorded in [2] are given below. Recall that a ring is said to be reduced if it has no nonzero nilpotent elements. The set of minimal prime ideals of the ring A will be denoted by $\text{Min}(A)$. If B is a subset of a ring A and x is an element of B, define $V(B) = \{P \in \text{Min}(A) \,:\, P \subseteq B\}$, $V(x) = \{P \in \text{Min}(A) \,:\, x \in P\}$, and $D(B) = \text{Min}(A) \smallsetminus V(B)$, then the set $D(a) = \text{Min}(A) \smallsetminus V(a)$ form a basis for the usual Zariski topology.

Examples 3.1.2. (1) If I is a nonzero ideal in a reduced ring A, then $\text{Ann}(I)$ is a d-ideal, for we observe that $\text{Ann}(I) = \bigcap\{P \in \text{Min}(A) \,:\, P \in D(I)\}$. More generally if I is a d-ideal in A and S is a subset of A not contained in I, then $(I : S) = \{a \in A \,:\, aS \subseteq I\}$ is a d-ideal.

(2) If S is a multiplicatively closed set in a reduced ring A, then

$$O_s = \{a \in A \,:\, as = 0 \;\text{ for some }\; s \in S\} = \bigcup_{s \in S} \text{Ann}(s) = \sum_{s \in S} \text{Ann}(s)$$

is a d-ideal.

(3) Let I be any ideal in a reduced ring A. Put $S = 1 + I = \{1 + x : x \in I\}$, then the ideal O_s defined as in the previous example deserves to be called the d-trace of I and denoted by $T(I) = O_s$. Clearly $T(I) \subseteq I$.

(4) Let I be a d-ideal in a reduced ring A and S be any multiplicatively closed set in A with $I \cap S = \emptyset$. Now define

$$I_s = \{a \in A : sa \in I \;\text{ for some }\; s \in S\}$$

then $I_S = \bigcup_{s \in S}(I : s) = \sum_{s \in S}(I : s)$ is a d-ideal.

(5) Let A be a reduced ring, then each minimal ideal and the socle of A, which is the sum of all minimal ideals, are d-ideals.

In the next proposition we will show various ways of characterising these ideals. For use in the upcoming proposition, and elsewhere, we introduce the following notation. For any $a \in A$, let

$$P(a) = \bigcap\{Q \in \text{Min}(A) \,|\, a \in Q\}$$

and we call it a *basic d-ideal.*

It is shown in [29] that $P(a) = \text{Ann}^2(a)$.

Proposition 3.1.3. *The following are equivalent for an ideal I of a ring A.*

(1) I is a d-ideal in A.

(2) $P(a) \subseteq I$ for every $a \in I$.

(3) For $a, b \in A$, $P(a) = P(b)$ and $b \in I$ imply that $a \in I$.

(4) For $a, b \in A$, $V(a) = V(b)$ and $a \in I$ imply that $b \in I$.

(5) $a \in I$ implies that $\mathrm{Ann}^2(a) \subseteq I$.

Proof. (1) $\Rightarrow$ (2): Assume that I is a d-ideal. Let $a \in I$. We need to show that

$$P(a) = \mathrm{Ann}^2(a) \subseteq I.$$

Let $x \in P(a)$. By [29, Lemma 1.3], $\mathrm{Ann}^2(x) \subseteq \mathrm{Ann}^2(a)$. Therefore

$$\mathrm{Ann}^2(x) \subseteq \mathrm{Ann}^2(a) \quad \text{and} \quad a \in I$$

imply that $x \in I$ since I is a d-ideal. Since x is an arbitrary element of $P(a)$, it follows that $P(a) \subseteq I$.

(2) $\Rightarrow$ (3): Let $P(a) \subseteq I$ for every $a \in I$. Consider any x and y in A with $P(x) = P(y)$ and $y \in I$. We must show that $x \in I$. We have

$$x \in P(x) = P(y).$$

But by hypothesis $P(y) \subseteq I$ for every $y \in I$. Therefore $x \in I$.

(3) $\Rightarrow$ (4): Assume that (3) holds. Consider any x and y in A with $V(x) = V(y)$ and $y \in I$. We must show that $x \in I$. We have

$$x \in P(x) = \bigcap V(x) = \bigcap V(y) = P(y)$$

and $y \in I$, so by the stated condition, $x \in I$.

(4) $\Rightarrow$ (5): Assume that the stated condition holds and let $x \in I$. We must show that $\mathrm{Ann}^2(x) \subseteq I$. It is shown in [29, Lemma 1.3] that $V(x) = V(\mathrm{Ann}^2(x))$. We therefore have

$$V(x) = V(\mathrm{Ann}^2(x)) = V(\mathrm{Ann}^2(y)) = V(y) \quad \text{and} \quad x \in I,$$

so by the stated condition, $\mathrm{Ann}^2(x) \subseteq I$.

(5) $\Rightarrow$ (1): Assume that $\mathrm{Ann}^2(a) \subseteq I$ for every $a \in I$. To show that I is a d-ideal, consider any x and y in A with $\mathrm{Ann}^2(x) = \mathrm{Ann}^2(y)$ and $y \in I$. We must show that $x \in I$. But now we have

$$x \in \mathrm{Ann}^2(x) = \mathrm{Ann}^2(y).$$

Now, by hypothesis $\mathrm{Ann}^2(y) \subseteq I$ for every $y \in I$. So $x \in I$. Therefore I is a d-ideal. □

We now give the following useful results in terms of d-ideals and they are recorded in [2].

Remark 3.1.4. (1) Every ideal in a von Neumann regular ring is an intersection of minimal prime ideals, that is, it is a d-ideal.

(2) Each d-ideal I in a reduced ring is a radical ideal (that is, an intersection of prime ideals).

(3) The principal ideal is a d-ideal if and only if it is a basic d-ideal.

3.2 Miscellaneous results on d-ideals

The following results, which are presented in [2] without proofs, show how d-ideals behave under contraction and extension with respect to some natural homomorphisms.

Let $\phi\colon A \to B$ be a ring homomorphism, where A and B are reduced rings. Recall that by the contraction of an ideal J of B we mean the ideal $J^c = \phi^{-1}(J)$ and by the *extension* of an ideal J of A we mean the ideal $J^e = \phi(J)B$. If S is a multiplicatively closed set, then by $S^{-1}[A]$ we mean the *ring of fractions* of A with respect to S and if S is the set of all nonzero-divisors, then $S^{-1}[A]$ is called the *classical ring of quotients* of A and is denoted by Q.

Lemma 3.2.1. *Let $\phi\colon A \to B$ be a ring homomorphism, where A and B are reduced rings. Then every proper d-ideal of B contracts to a proper d-ideal of A if and only if every minimal prime ideal of A contracts to a d-ideal.*

Proof. ($\Rightarrow$): Suppose that every d-ideal of B contracts to a d-ideal of A and let $\phi^{-1}(I)$ be a d-ideal in A. To show that the minimal prime ideal I of B contracts to a d-ideal $\phi^{-1}(I)$ of A, consider a and b in B with $\mathrm{Ann}_B(\phi(a)) = \mathrm{Ann}_B(\phi(b))$ and $\phi(a) \in I$. We must show that $\phi(b) \in I$. Since $\phi^{-1}(I)$ is a d-ideal in A, the hypothesis says I is a d-ideal in B. So $\phi(b) \in I$.

($\Leftarrow$): Suppose that every minimal prime ideal of B contracts to a proper d-ideal of A and let J be a proper d-ideal in A. To show that $\phi^{-1}(J)$ is a d-ideal, consider x and y in A with $\operatorname{Ann}_A(x) = \operatorname{Ann}_A(y)$ and $y \in \phi^{-1}(J)$. Let M be a minimal prime ideal in B containing $\phi(y)$. Then $y \in \phi^{-1}(M)$ and since $\operatorname{Ann}(x) = \operatorname{Ann}(y)$ and $\phi^{-1}(M)$ is a d-ideal by hypothesis, then $x \in \phi^{-1}(M)$ which implies $\phi(x) \in M$. Thus we have shown that $\operatorname{Ann}_B(\phi(x)) = \operatorname{Ann}_B(\phi(y))$ and $\phi(x)$ is in the d-ideal J. So $\phi(y) \in J$ implies $y \in \phi^{-1}(J)$. Therefore $\phi^{-1}(J)$ is a d-ideal. □

The above result immediately yields the following.

Corollary 3.2.2. *If A is a reduced ring and $f\colon A \to S^{-1}[A]$ is the natural ring homomorphism, then every d-ideal of $S^{-1}[A]$ contracts to a d-ideal of A.*

Proposition 3.2.3. *If A is a reduced ring and $f\colon A \to S^{-1}[A]$ is the natural ring homomorphism and I is a d-ideal in A with $S \cap I = \emptyset$, then I^{ec} is also a d-ideal containing I.*

Proof. We just observe that $I^{ec} = \bigcup_{s\in S}(I : s) = \sum_{s\in S}(I : s) = I_s$. By example 3.1.2 (4), I^{ec} is a d-ideal. □

In order to present the next result, we will recall the definition of the Jacobson radical. Recall that the Jacobson radical of a ring A, denoted by $\operatorname{Jac}(A)$, is the intersection of all maximal ideals. The following results in [2] and [23] show that in a large class of rings, every d-ideal is a z-ideal. The proofs are recorded in [29, Proposition 2.12] and [1, Theorem 2.3].

Proposition 3.2.4. *The Jacobson radical $\operatorname{Jac}(A)$ of a ring A is zero if and only if every d-ideal is a z-ideal.*

Proof. ($\Rightarrow$): Let I be a d-ideal and suppose $\mathfrak{M}(x) = \mathfrak{M}(y)$ with $y \in I$. We claim that $x \in \operatorname{Ann}^2(y) \subseteq I$. For if $s \in \operatorname{Ann}(y)$ and $xs \neq 0$ then since $\operatorname{Jac}(A) = 0$ there is a maximal ideal M such that $xs \notin M$, that is $x \notin M$ and $s \notin M$. But $sy = 0$ and $s \notin M$ implies $y \in M$, so $x \in M$ which is a contradiction.

