Monochromatic cycle power partitions *

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Abstract

Improving our earlier result we show that for every integer $k \geq 1$ there exists a $c(k)$ such that in every 2-colored complete graph apart from at most $c(k)$ vertices the vertex set can be covered by $200k^2 \log k$ vertex disjoint monochromatic k-th powers of cycles.

1 Monochromatic partitions and powers of cycles

K_n is the complete graph on n vertices and $K_{n,n}$ is the complete bipartite graph between two sets of n vertices each. If $G_1, G_2, \ldots, G_r$ are graphs, then the **Ramsey number** $R(G_1, G_2, \ldots, G_r)$ is the smallest positive integer n such that if the edges of a complete graph K_n are partitioned into r disjoint color classes giving r graphs $H_1, H_2, \ldots, H_r$, then at least one of the subgraphs H_i $(1 \leq i \leq r)$ has a subgraph isomorphic to G_i. In this paper we will deal with 2-color Ramsey numbers (so $r = 2$) and we will think of color 1 as red and color 2 as blue. The k-th **power** of a cycle of length n, denoted C_n^k, is the graph obtained from C_n by joining every pair of vertices

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with distance at most k (counting edges) in C. For simplicity let us call the k-th power of a cycle a k-cycle.

Assume first that K_n is a complete graph on n vertices whose edges are colored with r colors ($r \geq 1$). How many monochromatic cycles are needed to partition the vertex set of K_n? This question received a lot of attention in the last few years. Throughout the paper, single vertices and edges are considered to be cycles. Let $p(r)$ denote the minimum number of monochromatic cycles needed to partition the vertex set of any r-colored K_n. It is not obvious that $p(r)$ is a well-defined function. That is, it is not obvious that there always is a partition whose cardinality is independent of the order of the complete graph. However, in [8] Erdős, Gyárfás and Pyber proved that there exists a constant c such that $p(r) \leq cr^2 \log r$ (throughout this paper log denotes natural logarithm). Furthermore, in [8] (see also [15]) the authors conjectured the following.

Conjecture 1. *$p(r) = r$.*

The special case $r = 2$ of this conjecture was asked earlier by Lehel and for $n \geq n_0$ was first proved by Łuczak, Rödl and Szemerédi [28]. Allen improved on the value of n_0 [1] and finally Bessy and Thomassé [4] proved the original conjecture for $r = 2$. For general r the current best bound is due to Gyárfás, Ruszinkó, Sárközy and Szemerédi [17] who proved that for $n \geq n_0(r)$ we have $p(r) \leq 100r \log r$. For $r = 3$ an approximate version of the conjecture was proved in [18] but surprisingly Pokrovskiy [29] found a counterexample to that conjecture. However, in the counterexample all but one vertex can be covered by r vertex disjoint monochromatic cycles. Thus a slightly weaker version of the conjecture still may still be true, namely that apart from a constant number of vertices the vertex set can be covered by r vertex disjoint monochromatic cycles.

Conjecture 2. *Let G be a r-colored graph. Then there exist a constant $c = c(r)$ and r vertex disjoint monochromatic cycles of G that cover at least $n - c$ vertices.*

Let us also note that the above problem was generalized in various directions; for hypergraphs (see [19] and [31]), for complete bipartite graphs (see [8] and [20]), for graphs which are not necessarily complete (see [3] and [30]) and for vertex partitions by monochromatic connected k-regular subgraphs (see [32] and [33]).

Another area that attracted a lot of interest is powers of cycles; in particular the famous Pósa-Seymour conjecture.

Conjecture 3. *If the minimum degree of a graph G on n vertices is at least $\frac{k}{k+1}n$, then G contains the k-th power of a Hamiltonian cycle, i.e. a Hamiltonian k-cycle.*

After a sequence of partial results ([9], [10], [11], [12], [13], [24]) with an application of the Regularity Lemma-Blow-up Lemma method we showed in [21] and [25] that the conjecture is true for graphs with $n \geq n_0$. Since we used the Regularity Lemma, the resulting n_0 was huge. Later for $k = 2$ in [27] we "deregularized" the proof, i.e. we eliminated the use of the Regularity Lemma from the proof and thus the resulting n_0 was much better. This was further improved in [7].

