RANK, COMBINATORIAL COST AND HOMOLOGY TORSION GROWTH IN HIGHER RANK LATTICES

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ABSTRACT. We investigate the rank gradient and growth of torsion in homology in residually finite groups. As a tool, we introduce a new complexity notion for generating sets, using measured groupoids and combinatorial cost.

As an application we prove the vanishing of the above invariants for Farber sequences of subgroups of right angled groups. A group is right angled if it can be generated by a sequence of elements of infinite order such that any two consecutive elements commute.

Most non-uniform lattices in higher rank simple Lie groups are right angled. We provide the first examples of uniform (co-compact) right angled arithmetic groups in $SL(n, \mathbb{R})$, $n \geq 3$ and $SO(p, q)$ for some values of p, q. This is a class of lattices for which the Congruence Subgroup Property is not known in general.

Using rigidity theory and the notion of invariant random subgroups it follows that both the rank gradient and the homology torsion growth vanish for an arbitrary sequence of subgroups in any right angled lattice in a higher rank simple Lie group.

1. INTRODUCTION

In this paper we deal with the asymptotic behavior of two delicate group invariants, the minimal size of a generating set (the rank), and the size of the torsion part of the first homology. In general, it is extremely hard to determine the rank, or even decide whether it is finite. Finite generation of lattices in connected semisimple Lie groups G was proved case by case by many different authors, notably in the works of Garland and Raghunathan [GR] for $\mathbb{R}$-rank$(G) = 1$ and of Kazhdan [K] for the case where all the factors of G have $\mathbb{R}$-rank ≥ 2. Torsion in homology is a more recent object of investigation, related to algebraic topology and number theory (see [Luc, BV]).

The asymptotic behavior of Betti numbers and more generally, representation multiplicities associated to lattices in Lie groups have been extensively studied since the 80's (see [DW, De] and the introduction of [7Sb]). In recent years, there has been new progress in this direction (see [7Sb, 7Sa, FL1, FL2, FLM]). These invariants can be effectively handled using the analytic side of the theory and limiting arguments, like the Lück approximation theorem [Luc]. For the rank and the torsion in homology, there do exist some analytic tools, namely the cost and the L^2-torsion. However, their efficiency is much more limited.

In this paper we develop new, geometric tools to estimate the rank and the torsion in homology for subgroups of finite index in a special class of groups. This class includes non-uniform as well as uniform lattices in simple Lie groups.

1.1. **Rank and homology torsion growth in right angled groups.** For a finitely generated group Γ let $d(\Gamma)$ denote the minimal number of generators (or rank) of Γ. For a subgroup $H \leq \Gamma$ of finite index let

$$r(\Gamma, H) = (d(H) - 1)/|\Gamma : H|.$$

The *rank gradient* of Γ with respect to a sequence (Γ_n) of finite index subgroups is defined to be

$$\mathrm{RG}(\Gamma, (\Gamma_n)) = \lim_{n \to \infty} r(\Gamma, \Gamma_n)$$

when this limit exists. This notion has been introduced by Lackenby [Lac] and further investigated in the literature for *chains* of subgroups. Recall that a chain in Γ is a decreasing sequence $\Gamma = \Gamma_0 > \Gamma_1 > \ldots$ of subgroups of finite index in Γ. In this case, it is easy to see that $r(\Gamma, \Gamma_n)$ is non-increasing and so the limit exists.

For a finitely generated group H let $\mathrm{trs}(H)$ denote the size of the torsion part of the Abelianization (or first homology group) H/H'. The *homology torsion growth* of Γ with respect to (Γ_n) is defined as

$$\mathrm{Trs}(\Gamma, (\Gamma_n)) = \lim_{n \to \infty} \frac{\ln \mathrm{trs}(\Gamma_n)}{|\Gamma : \Gamma_n|}$$

when this limit exists. It is easy to see that when Γ is finitely presented this sequence is bounded, see Lemma 27. The homology torsion growth has been analysed in [BV] and [Lk]. In this paper we deal only with the first homology.

The following definition is the central object of investigation of this paper.

Definition 1. *The group Γ is* right-angled *if it admits a finite generating list $\{\gamma_1, \ldots, \gamma_m\}$ of elements of infinite order such that $[\gamma_i, \gamma_{i+1}] = 1$ for $i = 1, \ldots, m - 1$.*

So, a right-angled group is a quotient of a right-angled Artin group with a connected defining graph where the order of the generators stay infinite. The importance of this class is reflected both in its ubiquity and in the strength of the results one can prove for such groups, mainly when combined together with rigidity theory. There are plenty of non-uniform and, as we shall prove, some uniform lattices in higher rank simple Lie groups that are right angled.

We now state our main vanishing theorems concerning rank and homology torsion growth for higher rank right angled lattices.

Theorem 2. *Let G be a simple real Lie group of real rank at least 2 and let Γ be a right-angled lattice in G. Then for any sequence of subgroups (Γ_n) in Γ such that $|\Gamma : \Gamma_n| \to \infty$ we have*

$$\mathrm{RG}(\Gamma, (\Gamma_n)) = 0.$$

One can attempt to further relax on the condition of having a common ambient lattice Γ in Theorem 2 and just consider arbitrary sequences of lattices in the fixed Lie group G. It was shown in [Ge1] that given a simple Lie group G and fixing the Haar measure on G, there exists a constant C such that

$$d(\Gamma) \leq C \cdot \mathrm{Vol}(G/\Gamma)$$

for every lattice $\Gamma \leq G$. In higher rank, the growth of rank seems to be sub-linear in the volume. We make the following provocative conjecture.

Conjecture 3. *Let G be a higher rank simple Lie group with a fixed Haar measure. Let Γ_n be a sequence of lattices in G with $\mathrm{Vol}(G/\Gamma_n) \to \infty$. Then*

$$\lim_{n \to \infty} \frac{d(\Gamma_n) - 1}{\mathrm{Vol}(G/\Gamma_n)} = 0.$$

In rank 1, this is false in general. Indeed, if G admits a lattice Γ that surjects on a non-abelian free group F, then the set of finite index subgroups in F leads to a counterexample. (For instance, the groups $SO(n, 1)$ ($n \geq 2$) admit such co-compact as well as non-co-compact lattices). However, our conjecture does admit a natural generalization to arbitrary Lie groups, using the framework of invariant random subgroups. See Section 8 for details.

