Chapter 1
Bernstein-type polynomials on several intervals

J. Szabados[*]

Dedicated to the memory of Professor Qazi Ibadur Rahman

Abstract We construct the analogues of Bernstein polynomials on the set J_s of s finitely many intervals. Two cases are considered: first when there are no restrictions on J_s, and then when J_s has a so-called T-polynomial. On such sets we define approximating operators resembling the classic Bernstein polynomials. Reproducing and interpolation properties, as well as estimates for the rate of convergence are given.

Keywords: Bernstein polynomial, T-polynomial, set of intervals, rate of convergence
2000 Mathematics Subject Classification: 41A10, 41A25

1.1 Introduction

For any $s \geq 1$ let
$$0 = a_1 < b_1 < \cdots < a_s < b_s = 1$$
be a finite partition of the interval $[0,1]$, let $I_j := [a_j, b_j]$, $j = 1, \ldots, s$, and let
$$J_s := \bigcup_{j=1}^{s} I_j, \quad I_j := [a_j, b_j]$$

J. Szabados
Alfréd Rényi Institute of Mathematics,
P.O.B. 127, H-1364 Budapest, Hungary
e-mail: szabados.jozsef@renyi.mta.hu

be the corresponding set of s pairwise disjoint intervals, and denote by Π_n the set of polynomials of degree at most n. Denote by $C(J_s)$ the set of continuous functions on J_s.

Some basic problems of approximation theory, like Markov–Bernstein inequalities or Lagrange interpolation have been considered for functions $f \in C(J_s)$; see for example [4] or [2]. It is equally important to establish analogues of classic operators on this set. In this context we will be concerned with generalizations of Bernstein polynomial.

1.2 The general case

The simplest construction for a general set J_s is the following. If $f \in C(J_s)$ is not differentiable, then extend it to $[0,1] \backslash J_s$ by defining it as a linear function in each $[b_j, a_{j+1}]$, $j = 1, \ldots, s-1$, continuously connecting the different parts of the original function. If $f' \in C(J_s)$ then in each $[b_j, a_{j+1}]$, $j = 1, \ldots, s-1$, define f as the third degree Hermite polynomial satisfying

$$H_j(b_j) = f(b_j), \ H_j'(b_j) = f'(b_j), \ H_j(a_{j+1}) = f(a_{j+1}), \ H_j'(a_{j+1}) = f'(a_{j+1}),$$

$1 \leq j \leq s-1$. Then evidently the modulus of continuity $\omega(\tilde{f}, t) \leq c\omega(f, t)$ in the first case, and $\omega(\tilde{f}', t) \leq c\omega(f', t)$ in the second case. Thus considering the ordinary Bernstein polynomials for the extended function $\tilde{f}$ on $[0,1]$, we obtain the usual error estimates $c\omega(f, 1/\sqrt{n})$ and $\omega(f', 1/\sqrt{n})/\sqrt{n}$ in the corresponding cases, respectively. (In the latter case besides function values, derivative values at the endpoints I_j of the function also appear in the construction.)

This method has two disadvantages. First, in the classic case the difference between the number of data used in the construction and the degree of the Bernstein polynomials is 1, while here it is cn. Second, the classic Bernstein polynomials interpolate at the endpoints of the interval, while here not.

1.3 The case of T-polynomials

It is well-known that given arbitrary positive integer m, a set J_s always has a so-called Chebyshev polynomial of degree m, i.e. a polynomial which attains its maximum and minimum on J_s at consecutive $m+1$ points in J_s. If, in addition, this polynomial attains its maximum or minimum on J_s at the endpoints of I_j, $j = 1, \ldots, s$, then we say that this is a T-polynomial. Of course, this additional property does not hold for all sets J_s. It was F. Peherstorfer who introduced this notion, and thoroughly investigated its existence and properties (see [5], [6]).

It will be convenient to assume that a T-polynomial has a minimum 0 and maximum 1 on J_s.

Assume that J_s has a T-polynomial $p(x) \in \Pi_m$, $m \geq s$ normalized such that $p(0) = 0$. For $n \in \mathbb{N}$, let $x_{k1} < \cdots < x_{km_k}$ be defined by

$$p(x_{ki}) = \frac{k}{n}, \qquad i = 1, \ldots, m_k; \quad k = 0, \ldots, n$$

where

$$m_k = \begin{cases} m + s - \left[\dfrac{m+s}{2}\right], & \text{if} \quad k = 0, \\ m, & \text{if} \quad k = 1, \ldots, n-1, \\ \left[\dfrac{m+s}{2}\right], & \text{if} \quad k = n. \end{cases}$$

The existence of such x_{ki}'s follows from the properties of T-polynomials (they are monotone between two adjacent extremal values).