($\Leftarrow$): Suppose $a \neq 0$ belongs to $\operatorname{Jac}(A)$ and take a minimal prime ideal P of A not containing a. Then P is a d-ideal and it is not a z-ideal for $\mathfrak{M}(a) = \mathfrak{M}(0)$ and a does not belong to P. □

Aliabad, Azarpanah and Karamzadeh in [2] mention that not every z-ideal is a d-ideal. For every maximal ideal in any ring with identity is a z-ideal and not every maximal ideal consists entirely of zero-divisors. Kaplansky in [23] gives the following result, which proves that not all z-ideals are d-ideals.

Theorem 3.2.5. *Let A be a ring and M a maximal ideal in the polynomial ring $A[x]$. Then M cannot consist entirely of zero-divisors.*

Proof. Assume the contrary. Then $x \notin M$ since x is not a zero-divisor. Hence $\langle x, M \rangle = A[x]$ and we write $1 = xf + g$ with $f \in A[x]$ and $g \in M$. But clearly $g = 1 - xf$ cannot be a zero-divisor. □

A d-ideal P in a ring A is a *maximal d-ideal* if P is a proper d-ideal and there is no proper d-ideal S with $P \subsetneq S \subsetneq A$. Recall that an inductive set is a non-empty partially ordered set in which every element has a successor. The following results in [2] show that prime d-ideals are key elements in the concepts of d-ideals.

Lemma 3.2.6. *Let A be a ring and $\mathrm{Ann}(S_i) \subseteq \mathrm{Ann}(T_i)$, $i = 1, 2, \ldots, n$, where S_i and T_i are subsets of A, for every i. Then $\mathrm{Ann}(S_1 S_2 \ldots S_n) \subseteq \mathrm{Ann}(T_1 T_2 \ldots T_n)$.*

Theorem 3.2.7. *Let A be a reduced ring and I be a d-ideal, then every prime ideal, minimal over I is a prime d-ideal.*

Proof. Let P be a prime ideal, minimal over I and $\mathrm{Ann}(a) \subseteq \mathrm{Ann}(b)$ where $a \in P$ and $b \in A$. Now since P/I is minimal prime ideal in A/I and A/I is a reduced ring, there exists $0 \neq c + I$ in A/I with $c \notin P$ and $ac \in I$. Now by the previous lemma, $\mathrm{Ann}(ac) = \mathrm{Ann}(bc)$. Since I is a d-ideal and $ac \in I$, we have $bc \in I \subseteq P$. But $c \notin P$, that is $b \in P$. □

The following corollaries are now immediate.

Corollary 3.2.8. *If $f\colon A \to A/I$ is a natural epimorphism, where A is a reduced ring and I is a d-ideal in A, then every d-ideal of A/I contracts to a d-ideal in A.*

Proof. Suppose that K is a d-ideal in A. Consider x and y in A such that $\mathrm{Ann}(x) = \mathrm{Ann}(y)$ and $x + I \in f(K)$. We must show that $y + I \in f(K)$. Since K is a d-ideal and $f\colon A \to A/I$ is the natural epimorphism then $f(y) = y + I \in f(K)$. Therefore $f(K)$ is a d-ideal in A/I. □

Corollary 3.2.9. *An ideal I in a reduced ring A is a d-ideal if and only if it is an intersection of prime d-ideals.*

Corollary 3.2.10. *If A is a reduced ring, then every maximal d-ideal is a prime d-ideal.*

Corollary 3.2.11. *Let A be a reduced ring and P be a prime ideal in A, then either P is a d-ideal or contains a maximal d-ideal which is a prime d-ideal.*

Proof. Put $\mathcal{K} = \{I \subseteq A \,|\, I \text{ is a } d\text{-ideal and } I \subseteq P\}$. Then $\{0\} \in \mathcal{K}$ and $\mathcal{K}$ is inductive so by Zorn's lemma, $\mathcal{K}$ has a maximal element. Let I be one. Then $I = P$ iff P is a prime d-ideal. If $I \subsetneqq P$ then there exists a prime ideal Q minimal with respect to containing I and contained in P; $Q \neq P$ since Q will be a d-ideal (Theorem 3.2.7). Moreover, either $Q = I$ in which case I is prime, or $I \subsetneqq Q$, which contradicts the maximality of I. □

Definition 3.2.12. According to [2], a ring A satisfies *property R* if each finitely generated ideal of A consisting of zero-divisors has a nonzero annihilator. A is said to have the *annihilator condition* or briefly A satisfies *a.c* if for each finitely generated ideal I of A there exists an element $b \in A$ with $\text{Ann}(I) = \text{Ann}(b)$. If this element $b \in A$ can be chosen to be an element in I, we say A satisfies *strong a.c.*

It is well known that most rings satisfy some of these properties. For example, Noetherian rings, see [23, page 56], $C(X)$, see [13], zero-dimensional rings (each prime ideal is maximal), the polynomial ring $A[x]$ and rings whose classical ring of quotients are regular. See [15] and [29] for examples of rings with property R. We also observe that $A[x]$, where A is a reduced ring, $C(X)$ and Bezout rings (rings where finitely generated ideals are principal) and many other rings satisfy a.c. see [17], [28] and [33]. Moreover, the latter two classes of rings satisfy strong a.c. The reader is referred to [17] for various examples and counter examples of rings with these properties.

The next results in [2] show that d-ideals are indispensable in dealing with ideals consisting of zero-divisors.

Theorem 3.2.13. *If A is a reduced ring satisfying property R and I is an ideal consisting of zero-divisors, then I is contained in a d-ideal.*

Proof. We define $I_0 = I, I_1 = \sum_{x \in I_0} \text{Ann}^2(x)$ and if α is a limit ordinal, we define $I_\alpha = \bigcup_{\beta < \alpha} I_\beta$, where β is an ordinal, and finally if $\alpha = \beta + 1$, we define $I_\alpha = \sum_{x \in I_\beta} \text{Ann}^2(x)$. Then we get an ascending chain $I_0 \subseteq I_1 \subseteq \cdots \subseteq I_\alpha \subseteq I_{\alpha+1} \subseteq \ldots$ and since A is a set, there exists the smallest ordinal α such that $I_\alpha = I_\gamma$, for all $\gamma \geq \alpha$. We claim that I_α is a proper ideal which is also a d-ideal. If I_α is a proper ideal, it is certainly a d-ideal, for $I_\alpha = I_{\alpha+1} = \sum_{x \in I_\alpha} \mathrm{Ann}^2(x)$. This means that $\mathrm{Ann}^2(x) \subseteq I_\alpha$, for all $x \in I_\alpha$ and therefore by Proposition 3.1.3, we are through. Thus it remains to be shown that I_α is proper ideal. To see this, it suffices to show that each I_α, $\forall\alpha$ consists entirely of zero-divisors. We proceed by transfinite induction (an extension of mathematical induction to well-ordered sets) on α. For $\alpha = 0$, it is evident. Let us assume it is true for all ordinals $\beta < \alpha$ and prove it for α. If α is a limit ordinal, then $I_\alpha = \bigcup_{\beta<\alpha} I_\beta$ and therefore I_α consists of zero-divisors. Now let $\alpha = \beta + 1$ be a nonlimit ordinal, then $I_\alpha = \sum_{x \in I_\beta} \mathrm{Ann}^2(x)$. We must show that each element a of I_α is a zero-divisor. Put $a = a_1 + a_2 + \cdots + a_n$, where $a_i \in \mathrm{Ann}^2(x_i)$, $x_i \in I_\beta, i = 1, 2, \ldots, n$. But by the induction hypothesis each element of I_β is a zero-divisor. Now by the property R, there exists $0 \neq b \in \mathrm{Ann}(x_1A + x_2A + \cdots + x_nA)$, that is, $ab = 0$. □

Corollary 3.2.14. *If A is reduced with property R, then every maximal ideal consisting only of zero-divisors is a d-ideal.*

Corollary 3.2.15. *If A is a reduced ring with property R, and I is an ideal of A consisting of zero-divisors, then there is the smallest d-ideal containing I and also there is a maximal d-ideal containing I which is also a prime d-ideal.*

In the case of f-rings, the proof of the previous theorem does not use ordinals. This was proved by Ighedo in her PhD thesis [20]. We reproduce the proof. An ideal of a ring A is *singular* if it consists entirely of zero-divisors. A set Λ is a *directed set* if and only if there is a relation $\leq$ on Λ satisfying:

(a) $\lambda \leq \lambda$, for each $\lambda \in \Lambda$,

(b) if $\lambda_1 \leq \lambda_2$ and $\lambda_2 \leq \lambda_3$, then $\lambda_1 \leq \lambda_3$,

(c) if $\lambda_1, \lambda_2 \in \Lambda$ then there is some $\lambda_3 \in \Lambda$ with $\lambda_1 \leq \lambda_3$, $\lambda_2 \leq \lambda_3$.

Theorem 3.2.16. *Let A be a reduced f-ring and I be a singular ideal of A. Then the set*

$$J = \bigcup \{\mathrm{Ann}^2(a) \mid a \in I\}$$

is the smallest d-ideal of A containing I.

Proof. Let us show first that the family $\{\mathrm{Ann}^2(a) \mid a \in I\}$ is directed. Let $a, b \in I$. We claim that $\mathrm{Ann}^2(a) \cup \mathrm{Ann}^2(b) \subseteq \mathrm{Ann}^2(a^2 + b^2)$. To verify this it suffices to show that

$$\mathrm{Ann}(a^2 + b^2) \subseteq \mathrm{Ann}(a) \cap \mathrm{Ann}(b).$$

Let $r \in \mathrm{Ann}(a^2 + b^2)$. Then $r(a^2 + b^2) = 0$, which implies $(ra)^2 + (rb)^2 = 0$. Since squares are positive in f-rings, this implies $(ra)^2 = (rb)^2 = 0$, and hence $ra = rb = 0$ since A is reduced. Therefore $r \in \mathrm{Ann}(a) \cap \mathrm{Ann}(b)$. Thus, J is d-ideal which clearly contains I. To show that it is the smallest such, consider any d-ideal K of A which contains I. Let $u \in J$. Then $u \in \mathrm{Ann}^2(a)$ for some $a \in I$. But $a \in K$ since K is a d-ideal, so $u \in K$, and hence $J \subseteq K$. □

We also have the following.

