Nonlinear Projection-Based Approach for Generating Compact Models of Nonlinear Thermal Networks

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Abstract

A nonlinear projection-base approach for generating compact models of nonlinear thermal networks is proposed. This approach is an extension of Galerkin's method, based on the theory of kernels. High accuracy for large temperature variations and high compactness of the generated models can be obtained.

Keywords: Nonlinear Heat Diffusion Equation, Thermal Networks, Compact Models

1 Introduction

Different methods have been proposed in literature for generating compact models of thermal networks when thermal conductivity and volumetric heat capacity are independent on temperature [1]. However as it has been shown in [2], for heat diffusion in electronics component and packages, the temperature dependency of thermal conductivity and volumetric heat capacity can be neglected only for small temperature variations. For large temperature variations instead the temperature dependence of thermal conductivity and volumetric heat capacity has to be taken into account. In this case the question of generating compact models of nonlinear thermal networks strongly arises. This is still an open question. The most common approach [3] is that of transforming the nonlinear heat diffusion equation into a linear heat diffusion equation by means of a Kirchhoff's transformation. In this manner the problem of generating a compact model of a nonlinear thermal network is reduced to the problem of generating a compact model of a linear thermal network. However this approach is rigorous only for very particular temperature dependences of thermal conductivity and volumetric heat capacity [4]. In the general case this approach is empirical and, as shown in this paper, usually introduces inaccuracies which cannot be controlled. In this case a different approach is needed.

In this paper a nonlinear projection-base approach for generating compact models of nonlinear thermal networks is proposed. This approach is an extension of Galerkin's method, based on the theory of kernels [5]. As shown in numerical results, high accuracy for large temperature variations and high compactness of the generated compact models can be obtained.

The rest of this paper is organized as follows. In section 2 and 3 nonlinear thermal networks are introduced. Compact models by Kirchhoff's transformation are discussed in section 4. Galerkin's method and the nonlinear projection approach for generating compact models are presented in sections 5, 6. Numerical results are shown in section 7.

2 Nonlinear Thermal Networks

The relation between the generated power density $F(\mathbf{r}, t)$ and the temperature *rise* $u(\mathbf{r}, t)$ with respect to a fixed ambient temperature, in a spatial region Ω, is ruled by the heat diffusion equation

$$c(\mathbf{r}, u(\mathbf{r}, t))\frac{\partial u}{\partial t}(\mathbf{r}, t) + \\ + \nabla \cdot (-k(\mathbf{r}, u(\mathbf{r}, t))\nabla u(\mathbf{r}, t)) = F(\mathbf{r}, t) \quad (1)$$

in which $c(\mathbf{r}, u(\mathbf{r}, t))$ is the volumetric heat capacity, $k(\mathbf{r}, u(\mathbf{r}, t))$ is the thermal conductivity and $F(\mathbf{r}, t)$ is the generated power density. Eq. (1) is completed by the conditions on the boundary $\partial\Omega$ of Ω and by the conditions at the initial time instant for the temperature rise $u(\mathbf{r}, t)$. Boundary conditions are assumed of Robin's type

$$-k(\mathbf{r}, u(\mathbf{r}, t))\frac{\partial u}{\partial \nu}(\mathbf{r}, t) = h(\mathbf{r}, u(\mathbf{r}, t))u(\mathbf{r}, t), \quad (2)$$

$h(\mathbf{r}, u(\mathbf{r}, t))$ being the heat transfer coefficient and $\boldsymbol{\nu}(\mathbf{r})$ the outward unit vector normal to $\partial\Omega$. The initial conditions for the temperature rise $u(\mathbf{r}, t)$ are assumed to be zero.

As shown in [2], typical expressions of volumetric heat capacity and thermal conductivity have the form

$$\beta\, e^{\alpha\, u(\mathbf{r}, t)} \quad (3)$$

with proper choices of parameters α and β. For small values of the generated power densities $G(\mathbf{r}, t)$, the dependence on temperature of volumetric heat capacity, thermal conductivity and heat exchange coefficient can be neglected and the heat diffusion problem becomes linear. Otherwise the heat diffusion problem is *nonlinear*.

As shown in [7], linear heat diffusion problems satisfy the positivity, reciprocity and passivity properties. Nonlinear heat diffusion problems do not satisfy the reciprocity property, but satisfy the *positivity* and *passivity* properties.

