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On one Dubinin extremal problem

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Let $\mathbb{N}$, $\mathbb{R}$ be a sets of natural and real numbers, respectively, $\mathbb{C}$ be the complex plane, $\overline{\mathbb{C}}=\mathbb{C}\bigcup\{\infty\}$ be a one point compactification. Let B be a domain in $\mathbb{C}$, $a\in B$ be a point in B and $r(B,a)$ be a inner radius of the domain $B\subset\overline{\mathbb{C}}$ with respect to the point $a\in B$. Inner radius is a generalization of conformal radius for multiply connected domains. Consider an extremal problem which was formulated in 1994 in the paper of Dubinin in the journal "Russian Mathematical Surveys" in the list of unsolved problems (see, for example, [1]).

Problem 1. Consider the product $I_n(\gamma)=r^{\gamma}\left(B_0,0\right)\prod\limits_{k=1}^{n}r\left(B_k,a_k\right)$, where B_0, B_1,..., B_n ($n\geqslant 2$) are pairwise disjoint domains in $\overline{\mathbb{C}}$, $a_0=0$, $|a_k|=1$, $k=\overline{1,n}$ and $0<\gamma\leqslant n$. Show that it attains its maximum at a configuration of domains B_k and points a_k possessing rotational n-symmetry. This problem has a solution only if $\gamma\leqslant n$ as soon as $\gamma=n+\varepsilon$, $\varepsilon>0$, the Problem 1 has no solution. Currently it is not solved in general only special results are known. Let

$$I_n^0(\gamma)=r^{\gamma}\left(D_0,0\right)\prod_{k=1}^{n}r\left(D_k,d_k\right), \tag{1}$$

where d_k and D_k, $k=\overline{0,n}$, are, respectively, poles and circular domains of the quadratic differential

$$G(w)dw^2=-\frac{(n^2-\gamma)w^n+\gamma}{w^2(w^n-1)^2}dw^2.$$

Denote

$$Q_n(\gamma)=\frac{\left[2^n\frac{2}{\sqrt{\gamma}}\left(2-\frac{2}{\sqrt{\gamma}}\right)^{n-1}(n-1)^{-(n-1)}\right]^{1-\frac{\gamma}{n}}}{\left(\frac{4}{n}\right)^n\frac{\left(\frac{4\gamma}{n^2}\right)^{\frac{\gamma}{n}}}{\left(1-\frac{\gamma}{n^2}\right)^{n+\frac{\gamma}{n}}}\left(\frac{1-\frac{\sqrt{\gamma}}{n}}{1+\frac{\sqrt{\gamma}}{n}}\right)^{2\sqrt{\gamma}}}. \tag{2}$$

Theorem 1 (1). *Let $n\in\mathbb{N}$, $n\geqslant 6$ be a fixed natural number and γ, $\gamma\geqslant 1$ be a real number. Then for any configuration of domains B_k and points a_k ($k=\overline{0,n}$) satisfying a condition of Problem 1 and also provided that $\alpha_0>\frac{2}{\sqrt{\gamma}}$, the following sharp estimate holds*

$$\frac{r^{\gamma}(B_0,a_0)\prod\limits_{k=1}^{n}r(B_k,a_k)}{I_n^0(\gamma)}\leqslant Q_n(\gamma),$$

where $I_n^0(\gamma)$ and $Q_n(\gamma)$ are defined by the relations (1) and (2). If γ_n^0 be a root of the equation $Q_n(\gamma)=1$ then for an arbitrary γ_n such that $1\leqslant\gamma_n<\gamma_n^0$ the following inequality holds

$$\frac{I_n(\gamma_n)}{I_n^0(\gamma_n)}<1.$$

Note that using Theorem 1 we can solve the Problem 1 for $n\geqslant 6$, $\gamma=\gamma_n^0$, and indicate its solution for an arbitrary γ_n such that $1<\gamma_n<\gamma_n^0$.

1. Denega I. V., Zabolotnii Ya. V. Estimates of products of inner radii of non-overlapping domains in the complex plane. Complex Variables and Elliptic Equations, Feb 2017, http://www.tandfonline.com/eprint/ACymqAMAceewIG54tcCJ/full.