Note that Theorem 2 together with the existence results (Theorem 5 and Theorem 6) below provide the first examples of sequences of cocompact lattices in a simple Lie group of higher rank where the rank grows sublinearly in the volume.

We now turn to the main result concerning torsion.

Theorem 4. *Let G, Γ and (Γ_n) be as in Theorem 2. Then*

$$\mathrm{Trs}(\Gamma, (\Gamma_n)) = 0.$$

Note that in this situation, Γ_n has property (T) and so $\mathrm{trs}(\Gamma_n)$ equals the size of the Abelianization of Γ_n.

For *non-uniform* lattices, the analogues of both Theorem 2 and Theorem 4 can be derived from the existing literature, using Raghunathan's theorem [R1, R2] that such lattices satisfy the so-called Congruence Subgroup Property (CSP), adding a result of Sharma and Venkataramana [ShV] and some finite group theory. We will do this in Section 5.1 where we also give effective bounds on both the torsion and the rank when the CSP holds. However, the CSP is far from being proved in the cocompact (uniform) case and so the above, arithmetic methods fail. Our proofs for Theorems 2 and 4 are inherently geometric and do not use arithmetic arguments.

1.2. **Existence of cocompact right angled lattices.** We prove the existence of co-compact higher rank right angled lattices in various classes of simple Lie groups. In particular we obtain:

Theorem 5. *The groups $SL_n(\mathbb{R})$ admit right angled co-compact lattices for $n > 2$.*

For a detailed description of these examples, see Section 6.2. In particular, it is not known whether these lattices satisfy the CSP (or bounded generation). In addition, we demonstrate our method for constructing right angled lattices in some special orthogonal groups where the geometric picture is more transparent:

Theorem 6. *Let $K = \mathbb{Q}(\sqrt{2})$ and denote its ring of integers $\mathcal{O}_K = \mathbb{Z}[\sqrt{2}]$. For an integer $n \geq 7$ let*

$$f(x_1, \ldots, x_{n+2}) = x_1^2 + \ldots + x_n^2 - \sqrt{2}x_{n+1}^2 - \sqrt{2}x_{n+2}^2$$

be a quadratic form on $V = K^{n+2}$. Let $\mathbb{G} = SO(f)$ and $\Gamma = \mathbb{G}(\mathcal{O}_K)$. Then Γ is a uniform lattice in $G = \mathbb{G}(\mathbb{R}) \cong SO(n, 2)$ admitting a finite index right-angled subgroup.

It is natural to ask at this point which simple Lie groups admit right-angled cocompact lattices. It is clear that by our method such lattices can be constructed in every orthogonal group $SO(p, q)$ as long as $q \geq 2$ and p is not too small.

These new examples are interesting in their own right, as they seem to possess properties that usually reflect higher $\mathbb{Q}$-rank, i.e. properties of non-uniform arithmetic groups. Thus they might be suitable candidates to study various notions that are known to hold for non-uniform lattices but are widely open for uniform lattices, such as bounded generation.

1.3. **Combinatorial cost and sofic approximations.** The proofs of Theorem 2 and Theorem 4 use combinatorial cost, a notion introduced by Elek [El]. This is a metric invariant of a sequence of finite graphs G_n. A *rewiring* of G_n is another sequence of graphs H_n on the same vertex set as of G_n, such that the bi-Lipshitz distortion of the identity map stays bounded in n. The combinatorial cost $cc(G_n)$ is the infimum of the edge densities of possible rewirings of G_n.

A theorem of Gaboriau [Gab] says that every free action of a right angled group has cost 1. Adapting his proof to the finite graph setting gives us the following effective version.

Theorem 7. *Let Γ be a right angled group and let (G_n) be a sofic approximation of Γ. Then there exists $\varepsilon_k \to 0$ and rewirings of (G_n) of edge density $1 + \varepsilon_k$ with bi-Lipshitz distortion that is polynomial in $1/\varepsilon_k$. In particular, the combinatorial cost of (G_n) equals 1.*

For the notion of a sofic approximation see Section 3. We shall say that a graph sequence satisfying the conclusion of Theorem 7 has *polynomial distortion*. One can define subexponential distortion of graph sequences in a similar fashion.

Let G be a locally compact group. Then for any closed subgroup H of G where the normalizer of H in G has finite covolume, one can define the invariant random subgroup μ_H by taking a uniform random conjugate of H (against the Haar measure). A sequence of subgroups $\Gamma_n \leq G$ is *Farber*, if μ_{Γ_n} weakly converges to μ_1. When G is a discrete group generated by a finite symmetric set S, then Γ_n is Farber, if and only if the Schreier graphs $\mathrm{Sch}(G/\Gamma_n, S)$ give a sofic approximation of G.

Our next theorem connects the rank gradient of a Farber sequence to the combinatorial cost of the corresponding sofic approximation.

Theorem 8. *Let Γ be a group generated by the finite symmetric set S containing the identity and let (Γ_n) be a Farber sequence of subgroups of finite index in Γ such that $\mathrm{RG}(\Gamma, (\Gamma_n))$ exists. Then*

$$\mathrm{RG}(\Gamma, (\Gamma_n)) \leq cc(\mathrm{Sch}(\Gamma, \Gamma_n, S)) - 1.$$

If (Γ_n) is a chain, then there is equality above.

The proof goes through interpreting the rank of Γ_n as the minimal number of generators for the measured groupoid associated to the action of Γ on Γ/Γ_n. This approach was initiated in [AbN].

The next result controls the homology torsion for sofic approximations where there are effective rewirings to get close to the combinatorial cost.

Theorem 9. *Let Γ be a finitely presented group. Let (Γ_n) be a Farber sequence of subgroups in Γ such that the corresponding sofic approximation of Γ has subexponential distortion. Then the homology torsion growth*

$$\mathrm{Trs}(\Gamma, (\Gamma_n)) = 0.$$

Looking at the rank of a subgroup through combinatorial cost (using the cost of the measured groupoid associated to an action) seems to be the right way to define generating sets of the stabilizer of 'low complexity'. We expect more applications of this point of view.