Proposition 3.2.17. *If A is a reduced ring with strong a.c, then an ideal I in A is a d-ideal if and only if $I[x]$ is a d-ideal in $A[x]$.*

Proof. ($\Rightarrow$): Let I be a d-ideal in A and $\mathrm{Ann}(f) = \mathrm{Ann}(g)$, where $f \in I[x]$, $g \in A[x]$. Now put $f = \sum\limits_{i=0}^{n} a_i x^i$, $g = \sum\limits_{i=0}^{m} b_i x^i$ and $K = (a_0, a_1, \ldots, a_n)$, $B = (b_0, b_1, \ldots, b_m)$, then $h \in \mathrm{Ann}(f)$ if and only if $KC = \{0\}$, where $C = (c_0, c_1, \ldots, c_k)$ and $h = \sum\limits_{i=0}^{k} c_i x^i$ (note that if P is a prime ideal in A, then $hf = 0 \in P[x]$ implies that $h \in P[x]$ or $f \in P[x]$, that is, $KC \subseteq P$). This means that $\mathrm{Ann}(K) = \mathrm{Ann}(B)$. But A satisfies strong $a.c.$, that is $\mathrm{Ann}(K) = \mathrm{Ann}(a)$ for some $a \in K$. Hence $\mathrm{Ann}(a) = \mathrm{Ann}(B) = \bigcap\limits_{i=0}^{m} \mathrm{Ann}(b_i)$, that is, $\mathrm{Ann}(a) \subseteq \mathrm{Ann}(b_i)$, $i = 0, 1, \ldots, m$ and therefore $a \in I$ implies that $b_i \in I$, for all $i = 0, 1, \ldots, m$. This shows that $g \in I[x]$.

($\Leftarrow$): Let $I[x]$ be a d-ideal in $A[x]$ and let $\mathrm{Ann}(a) = \mathrm{Ann}(b), a \in I, b \in A$. Then

$$\mathrm{Ann}(ax) = \mathrm{Ann}(bx)$$

in $A[x]$ and $ax \in I[x]$ implies that $bx \in I[x]$, that is, $b \in I$. □

We know that $P(b) \subseteq \mathrm{Ann}(a)$ for every $b \in \mathrm{Ann}(a)$, for if $a \neq 0$, $\mathrm{Ann}(a)$ is a d-ideal in a reduced ring A. We show next when equality holds.

Lemma 3.2.18. *If A is a reduced ring and $b \in \mathrm{Ann}(a)$, then $\mathrm{Ann}(a) = P(b)$ if and only if $a + b$ is not a zero-divisor in A.*

Proof. If $\mathrm{Ann}(a) = P(b)$, then let $a + b$ belong to a minimal prime ideal P of A and seek a contradiction. We consider two cases. First, let $a \in P$, then $b \in P$ implies that $P(b) \subseteq P$, that is, $P(b) = \mathrm{Ann}(a) \subseteq P$, which is impossible, for we also have $a \in P$. Now let $a \notin P$, then we must have $b \notin P$, that is, $\mathrm{Ann}(a) = P(b) \nsubseteq P$, which is again impossible. Conversely, if $a + b$ is not a zero-divisor in A, we are to show that $\mathrm{Ann}(a) \subseteq P(b)$. Let P be a minimal prime ideal with $b \in P$, then $a + b \notin P$ implies that $a \notin P$, that is, $\mathrm{Ann}(a) \subseteq P$. Hence $\mathrm{Ann}(a) \subseteq P(b)$ and we are through. □

Rings whose classical rings of quotients are regular have been characterised in various ways, see [17] and [33]. Using the concept of d-ideals, we state the following results given in [2].

Proposition 3.2.19. *Let A be a reduced ring, then the following statements are equivalent.*

(1) Minimal prime ideals of A are the only prime ideals consisting of zero-divisors.

(2) A satisfies property R and each prime d-ideal in A is a minimal prime ideal.

(3) The classical ring of quotients of A is a reqular ring.

(4) $\mathrm{Ann}(a)$ *is a basic d-ideal for all $a \in A$.*

Proof. (1) ⇒ (2): It suffices to show that A satisfies property R. Let I be a nonzero finitely generated ideal consisting of zero-divisors. Hence $I \subseteq \bigcup\{P : P \in \mathrm{Min}(A)\}$, that is, $I \cap S = \emptyset$ where $S = \bigcap\{A \setminus P : P \in \mathrm{Min}(A)\}$. Now there exists a prime ideal $P' \supseteq I$ with $P' \cap S = \emptyset$. This shows that P' consists of zero-divisors and therefore by our hypothesis, P' must be a minimal prime ideal, that is, $\mathrm{Ann}(I) \nsubseteq P'$ and therefore $\mathrm{Ann}(I) \neq \{0\}$.

(2) ⇒ (3): Let Q be the classical ring of quotients of A and M be a maximal ideal of Q. Then since every element of Q is either a zero-divisor or a unit, M consists of zero-divisors, that is, by Corollary 3.2.14, M is a d-ideal. Thus by Corollary 3.2.18, $M^c = A \cap M$ is a prime d-ideal which must be a minimal prime ideal. This shows that M is a minimal prime ideal. Hence each prime ideal of Q is maximal and since Q is reduced, it is a regular ring.

(3) ⇒ (4): Let Q be a classical ring of quotients of A, then if $a \in A$ we have $\mathrm{Ann}_Q(a) = eQ$, for some idempotent $e \in Q$, since Q is a regular ring. But

$$eQ = \mathrm{Ann}_Q(1 - e) = \bigcap\{P \in \mathrm{Min}(Q) : e \in P\}.$$

Now let $e = \frac{b}{s}$, where $b, s \in A$ and s is a nonzero-divisor, then $e \in P$ if and only if $b \in P$, for all $P \in \mathrm{Min}(Q)$. Hence $\mathrm{Ann}_A(a) = \mathrm{Ann}_Q(a) \cap A = \bigcap\{P \in \mathrm{Min}(A) : b \in P\} = P(b)$.

(4) $\Rightarrow$ (1): Let P be a prime ideal consisting of zero-divisors. We must show that for each $a \in P$, $\mathrm{Ann}(a) \nsubseteq P$. Let us assume that $a \in P$ and $\mathrm{Ann}(a) \subseteq P$ and seek a contradiction. By part (4), we have $\mathrm{Ann}(a) = P(b)$, for some $b \in A$, that is, $a + b \in P$. But by Lemma 3.2.18, $a + b$ is not a zero-divisor, which is a contradiction. □

Definition 3.2.20. A reduced ring A is called an *almost regular ring* if A has the property R and is its own classical ring of quotients.

Clearly every regular ring is almost regular but not conversely. The following is a characterisation of the d-trace of an ideal in almost regular rings, see Example 3.1.2 (3).

Proposition 3.2.21. *Let A be an almost regular ring and I be any ideal of A, then*

$$T(I) = \{c \in A : \exists a \in I, \exists b \in A \text{ with } V(a) \subseteq D(b) \subseteq V(c)\}.$$

Proof. Let $x \in T(I)$, then $x(1 + y) = 0$ for some $y \in I$. Now if P is a minimal prime ideal containing y, then $1 + y \notin P$, that is, $x \in P$. Hence we have $V(y) \subseteq D(1 + y) \subseteq V(x)$. Conversely, if $b, c \in A$ and $a \in I$ are given with $V(a) \subseteq D(b) \subseteq V(c)$, then we are to show that $c \in T(I)$. We first observe that $bc = 0$, for $\mathrm{Ann}(b) = \bigcap_{P \in D(b)} P$ and $D(b) \subseteq V(c)$. Moreover $\langle a \rangle + \langle b \rangle = A$, for otherwise $\langle a \rangle + \langle b \rangle$ consists of zero-divisors and by property R, we must have $V(\langle a \rangle + \langle b \rangle) \neq \emptyset$ which is impossible, for $V(a) \subseteq D(b)$. Now $\langle a \rangle + \langle b \rangle = A$ implies that $I + \mathrm{Ann}(c) = A$. Hence $\exists x \in I$ with $1 - x \in \mathrm{Ann}(c)$, that is, $c \in \mathrm{Ann}(1 + x) \subseteq T(I)$. □

We observed in the previous chapter that every ideal has a z-cover, by which we mean the smallest z-ideal containing it. We shall now show that similarly every ideal has a d-cover.

Definition 3.2.22. Let I be an ideal consisting of zero-divisors in a ring A. Then the smallest d-ideal containing I is called the *d-cover* of I.

The d-cover of I is denoted by I_d and if it does not exist we put $I_d = A$.

The next proposition reveals some properties of the d-cover in rings with strong annihilator condition.

Proposition 3.2.23. *Let A be a reduced ring with strong annihilator condition and I, J be ideals in A consisting of zero-divisors, then we have the following.*

(1) $I_d = \{a \in A : \exists b \in A \text{ with } \mathrm{Ann}(b) \subseteq \mathrm{Ann}(a)\}$.

(2) $(I \cap J)_d = I_d \cap J_d$.

(3) If A is an almost regular ring, then $T(I_d) = T(I)_d = T(I)$, where $T(I)$ is the d-trace of I.

Proof. (1) Clearly I_d contains the set on the right hand side. We claim that this set is an ideal and then clearly becomes a d-ideal containing I and we are through. To this end, let a, a' be in this set, then there exists $b, b' \in I$ with $\mathrm{Ann}(b) \subseteq \mathrm{Ann}(a)$ and $\mathrm{Ann}(b') \subseteq \mathrm{Ann}(a')$. Now since A has strong annihilator condition, there exists $c \in I$ such that

$$\mathrm{Ann}(c) = \mathrm{Ann}(\langle b\rangle + \langle b'\rangle) = \mathrm{Ann}(b) \cap \mathrm{Ann}(b') \subseteq \mathrm{Ann}(a + a').$$

This shows that $a + a'$ belongs to this set. We also note that $\mathrm{Ann}(a) \subseteq \mathrm{Ann}(ar)$, for every $r \in A$ implies that ar belongs to this set whenever a does.

(2) Since $I \subseteq I_d$ and $J \subseteq J_d$, by the first part we have $(I \cap J)_d \subseteq I_d \cap J_d$.

Conversely, if $x \in I_d \cap J_d$, then there exist $a \in I$, $b \in J$ such that $\mathrm{Ann}(a) \subseteq \mathrm{Ann}(x)$ and $\mathrm{Ann}(b) \subseteq \mathrm{Ann}(x)$, by part (1). Therefore by Lemma 3.2.6, we have

$$\mathrm{Ann}(ab) \subseteq \mathrm{Ann}(x^2) = \mathrm{Ann}(x).$$

Now $ab \in I \cap J$ and by part (1) imply that $x \in (I \cap J)_d$.