Property 1 (Positivity) *For non-negative power densities $F(\mathbf{r}, t)$, the temperature rises $u(\mathbf{r}, t)$ are non-negative.*

Property 2 (Passivity) *A non-negative function $W(t)$ exists such that for each time $t_1 \leq t_2$*

$$W(t_2) \leq W(t_1) + \int_{t_1}^{t_2} dt \int_{\Omega} F(\mathbf{r}, t)u(\mathbf{r}, t)\, d\mathbf{r}. \quad (4)$$

A nonlinear thermal network can be defined from the nonlinear heat diffusion equations by introducing the powers and the temperature rises measured at its ports, as with a linear thermal network. The port powers $P_i(t)$, with $i = 1, \ldots, n$, elements of column vector $\mathbf{P}(t)$, determine $F(\mathbf{r}, t)$ as

$$F(\mathbf{r}, t) = \mathbf{f}^T(\mathbf{r})\mathbf{P}(t). \quad (5)$$

in which $\mathbf{f}(\mathbf{r})$ is a column vector of shape functions $f_i(\mathbf{r})$, with $i = 1, \ldots, n$. The port temperature rises $T_i(t)$, with $i = 1, \ldots, n$, elements of vector $\mathbf{T}(t)$, are defined by

$$\mathbf{T}(t) = \int_\sigma \mathbf{g}(\mathbf{r}) T(\mathbf{r}, t)\, d\mathbf{r}. \tag{6}$$

in which $\mathbf{g}(\mathbf{r})$ is a column vector of shape functions $g_i(\mathbf{r})$, with $i = 1, \ldots, n$. The resulting nonlinear thermal network in general does not preserve the positivity and passivity properties of the nonlinear heat diffusion property. However if

$$\mathbf{f}(\mathbf{r}) \geq \mathbf{0},$$
$$\mathbf{g}(\mathbf{r}) \geq \mathbf{0},$$

the positivity property is preserved in the form

Property 3 (Positivity) *For non-negative powers $\mathbf{P}(t)$, the temperature rises $\mathbf{T}(t)$ are non-negative.*

Besides if

$$\mathbf{f}(\mathbf{r}) = \mathbf{g}(\mathbf{r}). \tag{7}$$

the passivity property is preserved in the form

Property 4 (Passivity) *A non-negative function $W(t)$ exists such that, for each time $t_1 \leq t_2$,*

$$W(t_2) \leq W(t_1) + \int_{t_1}^{t_2} \mathbf{T}^T(t)\mathbf{P}(t)\, d\tau. \tag{8}$$

As shown in [7], preserving the Passivity Property 4 is crucial when using thermal networks in coupled analysis, such as in electro-thermal simulations.

The heat diffusion equation is a much more complex problem in the nonlinear case than in the linear case. In fact in general it cannot be even assured that the nonlinear thermal network has a solution, as shown in the following example.

Example 1
Let Ω be a cylinder of length L along direction x, area A, and thermal conductivity

$$k\, e^{\alpha u(x)},$$

with $\alpha < 0$. Power P is uniformly dissipated within the slab. On the lower face of the boundary $\partial\Omega$ the temperature is set to the ambient temperature. On the rest of the boundary $\partial\Omega$ the thermal flux is set to zero. By using Kirchhoff's transformation of Eq. (9), the stationary temperature rise distribution $u(x)$ within Ω can be computed in closed form to be

$$u(x) = \frac{1}{\alpha} \log\left(1 + \frac{\alpha}{2}\frac{L}{Ak}\left(1 - \left(\frac{x}{L}\right)^2\right) P\right),$$

and the nonlinear thermal network is ruled by

$$T = -\frac{2}{\alpha} I\left(\frac{\alpha}{2}\frac{L}{Ak} P\right),$$

in which

$$I(\beta) = 1 - \frac{\arctan\sqrt{\dfrac{-\beta}{1+\beta}}}{\sqrt{\dfrac{-\beta}{1+\beta}}},$$

Temperature rise $u(0)$ remains finite only for

$$P < -\frac{2}{\alpha}\frac{Ak}{L}.$$

Otherwise no solution exists. ■

Moreover, while in the linear case the analytic form of the solution of heat diffusion equation is known, in the nonlinear case the analytic form of the solution is in general unknown. A significant exception is the particular case in which

$$c(\mathbf{r}, u(\mathbf{r}, t)) = c(\mathbf{r})\, f(u(\mathbf{r}, t)),$$
$$k(\mathbf{r}, u(\mathbf{r}, t)) = k(\mathbf{r})\, f(u(\mathbf{r}, t)),$$
$$h(\mathbf{r}, u(\mathbf{r}, t)) = h(\mathbf{r})\, \frac{1}{u(\mathbf{r}, t)} \int_0^{u(\mathbf{r},t)} f(\tau)\, d\tau,$$