For an arbitrary $f(x) \in C(J_s)$, let

$$L_k(f, x) = \sum_{i=1}^{m_k} f(x_{ki}) \ell_{ki}(x) \in \Pi_{m_k - 1}, \qquad k = 0, 1, \ldots, n,$$

be the Lagrange interpolation polynomial with respect to the nodes x_{ki}. Here $\ell_{ki}(x) \in \Pi_{m_k - 1}$ are the fundamental polynomials with the property

$$\ell_{ki}(x_{kj}) = \delta_{ij}, \qquad i, j = 1, \ldots, m_k, \quad k = 0, 1, \ldots, n. \tag{1.1}$$

Consider the discrete linear operator

$$B_n(f, x) := \sum_{k=0}^{n} L_k(f, x) b_{nk}(p(x)), \qquad x \in J_s, \tag{1.2}$$

where

$$b_{nk}(x) = \binom{n}{k} x^k (1 - x)^{n-k}, \qquad k = 0, \ldots, n,$$

are the fundamental functions of the Bernstein polynomials. Evidently, $B_n(f, x) \in \Pi_{mn+m-1}$, and there are $mn + s$ function values used in the construction of the operator. This means that the difference between the number of function values and the degree of the operator is $m - s + 1$, i.e. independent of n, just like in case of the classic Bernstein polynomials. In this respect they are better than the polynomials defined in the previous section.

Although this is not a positive operator, it still has a bounded norm; this will follow from Theorem 1 below.

We mention two properties of this operator. The first one is about reproducing polynomials:

$$B_n(q, x) \equiv q(x) \qquad \text{if} \quad q \in \Pi_{m_n - 1} \quad \text{or} \quad q = p. \tag{1.3}$$

Namely, if $q \in \Pi_{m_n - 1}$, then $L_k(q, x) \equiv q(x)$, $k = 0, 1, \ldots, n$ by the reproducing property of Lagrange interpolation, and thus

$$B_n(q, x) = q(x) \sum_{k=0}^{n} b_{nk}(p(x)) = q(x).$$

And if $q = p$, then $L_k(p, x) \equiv \dfrac{k}{n}$, $k = 0, 1, \ldots, n$, whence

$$B_n(p, x) = \sum_{k=0}^{n} \frac{k}{n} b_{nk}(p(x)) = p(x),$$

since the classic Bernstein polynomials reproduce linear functions. Again, this is better than the corresponding reproducing property of the classic Bernstein polynomials (which reproduce only linear functions).

The second property is about interpolation:

$$B_n(f, x_{ki}) = f(x_{ki}), \qquad i = 1, \ldots, m_k; \quad k = 0 \text{ or } n. \tag{1.4}$$

Namely, $p(x_{0i}) = 0$, $i = 1, \ldots, m_0$, and thus

$$B_n(f, x_{0i}) = L_0(f, x_{0i}) = f(x_{0i}), \qquad i = 1, \ldots, m_0.$$

Similarly for the x_{ni}'s. Notice that the polynomials defined in the previous section interpolate only at 0 and 1.

Now we state a pointwise convergence estimate. Let

$$\varphi(x) = \sqrt{(x - a_j)(b_j - x)} \qquad \text{if} \quad x \in I_j, \ j = 1, \ldots, s,$$

and define the Ditzian–Totik modulus of continuity (cf. [1], Ch. 1) as

$$\omega_\varphi(f, t) = \sup_{0 < h \le t} \|\Delta_{h\varphi(x)} f(x)\|_{J_s}$$

where the difference is meant to be zero if any of the arguments is outside J_s, and we assume that t is so small that both $x \pm \varphi(x)$ fall into the same interval I_j. Further let

$$V(f) = \sup_{x, y \in J_s} |f(x) - f(y)|.$$

Theorem 1. *For an arbitrary $f \in C(J_s)$ we have*

$$\|f(x) - B_n(f, x)\|_{J_s} \le c\omega_\varphi\left(f, \frac{1}{\sqrt{n}}\right) + c\frac{V(f)}{\sqrt{n}}.$$

(Here and in what follows, c will always denote a positive constant depending on J_s and m, but independent of n, not necessarily the same at each occurrence.) Note that the second term in the error estimate cannot be dropped, since for functions which are constant on each interval I_j, the modulus of continuity is zero, while the operator does not reproduce such functions.