(3) Since $I \subseteq I_d$, we infer that $T(I) \subseteq T(I_d)$. Conversely, if $z \in T(I_d)$ then by Proposition 3.2.21, there exist $x \in I_d$, $y \in A$ such that $V(x) \subseteq D(y) \subseteq V(z)$. But by part (1), $x \in I_d$ means that there exists $b \in I$ with $\mathrm{Ann}(b) \subseteq \mathrm{Ann}(x) \subseteq \mathrm{Ann}(z)$ (note that $V(x) \subseteq V(z)$). Hence $V(b) \subseteq V(x) \subseteq D(y) \subseteq V(z)$, that is, $z \in T(I)$, by Proposition 3.2.21. Hence $T(I_d) = T(I)$ and it is evident that $T(I)_d = T(I)$, for $T(I)$ is a d-ideal. □

The above result immediately yields the following.

Corollary 3.2.24. *Let A be a reduced ring with strong annihilator condition, then any ideal I of A consisting of zero-divisors is contained in a d-ideal and* $I_d = \bigcup_{b \in I} P(b) = \sum_{b \in I} P(b)$.

3.3 Sums of d-ideals

We observed in the previous chapter that in the class of f-rings, given sufficient conditions, the sum of z-ideals is a z-ideal. Similarly, in this section we will discuss, mainly in the class of f-rings, sufficient conditions for the sum of d-ideals to be a d-ideal. For the following results, our main reference is [27].

In the previous section, we showed that if A is an f-ring with bounded inversion in which the sum of any two minimal prime ℓ-ideals is a prime z-ideal, then the sum of any two prime ℓ-ideals not in a chain is a z-ideal. Now we deduce from the result mentioned above the following theorem in terms of d-ideals

Theorem 3.3.1. *Let A be an f-ring with bounded inversion. Suppose that the sum of any two minimal prime ℓ-ideals of A is a prime d-ideal. Then the sum of any two prime ℓ-ideals not in a chain is a d-ideal. .*

Proof. Suppose X and Y are prime ℓ-ideals of A. Let K_1 and K_2 be minimal prime ℓ-ideals such that $K_1 \subseteq X$ and $K_2 \subseteq Y$. By hypothesis, $K_1 + K_2$ is a prime d-ideal. Since $K_1 \subseteq X$ and $K_2 \subseteq Y$, it follows that $K_1 + K_2 \subseteq X + Y$.

On the other hand, $K_1 + K_2$ is a prime ℓ-ideal containing K_1 and K_2 so is in a chain with both X and Y by Lemma 2.3.7. Since X and Y are not in a chain, both X and Y must be contained in $K_1 + K_2$ whence $K_1 + K_2 = X + Y$ is a prime d-ideal. □

Mason in [29, Theorem 2.5] proves that the prime ideals minimal with respect to containing a d-ideal I are themselves d-ideals. With very slight modifications to the proof of Mason's results, we give the next result in the context of f-rings, recorded in [27], and we will use it in the proof of the next theorem.

Lemma 3.3.2. *If A is an f-ring and $P \subseteq A$ is minimal in the class of prime ℓ-ideals containing a d-ideal I which is an ℓ-ideal, then P is also a d-ideal.*

The following results are recorded in [27].

Theorem 3.3.3. *Let A be a f-ring in which minimal prime ℓ-ideals are square dominated. Suppose that the sum of any two minimal prime ℓ-ideals of A is a d-ideal. Then the sum of any two d-ideals which are ℓ-ideals of A is a d-ideal.*

Proof. Suppose M and N are d-ideals which are ℓ-ideals. Then M and N are radical ℓ-ideals since every d-ideal is a radical ℓ-ideal, and by Theorem 2.3.4, $M+N$ is a radical ℓ-ideal. We will show that $M+N$ is the intersection of d-ideals. To do so, we let $z \in A$ such that $z \notin M+N$, and we will show there is a d-ideal containing $M+N$ but not z. Since $M+N$ is a radical ℓ-ideal, it is the intersection of prime ℓ-ideals. So there is a prime ℓ-ideal P containing $M+N$ but not z. Let $P_1, P_2 \subseteq P$ be prime ℓ-ideals minimal with respect to containing M and N respectively. By Lemma 3.3.2, P_1, P_2 are prime d-ideals. It follows from Theorem 3.3.1 that $P_1 + P_2$ is a d-ideal. Also, $M + N \subseteq P_1 + P_2$ and $z \notin (P_1 + P_2)$ since $P_1 + P_2 \subseteq P$. □

It is known that for any element a of an f-ring A, $\mathrm{Ann}(a)$ is a d-ideal. Recall that a prime ℓ-ideal P of a commutative reduced f-ring is minimal if and only if $a \in P$ implies there is a $b \notin P$ such that $ab = 0$. By the preceeding results the following two corollaries are immediate.

Corollary 3.3.4. *Let A be a reduced f-ring in which minimal prime ℓ-ideals are square dominated. Suppose that for every $a, b \in A^+$, $\mathrm{Ann}(a) + \mathrm{Ann}(b)$ is a d-ideal. Then the sum of any two d-ideals which are ℓ-ideals of A is a d-ideal.*

Proof. We only need show that the sum of any two minimal prime ℓ-ideals is a d-ideal. Let C and D be minimal prime ℓ-ideals and $\mathrm{Ann}(a) + \mathrm{Ann}(b)$ be a d-ideal for every $a, b \in A^+$. We need to show that the sum of any two minimal prime ℓ-ideals is a d-ideal. Suppose that for any $a, b \in A$, $\mathrm{Ann}(a) = \mathrm{Ann}(b)$ and $b \in C + D$. Then $b = c + d$ for some $c \in C$ and $d \in D$. Also, there is $c_1, d_1 \in A^+$ such that $c_1 \notin C, d_1 \notin D$, and $cc_1 = 0, dd_1 = 0$. So $b = c + d \in \mathrm{Ann}(c_1) + \mathrm{Ann}(d_1)$. By hypothesis, $\mathrm{Ann}(c_1) + \mathrm{Ann}(d_1)$ is a d-ideal. So $a \in \mathrm{Ann}(c_1) + \mathrm{Ann}(d_1) \subseteq C + D$. □

In [26, Lemma 2.5] it is shown that in a commutative, reduced normal f-ring with identity element, every minimal prime ℓ-ideal of A is square dominated. We therefore conclude this chapter with the following result from [2].

Corollary 3.3.5. *Let A be a reduced normal f-ring with identity element. In A, the sum of any two d-ideals which are ℓ-ideals is a d-ideal.*

Proof. In view of the fact that minimal prime ℓ-ideals of A are square dominated and in light of Theorem 3.3.3, we only need to show that the sum of any two minimal prime ℓ-ideals is a d-ideal. So let C and D be minimal prime ℓ-ideals. We will show that if $C \neq D$, then $C + D = A$. If

$C \neq D$, then there is an element $c \in C \setminus D$. Since C is a minimal prime ℓ-ideal, there is an element $d \notin C$ such that $cd = 0$. Then $c \wedge d = 0$, and $\mathrm{Ann}(c) + \mathrm{Ann}(d) = A$. But $\mathrm{Ann}(d) \subseteq D$, $\mathrm{Ann}(c) \subseteq C$. So $A = C + D$. □

Chapter 4

Higher order d-ideals of commutative rings

This chapter will consist of new concepts which have hitherto not been considered. We will introduce ideals that resemble d-ideals, called higher order d-ideals, and the definition we use is motivated by the definition of higher order z-ideals in [11].

Dube and Ighedo in [11] prove that the direct product of finitely many rings is z-terminating if and only if each factor is z-terminating. In the first section of this chapter we extend the result by proving that the direct product of infinitely many rings is z-terminating if each factor is z-terminating.

4.1 Higher order z-ideals of commutative rings

We recall from [11] the definition of a z^n-ideal. Let n be a positive integer. An ideal I of a ring A is a *z^n-ideal* if for any $a, b \in A$, $\mathfrak{M}(a) = \mathfrak{M}(b)$ and $b^n \in I$ imply $a^n \in I$. We denote by $\mathfrak{Z}^n(A)$ the set of all z^n-ideals of A. In particular, $\mathfrak{Z}(A)$ denotes the set of all z-ideals of A.

It is also defined in [11] that a ring A is *z-terminating* if there is a positive integer n such that for every $m \geq n$, each z^m-ideal is a z^n-ideal.

We state the result given in [11] below.

Theorem 4.1.1. *The direct product of finitely many rings is z-terminating if and only if each* *factor is z-terminating.*

Next we extend the above result.

Let $\{A_\lambda \mid \lambda \in \Lambda\}$ be a family of commutative rings with identity. For each fixed $k \in \Lambda$, let $\pi_k \colon \prod_\lambda A_\lambda \to A_k$ be the projection of the product ring $\prod_\lambda A_\lambda$ to the ring A_k. This means that if we write an arbitrary element of $\prod_\lambda A_\lambda$ as $a = (a_\lambda)$, then $\pi_k(a) = a_k$.

