Canonical Forms of Multi-Port Dynamic Thermal Networks

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Abstract

In this paper it is shown that multi-port dynamic thermal networks admit four canonical representations which generalize the four canonical representations of passive lumped RC networks: Foster I and II canonical forms, Cauer I and Cauer II canonical forms. In particular the generalized Foster I canonical form is equivalent to the time-constant representation and the generalized Cauer I canonical form is a passive multi-conductor RC transmission line.

Keywords: Multi-Port Thermal Networks, Time-Constant Representation, Structure Function.

1 Introduction

Thermal networks are widely used for modeling heat diffusion in components and packages. At first thermal networks have been proposed for modeling static heat diffusion [1]. More recently thermal networks have been proposed also for modeling dynamic heat diffusion [2, 3].

The question of determining the canonical forms of *one-port* dynamic thermal networks has been considered in [2, 4–6]. It has been shown that *passive* one-port dynamic thermal networks admit four canonical forms which are the generalizations of Foster I, Foster II, Cauer I and Cauer II canonical forms of passive one-port RC lumped networks. In particular the generalized Foster I canonical form is equivalent to the time-constant representation [4, 5], while the Cauer I canonical form is equivalent to the structure function representation [2, 5, 6]

The question of determining canonical forms of multi-port dynamic thermal networks has not been tackled in literature yet. In this paper it is shown that all the results proved for passive one-port dynamic thermal networks can be extended to passive multi-port dynamic thermal networks. As a result the Foster I, Cauer I, Foster II and Cauer II representations of passive multi-port RC lumped networks are extended one-port to multi-port passive dynamic thermal networks.

In particular the generalized Foster I canonical form is equivalent to the time-constant representation. Besides the generalized Cauer I canonical form is a passive *multi-conductor* RC transmission line.

It is also shown how all multi-port dynamic thermal networks can be represented by passive multi-port dynamic thermal networks. Thus the four canonical forms of passive multi-port dynamic thermal networks can be applied to all multi-port dynamic thermal networks.

The rest of this paper is organized as follows. In Section 2 multi-port dynamic thermal networks are introduced. In Sections 3, 4, preliminary results on passive multi-port dynamic thermal networks are presented. The four canonical forms are shown in Sections 5, 6 and 8, 9. An application example is presented in Sections 7 and 10.

2 Multi-Port Dynamic Thermal Networks

In a *bounded* spatial region Ω, the relation between the power density $F(\mathbf{r}, t)$ and the temperature rise $u(\mathbf{r}, t)$ with respect to ambient temperature, functions of the position vector $\mathbf{r}$ and of the time instant t, is ruled by the heat conduction equation

$$\nabla \cdot (-k(\mathbf{r})\nabla u(\mathbf{r}, t)) + c(\mathbf{r})\frac{\partial u}{\partial t}(\mathbf{r}, t) = F(\mathbf{r}, t), \qquad (1)$$

in which $c(\mathbf{r})$ is the volumetric heat capacity and $k(\mathbf{r})$ is the thermal conductivity. Eq. (1) is completed by conditions on the boundary of Ω, $\partial\Omega$, and by initial condition for the temperature rise $u(\mathbf{r}, t)$. The boundary conditions, assumed of Robin's type, are

$$-k(\mathbf{r})\frac{\partial u}{\partial \nu}(\mathbf{r}, t) = h(\mathbf{r})u(\mathbf{r}, t), \qquad (2)$$

in which $h(\mathbf{r})$ is the heat transfer coefficient and $\boldsymbol{\nu}(\mathbf{r})$ is the outward unit vector normal to $\partial\Omega$. Here $h(\mathbf{r})$ is *not* assumed to be identically zero over $\partial\Omega$, that is pure Neumann's boundary conditions are excluded. The initial condition is assumed to be zero

$$u(\mathbf{r}, 0) = 0. \qquad (3)$$

This is by no means a limitation. In fact any heat diffusion problem with non-zero initial condition

$$u(\mathbf{r}, 0) = U(\mathbf{r})$$

and power density $F(\mathbf{r}, t)$ can be represented by an equivalent heat diffusion problem with zero initial condition and power density

$$F(\mathbf{r}, t) + c(\mathbf{r})\, U(\mathbf{r})\, \delta(t).$$

The heat diffusion problem defined by Eqs. (1), (2), (3), satisfies the following main physical properties:

Theorem 1 (Passivity) *A non-negative function $W(t)$ exists such that, for each time $t_1 \le t_2$,*

$$W(t_2) \le W(t_1) + \int_{t_1}^{t_2} dt \int_\Omega F(\mathbf{r}, t)u(\mathbf{r}, t)\, d\mathbf{r}.$$