For the proof we need the following

Lemma 1. *Let (α, β) be one of the pairs of numbers $(0, 1), (1/2, 1), (1, 1),$ $(1, 2), (3/2, 2)$. Then*

$$\sum_{k=1}^{[n/2]} \left(\frac{n}{k}\right)^\alpha \left|x - \frac{k}{n}\right|^\beta b_{nk}(x) \leq \begin{cases} c\left(\dfrac{x}{n}\right)^{\frac{\beta-\alpha}{2}}, & if \quad 0 \leq x \leq 1/n, \\ c\dfrac{x^{\beta/2-\alpha}}{n^{\beta/2}}, & if \quad 1/n \leq x \leq 1. \end{cases}$$

Proof. By Cauchy–Schwarz inequality we get for $0 \leq x \leq 1/n$,

$$T := \sum_{k=1}^{[n/2]} \left(\frac{n}{k}\right)^\alpha \left|x - \frac{k}{n}\right|^\beta b_{nk}(x) \leq \left(\sum_{k=1}^{[n/2]} \left(\frac{n}{k}\right)^{2\alpha} \left(x - \frac{k}{n}\right)^{2\beta} b_{nk}(x)\right)^{1/2}$$

$$\leq \left(\sum_{k=1}^{[n/2]} \left(x - \frac{k}{n}\right)^{2\beta-2\alpha} b_{nk}(x)\right)^{1/2}.$$

Now we use the inequalities

$$\sum_{k=0}^{n} \left|x - \frac{k}{n}\right|^{2\gamma} b_{nk}(x) \leq c\left(\frac{x}{n}\right)^\gamma, \tag{1.5}$$

where $\gamma = 1/2$ or 1 and $0 \leq x \leq 1$, or $\gamma = 2$ and $1/n \leq x \leq 1$. These follow from well-known relations (cf. Lorentz [3], p. 14). Hence we obtain the first statement of the lemma.

Next let $1/n \leq x \leq 1$. We obtain, using again (1.5),

$$T \leq \left(\sum_{k=1}^{[n/2]} \left(\frac{n}{k}\right)^{2\alpha} \left(x - \frac{k}{n}\right)^{2\beta} b_{nk}(x)\right)^{1/2} \leq \frac{1}{x^\alpha}\left(\sum_{k=1}^{[n/2]} \left(x - \frac{k}{n}\right)^{2\beta} b_{n+2\alpha,k}(x)\right)^{1/2}$$

$$\leq \frac{c}{x^\alpha}\left(\sum_{k=1}^{[n/2]} \left(x - \frac{k}{n+2\alpha}\right)^{2\beta} b_{n+2\alpha,k}(x)\right)^{1/2} + c\frac{1}{n^{\beta-\alpha}} \leq c\frac{x^{\beta/2-\alpha}}{n^{\beta/2}}. \quad \square$$

Proof of Theorem 1. Let $x \in (\xi, \eta) \subset I_j = [a_j, b_j]$ be such that $p(\xi) = 0, p(\eta) = 1$ and $p'(x) > 0$ (the case $p'(x) < 0$ can be handled similarly). Let

$p(x_{kt}) = \dfrac{k}{n}$ where $x_{kt} \in [\xi, \eta]$. Let

$$S_k = \begin{cases} \{t\}, & \text{if} \quad \xi = a_j, \\ \{t-1, t\}, & \text{if} \quad \xi > a_j. \end{cases}$$

Since both operators L_k and the classic Bernstein polynomials reproduce constants, we get

$$|f(x) - B_n(f, x)| \leq \sum_{k=0}^{n} \sum_{i=1}^{m_k} |f(x) - f(x_{ki})| \cdot |\ell_{ki}(x)| b_{nk}(p(x))$$

$$\leq c \sum_{k=0}^{n} \left\{ \omega_\varphi \left(f, \frac{1}{\sqrt{n}} \right) \sum_{i \in S_k} \left[\frac{\sqrt{n}}{\varphi(x)} (x - x_{ki}) + 1 \right] |\ell_{ki}(x)| \right.$$

$$\left. + V(f) \sum_{i \notin S_k} |\ell_{ki}(x)| \right\} b_{nk}(p(x)).$$

Here we used the inequality $\varphi \left(\dfrac{x + x_{ki}}{2} \right) \geq c\varphi(x), \; i \in S_k$.

We estimate the right hand side sum for $0 \leq k \leq n/2$; the other part can be handled similarly. Then it is sufficient to consider the case $0 \leq p(x) \leq 4/5$, since for $4/5 \leq p(x) \leq 1$ we get

$$\sum_{k=0}^{[n/2]} \left(\frac{n}{k} \right)^\alpha b_{nk}(p(x)) \leq n^\alpha (1 - p(x))^{n/2} \sum_{k=0}^{n} \binom{n}{k} p(x)^k$$

$$\leq n^\alpha 2^n (1/5)^{n/2} = n^\alpha (4/5)^{n/2}.$$

This geometric convergence to zero implies that all similar sums to be handled subsequently are minor compared to the estimates for $0 \leq p(x) \leq 4/5$. Also, it is sufficient to consider the case when $\eta < b_j$, since $\eta = b_j$ corresponds to the case $\xi = a_j$.