It is a known result in commutative rings with identity that an ideal of the direct product $\prod_\lambda A_\lambda$ is maximal if and only if it is of the form $\pi_\ell^{-1}[M]$, for some index ℓ and $M \in \operatorname{Max}(A_\ell)$. That is

$$\operatorname{Max}\Big(\prod_\lambda A_\lambda\Big) = \Big\{\pi_\ell^{-1}[M] \mid \ell \in \Lambda \text{ and } M \in \operatorname{Max}(A_\ell)\Big\}.$$

Theorem 4.1.2. *Suppose $\{A_\lambda \,|\, \lambda \in \Lambda\}$ is a family of z-terminating rings, and suppose there exists $m \in \mathbb{N}$ such that termination for each A_λ occurs at or before stage m. Then $\prod_\lambda A_\lambda$ is z-terminating.*

Proof. We first show that if $\{A_\lambda \mid \lambda \in \Lambda\}$ is a family of commutative rings with identity, and $n \in \mathbb{N}$, for each λ if J_λ is an ideal of A_λ, then

$$\prod_\lambda J_\lambda \text{ is a } z^n\text{-ideal if and only if each } J_\lambda \text{ is a } z^n\text{-ideal.}$$

($\Rightarrow$): Suppose $\prod_\lambda J_\lambda$ is a z^n-ideal in $\prod_\lambda A_\lambda$. Fix an index $\ell \in \Lambda$, and consider $x, y \in J_\ell$ such that $\mathfrak{M}(x) = \mathfrak{M}(y)$ and $y^n \in J_\lambda$. Consider the elements $\bar{x} = (x_\lambda)$ and $\bar{y} = (y_\lambda)$ of $\prod_\lambda A_\lambda$ given by

$$x_\lambda = \begin{cases} x & \text{if } \lambda = \ell, \\ 0 & \text{if } \lambda \neq \ell \end{cases}$$

and

$$y_\lambda = \begin{cases} y & \text{if } \lambda = \ell, \\ 0 & \text{if } \lambda \neq \ell. \end{cases}$$

We claim that in the ring $\prod_\lambda A_\lambda$, $\mathfrak{M}(\bar{x}) \supseteq \mathfrak{M}(\bar{y})$. To see this, let $\mathfrak{n} \in \mathfrak{M}(\bar{y})$. Then $\mathfrak{n}$ is a maximal ideal of $\prod_\lambda A_\lambda$ containing $\bar{x}$. Therefore there is an index $k \in \Lambda$ and a maximal ideal M of A_k such that $\mathfrak{n} = \pi_k^{-1}[M]$. If $k \neq \lambda$, then $\bar{x} \in \mathfrak{n}$. On the other hand, if $k = \ell$, then $y \in M$ which implies $x \in M$ since $\mathfrak{M}(x) = \mathfrak{M}(y)$, hence $\bar{x} \in \mathfrak{n}$, so that $\mathfrak{n} \in \mathfrak{M}(\bar{x})$. Now, $\bar{y}^n \in \prod_\lambda J_\lambda$, and since $\prod_\lambda J_\lambda$ is a z^n-ideal, we have $\bar{x}^n \in \prod_\lambda J_\lambda$, which implies $x^n \in J_\lambda$ showing that J_λ is a z^n-ideal.

($\Leftarrow$): Suppose J_λ is a z^n-ideal in $\{A_\lambda \mid \lambda \in \Lambda\}$. Let $\bar{x} = x_\lambda$ and $\bar{y} = y_\lambda$ be elements of $\prod_\lambda A_\lambda$ such that $\mathfrak{M}(x_\lambda) = \mathfrak{M}(y_\lambda)$ and $y_\lambda^n \in \prod_\lambda J_\lambda$. We must show that $\mathfrak{M}(x_\lambda) \supseteq \mathfrak{M}(y_\lambda)$. Consider any $\mathfrak{n} \in \mathfrak{M}(y_\lambda)$. Then $\mathfrak{n}$ is a maximal ideal of $\prod_\lambda A_\lambda$ containing y_λ, and hence x_λ. Therefore $\mathfrak{n} \in \mathfrak{M}(x_\lambda)$. Since J_λ is a z^n-ideal and $y_\lambda^n \in J_\lambda$, it follows that $x_\lambda^n \in J_\lambda$. Hence $x_\lambda^n \in \prod_\lambda J_\lambda$, which proves that $\prod_\lambda J_\lambda$ is a z^n-ideal.

Suppose $\{A_\lambda \mid \lambda \in \Lambda\}$ are z-terminating. Pick $m \in \mathbb{N}$ such that termination for each A_λ occurs at or before stage m. We claim that

$$\mathfrak{Z}^m\left(\prod_\lambda A_\lambda\right) = \mathfrak{Z}^{m+1}\left(\prod_\lambda A_\lambda\right) = \cdots .$$

Let $\prod_\lambda J_\lambda \in \mathfrak{Z}^{m+1}(\prod_\lambda A_\lambda)$. Then, as proved above, $J_\lambda \in \mathfrak{Z}^{m+1}(A_\lambda)$ which implies $J_\lambda \in \mathfrak{Z}^m(A_\lambda)$ for each $\lambda \in \Lambda$. A simple induction argument shows that $\mathfrak{Z}^m\left(\prod_\lambda A_\lambda\right) = \mathfrak{Z}^{m+i}\left(\prod A_\lambda\right)$ for all i. Therefore $\prod_\lambda A_\lambda$ is z-terminating. $\square$

4.2 A tower of d-like ideals

We start by defining the ideals that will form the main study in this chapter.

Definition 4.2.1. Let n be a positive integer. An ideal I of a ring A is a d^n*-ideal* if for any $a, b \in A$, $\mathrm{Ann}(a) = \mathrm{Ann}(b)$ and $a^n \in I$ imply $b^n \in I$. Equivalently, I is a d^n-ideal if and only if for any $a, b \in A$, $\mathrm{Ann}^2(a) = \mathrm{Ann}^2(b)$ and $a^n \in I$ imply $b^n \in I$.

Let $\mathfrak{D}(A)$ denote the set of all d-ideals of A and $\mathfrak{D}^n(A)$ denote the set of all d^n-ideals of A.

It is clear that every d-ideal is a d^n-ideal for every $n \in \mathbb{N}$, so that, indeed, these ideals generalise d-ideal in a natural way. We show that, for every $n \in \mathbb{N}$, every d^n-ideal is a z^n-ideal in a ring with zero Jacobson radical. We start with the following lemma.

Lemma 4.2.2. *Let A be a ring with zero Jacobson radical. For any $a, b \in A$, $\mathfrak{M}(a) = \mathfrak{M}(b)$ implies* $\mathrm{Ann}(a) = \mathrm{Ann}(b)$*.*

Proof. Let $x \in \mathrm{Ann}(a)$. Then $ax = 0$. Suppose, by way of contradiction, that $bx \neq 0$. Then since $\mathrm{Jac}(A) = 0$, there is a maximal ideal M such that $bx \notin M$. Since M is prime, this implies $b \notin M$ and $x \notin M$. Since $ax = 0 \in M$, we therefore have $a \in M$, and have $b \in M$ as $\mathfrak{M}(a) = \mathfrak{M}(b)$. Thus we have reached a contradiction. $\square$

Corollary 4.2.3. *If A has zero Jacobson radical, then for any $n \in \mathbb{N}$, $\mathfrak{D}^n(A) \subseteq \mathfrak{Z}^n(A)$.*

Proof. Let $I \in \mathfrak{D}^n(A)$. Consider any x and y in A with $\mathfrak{M}(x) = \mathfrak{M}(y)$ and $y^n \in I$. Since A has zero Jacobson radical, then $\mathfrak{M}(x) = \mathfrak{M}(y)$ implies $\mathrm{Ann}(x) = \mathrm{Ann}(y)$. It follows that $x^n \in I$ since I is a d^n-ideal. Therefore $I \in \mathfrak{Z}^n(A)$. $\square$

Henceforth all rings are assumed to have zero Jacobson radical. In [29], Mason proves that a d-ideal consist entirely of zero-divisors. We show that this holds for higher order d-ideals.

Proposition 4.2.4. *For every $n \in \mathbb{N}$, any d^n-ideal consists entirely of zero-divisors.*

Proof. Let I be a d^n-ideal. Suppose I does not consist entirely of zero-divisors. Let $u \in I$ be a nonzero-divisor. Then u^n is a nonzero-divisor because if for $w \in A$, $wu^n = 0$, then $(wu)^n = 0$, implying $wu = 0$ since the ring is reduced. This then imply $w = 0$. Thus $\mathrm{Ann}(u) = \mathrm{Ann}(1)$, but $u^n \in I$; so $1 \in I$, contradicting that I is a proper ideal. $\square$

They can also be characterised similarly to d-ideals in terms of intersections of minimal prime ideals. Recall that $M^n(a) = \{x^n \mid x \in M(a)\}$ for every $n \in \mathbb{N}$ and $a \in A$. Let A be a ring, a be an element of A and $n \in \mathbb{N}$. We recall that

$$P(a) = \bigcap \{\, Q \in \mathrm{Min}(A) \mid a \in Q \,\}$$

and we define

$$P^n(a) = \{\, x^n \mid x \in P(a) \,\}.$$

It is clear that $P^n(a) \subseteq P(a)$ for every $n \in \mathbb{N}$.

Proposition 4.2.5. *Let A be a ring and n be a positive integer. The following are equivalent for an ideal I of a ring A.*

(1) I is a d^n-ideal.

(2) $P^n(a) \subseteq I$ *for every* $a^n \in I$.

(3) For $a, b \in A$, $P^n(a) = P^n(b)$ *and* $b^n \in I$ *imply that* $a^n \in I$.

(4) For $a, b \in A$, $V(a) = V(b)$ *and* $a^n \in I$ *imply that* $b^n \in I$.

(5) $a^n \in I$ *implies that* $\operatorname{Ann}^2(a) \subseteq I$.

Proof. (1) ⇒ (2): Assume that I is a d^n-ideal. Let $a^n \in I$. We need to show that $P^n(a) \subseteq I$. Let $x^n \in P^n(a)$. Since $P^n(a) \subseteq P(a) = \operatorname{Ann}^2(a)$, then

$$\operatorname{Ann}^2(x^n) \subseteq \operatorname{Ann}^2(a).$$

Since A is reduced, $\operatorname{Ann}^2(x^n) = \operatorname{Ann}^2(x)$. Therefore $\operatorname{Ann}^2(x) \subseteq \operatorname{Ann}^2(a)$ and $a^n \in I$ imply that $x^n \in I$ since I is a d^n-ideal. Since x^n is an arbitrary element of $P^n(a)$, it follows that $P^n(a) \subseteq I$.

(2) ⇒ (3): Let $P^n(a) \subseteq I$ for every $a^n \in I$. Consider any x and y in A with $P^n(x) = P^n(y)$ and $y^n \in I$. We must show that $x^n \in I$. We have

$$x^n \in P^n(x) = P^n(y).$$

But by hypothesis $P^n(y) \subseteq I$ for every $y^n \in I$. Therefore $x^n \in I$.

(3) ⇒ (4): Assume that (3) holds. Consider x and y in A with $V(x) = V(y)$ and $y^n \in I$. We must show that $x^n \in I$. We have

$$x^n \in P^n(x) \subseteq \bigcap V(x) = \bigcap V(y) \supseteq P^n(y)$$

and $y^n \in I$, so by the stated condition, $x^n \in I$.