In [2] Allen, Brightwell and Skokan studied the Ramsey number $R(C_n^k, C_n^k)$ where again C_n^k is a k-cycle on n vertices. They proved the following lower bounds

$$R(C^k_{(k+1)t}, C^k_{(k+1)t}) \geq t(k+1)^2 - 2k \text{ for } k \geq 2,$$

and

$$R(C^k_{(k+1)t+r}, C^k_{(k+1)t+r}) \geq (k+1)((k+2)t + 2r - 2) + r \text{ for } k \geq 2, 1 \leq r \leq k,$$

and they conjectured that these bounds are, at least asymptotically, optimal. However, they were able to prove only the following upper bound

$$R(C_n^k, C_n^k) \leq \left(2\chi(C_n^k) + \frac{2}{\chi(C_n^k)} \right) n + o(n), \tag{1}$$

(where $\chi(G)$ denotes the chromatic number of graph G) which differs from the lower bounds by a multiplicative factor slightly greater than 2.

A natural question (first asked by András Gyárfás) is to combine the above two areas and ask how many monochromatic k-cycles are needed to partition the vertex set of a 2-colored K_n. In an earlier paper [14] (as a consequence of a more general theorem) we showed that $2^{ck \log k}$ monochromatic k-cycles are enough. Here we improve this significantly to $ck^2 \log k$ but the price we have to pay is that a constant number of vertices might be left uncovered (similarly to Conjecture 2).

Theorem 1. *For every integer $k \geq 1$ there exists a $c(k)$ such that in every 2-colored complete graph apart from at most $c(k)$ vertices the vertex set can be covered by $200k^2 \log k$ vertex disjoint monochromatic k-cycles.*

Unfortunately, the number $c(k)$ of uncovered vertices is quite large in terms of k; it is a Regularity Lemma-type quantity. If possible, it would be desirable to eliminate these uncovered vertices. Furthermore, we believe that in light of (1) (since the Ramsey number is $O(kn)$) the right number of k-cycles is probably linear in k. Finally, it would be interesting to extend this problem for more than 2 colors.

2 Notation and tools

For basic graph concepts see the monograph of Bollobás [5].
$V(G)$ and $E(G)$ denote the vertex-set and the edge-set of the graph G. (A, B, E) denotes a bipartite graph $G = (V, E)$, where $V = A \cup B$, and $E \subset A \times B$. For a graph G and a subset U of its vertices, $G|_U$ is the restriction to U of G. $N(v)$ is the set of neighbors of $v \in V$. Hence $|N(v)| = deg(v) = deg_G(v)$, the degree of v. $\delta(G)$ stands for the minimum, and $\Delta(G)$ for the maximum degree in G. For $A \subset V(G)$ we write $N(A) = \cap_{v \in A} N(v)$, the set of common neighbours. $N(x, y, z, ...)$ is shorthand for $N(\{x, y, z, ...\})$. When A, B are subsets of $V(G)$, we denote by $e(A, B)$ the number of edges of G with one endpoint in A and the other in B. In particular, we write $deg(v, U) = e(\{v\}, U)$ for the number of edges from v to U. For non-empty A and B,

$$d(A, B) = \frac{e(A, B)}{|A||B|}$$

is the **density** of the graph between A and B. The density of the graph G is $d(G) = |E(G)|/\binom{n}{2}$. We say that the graph G is δ-**dense** if $d(G) \geq \delta$.

Definition 1. *The bipartite graph $G = (A, B, E)$ is ε-**regular** if*

$$X \subset A,\ Y \subset B,\ |X| > \varepsilon|A|,\ |Y| > \varepsilon|B| \quad \textit{imply} \quad |d(X, Y) - d(A, B)| < \varepsilon,$$

otherwise it is ε-irregular.

We will often say simply that "the pair (A, B) is ε-regular" with the graph G implicit.

Definition 2. *(A, B) is (ε, δ)-**super-regular** if it is ε-regular and*

$$deg(a) > \delta|B|\ \forall\, a \in A, \qquad deg(b) > \delta|A|\ \forall\, b \in B.$$

Definition 3. *Given a k-partite graph $G = (V, E)$ with k-partition $V = V_1 \cup \ldots \cup V_k$, the cylinder $V_1 \times \ldots \times V_k$ is (ε, δ)-super-regular if all the $\binom{k}{2}$ pairs of subsets (V_i, V_j), $1 \leq i < j \leq k$, are (ε, δ)-**super-regular**. Given $\kappa > 0$, the cylinder $V_1 \times \ldots \times V_k$ is κ-**balanced** if $|V_j| \leq \kappa|V_i|$ for all $1 \leq i, j \leq k$.*

We need a 2-edge-colored version of the Regularity Lemma [34].[1]

Lemma 1. *For every integer m_0 and positive ε, there is an $M_0 = M_0(\varepsilon, m_0)$ such that for $n \geq M_0$ the following holds. For any n-vertex graph G, where $G = G_1 \cup G_2$ with $V(G_1) = V(G_2) = V$, there is a partition of V into $\ell + 1$ clusters $V_0, V_1, \ldots, V_\ell$ such that*

[1]For background, this variant and other variants of the Regularity Lemma see [26].