Let first $k = 0$. Then we have $|\ell_{0i}(x)| \leq c, \; i = 0, 1, \ldots, m_0$, and

$$\frac{(x - x_{0i})|\ell_{0i}(x)|}{\varphi(x)} \leq c \frac{x - x_{0t}}{\varphi(x)} \leq c\sqrt{p(x)}, \qquad i = 1, \ldots, m_0.$$

Hence

$$\left(\frac{\sqrt{n}}{\varphi(x)} (x - x_{0i}) + 1 \right) |\ell_{0i}(x)| b_{n0}(p(x)) \leq c(\sqrt{np(x)}(1 - p(x))^n + c \leq c.$$

Now let $1 \leq k \leq n/2$. In order to complete the proof of Theorem 1 we need to prove the following estimates:

$$A := \sum_{k=1}^{[n/2]} \sum_{i \in S_k} |\ell_{ki}(x)| b_{nk}(p(x)) \le c \,,$$

$$B := \sum_{k=1}^{[n/2]} \sum_{i \in S_k} |x - x_{ki}| \cdot |\ell_{ki}(x)| b_{nk}(p(x)) \le \frac{\varphi(x)}{\sqrt{n}} \,,$$

$$C := \sum_{k=1}^{[n/2]} \sum_{i \notin S_k} |\ell_{ki}(x)| b_{nk}(p(x)) \le \frac{c}{\sqrt{n}} \,.$$

We distinguish two cases.

Case 1: $S_k = \{t\}$. Then $|\ell_{kt}(x)| \le c$, whence $A \le c \sum_{k=1}^{[n/2]} b_{nk}(p(x)) \le c$. On the other hand, $|x - x_{kt}| \le c \left| p(x) - \dfrac{k}{n} \right|$, therefore by Lemma 1 applied with $\alpha = 0, \beta = 1$ yields

$$B \le c \sum_{k=1}^{[n/2]} |x - x_{kt}| b_{nk}(p(x)) \le c \sum_{k=1}^{[n/2]} \left| p(x) - \frac{k}{n} \right| b_{nk}(p(x)) \le c \sqrt{\frac{p(x)}{n}} \le c \frac{\varphi(x)}{\sqrt{n}} \,.$$

Finally, using $|\ell_{ki}(x)| \le c \sqrt{\dfrac{n}{k}} |x - x_{kt}|$, $i \notin S_k$, and Lemma 1 with $\alpha = \dfrac{1}{2}, \beta = 1$,

$$C \le c \sum_{k=1}^{[n/2]} \sqrt{\frac{n}{k}} |x - x_{kt}| b_{nk}(p(x)) \le c \sum_{k=1}^{[n/2]} \sqrt{\frac{n}{k}} \left| p(x) - \frac{k}{n} \right| b_{nk}(p(x)) \le \frac{c}{\sqrt{n}} \,.$$

Case 2: $S_k = \{t-1, t\}$. Then

$$|\ell_{kt}(x)| \le c \frac{x - x_{k,t-1}}{x_{kt} - x_{k,t-1}} \le 1 + c \sqrt{\frac{n}{k}} |x - x_{kt}| \,,$$

$$|\ell_{k,t-1}(x)| \le c \frac{|x - x_{kt}|}{x_{kt} - x_{k,t-1}} \le c \sqrt{\frac{n}{k}} |x - x_{kt}| \,.$$

$$|\ell_{ki}(x)| \le c \frac{|x - x_{kt}|(x - x_{k,t-1})}{x_{ki} - x_{k,i \pm 1}} \le c |x - x_{kt}| + \sqrt{\frac{n}{k}} (x - x_{kt})^2, \qquad i \notin S_k \,.$$

We also need the inequalities

$$|x-x_{kt}| \leq \begin{cases} c\sqrt{\dfrac{n}{k}}\left(p(x) - \dfrac{k}{n}\right), & \text{if} \quad 1 \leq k \leq n\min(p(x), 1/2), \\[2.5em] c\min\left(\sqrt{\dfrac{k}{n}}, \dfrac{1}{\sqrt{p(x)}}\left(\dfrac{k}{n} - p(x)\right)\right) & \text{if} \quad n\min(p(x), 1/2) < k \leq n/2, \end{cases}$$

and

$$0 < x - x_{k,t-1} \leq c|x - x_{kt}| + c\sqrt{\dfrac{k}{n}}\,.$$

These relations can be easily seen by using the mean value theorem and considering that $p'(x) \sim \sqrt{p(x)}$.