1.4. **Rigidity.** The final ingredient of the proof of Theorem 2 as well as of Theorem 4 comes from [7Sb]. The following is implicitly proved there, using the notion of invariant random subgroups.

Theorem 10. *Let Γ be a lattice in a higher rank simple Lie group. Then any sequence Γ_n of finite index subgroups with $|\Gamma : \Gamma_n| \to \infty$ is Farber.*

The vanishing of the rank gradient, Theorem 2, now follows from the combination of Theorem 10 together with the soft form of Theorem 7 and Theorem 8. The vanishing of the homology torsion growth, Theorem 4, follows from Theorem 10 together with the effective form of Theorem 7, which gives polynomial distortion, together with Theorem 9.

1.5. **Algebraic groups over non-Archimedean local fields.** Our methods and ideas can also be applied to lattices in higher rank algebraic groups over local fields. We demonstrate this by proving the following:

Theorem 11. *Let $\mathbb{F}$ be a finite field and let $\Gamma = SL(n, \mathbb{F}[t])$ where $n > 2$ and $\mathbb{F}[t]$ is the ring of polynomials over $\mathbb{F}$. Then $\lim_{i \to \infty} r(\Gamma, \Gamma_i) = 0$ for every sequence of subgroups Γ_i of Γ with $|\Gamma : \Gamma_i| \to \infty$.*

What makes it possible to apply the rigidity part of the argument is the recent extension of Stuck–Zimmer theorem for the context that was proved by A. Levit [Levit] and the analysis of invariant random subgroups in non-archimedean groups that was carried out in [GL].

The structure of the paper is as follows. In Section 2 we recall the notion of invariant random subgroups from [7Sb] and deduce Theorem 10. Section 3 covers defintion and basic properties of sofic aproximations. The notion of combinatorial cost and its relation to rank gradient is discussed in Section 4 where we prove Theorem 8. Section 5 applies these results to torsion homology growth and proves Theorem 9. We also obtain some explicit bounds on the growth in case when Γ is an arithmetic group. In Section 6 we construct the right angled lattices, proving Theorems 5 and 6. Section 7 discusses the case of lattices over non-archimedean local fields and proves Theorem 11. In the final Section 8 we list open problems and suggest further directions for investigation motivated by our results.

2. The role of IRS and rigidity.

Recall that an IRS on Γ is a conjugacy invariant probability measure on the space Sub_Γ of subgroups of Γ equipped with the Chabauty topology. See [AGV] where the notion was introduced. We denote by $\mathrm{IRS}(\Gamma)$ the set of all IRS on Γ. This is a compact space under the weak* topology. Every subgroup Δ whose normalizer is of finite index in Γ induces an IRS, denoted μ_Δ, which is the uniform measure on the conjugacy class of Δ.

We say that a chain (Γ_n) in Γ is *Farber*, if the action of Γ on the boundary of its coset tree $T = T(\Gamma, (\Gamma_n))$ is essentially free, that is, if almost every element of

∂T has trivial stabilizer in Γ. This is the case e.g., when the chain is normal with trivial intersection. More generally:

Lemma 12. *Let (Γ_n) be a chain of finite index subgroups of Γ. Then (Γ_n) is Farber iff $\mu_{\Gamma_n} \to \mu_{1_\Gamma}$ in the weak star topology on $Prob(Sub_\Gamma)$.*

Thus the notion of IRS μ_n converging to the trivial IRS μ_{1_Γ} is a generalization of a Farber chain.

Proof. Since Γ is countable the chain (Γ_n) is Farber if and only if any $g \in \Gamma\backslash\{1\}$ fixes a set of measure zero in $\partial T(\Gamma, (\Gamma_n))$ i.e.

$$\frac{|\{\Gamma_n a \mid a \in \Gamma, \ g \in (\Gamma_n)^a\}|}{|\Gamma : \Gamma_n|} \to 0 \text{ with } n \to \infty.$$

We can interpret the fraction on the left hand side as the probability that the element g lies in a μ_{Γ_n}-random subgroup from Sub_Γ. Again countability of Γ gives that this condition is equivalent to $\mu_{\Gamma_n} \to \mu_{1_\Gamma}$ in the weak* topology. $\square$

The following result is implicit in [7Sb].

Theorem 13. *Let G be a higher rank simple Lie group and $\Gamma \le G$ a lattice. Let $\Gamma_n \le \Gamma$ be a sequence of finite index subgroups with $|\Gamma : \Gamma_n| \to \infty$. Then the sequence of IRS's μ_{Γ_n} converges to μ_{1_Γ} in $Prob(Sub_\Gamma)$.*

More precisely by [7Sb, Theorem 4.2] the IRS $\mu_{\Gamma_n}^G$ in G corresponding to the lattices $\Gamma_n \le G$ converge to μ_{1_G} as measures on Sub_G. However $\mu_{\Gamma_n}^G$ are induced from μ_{Γ_n} (see [7Sb] for details about inducing IRS from a lattice to the ambient group). Hence Theorem 13 is a consequence of:

Lemma 14. *Let G and $\Gamma \le G$ be as above. Let μ_n be IRSs on Γ and denote by $\mu_n^G = Ind_\Gamma^G(\mu_n)$ the induced IRSs on G. Then μ_n^G converges to μ_{1_G} if and only if μ_n converges to μ_{1_Γ}.*

Proof. For an element $\gamma \in \Gamma$ denote by $m_n(\gamma)$ the μ_n probability that γ belongs to a random subgroup, i.e.

$$m_n(\gamma) = \mu_n(\{\Delta \in Sub_\Gamma : \gamma \in \Delta\}).$$

Since Γ is countable we have that $\mu_n \to \mu_{1_\Gamma}$ iff $m_n(\gamma) \to 0$ for every $\gamma \in \Gamma \setminus \{1\}$. Moreover, since every IRS in G (as well as in Γ) without an atom on the trivial subgroup is supported on Zariski dense subgroups (see [7Sb, Theorem 2.6]) we have that $\mu_n \to \mu_{1_\Gamma}$ iff $m_n(\gamma) \to 0$ for every semisimple element $\gamma \in \Gamma \setminus \{1\}$.