Theorem 2 (Reciprocity) *Let $\mathsf{u}_1(\mathbf{r}, s)$, $\mathsf{u}_2(\mathbf{r}, s)$ be the Laplace transforms of the temperature rises due to the power densities whose Laplace transforms are $\mathsf{F}_1(\mathbf{r}, s)$, $\mathsf{F}_2(\mathbf{r}, s)$ respectively. It results in*

$$\int_\Omega \mathsf{F}_1(\mathbf{r}, s)\mathsf{u}_2(\mathbf{r}, s)\, d\mathbf{r} = \int_\Omega \mathsf{F}_2(\mathbf{r}, s)\mathsf{u}_1(\mathbf{r}, s)\, d\mathbf{r}.$$

A multi-port dynamic thermal network can be defined from the heat diffusion problem, by introducing the powers and the temperature rises measured at its ports. The powers $P_i(t)$, with $i = 1, \ldots, n$, elements of column vector $\mathbf{P}(t)$, determine $F(\mathbf{r}, t)$ as

$$F(\mathbf{r}, t) = \mathbf{f}^T(\mathbf{r})\mathbf{P}(t) \tag{4}$$

in which $\mathbf{f}(\mathbf{r})$ is a column vector of shape functions $f_i(\mathbf{r})$, with $i = 1, \ldots, n$. The temperature rise $T_i(t)$, with $i = 1, \ldots, n$, elements of column vector $\mathbf{T}(t)$, are defined by

$$\mathbf{T}(t) = \int_\Omega \mathbf{g}(\mathbf{r}) u(\mathbf{r}, t)\, d\mathbf{r}. \tag{5}$$

in which $\mathbf{g}(\mathbf{r})$ is a column vector of shape functions $g_i(\mathbf{r})$, with $i = 1, \ldots, n$.

The multi-port dynamic thermal network defined by Eqs. (1)-(5) in general does not preserve the main physical properties of the heat diffusion problem: reciprocity and passivity. However these physical properties are preserved if

$$\mathbf{f}(\mathbf{r}) = \mathbf{g}(\mathbf{r}). \tag{6}$$

In fact in this case it results in

Theorem 3 (Passivity) *A non-negative function $W(t)$ exists such that for each time $t_1 \leq t_2$*

$$W(t_2) \leq W(t_1) + \int_{t_1}^{t_2} \mathbf{P}^T(t)\mathbf{T}(t)\, dt.$$

Theorem 4 (Reciprocity) *Let $\mathsf{T}_1(s)$, $\mathsf{T}_2(s)$ be the Laplace transforms of the temperature rise vectors due to the power vectors whose Laplace transforms are $\mathsf{P}_1(s)$, $\mathsf{P}_2(s)$ respectively. It results in*

$$\mathsf{P}_1^T(s)\mathsf{T}_2(s) = \mathsf{P}_2^T(s)\mathsf{T}_1(s).$$

As shown in [5] limitedly to the case of one-port dynamic thermal networks, only if passivity and reciprocity hold canonical forms of dynamic thermal networks can be given. Thus hereafter it will be assumed that Eq. (6) holds or equivalently that the multi-port dynamic thermal network is *passive*. It can be observed that this is not a limitation. In fact a multi-port dynamic thermal network whose powers are defined by $\mathbf{f}(\mathbf{r})$ and whose temperature rises are defined by $\mathbf{g}(\mathbf{r})$ has port responses equal to a subset of the port responses of the passive multi-port dynamic thermal network whose shape functions for both powers and temperature rises are all the *distinct* elements of $\mathbf{f}(\mathbf{r})$ and $\mathbf{g}(\mathbf{r})$. As a result, a generic n-port dynamic thermal network can always be substituted by a passive N-port dynamic thermal network with $n \leq N \leq 2n$.

Hereafter it is also assumed that the shape functions in $\mathbf{f}(\mathbf{r}) = \mathbf{g}(\mathbf{r})$ are *linearly independent*. Again this is not a limitation. In fact if $\mathbf{f}(\mathbf{r}) = \mathbf{g}(\mathbf{r})$ are linearly dependent, it results in

$$\mathbf{g}(\mathbf{r}) = \mathbf{R}\hat{\mathbf{g}}(\mathbf{r}),$$

in which $\hat{\mathbf{g}}(\mathbf{r})$ is a vector of $\hat{n} < n$ linearly independent shape functions and $\mathbf{R}$ is an $n \times \hat{n}$ rectangular matrix. Thus it results in

$$\mathbf{T}(t) = \mathbf{R}\hat{\mathbf{T}}(t) \tag{7}$$

$$\hat{\mathbf{P}}(t) = \mathbf{R}^T\mathbf{P}(t) \tag{8}$$

and

$$F(\mathbf{r}, t) = \hat{\mathbf{g}}^T(\mathbf{r})\hat{\mathbf{P}}(t) \tag{9}$$

$$\hat{\mathbf{T}}(t) = \int_\Omega \hat{\mathbf{g}}(\mathbf{r}) u(\mathbf{r}, t)\, d\mathbf{r} \tag{10}$$