- $m \leq \ell \leq M$, $|V_1| = |V_2| = \ldots = |V_\ell|$, $|V_0| < \varepsilon n$,
- *apart from at most* $\varepsilon\binom{\ell}{2}$ *exceptional pairs, all pairs* $G_s|_{V_i \times V_j}$ *are* ε*-regular, where* $1 \leq i < j \leq \ell$ *and* $1 \leq s \leq 2$.

Our other main tool is the Blow-up Lemma (see [22, 23]).

Lemma 2. *Given a graph* R *of order* r *and positive parameters* δ, Δ, κ*, there exists an* $\varepsilon > 0$ *such that the following holds. Let us replace the vertices of* R *with pairwise disjoint sets* $V_1, V_2, \ldots, V_r$ *(blowing up) such that the resulting cylinder is* κ*-balanced. We construct two graphs on the same vertex-set* $V = \cup V_i$*. The graph* $R(N)$ *is obtained by replacing all edges of* R *with copies of the complete bipartite graph* $K_{N,N}$*, and a sparser graph* G *is constructed by replacing the edges of* R *with some* (ε, δ)*-super-regular pairs. If a graph* H *with* $\Delta(H) \leq \Delta$ *is embeddable into* $R(N)$ *then it is already embeddable into* G*.*

Note, that the original Blow-up Lemma was stated for balanced sets but it is not hard to generalize the statement for κ-balanced sets (see e.g. the recent extended, full version of the Blow-up Lemma, Theorem 1.4 in [6]).

In particular we will need the following consequence of the Blow-up Lemma.

Lemma 3. *Given an integer* $k \geq 2$ *and a positive parameter* δ*, there exist an* $\varepsilon > 0$ *and* n_0 *such that the following holds. Let* $k' = 4k(k-1)$ *and let* $G = (V, E)$ *be a* k'*-partite graph on* $n \geq n_0$ *vertices with* k'*-partition* $V = V_1 \cup \ldots \cup V_{k'}$*, where the cylinder* $V_1 \times \ldots \times V_{k'}$ *is* (ε, δ)*-super-regular and 2-balanced. Let* $v_1 \in V_j$ *and* $v_2 \in V_{j'}$ *where* $1 \leq j, j' \leq k'$ *and* $j \neq j'$*. Then there is a Hamiltonian* k*-path in* G *connecting* v_1 *and* v_2*.*

A similar lemma is implicit in [25]. See also Lemma 7 in [14] for a similar lemma (in a more general situation). To sketch the proof for completeness, note that by the Blow-up Lemma (applied with $R = K_{k'}$, $\Delta = 2k$ and $\kappa = 2$) it is enough to check this for the complete (k')-partite graph. The statement is clearly true (since $k' \geq k+1$) if the cylinder $V_1 \times \ldots \times V_{k'}$ is *perfectly* balanced (i.e. 1-balanced). Indeed, we think of the (k')-partite graph as a cycle $(V_1, \ldots, V_{k'}, V_1)$. By reordering the sets we may assume that $v_1 \in V_1$ and $v_2 \in V_{k'}$. Then we go around the cycle $|V_i|$ times always selecting one vertex from the next V_i. If the cylinder is only 2-balanced, then first we have to eliminate the discrepancies among the sizes of the sets. Assume by reordering that $v_1 \in V_1$ and $v_2 \in V_{k+2}$. First we equalize the number of vertices in the first $(k+2)$ sets by moving vertices from $V_i, i \geq k+3$ to these sets. We have enough vertices for this task, using $4k(k-1) \geq 2(k+2)$ $(k \geq 2)$ and the fact that the cylinder is 2-balanced and thus $|V_i| + |V_j| \geq |V_l|$ for all $1 \leq i, j, l \leq k'$. Then we move more vertices from these sets in blocks of size $(k+2)$ such that we add one vertex to each of the first $(k+2)$ clusters and thus preserving the property that they have equal sizes. Finally the remaining fewer than $(k+2)$ vertices are added to the first $(k+2)$ sets in such a way that each set gets at most vertex. Thus now all the vertices are in the first $(k+2)$ sets and the difference in the sizes of the sets is at most 1. Finally we can eliminate this difference by a simple greedy procedure by going around the cycle $(V_1, \ldots, V_{k+2}, V_1)$ a few times and always leaving out the set with the smallest size (this is where we need at least $(k+2)$ sets). Once the sizes are equal, we can follow the approach above.