Suppose that this is not the case and let $\gamma \in \Gamma \setminus \{1\}$ be a semisimple element such that $m_n(\gamma)$ does not tend to 0. Since γ is semisimple, its conjugacy class γ^G is closed in G and hence the set

$$\Theta(\gamma) := \{H \in Sub_G : \gamma^G \cap H \ne \emptyset\}$$

is compact in Sub_G. Let ψ be a continuous non-negative function on Sub_G equals to 1 on $\Theta(\gamma)$ and vanishes on 1_G. Then $\int \psi d\mu_n^G \ge m_n(\gamma)$ and hence does not tend to

$$0 = \psi(\langle 1_G \rangle) = \int \psi d\mu_{1_G},$$

implying that μ_n^G does not converge to μ_{1_G}.

Let us now prove the converse. Let Ω be a fundamental domain for Γ in G and suppose that the Haar measure of G is normalized so that Ω has measure 1. The definition of induced measure can be expressed by an integral as follows:

$$\mu_n^G(f) = \int_{G/\Gamma} \mu_n(f^g|_{\mathrm{Sub}_\Gamma}) d(g\Gamma) = \int_\Omega \Big(\int_{\mathrm{Sub}_\Gamma} f^w(\Delta) d\mu_n(\Delta) \Big) dw,$$

where f is an arbitrary continuous function on Sub_G and

$$f^w(H) := f(w^{-1}Hw) \text{ for } w \in G, \ H \in \mathrm{Sub}_G.$$

Note that although f^g is not well defined for $g \in G/\Gamma$, the quantity $\mu_n(f^g|_{\mathrm{Sub}_\Gamma})$ is well defined since μ_n is Γ-conjugacy invariant.

Suppose that μ_n^G does not converge to μ_{1_G}. Then there is a continuous function $f : \mathrm{Sub}_G \to \mathbb{R}^{\geq 0}$ such that $f(\langle 1 \rangle) = 0$ and $\mu_n^G(f) \not\to 0$. By the dominated convergence theorem it follows that there is a set of positive measure $U \subset \Omega$ such that $\int_{\mathrm{Sub}_\Gamma} f^u(\Delta) d\mu_n(\Delta)$ does not converge to 0 for every $u \in U$. Picking $u_0 \in U$, and regarding f^{u_0} as a function on the subspace $\mathrm{Sub}_\Gamma \subset \mathrm{Sub}_G$ we have that

$$\mu_n(f^{u_0}) = \int_{\mathrm{Sub}_\Gamma} f^{u_0}(\Delta) d\mu_n(\Delta) \not\to 0 = f(\langle 1 \rangle) = f^{u_0}(\langle 1 \rangle) = \mu_{1_\Gamma}(f^{u_0}),$$

implying that $\mu_n \not\to \mu_{1_\Gamma}$.

$\square$

3. Sofic approximations

A countable group Γ is *sofic*, if for every finite $B \subseteq \Gamma$ and $\varepsilon > 0$ there exists a finite set V and a map $\sigma : \Gamma \to \mathrm{Sym}(V)$ such that

(Sof) $\quad |\{v \in V \mid v\sigma(a) \neq v, v\sigma(a)\sigma(b) = v\sigma(ab) \ (a, b \in B, a \neq e)\}| > (1 - \varepsilon)|V|$

That is, if the pair (V, σ) looks like a free action of Γ from most $v \in V$.

The notion was invented by Gromov [Gr] and further clarified by Weiss [Weiss] as a common generalization of amenable groups and residually finite groups. As of now, there is no countable group that is known to be non-sofic.

A *sofic approximation of* Γ is a sequence of pairs (V_n, σ_n) such that for every finite subset B of Γ there exists $\varepsilon_n \to 0$ such that (Sof) holds for $V = V_n$, $\sigma = \sigma_n$ and $\varepsilon = \varepsilon_n$ $(n \geq 1)$.

When Γ is generated by a finite set S, one can visualize a sofic approximation by drawing the Schreier graphs $\mathrm{Sch}(V_n, S, \sigma_n)$ as follows. The vertex set equals V_n and for all $x \in V_n$ and $s \in S$ there is an s-labeled edge going from x to $x\sigma_n(s)$. Let $\mathrm{Cay}(\Gamma, S)$ denote the Cayley graph of Γ with respect to S. In a graph G let $B_R(G, v)$ denote the R-ball centered at the vertex v. For two labelled graphs G and G' and vertices $v \in G, v' \in G'$ we write $B_R(G, v) \cong B_R(G', v')$ when $B_R(G, v)$ and $B_R(G', v')$ are isomorphic as labelled graphs rooted at v and v' respectively. Applying (Sof) gives the following.

Lemma 15. *Let Γ be generated by the finite symmetric set and let (V_n, σ_n) be a sofic approximation for Γ. Let $G_n = \mathrm{Sch}(V_n, S, \sigma_n)$ and let $G = \mathrm{Cay}(\Gamma, S)$. Then for every $R > 0$, we have*

$$\lim_{n \to \infty} \mathcal{P}(B_R(G_n, v) \cong B_R(G, e)) = 1.$$

where $v \in V(G_n)$ is uniform random.

Clearly, every sequence of S-labeled graphs as above defines a sofic approximation. In other words, a sequence of S-labeled graphs is realized by a sofic approximation if and only if it Benjamini-Schramm converges to the Cayley graph $\mathrm{Cay}(\Gamma, S)$.

Recall that if Γ is a group and $H \leq \Gamma$ is such that the normalizer $N_\Gamma(H)$ has finite index in Γ, then μ_H denotes the uniform probability measure on the set of Γ-conjugates of H.

For a finite set of symbols S let F_S denote the free group with alphabet S. In the following, we give an alternate definition of soficity using invariant random subgroups.