The passive n-port dynamic thermal network is then the connection of a multi-port transformer [7] defined by Eqs. (7), (8) to a passive $\hat{n}$-port dynamic thermal network, defined by Eqs. (9), (10), with $\hat{n} < n$ linearly independent shape functions.

3 Solutions of Passive Multi-Ports Dynamic Thermal Networks

The solution of Eqs. (1), (2), (3) can be expressed by the series expansion [5]

$$T(\mathbf{r}, t) = \sum_{j}{}_{1}^{\infty} a_j(t) z_j(\mathbf{r}) \tag{11}$$

in which $z_j(\mathbf{r})$ are the eigenfunctions of the eigenvalue problem associated to the thermal problem

$$\nabla \cdot (-k(\mathbf{r})\nabla z_j(\mathbf{r})) = \lambda_j c(\mathbf{r}) z_j(\mathbf{r}) \qquad \mathbf{r} \in \Omega, \tag{12}$$

with boundary conditions

$$-k(\mathbf{r})\frac{\partial z_j}{\partial \nu}(\mathbf{r}) = h(\mathbf{r}) z_j(\mathbf{r}) \qquad \mathbf{r} \in \partial\tau. \tag{13}$$

The eigenvalues λ_j are real, positive and constitute a divergent, monotonically increasing sequence. The eigenfunctions $z_j(\mathbf{r})$ are real functions of position satisfying the orthonormality relations

$$\int_\Omega c(\mathbf{r}) z_j(\mathbf{r}) z_k(\mathbf{r})\, d\mathbf{r} = \delta_{jk} \tag{14}$$

in which δ_{jk} is Kronecher's delta. Coefficients $a_j(t)$ in Eq. (11) are solutions of equations

$$\frac{d}{dt} a_j(t) + \lambda_j a_j(t) = \int_\Omega z_j(\mathbf{r}) G(\mathbf{r}, t)\, d\mathbf{r}, \tag{15}$$

with zero initial conditions. The solutions to these initial value problems are

$$a_j(t) = e^{-\lambda_j t} * \int_\Omega z_j(\mathbf{r}) G(\mathbf{r}, t)\, d\mathbf{r}, \tag{16}$$

in which $*$ is the convolution operator in the time domain.

The solution of the passive multi-port dynamic thermal network can then be expressed as follows. From Eqs. (4), (16) it results in

$$a_j(t) = e^{-\lambda_j t} * \mathbf{\Gamma}_j^T \mathbf{P}(t). \tag{17}$$

in which

$$\mathbf{\Gamma}_j = \int_\Omega z_j(\mathbf{r}) \mathbf{g}(\mathbf{r})\, d\mathbf{r}. \tag{18}$$

From Eqs. (5), (11), it results in

$$\mathbf{T}(t) = \sum_{j}{}_{1}^{\infty} \mathbf{\Gamma}_j a_j(t). \tag{19}$$

Thus substituting Eq. (17) into Eq. (19), it follows

$$\mathbf{T}(t) = \mathbf{Z}(t) * P(t)$$

in which

$$\mathbf{Z}(t) = \sum_{1}^{\infty}{}_{j} \mathbf{\Gamma}_j \mathbf{\Gamma}_j^T e^{-\lambda_j t} \tag{20}$$

is the *power impulse thermal response matrix* of the passive multi-port dynamic thermal network. Taking the Laplace transform of Eq. (20) it also follows

$$\mathsf{T}(s) = \mathsf{Z}(s)\mathsf{P}(s)$$

in which

$$\mathsf{Z}(s) = \sum_{1}^{\infty}{}_{j} \frac{\mathbf{\Gamma}_j \mathbf{\Gamma}_j^T}{s + \lambda_j} \tag{21}$$

is the *thermal impedance matrix* of the multi-port dynamic thermal network.

4 Preliminary Results on Passive Multi-Port Dynamic Thermal Networks

Passive multi-port dynamic thermal networks are a generalization of passive multi-port lumped RC networks. In fact their impedance matrices satisfy properties common to passive multi-port lumped RC networks.