$f(\tau)$ being a positive function. By Kirchhoff's transformation

$$\upsilon(\mathbf{r}, t) = \int_0^{u(\mathbf{r},t)} f(\tau)\, d\tau \tag{9}$$

the nonlinear heat diffusion equation in $u(\mathbf{r}, t)$ is transformed into the linear heat diffusion equation in $\upsilon(\mathbf{r}, t)$

$$c(\mathbf{r})\frac{\partial \upsilon}{\partial t}(\mathbf{r}, t) + \nabla \cdot (-k(\mathbf{r})\nabla \upsilon(\mathbf{r}, t)) = F(\mathbf{r}, t) \tag{10}$$

with boundary conditions

$$-k(\mathbf{r})\frac{\partial \upsilon}{\partial \nu}(\mathbf{r}, t) = h(\mathbf{r})\upsilon(\mathbf{r}, t)$$

and zero initial conditions. The solution to the nonlinear heat diffusion equation is then obtained by inverting Eq. (9) as a function of $\upsilon(\mathbf{r}, t)$

$$u(\mathbf{r}, t) = K(\upsilon(\mathbf{r}, t)).$$

The nonlinear thermal network has thus equations

$$F(\mathbf{r}, t) = \mathbf{f}^T(\mathbf{r})\mathbf{P}(t),$$
$$\mathbf{T}(t) = \int_\Omega \mathbf{g}(\mathbf{r}) K(\upsilon(\mathbf{r}, t)).$$

3 Nonlinear Discretized Thermal Networks

The heat diffusion equation, in discretized form, is

$$\mathbf{C}(\mathbf{u}(t))\frac{d\mathbf{u}}{dt}(t) + \mathbf{K}(\mathbf{u}(t))\mathbf{u}(t) = \mathbf{F}(t) \tag{11}$$

in which the $M \times 1$ vector $\mathbf{u}(t)$ is formed by the degrees of freedom for the discretized temperature rise. The $M \times M$ matrices $\mathbf{K}(\mathbf{u}(t))$ and $\mathbf{C}(\mathbf{x}(t))$ are the stiffness and mass matrices respectively, at least in finite elements. The $M \times 1$ vector $\mathbf{F}(t)$ is the discretized power density.

We note that it can be written

$$\mathbf{C}(\mathbf{u}(t)) = \sum_{i}{}_{1}^{M} \gamma_i(\mathbf{e}_i^T \mathbf{u}(t))\, \mathbf{C}_i \tag{12}$$

$$\mathbf{K}(\mathbf{u}(t)) = \sum_{i}{}_{1}^{M} \kappa_i(\mathbf{e}_i^T \mathbf{u}(t))\, \mathbf{K}_i, \tag{13}$$

in which $\mathbf{C}_i$, $\mathbf{K}_i$ are $M \times M$ symmetric positive semi-definite matrices, $\gamma_i(\cdot)$, $\kappa_i(\cdot)$ are positive functions and $\mathbf{e}_i$ is the i-th column of the $\mathbf{I}_M$ identity matrix, with $i = 1\ldots, M$.

Only with careful choices of the discretization scheme the discretized heat diffusion equation preserves the Positivity Property in the form

Property 5 (Positivity) *For non-negative discretized power densities $\mathbf{F}(t)$, the discretized temperature rises $\mathbf{u}(t)$ are non-negative.*

However all stable discretization schemes have symmetric positive definite stiffness and mass matrices, which assures the Passivity Property in the form

Property 6 (Passivity) *A non-negative function $W(t)$ exists such that, for each time $t_1 \le t_2$,*

$$W(t_2) \le W(t_1) + \int_{t_1}^{t_2} \mathbf{u}^T(t)\mathbf{F}(t)\, d\tau. \tag{14}$$

The thermal network, in discretized form, has equations

$$\mathbf{F}(t) = \mathbf{F}\mathbf{P}(t) \tag{15}$$

$$\mathbf{T}(t) = \mathbf{G}^T\mathbf{u}(t), \tag{16}$$

$\mathbf{F}$ and $\mathbf{G}$ being $M \times n$ matrices. If Positivity Property 5 is satisfied and if

$$\mathbf{F} \ge \mathbf{0}$$
$$\mathbf{G} \ge \mathbf{0},$$

then also the discretized thermal network satisfies Positivity Property 3. If Passivity Property 6 is satisfied and if

$$\mathbf{F} = \mathbf{G} \tag{17}$$

then the discretized thermal network satisfies the Passivity Property 4

In the particular case in which the nonlinear heat diffusion equation can be transformed into a linear heat diffusion problem by Kirchhoff's transformation, instead of discretizing the nonlinear heat diffusion equation (1), the transformed linear diffusion equation (10) could be discretized in the form

$$\mathbf{C}\frac{d\boldsymbol{v}}{dt}(t) + \mathbf{K}\boldsymbol{v}(t) = \mathbf{F}(t), \tag{18}$$

in which the $M \times 1$ vector $\boldsymbol{v}(t)$ is formed by the freedom degrees of $v(\mathbf{r}, t)$. The discretized temperature rise vector $\mathbf{u}(t)$ is then

$$\mathbf{u}(t) = \sum_{1}^{M}{}_i\, K(\mathbf{e}_i^T\boldsymbol{v}(t))\, \mathbf{e}_i. \tag{19}$$

Moreover

$$\mathbf{F}(t) = \mathbf{F}\mathbf{P}(t) \tag{20}$$

$$\mathbf{T}(t) = \sum_{1}^{M}{}_i\, K(\mathbf{e}_i^T\boldsymbol{v}(t))\, (\mathbf{G}^T\mathbf{e}_i) \tag{21}$$

We note that Eqs. (18), (20), (21) define a thermal network which, as it can be verified, can be assured to satisfy the Positivity Property 3 but cannot be assured to satisfy the Passivity Property 4.