Lemma 16. *Let Γ be a finitely generated group. Then Γ is sofic, if and only if there exists a finitely generated free group F, a normal subgroup $N \lhd F$ with $\Gamma \simeq F/N$ and a sequence of subgroups $H_n \leq F$ of finite index such that*

$$\lim_{n\to\infty} \mu_{H_n} = \mu_N.$$

Proof. Assume Γ is sofic. Let S be a finite symmetric generating set of Γ. Let $\Phi : F_S \to \Gamma$ be the canonical homomorphism with kernel $N = \ker \Phi$. Let (V_n, σ_n) be a sofic approximation of Γ. By changing the value of σ_n on a subset of V_n of proportion which tends to zero with n, one can assume that $\sigma_n(s^{-1}) = \sigma_n(s)^{-1}$, $(s \in S, \ n \geq 1)$. Then the maps $\sigma_n : S \to \mathrm{Sym}(V_n)$ extend to homomorphisms $\Psi_n : F_S \to \mathrm{Sym}(V_n)$. Again, by changing σ_n on a subset of V_n of small proportion, one can assume that these actions are transitive. Let $v_n \in V_n$ and let H_n be the stabilizer of v_n with respect to the action defined by Ψ_n. Let K_n be a uniform random conjugate of H_n, that is, a random subgroup of F_S with distribution μ_{H_n}. Then K_n is the stabilizer of a uniform random element of V_n. If $g \in N$, then $g \in \ker \Psi_n$, so

$$\mathcal{P}(g \in K_n) = 1 \ \ (n \geq 1).$$

If $g \notin N$, then $\Phi(g) \neq 1$. Let $n_0 > 0$ such that $\Phi(g) \in S^{n_0}$. Then by (Sof), we have

$$\mathcal{P}(g \in K_n) = \frac{|\{x \in V_n \mid x\sigma_n(\Phi(g)) = x\}|}{|V_n|} \leq \varepsilon_n \ \ (n \geq n_0).$$

Since $\varepsilon_n \to 0$, we get that $\mathcal{P}(g \in K_n) \to 0$. This proves that μ_{H_n} weakly converges to μ_N.

Assume now that there exists a free group F with finite free generating set S, a normal subgroup $N \lhd F$ with $\Gamma \simeq F/N$ and a sequence of subgroups $H_n \leq F$ of finite index such that $\lim_{n\to\infty} \mu_{H_n} = \mu_N$. Let K_n be a uniform random conjugate of H_n. Let $\Phi : F \to \Gamma$ be the quotient map by N. For $g \in \Gamma$ choose $\phi(g) \in \Phi^{-1}(g)$.

Let $V_n = F/H_n$ ($n \geq 1$). Then F acts on V_n by the right coset action. We define $\sigma_n : \Gamma \to \mathrm{Sym}(V_n)$ as

$$x\sigma_n(g) = x\phi(g) \ (x \in V_n).$$

We claim that (V_n, σ_n) is a sofic approximation for Γ. Let $B \subseteq \Gamma$ be a finite subset. For $e \neq a \in B$ we have $\phi(a) \notin N$, so by weak convergence, we have

$$\lim_{n\to\infty} \frac{|\{x \in V_n \mid x\sigma_n(a) = x\}|}{|V_n|} = \lim_{n\to\infty} \mathcal{P}(\phi(a) \in K_n) = 0.$$

For $a, b \in B$ we have $w = \phi(a)\phi(b)\phi(ab)^{-1} \in N$, so by weak convergence, we have

$$\lim_{n\to\infty} \frac{|\{x \in V_n \mid x\phi(a)\phi(b) = x\phi(ab)\}|}{|V_n|} = \lim_{n\to\infty} \mathcal{P}(w \in K_n) = 1.$$

The claim holds. □

4. Combinatorial cost and rank gradient

For a graph G let d_G denote the graph metric on $V(G)$. For $G = (V, E)$ and $G' = (V, E')$ we define the *bi-Lipschitz distance of G and G'* as

$$d_L(G, G') = \max \left\{ \max_{(x,y) \in E} d_{G'}(x, y), \max_{(x,y) \in E'} d_G(x, y) \right\}.$$

Note that d_L equals the bi-Lipschitz distortion of the identity map $V \to V$ between the metric spaces (V, d_G) and $(V, d_{G'})$.

Now we define the combinatorial cost of a graph sequence. Let $G_n = (V_n, E_n)$ be a sequence of finite graphs.

Definition 17. *We say that the graph sequence $G'_n = (V_n, E'_n)$ is a* rewiring *of G_n (denoted by $(G_n) \sim (G'_n)$) if $d_L(G_n, G'_n)$ is bounded in n.*

Let the *edge measure* of the graph sequence (G_n) be

$$e(G_n) = \lim_{n \to \infty} \inf \frac{|E_n|}{|V_n|}.$$

Definition 18. *Let the combinatorial cost of (G_n) be*

$$\mathrm{cc}(G_n) = \inf_{(G'_n) \sim (G_n)} e(G'_n).$$

Remark: These notions were introduced by Elek in [El]. Note that for the purpose of the current paper it would be slightly more natural to define the edge measure as a lim sup. However, we decided not to change the original definition of Elek. As a result at some points we are led to make extra assumptions, for instance in Theorem 8 we have to assume that the rank gradient exists. See also Remark 29.

We now prove a vanishing theorem on the combinatorial cost of sofic approximations of right angled groups. In the proof we adapt Gaboriau's method [Gab] who used it to prove that higher rank semisimple non-uniform lattices have fixed price 1.

Theorem 19. *Let Γ be a right angled group and let (V_n, σ_n) be a sofic approximation of Γ. Then*

$$\mathrm{cc}(\mathrm{Sch}(V_n, S, \sigma_n)) = 1$$

for any finite symmetric generating set S of Γ.