Furthermore, as in most applications of the Blow-up Lemma, the statement of Lemma 3 remains true if we *a priori* restrict the possible positions of the first k vertices of the k-path after v_1 (and similarly for the last k vertices of the k-path before v_2) to subsets of clusters V (within the neighborhood of v_1) that are still sufficiently large compared to $\varepsilon|V|$. Again see the full version of the Blow-up Lemma, Theorem 1.4 in [6], which includes this generalization.

We will also need (1). For completeness we restate it in the following lemma.

Lemma 4 (Theorem 12 in [2]). *We have*

$$R(C_n^k, C_n^k) \leq \left(2\chi(C_n^k) + \frac{2}{\chi(C_n^k)}\right) n + o(n).$$

We will need to apply a similar statement in the reduced graph. However, as in many applications of the Regularity Lemma, one has to handle a few irregular pairs and the corresponding edges will not be present in the reduced graph. We say that the graph G on n vertices is ε-**perturbed** if it is almost complete, $(1-\varepsilon)$-dense, i.e. at most $\varepsilon\binom{n}{2}$ edges are missing. We cannot apply Lemma 4 in the reduced graph because in Lemma 4 we have a 2-colored complete graph, yet the reduced graph will be a 2-colored ε-perturbed graph only. Thus we need a perturbed version of Lemma 4, this was also worked out in [2].

Lemma 5 (Lemma 32 in [2]). *For every integer $k \geq 1$ there exist an $\varepsilon > 0$ and $n_0 = n_0(k)$, such that if we 2-color the edges of a $(1-\varepsilon)$-dense graph G on $n \geq n_0$ vertices, then G will contain a monochromatic k-cycle of length at least $\frac{n}{2k+5}$.*

Note that in Lemma 32 in [2] this is stated for a graph G with minimum degree at least $(1-\varepsilon)n$ instead of a $(1-\varepsilon)$-dense graph, but with a standard "trimming" lemma (see e.g. Lemma 9 in [16]) it is easy to move from the density condition to the minimum degree condition. Furthermore, in Lemma 32 in [2] this is stated for a k-path, but again it is easy to see that the proof goes through for a k-cycle as well.

3 Sketch of the proof of Theorem 1

We will use the absorbing-greedy method introduced in [8] that is used in most of the papers in this area. Here we have to iterate this technique as we will have an inductive argument in k (similarly to [14]).

To prove Theorem 1 we apply the 2-edge-colored version of the Regularity Lemma to our 2-edge-colored K_n. Then we introduce the so called **reduced graph** G^R, the $(1-\varepsilon)$-dense graph whose vertices are associated to the clusters and whose edges are associated to pairs that are ε-regular in both colors. The edges of the reduced graph will be colored with a color that appears on the majority of the edges between the two clusters.

Iterating the proof technique in [8], we establish the bound on the number of monochromatic k-cycles needed in the following steps.

- Step 1: Using Lemma 5 we find a sufficiently large monochromatic (say red) k'-cycle $C_1^{k'}$ in G^R with $k' = 4k(k-1)$ (for technical reasons we need this k' instead of k). We think of $C_1^{k'}$ as a collection of $(k'+1)$-cylinders.

- Step 2: We remove the vertices of $C_1^{k'}$ from G^R and we go back to the original graph (instead of the reduced graph). Using the Ramsey bound (Lemma 5) repeatedly, we greedily remove a number (depending on k) of vertex disjoint monochromatic k-cycles from the remainder in K_n until the number of leftover vertices is much smaller than the number of vertices associated to $C_1^{k'}$.

- Step 3: Then we want to iterate this procedure in the set of the leftover vertices but the progress that we made is that now it is sufficient to partition into red k-cycles and blue k- or $(k-1)$-cycles. We can use some of the vertices from $C_1^{k'}$ to "lift" a blue $(k-1)$-cycle back into a blue k-cycle in the original graph. In order to facilitate this lifting we need some special properties between the leftover vertices and $C_1^{k'}$. Indeed, we can guarantee that only those vertices remain in the leftover set which have mostly blue edges to most of the clusters in most of the cylinders. All other vertices are put back into the cylinders. With this extra property now we can iterate the procedure in the set of leftover vertices assuming that it has sufficiently many vertices so we can apply the Regularity Lemma. If not, then we stop, these are the uncovered vertices and indeed their number could be up to M_0 in the Regularity Lemma (Lemma 1). Otherwise we can continue with the iteration.

  Let us say that we have to partition with the pair $(k, k-1)$, i.e. we have to partition into red k-cycles and blue k- or $(k-1)$-cycles. So initially we have to partition with the pair (k, k) and in each iteration we decrease one of the components in the pair by 1. Thus in at most $2k-1$ iterations we arrive at a pair where one of the components (say the blue) is 0.