(4) ⇒ (5): Assume that the stated condition holds and let $x^n \in I$. We must show that $\operatorname{Ann}^2(x) \subseteq I$. It is shown in [29, Lemma 1.3] that $V(x) = V(\operatorname{Ann}^2(x))$. We therefore have

$$V(x) = V(\operatorname{Ann}^2(x)) = V(\operatorname{Ann}^2(y)) = V(y) \quad \text{and} \quad x^n \in I,$$

so by the stated condition, $\operatorname{Ann}^2(x) \subseteq I$.

(5) ⇒ (1): Assume that $\operatorname{Ann}^2(a) \subseteq I$ for every $a^n \in I$. To show that I is a d^n-ideal, consider any x and y in A with $\operatorname{Ann}^2(x) = \operatorname{Ann}^2(y)$ and $y^n \in I$. We must show that $x^n \in I$. But now we have

$$x^n \in \operatorname{Ann}^2(x) = \operatorname{Ann}^2(y).$$

Now, by hypothesis $\operatorname{Ann}^2(y) \subseteq I$ for every $y \in I$. So $x^n \in I$. Therefore I is a d^n-ideal. □

The following lemma records some elementary observations regarding d^n-ideals. Following [6], we say I is a *$\sqrt{d}$-ideal* in case $\sqrt{I}$ is a d-ideal. We define the set

$$\mathfrak{D}^{\text{rad}}(A) = \{I \subseteq A \mid I \text{ is a } \sqrt{d}\text{-ideal}\}.$$

We know that every d-ideal is a radical ideal, so that

$$\text{Rad}(A) \cap \mathfrak{D}(A) = \mathfrak{D}(A).$$

We are going to use this result in the proof of the following theorem.

Lemma 4.2.6. *Let A be a reduced ring and n be a positive integer. Then we have the following.*

(1) $\mathfrak{D}^n(A) \subseteq \mathfrak{D}^{n+1}(A)$.

(2) $\mathfrak{D}^n(A) \subseteq \mathfrak{D}^{rad}(A)$.

(3) $\text{Rad}(A) \cap \mathfrak{D}^n(A) = \mathfrak{D}(A)$.

Proof. (1) Let $I \in \mathfrak{D}^n(A)$. Consider any x and y in A with $\text{Ann}(x) = \text{Ann}(y)$ and $y^{n+1} \in I$. Since $2n \geq n+1$, we have $(y^2)^n = y^{2n} \in I$. Putting $w = y^2$, we see that x and w are elements of A such that $\text{Ann}(x) = \text{Ann}(w)$ and $w^n \in I$. Since $I \in \mathfrak{D}^n(A)$, it follows that $x^n \in I$, hence $x^{n+1} \in I$. Therefore $I \in \mathfrak{D}^{n+1}(A)$, showing that $\mathfrak{D}^n(A) \subseteq \mathfrak{D}^{n+1}(A)$.

(2) Let $I \in \mathfrak{D}^n(A)$. Consider any x and y in A such that $\text{Ann}(x) = \text{Ann}(y)$ and $y \in \sqrt{I}$. Pick $m \in \mathbb{N}$ such that $y^m \in I$. Then $(y^m)^n \in I$. Since A is reduced, we have that $\text{Ann}(x) = \text{Ann}(y^m)$ and $I \in \mathfrak{D}^n(A)$, it follows that $x^n \in I$. But this implies $x \in \sqrt{I}$; so $\sqrt{I}$ is a d-ideal. Thus, $I \in \mathfrak{D}^{\text{rad}}(A)$, which establishes the desired inclusion.

(3) Since every d-ideal is a radical ideal, then $\mathfrak{D}(A) \subseteq \text{Rad}(A) \cap \mathfrak{D}^n(A)$. It suffices to show that $\text{Rad}(A) \cap \mathfrak{D}^n(A) \subseteq \mathfrak{D}(A)$. So let I be a radical d^n-ideal. Consider any x and y in A such that $\text{Ann}(x) = \text{Ann}(y)$ and $y \in I$. Then $y^n \in I$, which implies $x^n \in I$ since I is a d^n-ideal, by hypothesis. But I is also a radical ideal, so $x \in I$, which shows that I is a d-ideal. □

We observed that $\mathfrak{D}^n(A) \subseteq \mathfrak{D}^{n+1}(A)$ in Lemma 4.2.6 above. We thus have an ascending chain

$$\mathfrak{D}(A) \subseteq \mathfrak{D}^2(A) \subseteq \mathfrak{D}^3(A) \subseteq \cdots \subseteq \mathfrak{D}^n(A) \subseteq \mathfrak{D}^{n+1}(A) \subseteq \cdots$$

of collections of ideals of A. We call it the *d-tower* of A. For brevity, we write

$$\mathfrak{D}^{\infty}(A) = \bigcup_{n=1}^{\infty} \mathfrak{D}^{n}(A),$$

and observe that $\mathfrak{D}^{\infty}(A) \subseteq \mathfrak{D}^{\mathrm{rad}}(A)$. We say an ideal of A is a *higher order d-ideal* if it belongs to $\mathfrak{D}^{\infty}(A)$. If there is a positive integer k such that

$$\mathfrak{D}^{k}(A) = \mathfrak{D}^{k+1}(A) = \mathfrak{D}^{k+2}(A) = \cdots ,$$

we say the d-tower *terminates.*

Mason [29, Theorem 2.5] proves that the prime ideals minimal with respect to containing a d-ideal I are themselves d-ideals. If P is a prime ideal minimal with respect to containing I is a d-ideal, then P minimal with respect to containing $\sqrt{I}$ is a d-ideal. Combining this with Mason's result, cited above, we deduce from Lemma 4.2.6 the following corollary.

Corollary 4.2.7. *A prime ideal minimal with respect to containing a higher order d-ideal is a d-ideal.*

We name the rings with terminating d-towers.

Definition 4.2.8. A ring A is *d-terminating* in case its d-tower terminates.

Before we give examples of d-terminating rings, recall that an f-ring A is 1-convex if for any $u, v \in A$ such that $0 \leq u \leq v$, there is a $w \in A$ such that $u = vw$. We also recall that a reduced f-ring has *square roots* if for every $u \geq 0$ there exists a (necessarily unique) $v \geq 0$ such that $v^2 = u$. In this case we write $u^{\frac{1}{2}}$. For any positive integer k, we denote $(u^{\frac{1}{2}})^k$ by $u^{\frac{1}{2^k}}$.

Examples 4.2.9. (1) Every von Neumann regular ring is d-terminating. By Remark 3.1.4 every ideal in von Neuman regular ring is an intersection of minimal prime ideals, that is, it is a d-ideal.

(2) The ring of integers is d-terminating, although (as shown in [11, Example 5]) it is not z-terminating. We will show, in fact that, for any positive integer n, the zero ideal is the only proper d^n-ideal of $\mathbb{Z}$. Recall that every ideal of $\mathbb{Z}$ is principal. Also, the generators can be taken to be non-negative integers.

Now suppose that for some positive integer k the ideal $\langle k\rangle$ is a d^n-ideal. Since $\mathrm{Ann}(k) = \mathrm{Ann}(k+1)$(they equal the zero ideal) and since $k^n \in \langle k\rangle$, we should then have $(k+1)^n \in \langle k\rangle$. Consequently, there is a non-negative integer l such that $(k+1)^n = kl$. Thus,

$$\begin{aligned} kl = (k+1)^n &= \sum_{r=0}^{n} \binom{n}{r} k^{n-r} \cdot 1^r \\ &= k^n + \binom{n}{1} k^{n-1} + \cdots + \binom{n}{n-1} k + 1 \\ &= ks + 1, \end{aligned}$$

for some positive integer s. This implies $k(l-s) = 1$, and since $l-s$ is an integer, this make $k = 1$, so that $\langle k\rangle = \mathbb{Z}$. Hence $\mathbb{Z}$ has no proper d^n-ideal except for the zero ideal. So the d-tower terminates right at the base.

(3) If a reduced 1-convex f-ring has square roots, then it is d-terminating. We show that in such an f-ring every higher order d-ideal is a d-ideal. Let n be a positive integer, and let I be a d^n-ideal in an f-ring A of the stated kind. Consider any $a, b \in A$ such that $\mathrm{Ann}(a) = \mathrm{Ann}(b)$ and $b \in I$. Choose $k \in \mathbb{N}$ such that $2^k \geq n$. Since A has square roots, $|a|^{\frac{1}{2^k}}$ exists in A, and, furthermore, $\mathrm{Ann}\left(|a|^{\frac{1}{2^k}}\right) = \mathrm{Ann}(b)$. Since $b^{2^k} \in I$, and I is a d^{2^k}-ideal, by Lemma 4.2.6, it follows that $|a| = \left(|a|^{\frac{1}{2^k}}\right)^{2^k} \in I$. Since I is an ℓ-ideal, this implies $a \in I$, showing that I is a d-ideal. Consequently, $\mathfrak{D}^n(A) = \mathfrak{D}(A)$ for every n, and hence A is d-terminating.

Recall that *Noetherian ring* is a ring in which every ideal is finitely generated. Using the ascending chain conditions and Lemma 4.2.6, we observed that $\mathfrak{D}^\infty(A) \subseteq \mathfrak{D}^{\mathrm{rad}}(A)$. We shall see that the equality $\mathfrak{D}^\infty(A) = \mathfrak{D}^{\mathrm{rad}}(A)$ is not enough for d-termination. We shall actually exhibit a large class of rings for which the stated equality holds, as we show next.

Definition 4.2.10. A ring A is *radically d-covered* if $\mathfrak{D}^\infty(A) = \mathfrak{D}^{\mathrm{rad}}(A)$.

A ring A is radically d-covered precisely when every $\sqrt{d}$-ideal in A is a higher order d-ideal. Examples of radically d-covered rings include Noetherian rings, as the following theorem shows.