Proof. Changing the generating set is a rewiring on the Schreier graph level, so we can assume that S is a right angled generating list. That is,

$$S = \{s_1, \dots, s_k\} \cup \{s_1^{-1}, \dots, s_k^{-1}\}$$

where all s_i have infinite order and $[s_i, s_{i+1}] = 1$ ($1 \le i < k$). Let

$$G_n = \mathrm{Sch}(V_n, S, \sigma_n) \text{ and } G = \mathrm{Cay}(\Gamma, S) \ (n \ge 1).$$

Note that $\{s_1, \dots, s_k\}$ may be a multiset. In that case, we take the corresponding edges of G_n and G with multiplicity. We will use the shortcut

$$xg = x\sigma_n(g) \ (g \in \Gamma, x \in V_n, 1 \le i < k, n \ge 1).$$

Let $R > 0$ be an even integer. We define a subgraph H_n of G_n as follows. Let $C_{n,i}$ be the subgraph spanned by the s_i-edges in G_n $(1 \leq i \leq k)$. The graph $C_{n,i}$ is a disjoint union of cycles. Let $X_{n,1} = V(G_n)$. For $1 < i \leq k$ let $X_{n,i} \subseteq V(G_n)$ be a maximal R-separated subset with respect to the graph metric $d_{C_{n,i-1}}$. Recall that a set is R-separated if every two distinct points in it has distance at least R. By maximality, for every $x \in V(G_n)$ there exists $y \in X_{n,i}$ with $d_{C_{n,i-1}}(x,y) \leq R$. The density of $X_{n,i}$ in a cycle of size at least R is at most $1/R$. Let

$$E_n = \{v \in V(G_n) \mid B_{R+1}(G_n, v) \not\cong B_{R+1}(G, e)\}.$$

be the set of exceptional vertices, where the sofic approximation is bad.

Let the edge sets

$$D_{n,i} = \{(x, s_i) \mid x \in X_{n,i}\} \ (1 \leq i \leq k)$$

and

$$F_n = \{(v, s_i) \mid v \in E_n, 1 \leq i \leq k\}.$$

Let the graph

$$H_n = F_n \cup \bigcup_{i=1}^{k} D_{n,i}.$$

That is, we keep all the s_1-edges in G_n and add the s_i-edges starting at $X_{n,i}$ $(1 < i \leq k)$ and then add all edges starting at the exceptional vertices.

We claim that (H_n) is a rewiring of (G_n) with bi-Lipschitz distortion at most $(2R + 1)^k$. Since H_n is a subgraph of G_n, we only have to check one Lipschitz condition. Let $x \in V(G_n)$, let $1 < m \leq k$ and let $y = xs_m$. We claim that $d_{H_n}(x, y) \leq (2R+1)^m$. We prove this by induction on m. For $m = 1$ this is trivial. Assume $m > 1$. Then there exists $z \in X_{n,m}$ such that $d_{C_{n,m-1}}(x, z) \leq R$, that is, $z = xs_{m-1}^l$ with some $0 \leq |l| \leq R$. If $x \in E_n$ is exceptional then $(x, s_m) \in F_n \subseteq H_n$. Otherwise, the $R+1$ ball around x in G_m is isomorphic to the $R+1$ ball in G. Since s_{m-1} and s_m commute in Γ, the word $s_{m-1}^l s_m s_{m-1}^{-l}$ is a walk of length at most $2R+1$ going from x to y in $C_{n,m-1} \cup D_{n,m}$. That is, $d_{C_{n,m-1} \cup D_{n,m}}(x, y) \leq 2R+1$. By induction, every s_{m-1}-edge in this walk can be substituted by a walk of length at most $(2R + 1)^{m-1}$ in H_n. This proves the claim and yields

$$d_L(G_n, H_n) \leq (2R + 1)^k$$

which implies that (H_n) is a rewiring of (G_n).

Since s_i has infinite order, for all $x \notin E_n$ the $C_{n,i}$-cycle of x has size at least R. On such cycles the density of $X_{n,i}$ is at most $1/R$ $(1 \leq i \leq k)$. This implies

$$E(H_n) \leq k \, |E_n| + |V(G_n)| + (k-1)(|E_n| + \frac{|V(G_n)| - |E_n|}{R})$$

which yields

$$\frac{|E(H_n)|}{|V(G_n)|} \leq 1 + \frac{k-1}{R} + (2k - 1 - \frac{k-1}{R})\frac{|E_n|}{|V(G_n)|}.$$

Lemma 15 gives $|E_n| / |V(G_n)| \to 0$, which gives

$$e(H_n) \leq 1 + \frac{k-1}{R}.$$

By chosing R arbitrarily large, we get $cc(G_n) = 1$.   □

The following corollary of the above proof directly implies Theorem 7.

Proposition 20. *Let Γ be a right angled group generated by a set of elements $S = \{x_1, \ldots, x_k\}$ of infinite order such that $[x_i, x_{i+1}] = 1$ for all $i = 1, \ldots, k-1$. Let (V_n, σ_n) be a sofic approximation of Γ and let $G_n = \mathrm{Sch}(V_n, S, \sigma_n)$.*

For every even integer $R \in \mathbb{N}$ there exist rewiring graphs H_n of G_n such that $d(G_n, H_n) \leq (2R+1)^k$ and

$$\frac{|E(H_n)|}{|V(G_n)|} \leq 1 + \frac{k-1}{R} + 2k\frac{|E_n|}{|V(G_n)|}$$

where

$$E_n = \{v \in V(G_n) \mid B_{R+1}(G_n, v) \not\cong B_{R+1}(\mathrm{Cay}(\Gamma, S), e)\}.$$

In particular

$$\limsup_n \frac{|E(H_n)|}{|V(G_n)|} \leq 1 + \frac{k-1}{R}.$$

Theorem 7 follows directly by setting $\epsilon_R = k/R$ and noting that

$$d(G_n, H_n) \leq (2R+1)^k = O(\epsilon_R^{-k}).$$

For a subgroup H of Γ let $\mathrm{Sch}(\Gamma, H, S)$ denote the Schreier diagram of Γ with respect to H and S. Recall that the vertex set of $\mathrm{Sch}(\Gamma, H, S)$ is the right coset space Γ/H and for every $s \in S$ and $x \in \Gamma/H$ there is an s-labeled edge going from x to xs. The metric coming from a Schreier diagram is defined by forgetting its labels and treating it as a simple graph.