Theorem 5

1. *Impedance matrix* $\mathsf{Z}(s)$ *with* $s = \sigma + i\omega$ *is* symmetric *and* positive real *[7]. That is, for* $\sigma > 0$,

   $\mathsf{Z}(s)$ *is analytic*,
   $\mathsf{Z}(\bar{s}) = \bar{\mathsf{Z}}(s)$,
   Re $\mathsf{Z}(s)$ *is positive definite*,

   in which the bar indicates the complex conjugate operator.

2. *Poles of* $\mathsf{Z}(s)$ *are simple, real, negative and form a divergent, monotonically decreasing sequence* $-\lambda_1, -\lambda_2, \ldots$.

3. *The residues at the poles of* $\mathsf{Z}(s)$ *are real, symmetric, positive semi-definite.*

4. *On the positive real axis* $-\mathsf{Z}'(\sigma)$ *is symmetric, positive definite.*

Since the shape functions defining the passive multi-port dynamic thermal network are linearly independent, an admittance matrix $\mathsf{Y}(s)$, inverse of $\mathsf{Z}(s)$, exists. Such admittance matrix satisfies the following properties common to that of passive multi-port lumped RC networks.

Theorem 6

1. *Matrix* $\mathsf{Y}(s)$ *is* symmetric *and* positive real *[7]. That is for* $\sigma > 0$

   $\mathsf{Y}(s)$ *is analytic,*
   $\mathsf{Y}(\bar{s}) = \bar{\mathsf{Y}}(s)$,
   Re $\mathsf{Y}(s)$ *is positive definite.*

2. *Poles of* $\mathsf{Y}(s)$ *are simple, real, negative and form a divergent, monotonically decreasing sequence* $-\mu_1, -\mu_2, \ldots$.

3. *The residues at the poles of* $\mathsf{Y}(s)/s$ *are real, symmetric, positive semi-definite.*

4. *On the positive real axis* $\mathsf{Y}'(\sigma)$ *is symmetric, positive definite.*

As a consequence of Theorems 5 and 6, a multi-port passive distributed thermal network can be approximated at $s \to 0$ by the parallel connection of a passive resistive multi-port and a passive capacitive multi-port and at $s \to \infty$ by a passive capacitive multi-port, as stated in the following

Theorem 7 *At* $s \to 0$, *the* $\mathsf{Z}(s)$ *impedance matrix converges to the impedance of the parallel connection of a passive resistive multi-port of resistance matrix* $\mathbf{R}_0$ *and of a passive capacitive multi-port of capacitance matrix* $\mathbf{C}_0$, *being*

$$\mathbf{R}_0 = \mathsf{Z}(0), \tag{22}$$
$$\mathbf{C}_0 = \mathsf{Y}'(0). \tag{23}$$

Theorem 8 *For* $s \to \infty$ *with* $\sigma > 0$, *the* $\mathsf{Y}(s)$ *admittance converges to the admittance of a passive capacitive multi-port of capacitance matrix*

$$\mathbf{C}_\infty = \lim_{s\to\infty} \frac{\mathsf{Y}(s)}{s} = \lim_{s\to\infty} \mathsf{Y}'(s). \tag{24}$$

The $\mathbf{R}_0$ matrix, the $\mathbf{C}_0$ matrix and the inverse of $\mathbf{C}_\infty$ matrix are hereafter referred to respectively as *total resistance matrix*, *total capacitance matrix* and *total elastance matrix* of the passive multi-port dynamic thermal network.

5 Generalized Foster I Canonical Form

As a consequence of Theorem 5

Theorem 9

$$\mathsf{Z}(s) = \sum_{1}^{\infty}{}_{j} \frac{\mathbf{r}_j}{1 + s/\lambda_j} = \sum_{1}^{\infty}{}_{j} \frac{\mathbf{e}_j}{s + \lambda_j} \tag{25}$$

in which $\mathbf{r}_j = \mathbf{e}_j/\lambda_j$ *are real, symmetric, positive semi-definite matrices.*

Eq. (25) defines an infinite network composed of ideal transformers, passive resistors and passive capacitors which generalizes the Foster I canonical form of a passive multi-port lumped RC network [7]. The resistance matrix of the series connections of the passive resistive multi-ports having $\mathbf{r}_j$ resistance matrices is the total resistance matrix $\mathbf{R}_0$. Thus the generalized Foster I canonical form defines partial resistance matrices of $\mathbf{R}_0$. Similarly the elastance matrix of the series connections of the passive capacitive multi-ports having $\mathbf{e}_j$ elastance matrices is the total elastance matrix, inverse of $\mathbf{C}_\infty$. Thus the generalized Foster I canonical form defines partial elastance matrices of the inverse of $\mathbf{C}_\infty$. The Foster I canonical form can be defined by the *cumulative resistance matrix*

$$\boldsymbol{\mathcal{R}}(\lambda) = \sum_{1}^{\infty}{}_{j} \mathbf{r}_j H(\lambda - \lambda_j),$$