4 Compact Models of Nonlinear Thermal Networks by Kirchhoff's Transformation

Compact thermal networks are lumped models of thermal networks with much lesser freedom degrees than discretized thermal networks. In the linear case, various effective approaches have been reported in literature for generating compact models of thermal networks. However in the nonlinear case only a few attempts have been reported in literature [2, 3, 6].

In case the nonlinear heat diffusion problem can be transformed into a linear heat diffusion problem by Kirchhoff's transformation, the nonlinear thermal network can be discretized as in Eqs. (18), (20), (21). In this way the problem of generating a compact model of the nonlinear thermal network is reduced to the problem of generating a compact model of the linear heat diffusion Eqs. (18), (20). The resulting compact thermal network however cannot be assured to satisfy the Passivity Property 4.

In the most general case, in which Kirchhoff's transformation is not applicable, the most common approach is that of approximating the nonlinear heat diffusion problem by a different nonlinear heat diffusion problem to which Kirchhoff's transformation is applicable. However this approach usually introduce errors which cannot be eliminated, as shown in the following example.

Example 2

Let Ω be a cylinder of length L along direction x and area A. For $0 \le x < L/2$ thermal conductivity is

$$k\, e^{\alpha u(x)},$$

with $\alpha < 0$, while for $L/2 \le x \le L$ thermal conductivity is k. Power P is uniformly dissipated within the cylinder. On the lower face of the boundary $\partial\Omega$ the temperature is set to the ambient temperature. On the rest of the boundary $\partial\Omega$ the thermal flux is set to zero. The stationary temperature rise distribution $u(x)$ within Ω can be determined in closed form to be

$$u(x) = \begin{cases} \dfrac{1}{\alpha}\log\left(e^{\frac{3}{8}\frac{\alpha L}{Ak}P} + \dfrac{1}{2}\dfrac{\alpha L}{Ak}\left(\dfrac{1}{4} - \left(\dfrac{x}{L}\right)^2\right)P\right), & x \le \dfrac{L}{2} \\ \dfrac{1}{2}\dfrac{\alpha L}{Ak}\left(1 - \left(\dfrac{x}{L}\right)^2\right)P, & x \ge \dfrac{L}{2}. \end{cases}$$

which is shown in Fig. 1. The nonlinear thermal network is then ruled by

$$T = \frac{1}{\alpha}\, I\left(\frac{\alpha L}{Ak}\, P\right),$$

in which

$$I(\beta) = \frac{7}{24}\beta - 1 + \frac{\arctan\sqrt{\dfrac{-\beta}{\beta + 8e^{\frac{3}{8}\beta}}}}{\sqrt{\dfrac{-\beta}{\beta + 8e^{\frac{3}{8}\beta}}}},$$

as shown in Fig. 2.

Both this temperature rise distribution and this nonlinear thermal network are now approximated by the nonlinear heat diffusion problem of Example 2 to which Kirchhoff's equation

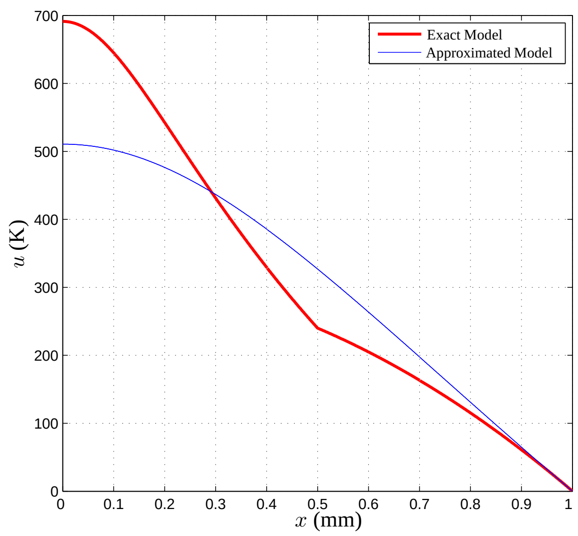

Figure 1: Exact and approximated stationary temperature rise distribution for $L = 1$ mm, $A = 0.1$ mm^2, $k = 150$W/m K, $\alpha = -4$ mK^{-1}, $P = 7$W.