Theorem 4.2.11. *Noetherian rings are radically d-covered.*

Proof. Let A be a Noetherian ring. We need to show that $\mathfrak{D}^{\mathrm{rad}}(A) \subseteq \mathfrak{D}^\infty(A)$. So let $I \in \mathfrak{D}^{\mathrm{rad}}(A)$. Then $\sqrt{I}$ is a d-ideal. Since A is Noetherian, there exist finitely many elements $a_1, \ldots, a_n$ in

A such that $\sqrt{I}$ is generated by the set $\{a_1, \ldots, a_n\}$. Choose positive integers $k_1, \ldots, k_n$ such that $a_i^{k_i} \in I$ for each $i = 1, \ldots, n$. Put $k = k_1 + \cdots + k_n$. We claim that $I \in \mathfrak{D}^k(A)$. To show this, consider any $x, y \in A$ with $\mathrm{Ann}(x) = \mathrm{Ann}(y)$ and $x^k \in I$. Then $x \in \sqrt{I}$, and since $\sqrt{I}$ is a d-ideal, we deduce that $y \in \sqrt{I}$. Since $\sqrt{I}$ is generated by the elements $a_1, \ldots, a_n$, there exist elements $u_1, \ldots u_n$ in A such that $y = u_1a_1 + \cdots + u_na_n$. Note that $(u_ia_i)^{k_i} \in I$, for each $i = 1, \ldots, n$. We prove by induction on n to show that $y^k \in I$. The result is trivial for $n = 1$. So assume $n > 1$. For brevity, we write $b_i = u_ia_i$, for $i = 1, 2, \ldots, n$. Then

$$\begin{aligned}(b_1 + \cdots + b_n)^k &= \big(b_1 + (b_2 + \cdots + b_n)\big)^k \\ &= \sum_{r=0}^{k} \binom{k}{r} b_1^r (b_2 + \cdots + b_n)^{k-r} \\ &= \sum_{r<k_1} \binom{k}{r} b_1^r (b_2 + \cdots + b_n)^{k-r} + \sum_{r\geq k_1} \binom{k}{r} b_1^r (b_2 + \cdots + b_n)^{k-r}\end{aligned}$$

The second summand is in I since $b^{k_1} \in I$. By the induction hypothesis, each term of the form $(b_2 + \cdots + b_n)^{k-r}$ is in I if $r < k_1$ because then $k - r \geq k_2 + \cdots + k_n$. It follows therefore that $(b_1 + \cdots + b_n)^{k_1+\cdots+k_n} \in I$. Thus, $y^k \in I$ since $k \geq n$. Therefore $I \in \mathfrak{D}^k(A)$. Consequently, $\mathfrak{D}^{\mathrm{rad}}(A) \subseteq \mathfrak{D}^\infty(A)$, and hence equality. □

4.3 Direct products and d-termination

In this section we examine the preservation and reflection of the d-terminating property by direct products, and by homomorphic images. In the proof of the following results we are going to use the fact that, for any $x = x_\lambda \in \prod_\lambda A_\lambda$, we have $\prod \mathrm{Ann}(x_\lambda) = \mathrm{Ann}(x)$ [36, Lemma 4.3.5].

Lemma 4.3.1. *Suppose $\{A_\lambda \mid \lambda \in \Lambda\}$ are d-terminating, and suppose there exist $m \in \mathbb{N}$ such that termination for each A_λ occurs at or before stage m. Then $\prod_\lambda A_\lambda$ is d-terminating.*

Proof. To start with, we will first show that if $\{A_\lambda \mid \lambda \in \Lambda\}$ is a family of commutative rings with identity, and $n \in \mathbb{N}$, then for each λ, if I_λ be an ideal of A_λ, then

$$\prod_\lambda I_\lambda \text{ is a } d^n\text{-ideal if and only if each } I_\lambda \text{ is a } d^n\text{-ideal.}$$

($\Rightarrow$): Suppose $\prod_\lambda I_\lambda$ is a d^n-ideal in $\prod_\lambda A_\lambda$. Fix an index $\ell \in \Lambda$ and consider $x, y \in I_\ell$ such that $\mathrm{Ann}(x) = \mathrm{Ann}(y)$ and $y^n \in I_\lambda$. Consider any $x = x_\lambda$ and $y = y_\lambda$ in $\prod_\lambda A_\lambda$ such that

$\mathrm{Ann}(x) = \mathrm{Ann}(y)$ and $y^n \in I_\lambda$. Then using [36, Lemma 4.3.5], $\mathrm{Ann}(x) = \mathrm{Ann}(y)$ and $y^n \in I_\lambda$ if and only if $\prod\limits_\lambda \mathrm{Ann}(x_\lambda) = \prod\limits_\lambda \mathrm{Ann}(y_\lambda)$ and $y_\lambda^n \in \prod\limits_\lambda I_\lambda$. By hypothesis $\prod\limits_\lambda I_\lambda$ is a d^n-ideal implying $x_\lambda^n \in \prod\limits_\lambda I_\lambda$. Hence $x^n \in I_\lambda$ which proves that I_λ is a d^n-ideal.

($\Leftarrow$): Suppose I_λ is a d^n-ideal in $\{A_\lambda \mid \lambda \in \Lambda\}$. Let $\bar{x} = x_\lambda$ and $\bar{y} = y_\lambda$ be elements of $\prod\limits_\lambda A_\lambda$ such that $\mathrm{Ann}(x_\lambda) = \mathrm{Ann}(y_\lambda)$ and $y_\lambda^n \in \prod\limits_\lambda I_\lambda$. We must show that $x_\lambda \in \prod\limits_\lambda I_\lambda$. Since $\mathrm{Ann}(\bar{x}) = \mathrm{Ann}(\bar{y})$ and $y_\lambda^n \in I_\lambda$ imply $x_\lambda^n \in I_\lambda$ by hypothesis and $y_\lambda^n \in \prod\limits_\lambda I_\lambda$, it follows that $x_\lambda^n \in \prod\limits_\lambda I_\lambda$. Therefore $\prod\limits_\lambda I_\lambda$ is a d^n-ideal.

Suppose $\{A_\lambda \,|\, \lambda \in \Lambda\}$ are d-terminating. Pick $m \in \mathbb{N}$ such that termination for each A_λ occurs at or before stage m. We claim that

$$\mathfrak{D}^m\left(\prod_\lambda A_\lambda\right) = \mathfrak{D}^{m+1}\left(\prod_\lambda A_\lambda\right) = \cdots .$$

Let $\prod\limits_\lambda I_\lambda \in \mathfrak{D}^{m+1}(\prod\limits_\lambda A_\lambda)$. Then, as proved above, $I_\lambda \in \mathfrak{D}^{m+1}(A_\lambda)$ which implies $I_\lambda \in \mathfrak{D}^m(A_\lambda)$ for each λ. A simple induction argument shows that $\mathfrak{D}^m\left(\prod\limits_\lambda A_\lambda\right) = \mathfrak{D}^{m+i}\left(\prod\limits_\lambda A_\lambda\right)$ for all i. Therefore $\prod\limits_\lambda A_\lambda$ is d-terminating. $\square$

We have established in the foregoing proof that if A and B are rings, and $m \in \mathbb{N}$, then

$$\mathfrak{D}^m(A \times B) = \mathfrak{D}^m(A) \times \mathfrak{D}^m(B).$$

A moment's reflection, taking into account Lemma 4.2.6 and the fact that, for any ideal I of A and any ideal J of B, $\sqrt{I \times J} = \sqrt{I} \times \sqrt{J}$, shows that

$$\mathfrak{D}^\infty(A \times B) = \mathfrak{D}^\infty(A) \times \mathfrak{D}^\infty(B) \quad \text{ and } \quad \mathfrak{D}^{\mathrm{rad}}(A \times B) = \mathfrak{D}^{\mathrm{rad}}(A) \times \mathfrak{D}^{\mathrm{rad}}(B).$$

These relations yield the following result.

Proposition 4.3.2. *The direct product of finitely many rings is radically d-covered if and only if each factor is radically d-covered.*

Proof. Assume that A and B are radically d-covered. Then

$$\begin{aligned} \mathfrak{D}^\infty(A \times B) &= \mathfrak{D}^\infty(A) \times \mathfrak{D}^\infty(B) \\ &= \mathfrak{D}^{\mathrm{rad}}(A) \times \mathfrak{D}^{\mathrm{rad}}(B) = \mathfrak{D}^{\mathrm{rad}}(A \times B). \end{aligned}$$

Therefore $A \times B$ is radically d-covered. Conversely, assume that $A \times B$ is radically d-covered. Then

$$\begin{aligned} \mathfrak{D}^{\infty}(A) \times \mathfrak{D}^{\infty}(B) &= \mathfrak{D}^{\infty}(A \times B) \\ &= \mathfrak{D}^{\mathrm{rad}}(A \times B) = \mathfrak{D}^{\mathrm{rad}}(A) \times \mathfrak{D}^{\mathrm{rad}}(B). \end{aligned}$$

Since none of these sets is empty, it follows that $\mathfrak{D}^{\infty}(A) = \mathfrak{D}^{\mathrm{rad}}(A)$ and $\mathfrak{D}^{\infty}(B) = \mathfrak{D}^{\mathrm{rad}}(B)$; which says A and B are radically d-covered. □

4.4 Homomorphic images

We show next a class of ring homomorphisms that map d-terminating (respectively, radically d-covered) rings onto rings with the same features. We give such homomorphisms the following name.

Definition 4.4.1. A ring homomorphism $\phi\colon A \to B$ is *strong* if it is surjective and for every minimal prime ideal P of A, there is a minimal prime ideal Q of B such that $\phi^{-1}[Q] = P$.

It is clear that an isomorphism is a strong homomorphism. In the following example we show that there are strong homomorphisms which are not isomorphisms.

Example 4.4.2. Let A be a ring which is not an integral domain, and which has exactly one minimal prime ideal. For instance, $\mathbb{Z}_4$ is such a ring. Denote by P the sole minimal prime ideal of A. The canonical map $\eta\colon A \to A/P$ is surjective, but it is not an isomorphism since A is not an integral domain and A/P is an integral domain as P is a prime ideal. Now, the zero ideal $\{0_{A/P}\}$ of A/P is a minimal prime ideal, and $\eta^{-1}(0_{A/P}) = \ker(\eta) = P$; which shows that η is strong.

In [11], Dube and Ighedo define a ring homomorphism to be *strict* if it is surjective and it contracts maximal ideals to maximal ideals. This they do in order to obtain homomorphisms that map z-terminating rings onto z-terminating rings. Since our aim in this section is to do likewise, but for d-termination, it is natural to compare strong homomorphisms with surjective ring homomorphisms that contract minimal prime ideals to minimal prime ideals. We show that the class of the latter kind of rings contains strictly the class of strong homomorphisms.