$H(\cdot)$ being Heaviside's step function, equivalent to the time-constant representation [4] or by the *cumulative elastance matrix*

$$\boldsymbol{\mathcal{E}}(\lambda) = \sum_{j}^{\infty}{}_{1} \mathbf{e}_j H(\lambda - \lambda_j).$$

6 Generalized Foster II Canonical Form

As a consequence of Theorem 6, the admittance matrix $\mathbf{Y}(s)$, can be represented as follows

Theorem 10

$$\mathbf{Y}(s) = s\mathbf{C}_\infty + \mathbf{R}_0^{-1} + \sum_{j=1}^{\infty} \frac{s\mathbf{c}_j}{1 + s/\mu_j} \tag{26}$$

in which $\mathbf{c}_j$ *are symmetric positive semi-definite.*

Eq. (26) defines an infinite network composed of ideal transformers passive resistors and passive capacitors which generalizes the Foster II canonical form of a passive multi-port lumped RC network. The resistance matrix of the passive resistive multi-port is $\mathbf{R}_0$. The capacitance matrix of the passive capacitive multi-port is $\mathbf{C}_\infty$. The capacitance matrix of the parallel connections of the capacitive multi-ports having $\mathbf{C}_\infty$ and $\mathbf{c}_j$ capacitance matrices is the total capacitance matrix $\mathbf{C}_0$. Thus the generalized Foster II canonical form defines partial capacitances of the total capacitance matrix $\mathbf{C}_0$. The Foster II canonical form can be defined by the *cumulative capacitance matrix*

$$\boldsymbol{\mathcal{C}}(\lambda) = \mathbf{C}_\infty + \sum_{j=1}^{\infty} \mathbf{c}_j H(\lambda - \mu_j).$$

and by $\mathbf{R}_0$.

7 Application Example: Part I

A cylinder Ω of length L, area A, thermal conductivity k and heat capacity c is considered. The powers $P_1(t)$, $P_2(t)$ are uniformly generated within the lower and upper halves of the cylinder respectively. On the lower and upper face of the boundary $\partial\Omega$ the temperature is set equal to the ambient temperature. On the rest of the boundary $\partial\Omega$ the thermal flux is set to zero. According to Eq. (6), the mean temperature rises in the lower and upper halves of the cylinder are the $T_1(t)$ and $T_2(t)$ temperature rise of a passive 2-port dynamic thermal network.

The thermal impedance matrix is

$$\mathbf{Z}(s) = \frac{L}{kA}\mathbf{K}\left(\frac{L^2c}{k}s\right) \tag{27}$$

in which

$$\mathbf{K}(p) = \frac{1}{p}\begin{bmatrix} f\left(\frac{\sqrt{p}}{4}\right) + f\left(\frac{\sqrt{p}}{2}\right) & f\left(\frac{\sqrt{p}}{4}\right) - f\left(\frac{\sqrt{p}}{2}\right) \\ f\left(\frac{\sqrt{p}}{4}\right) - f\left(\frac{\sqrt{p}}{2}\right) & f\left(\frac{\sqrt{p}}{4}\right) + f\left(\frac{\sqrt{p}}{2}\right) \end{bmatrix}$$

and

$$f(q) = 1 - \frac{\tanh q}{q}.$$

Thus from Eq. (27) and Eqs. (22), (23), (24) it results in

$$\mathbf{R}_0 = \frac{L}{kA}\begin{bmatrix} \frac{5}{48} & \frac{1}{16} \\ \frac{1}{16} & \frac{5}{48} \end{bmatrix},$$

$$\mathbf{C}_0 = LAc\begin{bmatrix} \frac{3}{5} & 0 \\ 0 & \frac{3}{5} \end{bmatrix},$$

$$\mathbf{C}_\infty = LAc\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

The generalized Foster I and II canonical forms of this thermal network can be determined analytically in closed form. Determining the Mittag-Leffler's partial fractions expansion of $\mathbf{K}(p)$, the Foster I canonical form follows

$$\lambda_j = \begin{cases} \frac{k}{L^2c}\pi^2(4k+1)^2 & j = 3k+1, \\ \frac{k}{L^2c}\pi^2(4k+2)^2 & j = 3k+2, \\ \frac{k}{L^2c}\pi^2(4k+3)^2 & j = 3k+3, \end{cases}$$

$$\mathbf{r}_j = \frac{\mathbf{e}_j}{\lambda_j} = \begin{cases} \frac{L}{kA}\frac{64}{\pi^4(4k+1)^4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} & j = 3k+1, \\ \frac{L}{kA}\frac{64}{\pi^4(4k+2)^4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} & j = 3k+2, \\ \frac{L}{kA}\frac{256}{\pi^4(4k+3)^4}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & j = 3k+3, \end{cases}$$

k being any natural number. The $\boldsymbol{\mathcal{R}}(\lambda)$ cumulative resistance matrix defining the Foster I canonical form is shown in Fig. 1.