- Step 4: This is where we have the lifting procedure. A 0-cycle is just an independent set. Then first we lift this independent set into a blue cycle, then we lift this cycle into a blue 2-cycle, etc. finally we get a blue k-cycle. Note that all other cycles found in subsequent iterations will be k-cycles; we have to use this lifting procedure only for the very last last cycle.

- Step 5: Using Lemma 3 repeatedly we find a monochromatic k-cycle spanning the remaining vertices of each $C_i^{k'}$.

Thus in summary, either we will actually get a partition of the whole graph, or in some iteration our leftover set is too small for the induction and then we are forced to leave it uncovered. The rest of the paper follows this outline; we discuss each step one by one.

4 Proof of Theorem 1

4.1 Step 1

We will assume that n is sufficiently large otherwise there is nothing to show. By the result of [4] we assume throughout that $k \geq 2$. We will use the following main parameters

$$0 < \varepsilon \ll \frac{1}{k} \leq \frac{1}{2}, \tag{2}$$

where $a \ll b$ means that a is sufficiently small compared to b. In order to present the results transparently we do not compute the actual dependencies, although it could be done.

Consider an 2-edge coloring of K_n. Let the red and blue subgraphs be G_1 and G_2, respectively. Apply the 2-color version of the Regularity Lemma (Lemma 1), with ε as in (2) and get a partition of $V(K_n) = V = \cup_{0 \leq i \leq \ell} V_i$, where $|V_i| = m, 1 \leq i \leq \ell$. As indicated above we define the reduced graph G^R: The vertices of G^R are $p_1, \ldots, p_l$, and we have an edge between vertices p_i and p_j if the pair $\{V_i, V_j\}$ is (ε, G_s)-regular for $s = 1, 2$. Thus we have a one-to-one correspondence $f : p_i \to V_i$ between the vertices of G^R and the clusters of the partition. Then,

$$|E(G^R)| \geq (1-\varepsilon)\binom{\ell}{2},$$

and thus indeed G^R is a $(1-\varepsilon)$-dense graph on ℓ vertices.

Define an red-blue edge-coloring (G_1^R, G_2^R) of G^R in the following way. The edge (p_i, p_j) is colored with a color s that contains the majority of the edges from $K(V_i, V_j)$, thus clearly $|E_{G_s}(V_i, V_j)| \geq \frac{1}{2}|V_i||V_j|$. Using Lemma 5 for our 2-colored $(1-\varepsilon)$-dense graph G^R with $k' = 4k(k-1)$ we can find a monochromatic k'-cycle of length at least $\frac{\ell}{2k'+5}$ in G^R. If this k'-cycle is red, then we denote it with $C_{1,1}^{k'}$, if it is blue then by $C_{2,1}^{k'}$.

Assume that we have the former case, so $C_{1,1}^{k'}$. Thus we have

$$|f(C_{1,1}^{k'})| \geq (1-\varepsilon)\frac{n}{2k'+5} \geq \frac{n}{4k'}. \tag{3}$$

We think of $C_{1,1}^{k'}$ as a collection of $(k'+1)$-cylinders: the first $(k'+1)$ clusters form the first such cylinder, the second $(k'+1)$ clusters form the second cylinder, etc. the last $(k'+1)$ clusters we can select form the last cylinder. Denote the number of cylinders by ℓ_1, then

$$\ell_1 \geq \lfloor \frac{\ell}{(k'+1)(2k'+5)} \rfloor \geq \frac{\ell}{4(k')^2}. \tag{4}$$

Denote the clusters in the i-th cylinder by $(V_1^i, V_2^i, \ldots, V_{k'+1}^i)$, where this is the order of the clusters on $C_{1,1}^{k'}$.

First we have to make the cylinders super-regular (so we can apply the Blow-up Lemma) by removing a small number of exceptional vertices. From each V_j^i we remove all exceptional vertices v for which

$$deg_{G_1}(v, V_{j'}^i) < \left(\frac{1}{2} - \varepsilon\right) m, \quad \text{for some } j' \neq j.$$

ε-regularity in G_1 guarantees that at most $k'\varepsilon|V_j^i|$ vertices are removed from each cluster V_j^i. The removed vertices are added back to the set of leftover vertices. Thus the resulting cylinders are $(\varepsilon', (1/2-\varepsilon'))$-super-regular and $(1+\varepsilon')$-balanced for some $\varepsilon \ll \varepsilon' \ll \frac{1}{k}$.