Using Lemma 1 with $\alpha = \beta = 1$ we have

$$A \leq c\sum_{k=1}^{[n/2]}\left(1 + \sqrt{\dfrac{n}{k}}|x - x_{kt}|\right)b_{nk}(p(x))$$

$$\leq c + c\sum_{k=1}^{[n\min(p(x),1/2)]}\dfrac{n}{k}\left(\dfrac{k}{n} - p(x)\right)b_{nk}(p(x)) + \sum_{k=[n\min(p(x),1/2)]}^{[n/2]}b_{nk}(p(x)) \leq c\,,$$

and since now $\varphi(x) \geq c$, Lemma 1 for $(\alpha, \beta) = (1/2, 1), (3/2, 2), (0, 1)$ gives

$$B \leq c\sum_{k=1}^{[n/2]}|x - x_{kt}|\left(1 + \sqrt{\dfrac{n}{k}}|x - x_{kt}|\right)b_{nk}(p(x))$$

$$\leq c\sum_{k=1}^{[n\min(p(x),1/2)]}\left\{\sqrt{\dfrac{n}{k}}\left(p(x) - \dfrac{k}{n}\right) + \left(\dfrac{n}{k}\right)^{3/2}\left(p(x) - \dfrac{k}{n}\right)^{2}\right\}b_{nk}(p(x))$$

$$+ \sum_{k=[n\min(p(x),1/2)]}^{[n/2]}\left\{\dfrac{1}{\sqrt{p(x)}}\left(\dfrac{k}{n} - p(x)\right)\right\}b_{nk}(p(x)) \leq \dfrac{c}{\sqrt{n}} \leq c\dfrac{\varphi(x)}{\sqrt{n}}\,.$$

Finally, using

$$|\ell_{ki}(x)| \leq c\left|\dfrac{(x - x_{kt})(x - x_{k,t-1})}{x_{ki} - x_{k,i\pm1}}\right| \leq c|x - x_{kt}| + c\sqrt{\dfrac{n}{k}}(x - x_{kt})^{2}, \qquad i \notin S\,.$$

we get

$$C \leq c\sum_{k=1}^{[n/2]}\left(|x - x_{kt}| + \sqrt{\dfrac{n}{k}}(x - x_{kt})^{2}\right)b_{nk}(p(x))\,.$$

But this leads to the same estimate as for B above. $\square$

Next, we state an equivalence result.

Theorem 2. *We have*

$$\|f - B_n(f)\|_{J_s} = O(n^{-\alpha/2}) \iff \omega_\varphi(f,t) = O(t^\alpha), \qquad 0 < \alpha < 1 \,.$$

For the proof we need two lemmas.

Lemma 2. *We have*

$$\|\varphi B_n'(f)\|_{J_s} \le c\sqrt{n}V(f)$$

for all $f \in C(J_s)$.

Proof of Lemma 2. We make the same assumptions on x and on the summation for k as in the proof of Theorem 1. Differentiating the identity

$$\sum_{k=0}^{n}\sum_{i=1}^{m_k} \ell_{ki}(x)b_{nk}(p(x)) = 1$$

we obtain

$$\sum_{k=0}^{n}\sum_{i=1}^{m_k} f(x)\ell_{ki}(x)b_{nk}'(p(x))p'(x) + \sum_{k=0}^{n}\sum_{i=1}^{m_k} f(x)\ell_{ki}'(x)b_{nk}(p(x)) = 0 \,.$$

Also, differentiating the operator $B_n(f,x)$ and subtracting the pervious relation we obtain

$$\varphi(x)|B_n'(f,x)| \le \varphi(x)V(f) \sum_{k=0}^{n}\sum_{i=1}^{m_k} \{|\ell_{ki}(x)|b_{nk}'(p(x))|p'(x)| + |\ell_{ki}'(x)|b_{nk}(p(x))\}$$

$$\le cV(f) \sum_{k=0}^{n}\sum_{i=1}^{m_k} \left\{ \sqrt{p(x)}|\ell_{ki}(x)|b_{nk}'(p(x)) + |\ell_{ki}'(x)|b_{nk}(p(x)) \right\} \,,$$

since

$$\varphi(x)|p'(x)| \le c\sqrt{p(x)} \,. \tag{1.6}$$

Thus we have to prove:

$$A := \sqrt{p(x)} \sum_{k=0}^{[n/2]} \left(\sum_{i=1}^{m_k} |\ell_{ki}(x)| \right) b_{nk}'(p(x)) \le c\sqrt{n}$$

and

$$B := \sum_{k=0}^{[n/2]} \left(\sum_{i=1}^{m_k} |\ell_{ki}'(x)| \right) b_{nk}(p(x)) \le c\sqrt{n} \,.$$

For $k = 0$ we have $|\ell_{ki}(x)| \le c$, and by the Markov inequality $|\ell_{ki}'(x)| \le c$ for all $i = 1, \ldots, m_0$, thus the contribution to A and B will be

$$n\sqrt{p(x)}(1 - p(x))^{n-1} \le c\sqrt{n} \quad \text{and} \quad c(1 - p(x))^n \le c \,,$$

respectively.