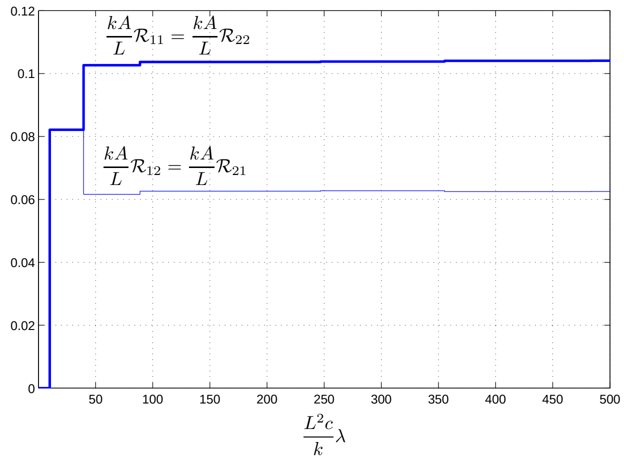

Figure 1: $\boldsymbol{\mathcal{R}}(\lambda)$ cumulative resistance matrix defining Foster I canonical form.

The admittance matrix $\mathbf{Y}(s)$ is

$$\mathbf{Y}(s) = \frac{kA}{L}\mathbf{H}\left(\frac{L^2c}{k}s\right) \tag{28}$$

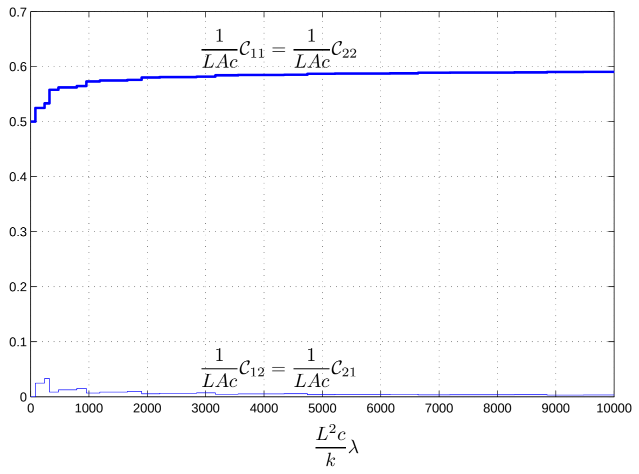

Figure 2: $\boldsymbol{\mathcal{C}}(\lambda)$ cumulative capacitance matrix defining Foster II canonical form.

in which

$$\mathbf{H}(p) = \frac{p}{4}\begin{bmatrix} \frac{1}{f\left(\frac{\sqrt{p}}{2}\right)} + \frac{1}{f\left(\frac{\sqrt{p}}{4}\right)} & \frac{1}{f\left(\frac{\sqrt{p}}{2}\right)} - \frac{1}{f\left(\frac{\sqrt{p}}{4}\right)} \\ \frac{1}{f\left(\frac{\sqrt{p}}{2}\right)} - \frac{1}{f\left(\frac{\sqrt{p}}{4}\right)} & \frac{1}{f\left(\frac{\sqrt{p}}{2}\right)} + \frac{1}{f\left(\frac{\sqrt{p}}{4}\right)} \end{bmatrix}$$

Determining the Mittag-Leffler's partial fractions expansion of $\mathbf{H}(p)$, the Foster II canonical form follows

$$\mu_j = \begin{cases} \frac{k}{L^2 c}\left(2\xi_{2k+1}\right)^2 & j = 3k+1, \\ \frac{k}{L^2 c}\left(2\xi_{2k+2}\right)^2 & j = 3k+2, \\ \frac{k}{L^2 c}\left(4\xi_{k+1}\right)^2 & j = 3k+3, \end{cases},$$

$$\mathbf{c}_j = \begin{cases} LAc\,\frac{1}{\xi_{2k+1}^2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} & j = 3k+1, \\ LAc\,\frac{1}{\xi_{2k+2}^2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} & j = 3k+2, \\ LAc\,\frac{1}{\xi_{k+1}^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & j = 3k+3, \end{cases},$$

k being any natural number and ξ_j being the j-th *positive* root of the equation

$$\tan\xi_j = \xi_j.$$

The $\boldsymbol{\mathcal{C}}(\lambda)$ cumulative capacitance matrix defining the Foster II canonical form is shown in Fig. 2.