Next we find connecting k-paths of length $2k$ between subsequent cylinders. The vertices of the i-th connecting path between the i-th and $(i+1)$-st cylinder come from the following clusters:

$$V_{k'-k+2}^i, \ldots, V_{k'}^i, V_{k'+1}^i, V_1^{i+1}, V_2^{i+1}, \ldots, V_k^{i+1}.$$

For $i = \ell_1$ we have $i+1 = 1$, so the connecting path may be a little longer as we might have some clusters between the last cylinder and the first cylinder (if the length is not divisible by $(k'+1)$).

Furthermore, for this connecting path we select a typical vertex from each of these clusters. Then in addition to being a k-path (so vertices at a distance at most

k are connected) the first k vertices on the connecting path will have a common neighborhood of size at least

$$\left(\frac{1}{2} - \varepsilon'\right)^k |V|$$

in each cluster V in the i-th cylinder that is different from the k clusters where these vertices come from and a similar statement holds for the last k vertices and the $(i+1)$-st cylinder. This property guarantees that we can continue the k-cycle within the cylinders (see the remark after Lemma 3). These connecting paths will be parts of the final k-cycle that spans the remainder of $f(C_{1,1}^{k'})$. We remove the internal vertices of these connecting paths from the cylinders.

Note that at this point we could have a red k-cycle spanning all remaining vertices of $f(C_{1,1}^{k'})$ by applying Lemma 3 in each cylinder. However, we postpone the construction of this red k-cycle until Step 5, since in Step 4 in the lifting procedure we might use some of the vertices from $f(C_{1,1}^{k'})$.

4.2 Step 2

Here we will use the Ramsey bound in Lemma 5 repeatedly. Indeed, in a 2-colored complete graph on n' vertices (where n' is sufficiently large) we can find a monochromatic k-cycle of length at least

$$\frac{n'}{2k+5} \tag{5}$$

(Note that by Lemma 5 this would be true even for a $(1-\varepsilon)$-dense graph.)

We go back from the reduced graph to the original graph and we remove the vertices in $f(C_{1,1}^{k'})$. We apply repeatedly the above to the 2-colored complete graph induced by $K_n \setminus f(C_{1,1}^{k'})$. This way we choose t vertex disjoint monochromatic k-cycles in $K_n \setminus f(C_{1,1}^{k'})$. We wish to choose t so that the remaining set R_1 of vertices in $K_n \setminus f(C_{1,1}^{k'})$ not covered by these t k-cycles has cardinality

$$|R_1| \leq \frac{|f(C_{1,1}^{k'})|}{(2k')^5}. \tag{6}$$

Since after t steps at most

$$(n - |f(C_{1,1}^{k'})|)\left(1 - \frac{1}{2k+5}\right)^t$$

vertices are left uncovered, we have to choose t to satisfy

$$(n - |f(C_{1,1}^{k'})|)\left(1 - \frac{1}{2k+5}\right)^t \leq \frac{|f(C_{1,1}^{k'})|}{(2k')^5}.$$

Using (3) this inequality is certainly true if

$$\left(1-\frac{1}{2k+5}\right)^t \leq \frac{1}{2(2k')^6},$$

which in turn is true using $1-x \leq e^{-x}$ if

$$e^{-\frac{t}{2k+5}} \leq \frac{1}{2(2k')^6}.$$

This shows that we can certainly choose $t = 100k\lfloor \log k \rfloor$.

4.3 Step 3

We will iterate this procedure (Steps 1 and 2) in R_1. However, before we start the second iteration certain vertices will be removed from R_1 and assigned back to the cylinders in $C_{1,1}^{k'}$. We say that a vertex $v \in R_1$ is *good* for the i-th cylinder if there are at least k clusters V in the i-th cylinder such that v has a large red degree into them, i.e.

$$deg_{G_1}(v, V) \geq \frac{1}{2k^2}|V|. \tag{7}$$

We say that $v \in R_1$ is *good* if it is good for at least $\frac{1}{2k^2}\ell_1$ cylinders in $C_{1,1}^{k'}$ (otherwise v is called *bad*). The good vertices are removed from R_1 and assigned back to cylinders for which they are good such that we distribute the good vertices among the cylinders as equally as possible. We can clearly assign the good vertices to the cylinders in such a way that each cylinder gets at most

$$\frac{|R_1|}{\frac{\ell_1}{2k^2}} \leq \frac{(\ell_1+1)(k'+1)\frac{n}{\ell}}{(2k')^5}\frac{2k^2}{\ell_1} \leq \frac{1}{8(k')^2}\frac{n}{\ell} \tag{8}$$

vertices (using (6)).