So let $1 \le k \le [n/2]$. Since

$$|b'_{nk}(p(x))| \le \frac{cn}{p(x)} \left| p(x) - \frac{k}{n} \right| b_{nk}(p(x)) , \qquad (1.7)$$

we have

$$A \le c\sqrt{n} + \frac{cn}{\sqrt{p(x)}} \sum_{k=1}^{[n/2]} \left(\sum_{i=1}^{m_k} |\ell_{ki}(x)| \right) \left| p(x) - \frac{k}{n} \right| b_{nk}(p(x)) .$$

From the proof of Theorem 1 we can see that

$$|\ell_{ki}(x)| \le c + c\sqrt{\frac{n}{k}} |x - x_{kt}|$$

$$\le c + \begin{cases} c\dfrac{n}{k} \left(p(x) - \dfrac{k}{n} \right), & \text{if} \quad 1 \le k \le n \min(p(x), 1/2), \\ 0, & \text{if} \quad n \min(p(x), 1/2) < k \le [n/2] , \end{cases}$$

$i = 1, \ldots, m_k; \; k = 1, \ldots, [n/2]$, where again x_{kt} is the nearest node to x on the right. Thus we obtain, using Lemma 1 with $\alpha = 1, \beta = 2$,

$$A \le c\sqrt{n} + \frac{cn}{\sqrt{p(x)}} \sum_{k=1}^{n \min(p(x), 1/2)} \frac{n}{k} \left(p(x) - \frac{k}{n} \right)^2 b_{nk}(p(x))$$

$$\le c\sqrt{n} + \frac{cn}{\sqrt{p(x)}} \left(\sqrt{\frac{p(x)}{n}} + \frac{1}{n} \right) \le c\sqrt{n} ,$$

since the sum appears only if $p(x) \ge 1/n$.

Finally, since $|\ell_{ki}(x)| \le c\sqrt{n}$, consequently $|\ell'_{ki}(x)| \le c\sqrt{n}$, we obtain

$$B \le c\sqrt{n} \sum_{k=0}^{n} b_{nk}(p(x)) \le c\sqrt{n} . \qquad \square$$

Lemma 3. *We have*

$$\|\varphi B'_n(f)\|_{J_s} \le c(\|\varphi f'\|_{J_s} + V(f))$$

for all $f' \in C(J_s)$.

Remark. The example of a piecewise constant function shows that the second term on the right hand side is necessary.

Proof of Lemma 3. Similarly as in the proof of Lemma 2, by (1.6) and (1.7) we obtain

$$\varphi(x)|B_n'(f,x)| \leq c\varphi(x)|p'(x)| \sum_{k=0}^{n} \left(\sum_{i=1}^{m_k} |f(x) - f(x_{ki})| \cdot |\ell_{ki}(x)| \right) |b_{nk}'(p(x))|$$

$$+ c\varphi(x) \sum_{k=0}^{n} \left(\sum_{i=1}^{m_k} |f(x) - f(x_{ki})| \cdot |\ell_{ki}'(x)| \right) b_{nk}(p(x)) .$$

$$(1.8)$$

Since $\varphi(x) \geq \sqrt{x - a_j}$, we have

$$\varphi(x)|f(x) - f(x_{ki})| = \varphi(x) \left| \int_{x_{ki}}^{x} f'(t)\, dt \right|$$

$$\leq c\varphi(x)\|\varphi f'\|_{J_s} \int_{x_{ki}}^{x} \frac{dt}{\varphi(t)} \leq c|x - x_{ki}| \cdot \|\varphi f'\|_{J_s}, \qquad i \in S_k . \qquad (1.9)$$

Hence (1.8) will take the form

$$\varphi(x)|B_n'(f,x)| \leq c\|\varphi f'\|_{J_s} \sum_{k=0}^{n} \left\{ \left(\sum_{i \in S_k} |(x - x_{ki})\ell_{ki}(x)| \right) p'(x)|b_{nk}'(p(x))| \right.$$

$$\left. + \left(\sum_{i \in S_k} |(x - x_{ki})\ell_{ki}'(x)| \right) b_{nk}(p(x)) \right\}$$

$$+ cV(f)\varphi(x) \sum_{k=0}^{n} \left\{ \left(\sum_{i \notin S_k} |\ell_{ki}(x)| \right) p'(x)|b_{nk}'(p(x))| + \left(\sum_{i \notin S_k} |\ell_{ki}'(x)| \right) b_{nk}(p(x)) \right\} .$$

Let

$$A_k := p'(x) \sum_{i \in S_k} |(x - x_{ki})\ell_{ki}(x)|, \quad B_k := \sum_{i \in S_k} |(x - x_{ki})\ell_{ki}'(x)| ,$$

$$C_k := p'(x)\varphi(x) \sum_{i \notin S_k} |\ell_{ki}(x)|, \quad D_k := \varphi(x) \sum_{i \notin S_k} |\ell_{ki}'(x)| .$$

Let first $k = 0$. Then $A_0 \sim C_0 \leq c(x - x_{0t})$, and both contribute

$$A_0|b_{n0}'(p(x))| \sim C_0|b_{n0}'(p(x))| \leq$$

$$\leq cnp'(x)(x - x_{0t})(1 - p(x))^{n-1} \leq cnp(x)(1 - p(x))^{n-1} \leq c .$$

Similarly, $B_0 \sim D_0 \leq c$, and their contribution is also majorized by constant.