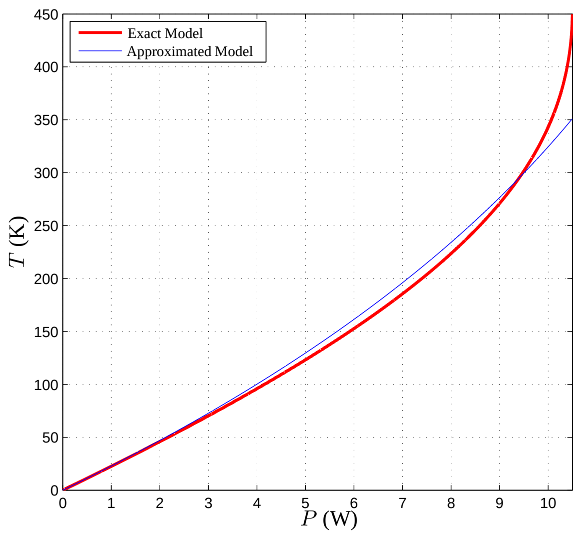

Figure 2: Exact and approximated characteristic line of nonlinear thermal network for $L = 1$ mm, $A = 0.1$ mm^2, $k = 150$W/m K, $\alpha = -4$ mK^{-1}.

is applicable. The α parameter of Example 2 is substituted by $\alpha/2$ in order to maximize accuracy. The resulting approximations are shown in Figs. 1, 2. ∎

A novel approach for generating compact models of nonlinear thermal networks is thus needed.

5 Compact Models of Nonlinear Thermal Networks by Galerkin's Method

As shown in [7], compact models of *linear* thermal networks can be effectively generated by Galerkin's method. In this technique, firstly the discretized temperature rise distribution in Eq. (11) is approximated by

$$\mathbf{u}(t) \approx \mathbf{U}\hat{\mathbf{u}}(t), \tag{22}$$

in which $\mathbf{U}$ is a proper $M \times \hat{m}$ matrix and $\hat{\mathbf{u}}(t)$ is a $\hat{m} \times 1$ vector of freedom degrees. Secondly Eqs. (11), (15), (16) are projected onto the space spanned by the columns of $\mathbf{U}$. It results in

$$\hat{\mathbf{C}}\,\frac{d\hat{\mathbf{u}}}{dt}(t) + \hat{\mathbf{K}}\,\hat{\mathbf{u}}(t) = \hat{\mathbf{F}}(t) \tag{23}$$

and

$$\hat{\mathbf{F}}(t) = \hat{\mathbf{F}}\mathbf{P}(t) \tag{24}$$

$$\mathbf{T}(t) = \hat{\mathbf{G}}^T\hat{\mathbf{u}}(t), \tag{25}$$

in which

$$\hat{\mathbf{C}} = \mathbf{U}^T\mathbf{C}\mathbf{U}, \tag{26}$$

$$\hat{\mathbf{K}} = \mathbf{U}^T\mathbf{K}\mathbf{U}, \tag{27}$$

$$\hat{\mathbf{F}} = \mathbf{U}^T\mathbf{F}. \tag{28}$$

$$\hat{\mathbf{G}} = \mathbf{U}^T\mathbf{G}. \tag{29}$$

If the discretized nonlinear thermal network satisfies the Passivity Property 4 also the compact model satisfies the Passivity Property 4. In general the Positivity Property 3 of the discretized nonlinear thermal network is not preserved by the compact model. Effective methods have been proposed in literature for choosing the $\mathbf{U}$ matrix in such a way that accurate compact models of small state-space dimension $\hat{m}$ are obtained, such as Multi-Point Moment Matching [7].

Galerkin's method can be applied not only to linear but also to nonlinear heat diffusion problems, proceeding as in [8,9]. In this way accurate compact models can be obtained also in the nonlinear case. However the following two questions arise:

1. An effective method is needed for choosing the $\mathbf{U}$ matrix in such a way that accurate compact models with small state-space dimension $\hat{m}$ are obtained.

2. It results in

$$\hat{\mathbf{C}}(\hat{\mathbf{u}}(t)) = \mathbf{U}^T\mathbf{C}(\mathbf{U}\hat{\mathbf{u}}(t))\mathbf{U}$$
$$\hat{\mathbf{K}}(\hat{\mathbf{u}}(t)) = \mathbf{U}^T\mathbf{K}(\mathbf{U}\hat{\mathbf{u}}(t))\mathbf{U}.$$

Thus the cost of evaluating the $\hat{\mathbf{C}}(\hat{\mathbf{u}}(t))$ and $\hat{\mathbf{K}}(\hat{\mathbf{u}}(t))$ matrices in the compact model is *not* in general reduced with respect to the cost of evaluating the matrices $\mathbf{C}(\mathbf{u}(t))$ and $\mathbf{K}(\mathbf{u}(t))$ in the discretized thermal network.

Some proposals have been given in literature for answering these two questions [9, 10]:

1. Columns of $\mathbf{U}$ are introduced in correspondence to values of $\mathbf{u}(t)$ at different time instants t and for different powers $\mathbf{P}(t)$. As a result, the number of columns $\hat{m}$ of the $\mathbf{U}$ matrix and equivalently the state-space dimension of the compact model tends to be large.