Next we will put the assigned good vertices on the connecting k-paths between the cylinders. Assume that vertex v was assigned to the i-th cylinder. We can extend the connecting k-path between the $(i-1)$-st and the i-th cylinder by $3k+1$ vertices such that $(2k+1)$-st vertex is v. Indeed, by ε-regularity and (2) we can extend the connecting path by k vertices from different clusters such that these k new vertices have a large common red neighborhood within the red neighborhoods of v in the k clusters V in the i-th cylinder that satisfy (7). Then we extend by k vertices from these red neighborhoods $N_{G_1}(v, V)$ of v (one from each), then we add v, and finally again k vertices from these red neighborhoods $N_{G_1}(v, V)$ in such a way that these last k vertices have a large common red neighborhood in all other clusters. Furthermore,

(7) and (8) imply that we may choose vertex disjoint extensions for the different good vertices. Again we remove the internal vertices of these extended connecting paths from the cylinders.

Note again that at this point we could have a red k-cycle spanning all remaining vertices of $f(C^{k'}_{1,1})$ by applying Lemma 3 in each cylinder of $C^{k'}_{1,1}$. However, for technical reasons again we postpone the construction of this red k-cycle.

For simplicity we still denote by R_1 the set of remaining bad vertices. Thus for each remaining vertex $v \in R_1$ (since it is bad) we have the following property

- For at least $\left(1 - \frac{1}{2k^2}\right) \ell_1$ cylinders in $C^{k'}_{1,1}$, apart from at most $(k-1)$ clusters, for all clusters V we have the following in blue

$$deg_{G_2}(v, V) \geq \left(1 - \frac{1}{2k^2}\right) |V|. \tag{9}$$

This property will be important later in the lifting procedure.

We may assume that $|R_1|$ is sufficiently large (so we can apply Lemma 1 again in R_1) otherwise these will be the uncovered vertices in the statement of the theorem. In fact, this is the only reason why we need the constant uncovered vertices in the theorem. Then we may iterate the process again in R_1. As indicated in Section 3 now it would be sufficient to partition into red k-cycles and blue k- or $(k-1)$-cycles, i.e. we have to partition with the pair $(k, k-1)$ as defined in Section 3. A blue $(k-1)$-cycle may be lifted into a k-cycle by using some vertices from $C^{k'}_{1,1}$. Note that at most one blue $(k-1)$-cycle will be used; all other cycles will be red or blue k-cycles.

So we repeat Steps 1 and 2 in R_1. We apply the 2-color Regularity Lemma (Lemma 1) in R_1, define the 2-colored $(1-\varepsilon)$-dense reduced graph and apply Lemma 5 to find a monochromatic k'-cycle in the reduced graph. Assuming we had $C^{k'}_{1,1}$ in the first iteration, now if this k'-cycle is red, then we denote it with $C^{k'}_{1,2}$, if it is blue, then we denote it with $C^{k'}_{2,1}$. In general in later iterations, if this regular k'-cycle is the i-th red k'-cycle, then we denote it by $C^{k'}_{1,i}$, if it is the i-th blue k'-cycle, then we denote it by $C^{k'}_{2,i}$.

Assume that we have $C^{k'}_{2,1}$ in the second iteration. We proceed as in Steps 1 and 2 but now we do everything in blue. We think of $C^{k'}_{2,1}$ as a collection of $(k'+1)$-cylinders, we make them super-regular and we find connecting k-paths between subsequent cylinders. Then we remove $C^{k'}_{2,1}$ from R_1 and we greedily remove vertex disjoint k-cycles from the leftover as in Step 2 until the remaining set R_2 of vertices has cardinality

$$|R_2| \leq \frac{|f(C^{k'}_{2,1})|}{(2k')^5}.$$

The number of monochromatic k-cycles used is again at most $100k \log k$.

Again we define good vertices in R_2 with respect to $C_{2,1}^{k'}$ and we remove them from R_2 and add them to the connecting k-paths between the cylinders in $C_{2,1}^{k'}$. For the remaining bad vertices in R_2 a similar property as (9) is true in red to $C_{2,1}^{k'}$. Again we may assume that $|R_2|$ is sufficiently large (so we can apply Lemma 1 again in R_2) otherwise these will be the uncovered vertices in the statement of the theorem.

Next we repeat the procedure in R_2, but now we have to partition with the pair $(k-1, k-1)$. We continue in this fashion, in each iteration we decrease one of the components of the pair by one. Indeed, if the regular k'-cycle found is red, then we decrease the blue component, if it is blue, then we decrease the red component by one.

In i ($\leq 2k-1$) iterations we arrive at a pair where one of the components (say the blue) is 0.

4.4 Step 4

Consider the final remaining set R_i (again keeping only the bad vertices). Since the blue component of the current pair is 0, and a 0-cycle is just an independent set, we just cover R_i with one independent set (we can think of it as blue). Then through k lifts we will lift this into a blue k-cycle by using vertices from $C_{1,1}^{k'}, C_{1,2}^{k'}, \ldots, C_{1,k}^{k'}$.