Now let $1 \leq k \leq n/2$. Assume first that $a_j = \xi < x < \eta$ (i.e. x is at the left "edge" of the interval I_j). Then

$$A_k \sim B_k \leq c \left| p(x) - \frac{k}{n} \right|, \quad C_k \leq c\sqrt{\frac{np(x)}{k}} \left| p(x) - \frac{k}{n} \right|, \quad D_k \sim \sqrt{\frac{np(x)}{k}} ,$$

Thus summing up for $1 \le k \le n$ using these estimates and the relation $|b'_{nk}(x)| \le c\frac{n}{x}\left|x - \frac{k}{n}\right| b_{nk}(x)$ we get

$$\sum_{k=1}^{n} A_k b'_{nk}(p(x)) \le c\frac{n}{p(x)} \sum_{k=1}^{n} \left(p(x) - \frac{k}{n}\right)^2 b_{nk}(p(x)) \le c$$

(cf. (1.5) with $\gamma = 2$),

$$\sum_{k=1}^{n} B_k b_{nk}(p(x)) \le c\sum_{k=1}^{n} \left|p(x) - \frac{k}{n}\right| b_{nk}(p(x)) \le c\sqrt{\frac{p(x)}{n}}$$

(cf. (1.5) with $\gamma = 1$),

$$\sum_{k=1}^{n} C_k|b'_{nk}(p(x))| \le c\frac{n}{p(x)} \sum_{k=1}^{n} \sqrt{\frac{np(x)}{k}} \left(p(x) - \frac{k}{n}\right)^2 b_{nk}(p(x))$$

$$\le \frac{cn}{\sqrt{p(x)}} \left(\sum_{k=1}^{n} \frac{k}{n}\left(p(x) - \frac{k}{n}\right)^2 b_{nk}(p(x)) \sum_{k=1}^{n}\left(p(x) - \frac{k}{n}\right)^2 b_{nk}(p(x))\right)^{1/2}$$

$$\le \frac{cn}{\sqrt{p(x)}} \left(\frac{1}{n} \cdot \frac{p(x)}{n}\right)^{1/2} \le c$$

(cf. Lemma 1 with $\alpha = 1$, $\beta = 2$), and

$$\sum_{k=1}^{n} D_k b_{nk}(p(x)) \le c\sum_{k=1}^{n} \sqrt{\frac{np(x)}{k}} b_{nk}(p(x)) \le \sqrt{p(x)} \sum_{k=1}^{n} \left(\frac{n}{k} b_{nk}(p(x))\right)^{1/2} \le c$$

(cf. Lemma 1 with $\alpha = 1$, $\beta = 0$).

Finally, let $a_j < \xi < x < \eta$, i.e. x is "inside" the interval I_j. Since now $p'(x) \le c\sqrt{p(x)}$, we have

$$A_k \sim C_k \le c\sqrt{p(x)}\left(\sqrt{\frac{n}{k}}(x - x_{kt})^2 + x - x_{kt}\right), \quad B_k \sim D_k \le c\sqrt{\frac{n}{k}}(x - x_{kt}) + 1.$$

Using previous estimates for $|x - x_{kt}|$ we get

$$\sum_{k=1}^{n} A_k|b'_{nk}(p(x))| \le c\frac{n}{\sqrt{p(x)}} \sum_{k=1}^{n} \left(\sqrt{\frac{n}{k}}(x - x_{kt})^2 + x - x_{kt}\right)\left|p(x) - \frac{k}{n}\right| b_{nk}(p(x))$$

$$\le \frac{cn}{\sqrt{p(x)}} \sum_{1 \le k \le np(x)} \left[\left(\frac{n}{k}\right)^{3/2}\left|p(x) - \frac{k}{n}\right|^3 + \sqrt{\frac{n}{k}}\left(p(x) - \frac{k}{n}\right)^2\right] b_{nk}(p(x))$$

$$+\frac{cn}{p(x)}\sum_{np(x)\leq k\leq n}\left(p(x)-\frac{k}{n}\right)^2 b_{nk}(p(x))\leq cn\sum_{k=1}^{n}\left(\frac{n}{k}\right)^2\left|p(x)-\frac{k}{n}\right|^3 b_{nk}(p(x))$$

$$+\frac{n}{\sqrt{p(x)}}\left(\sum_{k=1}^{n}\frac{n}{k}\left(p(x)-\frac{k}{n}\right)^2 b_{nk}(p(x))\right)^{1/2}+c$$

$$\leq cn\left(\frac{1}{n}+\frac{1}{\sqrt{x}n^{3/2}}\right)+\frac{1}{\sqrt{np(x)}}+c\leq c\,.$$