We will use the language of measured groupoids to analyze the connection between the rank gradient and the combinatorial cost. This goes back to [AbN] where the first and last authors expressed the rank gradient of a chain from the cost of the corresponding profinite measured groupoid.

Recall that a measured groupoid $\mathcal{G}$ is a Borel probability space $\mathcal{G}^0$ together with a Borel space of arrows $\mathcal{G}^1$, maps $i, t : \mathcal{G}^1 \to \mathcal{G}^0$ (the initial and terminal map) and a composition operation such that $\mathcal{G}$ is a groupoid with respect to these operations. We also assume that the underlying measure μ is invariant under the arrows.

Let the countable group Γ act on the Borel probability space (X, μ) by measure preserving maps. Set $\mathcal{G}^0 = (X, \mu)$ and $\mathcal{G}^1 = \{(x, g) \mid x \in X, g \in \Gamma\}$. Let $i((x,g)) = x$ and $t((x,g)) = xg$. Let the product $(x,g)(y,h)$ be defined if $y = xg$, in this case it equals (x, gh) and the inverse of (x, g) is defined to be (xg, g^{-1}). This defines the measured groupoid of the action. Using the initial map the measure μ can be naturally lifted to a Borel measure $\widetilde{\mu}$ on $\mathcal{G}^1$.

Let the *cost* of a measured groupoid $\mathcal{G}$ be

$$\mathrm{c}(\mathcal{G}) = \inf_{\substack{Y \subseteq \mathcal{G}^1 \text{ Borel} \\ Y \text{ generates } \mathcal{G}}} \widetilde{\mu}(Y)$$

the infimum of measures of generating Borel subsets. Here generation is meant in the usual algebraic sense. The notion of cost for measurable equivalence relations has been introduced by Levitt [Levitt] and analyzed in depth by Gaboriau [Gab].

A case of particular interest in this paper is when Γ acts on the right coset space Γ/H for some finite index subgroup H of Γ. Here μ and $\widetilde{\mu}$ are just counting measures, normalized by the index of H in Γ. In [AbN] the following is proved. Since we use a somewhat different language in [AbN], we sketch a proof here.

Lemma 21. *Let Γ be a countable group and let H be a subgroup of finite index in Γ. Let $\mathcal{G}$ be the measured groupoid of the right coset action of Γ on Γ/H. Then we have*

$$r(\Gamma, H) = \frac{d(H) - 1}{|G : H|} = \mathrm{c}(\mathcal{G}) - 1.$$

Proof. If R generates H then it is easy to see that

$$\{(H, r) \mid r \in R\}$$

together with any subset of $\mathcal{G}^1$ that connects $\mathcal{G}^0$ (e.g. a spanning tree) will generate $\mathcal{G}$. This gives

$$\mathrm{c}(\mathcal{G}) \leq \frac{1}{|\Gamma : H|} \left(|R| + (|\Gamma : H| - 1) \right)$$

which gives

$$r(\Gamma, H) \geq \mathrm{c}(\mathcal{G}) - 1.$$

Let $Y \subseteq \mathcal{G}^1$ be a subset. Look at the graph G on Γ/H defined by Y. Note that G may have multiple edges and loops. The fundamental group π_1 of G rooted at the vertex H admits a natural homomorphism Φ to H by evaluating loops by taking the product of edge labels coming from $\mathcal{G}^1$. It is easy to see that Y generates $\mathcal{G}$ if and only if G is connected and Φ is surjective. So, if Y generates $\mathcal{G}^1$, we have

$$d(\pi_1) \geq d(H).$$

Let Y be a generating subset of minimal measure $\mathrm{c}(\mathcal{G})$. The fundamental group π_1 is a free group of rank $|Y| - |\Gamma : H| + 1$, which yields

$$r(\Gamma, H) = \frac{d(H) - 1}{|\Gamma : H|} \leq \frac{|Y|}{|\Gamma : H|} - 1 = \mathrm{c}(\mathcal{G}) - 1.$$

The Lemma holds. $\square$

Let $\mathcal{G}$ be a measured groupoid of a p.m.p. action of Γ on (X, μ). Let the trivial subset

$$N = \left\{ (x, 1) \in \mathcal{G}^1 \mid x \in X \right\}.$$

Definition 22. *For two Borel subsets $A, B \subseteq \mathcal{G}^1$ let the distance between them be defined as*

$$d_L(A, B) = \inf \left\{ k \in \mathbb{N} \mid A \subseteq (B \cup B^{-1} \cup N)^k \text{ and } B \subseteq (A \cup A^{-1} \cup N)^k \right\}.$$

Of course, d_L may be infinite, even when both A and B generate $\mathcal{G}$.

Let H be a subgroup of Γ of finite index. Let $\mathcal{G}$ denote the measured groupoid defined by the right action of Γ on Γ/H, with underlying measure μ. Let $G = (V, F)$ be an abstract directed graph defined on the vertex set $V = \Gamma/H$. By a Γ-labeling of G we mean a map $\phi : F \to \Gamma$ such that for all $f = (x, y) \in F$, we have $x\phi(f) = y$. In other terms, ϕ is a map from F to $\mathcal{G}^1$ which is trivial on the $\mathcal{G}^0$ level. When G is a Schreier graph, the natural Γ-labeling means the labeling coming from the coset action.

The following lemma controls how a rewiring of a Schreier graph (viewed as an abstract graph) can be labeled and extended to be a generator for the measured groupoid. This lemma is essential to proving Theorem 8 and the estimates on the torsion homology.