Indeed, first we lift R_i into a blue 1-cycle (an ordinary cycle) by using vertices from $C_{1,k}^{k'}$. We have

$$|R_i| \leq \frac{|f(C_{1,k}^{k'})|}{(2k')^5}. \tag{10}$$

Take an arbitrary cyclic ordering $v_1, v_2, \ldots, v_{|R_i|}$ of the vertices in R_i. Between each v_j and v_{j+1}, $1 \leq j \leq |R_i|$, we will insert a vertex of $C_{1,k}^{k'}$ in such a way that it is connected in blue to both v_j and v_{j+1} and thus resulting in a blue cycle. For each $v_j \in R_i$ we have a similar property as in (9) to $C_{1,k}^{k'}$, i.e. for at least an $\left(1 - \frac{1}{2k^2}\right)$-fraction of the cylinders in $C_{1,k}^{k'}$ apart from at most $(k-1)$ clusters, for all clusters V we have the following in blue

$$deg_{G_2}(v_j, V) \geq \left(1 - \frac{1}{2k^2}\right)|V|. \tag{11}$$

Take a cylinder such that this is satisfied for both v_j and v_{j+1} and take a cluster V such that (11) is true in V for both v_j and v_{j+1} (we have at least $k' - 2(k-1)$ clusters to choose from). Then v_j and v_{j+1} have at least $\left(1 - \frac{1}{k^2}\right)|V|$ common blue neighbors in V; take one of them as the intermediate vertex between v_j and v_{j+1}. Furthermore, using (10), we can clearly guarantee that from each cluster V in $C_{1,k}^{k'}$ we do not use more than $\frac{1}{(2k')^4}|V|$ vertices and thus we may always select distinct intermediate vertices between v_j and v_{j+1}. Note that this blue cycle has an even number of vertices from the construction.

Then we lift this blue cycle into a blue 2-cycle by using vertices from $C^{k'}_{1,k-1}$, etc. Finally we lift the blue $(k-1)$-cycle into a blue k-cycle by using vertices from $C^{k'}_{1,1}$. Indeed, we divide the blue $(k-1)$-cycle into blocks of k vertices (from the construction the length of the $(k-1)$-cycle is divisible by k) and we will insert a vertex of $C^{k'}_{1,1}$ between two consecutive blocks of k vertices similarly as above. Consider two consecutive blocks of k vertices, so $2k$ vertices altogether. For each of these vertices (9) is satisfied, i.e. for at least $\left(1-\frac{1}{2k^2}\right)\ell_1$ cylinders in $C^{k'}_{1,1}$, apart from at most $(k-1)$ clusters, for all clusters V we have the following in blue

$$deg_{G_2}(v,V) \geq \left(1-\frac{1}{2k^2}\right)|V|.$$

Take a cylinder such that this is satisfied for each of the $2k$ vertices. Indeed, we have at least

$$\left(1-2k\frac{1}{2k^2}\right)\ell_1 = \left(1-\frac{1}{k}\right)\ell_1 \geq \frac{\ell_1}{2}$$

cylinders to choose from (using $k \geq 2$). Then in this cylinder we pick a cluster V such that for each of the $2k$ vertices we have a blue neighborhood in V of size $\left(1-\frac{1}{2k^2}\right)|V|$. Indeed, we have at least $k' - 2k(k-1) = 2k(k-1)$ clusters to choose from. Then the $2k$ vertices have at least

$$\left(1-2k\frac{1}{2k^2}\right)|V| = \left(1-\frac{1}{k}\right)|V| \geq \frac{|V|}{2}$$

common blue neighbors in V; take one of them as the intermediate vertex between the two blocks of k vertices. Furthermore, using (10), we can clearly guarantee that from each cluster V in $C^{k'}_{1,1}$ we do not use more than $\frac{1}{(2k')^4}|V|$ vertices and thus we may always select distinct intermediate vertices between the consecutive blocks of k vertices. This lifts the blue $(k-1)$-cycle into a blue k-cycle.

4.5 Step 5

Finally applying Lemma 3 within each cylinder we close the k-path such that we span the remaining vertices. Indeed clearly the cylinders are still $(\varepsilon', 1/4)$-super-regular and 2-balanced for some $\varepsilon \ll \varepsilon' \ll \frac{1}{k}$.

Thus the total number of vertex disjoint monochromatic k-cycles used in the cover is at most

$$(2k-1)100k\log k + 2k \leq 200k^2 \log k,$$

finishing the proof. $\square$

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