8 Generalized Cauer I Canonical Form

Let us consider a passive *multi-conductor* RC transmission line of length $\bar{x}$ in the x dimension described at a complex frequency s by equations

$$\frac{\partial\mathbf{V}}{\partial x}(x,s) = -\mathbf{r}(x)\mathbf{I}(x,s) \qquad (29)$$

$$\mathbf{e}(x)\frac{\partial\mathbf{I}}{\partial x}(x,s) = -s\mathbf{V}(x,s) \qquad (30)$$

in which $\mathbf{V}(x,s)$, $\mathbf{I}(x,s)$ are the Laplace transforms of the voltage and current $n\times 1$ vectors at x and $\mathbf{r}(x)$, $\mathbf{e}(x)$ are symmetric, positive semi-definite $n\times n$ matrices representing the resistance matrix density and the elastance matrix density of the line at x.

By introducing the impedance matrix $\mathbf{Z}(x,s)$ at each x along the line, Eqs. (29), (30) can be reduced to the single Riccati-type matrix equation

$$\frac{\partial\mathbf{Z}}{\partial x}(x,s) + \mathbf{r}(x) = s\mathbf{Z}(x,s)\mathbf{e}^{+}(x)\mathbf{Z}(x,s), \qquad (31)$$

in which $+$ is the pseudo-inverse operator. Thus, if the output port of the line is short-circuited, the impedance matrix $\mathbf{Z}(x,s)$ and, in particular, the input impedance matrix

$$\mathbf{Z}(s) = \mathbf{Z}(0,s) \qquad (32)$$

can be determined by solving Eq. (31) with boundary condition

$$\mathbf{Z}(\bar{x},s) = \mathbf{0}. \qquad (33)$$

The inverse problem can also be considered. By assigning the input impedance matrix $\mathbf{Z}(s)$ when the output port of the line is short-circuited, a passive multi-conductor RC transmission line can be determined, by solving Eqs. (31), (32), (33) for $\mathbf{r}(x)$ and $\mathbf{e}(x)$. In this way, as a consequence of Theorem 5, the following result can be proved

Theorem 11 *The $\mathbf{Z}(s)$ impedance matrix of a passive multi-port dynamic thermal network is the short-circuit input impedance matrix of a passive multi-conductor RC transmission line ruled by Eqs. (29), (30).*

Thus passive multi-port dynamic thermal networks can be represented by passive multi-conductor RC transmission lines with short-circuited output ports. This is the generalization of Cauer I canonical form of passive multi-port lumped RC networks to passive multi-port dynamic thermal networks.

As shown in [5], with passive one-port dynamic thermal networks, the RC transmission line is not uniquely determined. Similarly with passive multi-port dynamic thermal networks, the multi-conductor RC transmission line is not uniquely determined.

Such passive RC multi-conductor transmission line defines partial resistance matrices of the $\mathbf{R}_0$ total resistance matrix and can be decomposed into the shunt connection of a passive capacitive multi-port of capacitance matrix $\mathbf{C}_\infty$ and a second passive multi-conductor RC transmission line.

9 Generalized Cauer II Canonical Form

As a consequence of Theorem 6, it can be proved that a continued fraction expansion can be performed for the impedance matrix $\mathbf{Z}(s)$ of a passive multi-port dynamic thermal network, exactly as when determining the Cauer II canonical form of a passive multi-port lumped RC network [7].

Theorem 12 *Given the impedance matrix* $\mathbf{Z}(s)$ *of a passive multi-port dynamic thermal network, it results in*

$$\mathbf{Z}(s) = \left(\mathbf{r}_1^+ + \left(\frac{\mathbf{e}_1}{s} + \left(\mathbf{r}_2^+ + \left(\frac{\mathbf{e}_2}{s} + \dots\right)^+\right)^+\right)^+\right)^+ \quad (34)$$

in which $\mathbf{r}_1,\ \mathbf{r}_2,\dots$ *and* $\mathbf{e}_1,\ \mathbf{e}_2,\dots$ *are symmetric, positive semi-definite.*