2. The $\mathbf{C}(\mathbf{u}(t))$ and $\mathbf{K}(\mathbf{u}(t))$ matrices in the discretized heat diffusion equations are substituted by approximations in such a way that the cost of evaluating $\hat{\mathbf{C}}(\hat{\mathbf{u}}(t))$ and $\hat{\mathbf{K}}(\hat{\mathbf{u}}(t))$ is reduced with respect to the cost of evaluating $\mathbf{C}(\mathbf{u}(t))$ and $\mathbf{K}(\mathbf{u}(t))$. In this way however the compact model cannot be assured to satisfy the Passivity Property 4.

6 Nonlinear Projection-Based Approach

As shown in Section 3, a nonlinear heat diffusion equation which can be transformed into a linear heat diffusion problem by Kirchhoff's transformation can be discretized as in Eqs. (18), (20), (21). Thus by applying Galerkin's method to linear Eqs. (18), (20), it follows

$$\boldsymbol{\upsilon}(t) \approx \boldsymbol{\Upsilon}\hat{\boldsymbol{\upsilon}}(t),$$

in which $\boldsymbol{\Upsilon}$ has a small number of columns. Then from Eq. (19) it results in

$$\mathbf{u}(t) \approx \sum_{1}^{M}{}_{i} K((\boldsymbol{\Upsilon}^T\mathbf{e}_i)^T\hat{\boldsymbol{\upsilon}}(t))\,\mathbf{e}_i. \tag{30}$$

Thus in general $\mathbf{u}(t)$ cannot be approximated by Eq. (22), as in Galerkin's method, with a small number of freedom degrees. Instead it can be approximated by a nonlinear expression with a small number of freedom degrees.

Thus for a general nonlinear heat diffusion problem it is *proposed* to approximate $\mathbf{u}(t)$ by a nonlinear function, of the form suggested by Eq. (30),

$$\boxed{\mathbf{q}(\hat{\mathbf{u}}(t)) = \sum_{1}^{p}{}_{i} K\left(\hat{\mathbf{u}}_i^T\hat{\mathbf{u}}(t)\right)\,\mathbf{v}_i}, \tag{31}$$

in which $\hat{\mathbf{u}}_i$ and $\mathbf{v}_i$ are $\hat{m}\times 1$ and $M\times 1$ vectors respectively, with $i = 1,\ldots,p$. The theory of kernels [5] assures that the expression given in Eq. (31) can be used to approximate general nonlinear functions. To this aim $K(\cdot)$ is assumed to be a *positive semi-definite kernel* [5]. Besides, as in Eq. (30), it is assumed that $K(x)$ has the same order of x for $x\to 0$. Hereafter this property will turn out to be crucial. For instance it will be assumed

$$K(x) = \frac{e^{\alpha x}-1}{\alpha},$$

α being positive. Function $\mathbf{q}(\hat{\mathbf{u}}(t))$ in Eq. (30) is determined by means of the following three steps:

1. A representative $\mathbf{u}(t)$ is evaluated at representative time instants. In this way the $M\times 1$ vectors $\mathbf{u}_i$ with $i = 1\ldots,p$ are determined.

2. A nonlinear dimensionality reduction method, such as *Locally Linear Embedding* [11], is used to determine the $\hat{m}\times 1$ vectors $\hat{\mathbf{u}}_i$ corresponding to the $M\times 1$ vectors $\mathbf{u}_i$, with $i = 1\ldots,p$. In this step the $\hat{m}\times 1$ zero vector is imposed to correspond to the $M\times 1$ zero vector.

3. A nonlinear regression method, such as *Support Vector Machine* [5], is used to determine the $M\times 1$ vectors $\mathbf{v}_i$ of Eq. (31) in such a way that the $\mathbf{q}(\hat{\mathbf{u}}(t))$ function approximatively map the $\hat{m}\times 1$ vectors $\hat{\mathbf{u}}_i$ into the $M\times 1$ vectors $\mathbf{u}_i$.

In order to generate the compact model of the nonlinear discretized thermal network, a nonlinear projection of Eqs. (11), (15), (16) is *proposed*. It results in

$$\begin{aligned}&\hat{\mathbf{C}}(\hat{\mathbf{u}}(t))\,\frac{d}{dt}\mathbf{q}(\hat{\mathbf{u}}(t)) + \hat{\mathbf{K}}(\hat{\mathbf{u}}(t))\,\mathbf{q}(\hat{\mathbf{u}}(t)) = \hat{\mathbf{F}}(\hat{\mathbf{u}}(t))\mathbf{P}(t),\\ &\mathbf{T}(t) = \mathbf{G}^T\mathbf{q}(\hat{\mathbf{u}}(t)),\end{aligned}$$

in which

$$\begin{aligned}\hat{\mathbf{C}}(\hat{\mathbf{u}}(t)) &= \mathbf{Q}^T(\hat{\mathbf{u}}(t))\mathbf{C}(\mathbf{q}(\hat{\mathbf{u}}(t))),\\ \hat{\mathbf{K}}(\hat{\mathbf{u}}(t)) &= \mathbf{Q}^T(\hat{\mathbf{u}}(t))\mathbf{K}(\mathbf{q}(\hat{\mathbf{u}}(t))),\\ \hat{\mathbf{F}}(\hat{\mathbf{u}}(t)) &= \mathbf{Q}^T(\hat{\mathbf{u}}(t))\mathbf{F}.\end{aligned}$$