And finally,

$$\sum_{k=1}^{n}\left[\sqrt{\frac{n}{k}}+1\right]b_{nk}(p(x))\leq c\sum_{1\leq k\leq np(x)}\frac{n}{k}\left|p(x)-\frac{k}{n}\right|b_{nk}(p(x))$$

$$+\sum_{np(x)\leq k\leq n}b_{nk}(p(x))+c\leq c\,.$$

$\square$

Proof of Theorem 2. The implication "$\Longleftarrow$" follows from Theorem 1. The proof of the other direction is modelled after the proof of (9.3.3) in [1] for the classic Bernstein-type operators. Define the K-functional by

$$K_\varphi(f,t):=\inf(\|f-g\|_{J_s}+t\|\varphi f'\|_{J_s})\,,$$

where the infimum is taken over all functions g which are absolutely continuous on each interval I_j. By applying Lemmas 2 and 3 for fig and g, respectively, we get

$$K_\varphi\left(f,\frac{1}{\sqrt{n}}\right)\leq\|f-B_k(f)\|_{J_s}+\frac{1}{\sqrt{n}}\|\varphi B_k'(f)\|_{J_s}$$

$$\leq\|f-B_k(f)\|_{J_s}+\frac{c}{\sqrt{n}}\left(\sqrt{k}\|f-g\|_{J_s}+\|\varphi g'\|_{J_s}+c\right)$$

$$\leq\|f-B_k(f)\|_{J_s}+\frac{c}{\sqrt{k}}+c\sqrt{\frac{k}{n}}K_\varphi\left(f,\frac{1}{\sqrt{k}}\right)$$

$$\leq\frac{c}{k^{\alpha/2}}+c\sqrt{\frac{k}{n}}K_\varphi\left(f,\frac{1}{\sqrt{k}}\right),\qquad 0<\alpha<1\,.$$

Applying the Berens–Lorentz lemma (cf. Lemma 9.3.4 from [1]) with $r=1/2$ we get

$$K_\varphi\left(f,\frac{1}{\sqrt{n}}\right)\leq\frac{c}{n^{\alpha/2}},\quad\text{whence}\quad K_\varphi(f,t)\leq ct^\alpha,\qquad 0<\alpha<1\,.$$

This easily implies $\omega_\varphi(f,t)\leq t^\alpha$ (cf. the proof in Section 2.4 in [1]). $\square$

We do not know if the operator B_n is saturated with $O(n^{-1/2})$. The following example seems to support this conjecture.

Example. Let $m - s \geq 1$. Then there are $m - s$ extremal points of $p(x)$ inside J_s. Assume that there exists an extremal point such that $p(x) = 1$, and let

$$f_0(x) = \prod_{i=1}^{m_n}(x - x_{ni}) \in \Pi_{m_n}, \qquad m_n = \left[\frac{m+s}{2}\right].$$

Then

$$\|f_0(x) - B_n(f_0, x)\|_{J_s} \sim \frac{1}{\sqrt{n}}.$$

Here the upper estimate follows from Theorem 1. To prove the lower estimate, note that

$$L_k(f_0, x) = \begin{cases} f_0(x), & \text{if} \quad k = 0, \ldots, n-1, \\ 0, & \text{if} \quad k = n. \end{cases}$$

Thus

$$f_0(x) - B_n(f_0, x) = f_0(x)p(x)^n.$$

Now let $x_{nr} \in \mathrm{int}J_s$ be such that $p(x_{nr}) = 1$ and let $x = x_{nr} + \frac{1}{\sqrt{n}}$. Then

$$|f_0(x) - B_n(f_0, x)| \geq c(x - x_{nr})(1 - |p(x_{nr}) - p(x)|)^n$$

$$\geq c(x - x_{nr})(1 - c(x - x_{nr})^2)^n \geq \frac{c}{\sqrt{n}}.$$

References

1. Z. Ditzian and V. Totik, *Moduli of Smoothness*, Springer (1987).
2. A. Kroó and J. Szabados, Inverse polynomial mappings and Lagrange interpolation on several intervals, *J. Math. Anal. Appl.*, **436** (2016), 1165-1179.
3. G. G. Lorentz, *Bernstein Polynomials*, Chelsea Publishing Company (New York, 1986).
4. A. L. Lukashov and J. Szabados, The order of Lebesgue constant of Lagrange interpolation on several intervals, *Periodica Math. Hungar.*, DOI10.1007/s10998-015-0106-z
5. F. Peherstorfer, Orthogonal and Chebyshev polynomials on two intervals, *Acta Math. Hungar.*, **55** (1990), 245-278.
6. F. Peherstorfer, K. Schiefermayr, Description of extremal polynomials on several intervals and their computation.I, *Acta Math. Hungar.*, **83** (1999), 27-58.