Proposition 4.4.3. *Every strong homomorphism contracts minimal prime ideals to minimal prime ideals.*

Proof. Let $Q \in \mathrm{Min}(B)$. Then $\phi^{-1}[Q]$ is a prime ideal in A, and hence contains some minimal prime P. By strongness, there exists $R \in \mathrm{Min}(B)$ such that $P = \phi^{-1}[R]$. We show that $R \subseteq Q$, which will imply $R = Q$ by minimality. Let $r \in R$. Since ϕ is onto, there exists $a \in A$ such that $\phi(a) = r$. Then $a \in \phi^{-1}[R] = P \subseteq \phi^{-1}[Q]$. Therefore $\phi(a) \in Q$, that is, $r \in Q$, hence $R \subseteq Q$, where $R = Q$. Therefore $\phi^{-1}[Q] = P$, which is minimal prime. □

Here is an example of a surjective ring homomorphism that contracts minimal prime ideals to minimal prime ideals, which is however not strong.

Example 4.4.4. Let A be a von Neumann ring with two maximal ideals. For instance, if X is a discrete space with two points, then the ring $C(X)$ is von Neumann regular and it has two maximal ideals. Say the maximal ideals of A are M and N. Since A is von Neumann regular, M and N are actually minimal prime ideals. The canonical map $\eta\colon A \to A/M$ is surjective, and since A/M is a field, it has only one prime ideal; its zero ideal, whose contraction under η is M. Thus η contracts minimal prime ideals to minimal prime ideals. Since $N \neq M$, there is no minimal prime ideal whose contraction is N, so η is not strong.

We have some further observations regarding strong homomorphisms, which we now present.

Observations 4.4.5. When we say a homomorphism $\phi\colon A \to B$ takes minimal primes to minimal primes we mean that $\phi[P] \in \mathrm{Min}(B)$ for every $P \in \mathrm{Min}(A)$. Recall that a *local ring* is a ring with exactly one maximal ideal.

(a) A strong homomorphism takes minimal primes to minimal primes.

(b) A surjective homomorphism that takes minimal primes to minimal primes need not be strong.

Proof. (a) If $\phi\colon A \to B$ is a strong homomorphism and $P \in \mathrm{Min}(A)$, then there exists some $Q \in \mathrm{Min}(B)$ such that $P = \phi^{-1}[Q]$. We show that $\phi[P] = Q$. Let $q \in \phi[P]$. There exists $p \in A$ such that $\phi(p) = q$. Then $p \in P = \phi^{-1}[Q]$. Therefore $\phi(p) \in Q$, that is, $q \in Q$, hence $\phi[P] \subseteq Q$.

Conversely, let $q \in Q$. Since ϕ is surjective, there exists $p \in A$ such that $\phi(p) = q$. Then $p \in \phi^{-1}[Q] = P$. Therefore $\phi(p) \in \phi[P]$, that is, $q \in \phi[P]$, hence $Q \subseteq \phi[P]$. Thus $\phi[P] = Q$.

(b) Let A be a local ring which is not von Neumann regular, and denote by M its sole maximal ideal. Let P be any minimal prime ideal of A. Consider the canonical map $\eta\colon A \to A/M$. Since $P \subseteq M$, we have

$$\eta[P] = \{\eta(x) \mid x \in P\} = \{x + M \mid x \in P\} = \{M\} = \{0_{A/P}\},$$

which shows that η takes minimal primes to minimal primes since $\{0_{A/P}\}$ is a minimal prime ideal in A/M. We claim that η is not strong. Indeed, since A is not von Neumann regular, M is not a minimal prime ideal. It however contains a minimal prime ideal, Q, say. Thus,

$$\eta^{-1}[\{0_{A/M}\}] = \ker \eta = M \neq Q,$$

which shows that there is no minimal prime ideal of A/M whose contraction is Q. Therefore η is not strong. □

Recall from Lemma 3.2.1 that a ring homomorphism $\phi\colon A \to B$ contracts d-ideals to d-ideals if and only if it contracts minimal prime ideals to d-ideals. An almost verbatim arqument as employed in Lemma 3.2.1 to prove this establishes the following lemma.

Lemma 4.4.6. *Let $\phi\colon A \to B$ be a strong ring homomorphism, and let n be a positive integer. Then the following statements are equivalent.*

1. $\phi^{-1}[I] \in \mathfrak{D}^n(A)$ *for every* $I \in \mathfrak{D}^n(B)$.
2. $\phi^{-1}[P] \in \mathfrak{D}^n(A)$ *for every* $P \in \mathrm{Min}(B)$.
3. $\phi^{-1}[P] \in \mathfrak{D}(A)$ *for every* $P \in \mathrm{Min}(B)$.
4. $\phi^{-1}[I] \in \mathfrak{D}(A)$ *for every* $I \in \mathfrak{D}(B)$.

Proof. (1) ⇒ (2): This is so because every minimal prime ideal is a d-ideal, and therefore a d^n-ideal, by (1) of Lemma 4.2.6.

(2) ⇔ (3): This follows from the fact that $\phi^{-1}[P]$ is a minimal prime ideal for every minimal prime ideal P, and prime ideals are d^n-ideals precisely when they are d-ideals, by the third part of Lemma 4.2.6.

(3) ⇔ (4): See Lemma 3.2.1 for the proof.

(2) ⇒ (1): Exactly as in the proof of Lemma 3.2.1. □

We now give the following result.

Proposition 4.4.7. *Let $\phi\colon A \to B$ be a strong homomorphism.*

(a) If A is d-terminating, then so is B.

(b) If A is radically d-covered, then so is B.

Proof. (a) Let n be a positive integer such that $\mathfrak{D}^n(A) = \mathfrak{D}^{n+i}(A)$ for every $i = 1, 2, \ldots$. We aim to show that the same n works for B as well. Fix $i \in \mathbb{N}$, and let $I \in \mathfrak{D}^{n+i}(B)$. Since ϕ is strong, $\phi^{-1}[I] \in \mathfrak{D}^{n+i}(A)$, by Lemma 4.4.6, and hence $\phi^{-1}[I] \in \mathfrak{D}^n(A)$. Consider any $b_1, b_2 \in B$ such that $\operatorname{Ann}(b_1) = \operatorname{Ann}(b_2)$ and $b_2^n \in I$. Pick $a_1, a_2 \in A$ with $\phi a_1 = b_1$ and $\phi a_2 = b_2$. Then $\operatorname{Ann}(\phi a_1) = \operatorname{Ann}(\phi a_2)$ for every $\operatorname{Ann}(a_1) = \operatorname{Ann}(a_2)$, by Observation 4.4.5, since ϕ is strong. Now, $a_2^n \in \phi^{-1}[I]$ implies $a_1^n \in \phi^{-1}[I]$ since $\phi^{-1}[I]$ is a d^n-ideal. Consequently, $b_1^n = \phi(a_1)^n = \phi(a_1^n) \in I$. This shows that I is a d^n-ideal, and therefore $\mathfrak{D}^n(B) = \mathfrak{D}^{n+i}(B)$. Thus, B is d-terminating.

(b) Let $I \in \mathfrak{D}^{\mathrm{rad}}(B)$. Then $\sqrt{I}$ is a d-ideal, and therefore $\phi^{-1}\sqrt{I}$ is a d-ideal in A. But $\phi^{-1}\sqrt{I} = \sqrt{\phi^{-1}[I]}$; so, by hypothesis, there exists a positive integer n such that $\phi^{-1}[I] \in \mathfrak{D}^n(A)$. Exactly as above, this implies $I \in \mathfrak{D}^n(B)$, which shows that $\mathfrak{D}^{\mathrm{rad}}(B) \subseteq \mathfrak{D}^\infty(B)$, and hence the desired equality. □

Corollary 4.4.8. *If $A[\![x]\!]$ is d-terminating (respectively, radically d-covered) then A has the same property.*

Corollary 4.4.9. *If* $\operatorname{Jac}(A)$ *is contained in every higher order d-ideal of A, then A is d-terminating (respectively, radically d-covered) if and only if $A/\operatorname{Jac}(A)$ has the same property.*

Proof. For brevity, we write $J = \operatorname{Jac}(A)$. The left-to-right implication always holds since, as observed above, the canonical mapping $A \to A/J$ is strong.

Conversely, assume that A/J is d-terminating. Then there exists a positive integer n such that $\mathfrak{D}^n(A/J) = \mathfrak{D}^{n+i}(A/J)$ for every $i = 1, 2, \ldots$. We shall show that

$$\mathfrak{D}^n(A) = \mathfrak{D}^{n+1}(A) = \mathfrak{D}^{n+2}(A) = \cdots .$$

Fix $i \in \mathbb{N}$, and let $I \in \mathfrak{D}^{n+i}(A)$. By hypothesis, $J \subseteq I$, and therefore I/J is an ideal of A/J. Since ϕ is strong, $\phi^{-1}[I] \in \mathfrak{D}^{n+i}(A)$, by Lemma 4.4.6, and hence $\phi^{-1}[I] \in \mathfrak{D}^{n}(A)$. Consider any $y_1, y_2 \in A/J$ such that $\mathrm{Ann}(y_1) = \mathrm{Ann}(y_2)$ and $y_2^n \in I/J$. Pick $x_1, x_2 \in A$ with $\phi x_1 = y_1$ and $\phi x_2 = y_2$. Then $\mathrm{Ann}(\phi x_1) = \mathrm{Ann}(\phi x_2)$ for every $\mathrm{Ann}(x_1) = \mathrm{Ann}(x_2)$, by Observation 4.4.5, since ϕ is strong. Now, $x_2^n \in \phi^{-1}[I]$ implies $x_1^n \in \phi^{-1}[I]$ since $\phi^{-1}[I]$ is a d^n-ideal. Consequently, $y_1^n = \phi(x_1)^n = \phi(x_1^n) \in I/J$. Hence $I/J \in \mathfrak{D}^n(A/J)$. By Lemma 4.4.6, $I \in \mathfrak{D}^n(A)$ since I is the inverse image of I/J under the (strong) canonical map $A \to A/J$. Therefore A is d-terminating.

The proof for the result in parenthesis is similar. □

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