This expansion defines an infinite network composed of passive resistive multi-ports of resistance matrices $\mathbf{r}_1,\ \mathbf{r}_2,\dots$ and of passive capacitive multi-ports of elastance matrices $\mathbf{e}_1,\ \mathbf{e}_2,\dots$ which generalizes the Cauer II canonical form of a passive multi-port dynamic thermal network. The first resistance matrix $\mathbf{r}_1$ is the total resistance matrix $\mathbf{R}_0$, the first elastance matrix $\mathbf{e}_1$ is the inverse of the total capacitance matrix $\mathbf{C}_0$. The elastance matrix of the series connections of the passive capacitive multi-ports of elastance matrices $\mathbf{e}_1,\ \mathbf{e}_2,\dots$ is the inverse of matrix $\mathbf{C}_\infty$. Thus the Cauer II canonical form of a passive multi-port dynamic thermal networks defines partial elastance matrices of the inverse of matrix $\mathbf{C}_\infty$.

10 Application Example: Part II

From Eqs. (31), (32), (33), a generalized Cauer I canonical form of the passive multi-port dynamic thermal network of section 7, is

$$\mathbf{r}(x) = \begin{cases} \frac{L}{kA}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, & 0 \le x < 1 \\ \frac{L}{kA}\begin{bmatrix} \frac{5}{16} & \frac{3}{16} \\ \frac{3}{16} & \frac{5}{16} \end{bmatrix} \cdot \frac{1}{x^4}, & x \ge 1 \end{cases}$$

$$\mathbf{e}(x) = \left\{ \frac{1}{LAc}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \ x \ge 0 \right.$$

as shown in Fig. 3.

The Cauer II canonical form can be determined by performing a continued fraction expansion of $\mathbf{Z}(s)$. The first resistance matrices are

$$\mathbf{r}_1 = \frac{L}{kA}\begin{bmatrix} \frac{5}{48} & \frac{1}{16} \\ \frac{1}{16} & \frac{5}{48} \end{bmatrix},$$

$$\mathbf{r}_2 = \frac{L}{kA}\begin{bmatrix} \frac{5}{4032} & \frac{1}{1344} \\ \frac{1}{1344} & \frac{5}{4032} \end{bmatrix},$$

$$\mathbf{r}_3 = \frac{L}{kA}\begin{bmatrix} \frac{1}{7920} & \frac{1}{13200} \\ \frac{1}{13200} & \frac{1}{7920} \end{bmatrix},\dots$$

$\mathbf{r}_1$ being $\mathbf{R}_0$. The first elastance matrices are

$$\mathbf{e}_1 = \frac{1}{LAc}\begin{bmatrix} \frac{5}{3} & 0 \\ 0 & \frac{5}{3} \end{bmatrix},$$

$$\mathbf{e}_2 = \frac{1}{LAc}\begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{5} \end{bmatrix},$$

$$\mathbf{e}_3 = \frac{1}{LAc}\begin{bmatrix} \frac{13}{210} & 0 \\ 0 & \frac{13}{210} \end{bmatrix},\dots$$

$\mathbf{e}_1$ being the inverse of $\mathbf{C}_0$.

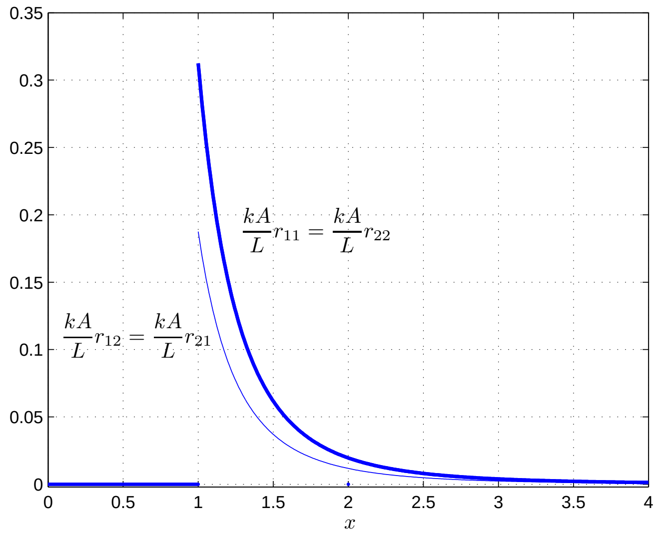

Figure 3: Resistance matrix of the Cauer I canonical form.

11 Conclusions

In this paper four canonical forms have been introduced for multi-port dynamic thermal networks, which generalize the four canonical forms of passive multi-port lumped RC networks. In particular it has been shown that the generalized Foster I canonical form is equivalent to the time-constant representation and that the generalized Cauer I canonical form is a passive multi-conductor RC transmission line.

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