The $M\times\hat{m}$ matrix $\mathbf{Q}(\hat{\mathbf{u}}(t))$ is chosen as

$$\boxed{\mathbf{Q}(\hat{\mathbf{u}}(t)) = \sum_{1}^{p}{}_{i}\frac{K\left(\hat{\mathbf{u}}_i^T\hat{\mathbf{u}}(t)\right)}{\left(\hat{\mathbf{u}}_i^T\hat{\mathbf{u}}(t)\right)}\,\mathbf{v}_i\hat{\mathbf{u}}_i^T}. \tag{32}$$

We note that $\mathbf{Q}(\hat{\mathbf{u}}(t))$ is well defined since $K(x)/x$ has the same order of 1 for $x\to 0$. As it can be straightforwardly proven, with this choice of $\mathbf{Q}(\hat{\mathbf{u}}(t))$ a compact model is obtained which preserves Passivity Property 4.

Moreover it results in

$$\begin{aligned}\hat{\mathbf{C}}(\hat{\mathbf{u}}(t))\,\frac{d}{dt}\mathbf{q}(\hat{\mathbf{u}}(t)) &= \sum_{1}^{p}{}_{i}\frac{d}{dt}K\left(\hat{\mathbf{u}}_i^T\hat{\mathbf{u}}(t)\right)\hat{\mathbf{u}}_i\cdot\\ &\cdot\left(\sum_{1}^{p}{}_{j}\frac{K\left(\hat{\mathbf{u}}_j^T\hat{\mathbf{u}}(t)\right)}{\left(\hat{\mathbf{u}}_j^T\hat{\mathbf{u}}(t)\right)}\,\mathbf{v}_i^T\mathbf{C}(\mathbf{q}(\hat{\mathbf{u}}(t)))\mathbf{v}_j\right)\\ \hat{\mathbf{K}}(\hat{\mathbf{u}}(t))\,\mathbf{q}(\hat{\mathbf{u}}(t)) &= \sum_{1}^{p}{}_{i}K\left(\hat{\mathbf{u}}_i^T\hat{\mathbf{u}}(t)\right)\hat{\mathbf{u}}_i\cdot\\ &\cdot\left(\sum_{1}^{p}{}_{j}\frac{K\left(\hat{\mathbf{u}}_j^T\hat{\mathbf{u}}(t)\right)}{\left(\hat{\mathbf{u}}_j^T\hat{\mathbf{u}}(t)\right)}\,\mathbf{v}_i^T\mathbf{K}(\mathbf{q}(\hat{\mathbf{u}}(t)))\mathbf{v}_j\right)\\ \hat{\mathbf{F}}(\hat{\mathbf{u}}(t)) &= \sum_{1}^{p}{}_{i}\frac{K\left(\hat{\mathbf{u}}_i^T\hat{\mathbf{u}}(t)\right)}{\left(\hat{\mathbf{u}}_i^T\hat{\mathbf{u}}(t)\right)}\,\hat{\mathbf{u}}_i(\mathbf{v}_i^T\mathbf{F})\end{aligned}$$

Besides

$$\mathbf{G}^T\mathbf{q}(\hat{\mathbf{u}}(t)) = \hat{\mathbf{G}}(\hat{\mathbf{u}}(t))\hat{\mathbf{u}}(t)$$

in which

$$\hat{\mathbf{G}}(\hat{\mathbf{u}}(t)) = \sum_{1}^{p}{}_{i}\frac{K\left(\hat{\mathbf{u}}_i^T\hat{\mathbf{u}}(t)\right)}{\left(\hat{\mathbf{u}}_i^T\hat{\mathbf{u}}(t)\right)}\,\hat{\mathbf{u}}_i(\mathbf{v}_i^T\mathbf{G})$$

Thus, as with Galerkin's method, the cost of evaluating the $\hat{\mathbf{C}}(\hat{\mathbf{u}}(t))$ and $\hat{\mathbf{K}}(\hat{\mathbf{u}}(t))$ matrices in the compact model is *not* in general reduced with respect to the cost of evaluating the matrices $\mathbf{C}(\mathbf{u}(t))$ and $\mathbf{K}(\mathbf{u}(t))$ in the discretized thermal network. In order to remove this drawback, approximations of the $\gamma_h(\mathbf{e}_h^T\mathbf{u}(t)$ and $\kappa_h(\mathbf{e}_h^T\mathbf{u}(t)$ functions in Eqs. (12), (13) are introduced, with $h = 1\ldots,M$, by means of a nonlinear regression method, such as Support Vector Machine [5]. Thus

$$\gamma_h(\mathbf{e}_h^T\mathbf{u}(t)) \approx \sum_{1}^{p}{}_{k} H(\hat{\mathbf{u}}_i^T\hat{\mathbf{u}}(t))\,m_{hk}, \tag{33}$$

$$\kappa_h(\mathbf{e}_h^T\mathbf{u}(t)) \approx \sum_{1}^{p}{}_{k} H(\hat{\mathbf{u}}_i^T\hat{\mathbf{u}}(t))\,n_{hk}, \tag{34}$$

in which $H(\cdot)$ is a positive semi-definite kernel, such as

$$H(x) = e^{\alpha x},$$

α being positive. In this way

$$\mathbf{v}_i^T \mathbf{C}(\mathbf{q}(\hat{\mathbf{u}}(t)))\mathbf{v}_j \approx \sum_1^p {}_k \, H(\hat{\mathbf{u}}_i^T \hat{\mathbf{u}}(t)) \cdot \mathbf{v}_i^T \hat{\mathbf{C}}_k \mathbf{v}_j$$

$$\mathbf{v}_i^T \mathbf{K}(\mathbf{q}(\hat{\mathbf{u}}(t)))\mathbf{v}_j \approx \sum_1^p {}_k \, H(\hat{\mathbf{u}}_i^T \hat{\mathbf{u}}(t)) \cdot \mathbf{v}_i^T \hat{\mathbf{K}}_k \mathbf{v}_j$$

in which, for $k = 1, \ldots, p$,

$$\hat{\mathbf{C}}_k = \sum_1^M {}_h \, \mathbf{C}_h m_{hk},$$

$$\hat{\mathbf{K}}_k = \sum_1^M {}_h \, \mathbf{C}_h n_{hk}$$

and the cost of evaluating the compact model is reduced to $O(p^3)$ flops. If the functions introduced in Eqs. (33), (34) are assured to be positive the Passivity Property 4 is preserved by the compact model.

7 Numerical Results

A simple application example has been considered. Let Ω be a cube whose edge has length 10^{-2}m. On the lower face of Ω the temperature is set equal to ambient temperature 300K. On the rest of the boundary $\partial\Omega$ the heat flux is set equal to zero. The volumetric heat capacity is assumed to be 10^6Jm^{-3}K^{-1}. The thermal conductivity is assumed to be given by Eq. (3) with $\beta = 10^2$Wm^{-1}K^{-1} and $\alpha = -10^{-3}$K^{-1}. Power $P(t)$ is uniformly dissipated in a cube of length $2 \cdot 10^{-3}$m beneath the upper face of Ω. A passive nonlinear thermal network has been defined by defining $T(t)$ according to Eq. (7). The heat diffusion equation problem has been discretized by means of $21\,952$ unknowns. A number $p = 12$ of discretized temperature rises, below 500K, have been selected from the thermal responses to different power steps. By means of the nonlinear projection approach, a nonlinear compact model of state-space dimension $\hat{m} = 4$ has been determined. A relative error smaller than 1% in the power step thermal responses of the thermal network has been observed, for temperature rises below 500K, as shown in Fig. 3.

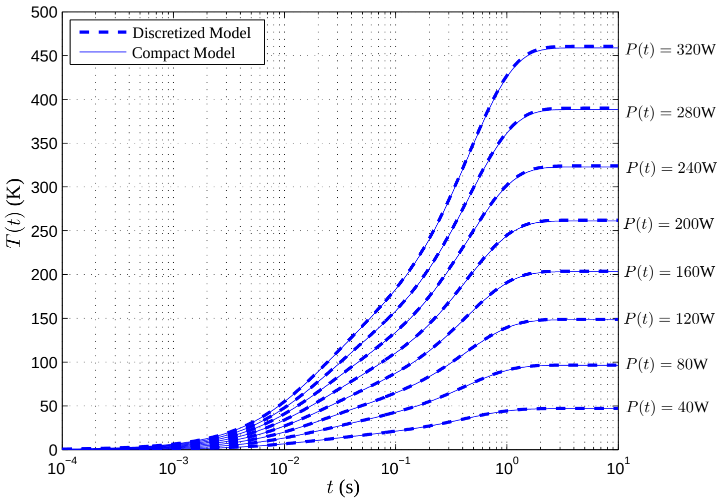

Figure 3: Power step thermal responses.

8 Conclusion

In this paper a nonlinear projection-based approach for generating compact models of nonlinear thermal networks has been proposed. In the numerical results, high accuracy for large temperature variations and high compactness of the generated models have been obtained.

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