Lemma 23. *Let Γ be a group generated by the finite symmetric set S. Let H be a subgroup of Γ of finite index with measured groupoid $\mathcal{G}$ and let K be the uniform random conjugate of H in Γ. Let $G = (V, E) = \mathrm{Sch}(\Gamma, H, S)$ with the natural Γ-labeling φ and let $G' = (V, F)$ be a graph with $D = d_L(G, G')$. Then there exists a Γ-labeling ϕ of G' and a subset $I \subseteq \mathcal{G}^1$ of measure*

$$\tilde{\mu}(I) \leq \sum_{g \in S^{D^2+1} \setminus \{1\}} \mathcal{P}(g \text{ is in } K)$$

such that

$$d_L(\varphi(E), \phi(F) \cup I) \leq D^2 + 1.$$

In particular, we have

$$r(\Gamma, H) \leq \frac{|F|}{|V|} - 1 + \sum_{g \in S^{D^2+1} \setminus \{1\}} \mathcal{P}(g \text{ is in } K).$$

Proof. By the definition of D, for each $f = (x, y) \in F$, there exists $k \leq D$ and a walk $e_1, \ldots, e_k \in E$ from x to y. Choose such a walk and let

$$\phi(f) = \varphi(e_1) \cdots \varphi(e_k).$$

Let $e = (x, y) \in E$. Then there exists $k \leq D$ and a walk $f_1, \ldots, f_k \in F$ from x to y. Choose such a walk. Let

$$g(e) = \varphi(e)\phi(f_k)^{-1} \cdots \phi(f_1)^{-1} \in \Gamma.$$

Let $I \subseteq \mathcal{G}$ defined by

$$I = \{(x, g(e)) \mid e = (x, y) \in E, g(e) \neq 1\}$$

Note that $g(e)$ stabilizes x. Let

$$J = \phi(F) = \{(x, \phi(f)) \mid f = (x, y) \in F\}.$$

We claim that $I \cup J \subseteq \varphi(E)^{D^2+1}$. For $f \in F$ we have $\phi(f) = \varphi(e_1) \cdots \varphi(e_k)$ which implies $J \subseteq \varphi(E)^D$. For $e = (x, y) \in E$ with $g(e) \neq 1$ we have $g(e) = \varphi(e)\phi(f_k)^{-1} \cdots \phi(f_1)^{-1}$. Since $\phi(f_k) \in E^D$ we get $(x, g(e)) \in \varphi(E)^{D^2+1}$ and so $I \subseteq \varphi(E)^{D^2+1}$. The claim holds.

We now claim that $\varphi(E) \subseteq (I \cup J)^{D+1}$. Let $e = (x, y) \in E$. If $g(e) = 1$ then $\varphi(e) = \phi(f_1) \cdots \phi(f_k)$ and thus $\varphi(e) \in J^D$. If $g(e) \neq 1$ then $\varphi(e) = g(e)\phi(f_1) \cdots \phi(f_k)$ and thus $\varphi(e) \in J^{D+1}$. The claim holds.

The two claims together give

$$d_L(\varphi(E), I \cup J) \leq D^2 + 1.$$

Since $\varphi(E)$ generates $\mathcal{G}$, this also implies that $I \cup J$ generates $\mathcal{G}$.

For every e, $g(e)$ stabilizes x and can be written as a word of length at most $D^2 + 1$ in S. Hence, the measure

$$\tilde{\mu}(I) \leq \sum_{g \in S^{D^2+1} \setminus \{1\}} \mathcal{P}(g \text{ is in } K)$$

and we have

$$\tilde{\mu}(J) = |F| / |V|.$$

By Lemma 21, $r(\Gamma, H) + 1$ equals the minimal $\widetilde{\mu}$-measure of a generating subset, thus we have

$$r(\Gamma, H) + 1 \leq \widetilde{\mu}(I \cup J) \leq \frac{|F|}{|V|} + \sum_{g \in S^{d^2+1} \setminus \{id\}} \mathcal{P}(g \text{ is in } K)$$

The lemma holds. □

We are ready to prove Theorem 8.

Proof of Theorem 8. Let $G_n = \text{Sch}(\Gamma, \Gamma_n, S)$ and let $cc = \text{cc}(G_n)$. Let K_n be the IRS with distribution μ_{Γ_n}.

Let $\varepsilon > 0$. Then there exists a rewiring (G'_n) of (G_n) such that $e(G'_n) \leq cc + \varepsilon$. Let $d > 0$ such that $d_L(G_n, G'_n) < d$ $(n \geq 1)$. Let $W = S^{d^2+1} \setminus \{id\}$. Lemma 23 implies that for all $n \geq 1$ we have

$$r(\Gamma, \Gamma_n) \leq \frac{|E(G'_n)|}{|G_n|} - 1 + \sum_{g \in W} \mathcal{P}(g \text{ is in } K_n).$$

Since (Γ_n) is Farber, for each $g \in W$, we have

$$\lim_{n \to \infty} \mathcal{P}(g \text{ is in } K_n) = 0.$$

So, taking a liminf on both sides and using that $\text{r}(\Gamma, (\Gamma_n))$ converges gives us

$$\text{RG}(\Gamma, (\Gamma_n)) \leq \text{cc}(\text{Sch}(\Gamma, \Gamma_n, S)) - 1$$

and the inequality is proved.

When (Γ_n) forms a chain, then the reverse inequality is proved in [El, Theorem 1.2]. We give a short proof here using the measured groupoid language.

Let $\mathcal{G}_n$ be the measured groupoid associated to the action of Γ on Γ/Γ_n. Assume that $r(\Gamma, \Gamma_{n_0}) \leq R$ for some n_0. Let $F \subseteq \mathcal{G}_{n_0}$ be a generating subset of measure at most $R + 1$ and let $E \subseteq \mathcal{G}_{n_0}$ be the standard generating set coming from S. Let $d > 0$ such that $F^d \supseteq E$ and $E^d \supseteq F$, where Y^d denotes all words of length at most d in $\mathcal{G}_{n_0}$.

Let $\Phi_n : \mathcal{G}_n \to \mathcal{G}_{n_0}$ $(n \geq n_0)$ denote the standard factor map from $\mathcal{G}_n$ to $\mathcal{G}_{n_0}$. This associates a Γ_n-coset to the unique Γ_{n_0}-coset containing it and is a groupoid homomorphism. Let $F_n = \Phi_n^{-1}(F)$ and $E_n = \Phi_n^{-1}(E)$. Then F_n has the same measure as F and we have $F_n^d \supseteq E_n$ and $E_n^d \supseteq F_n$. Let G'_n be the graph defined on Γ/Γ_n by the arrows F_n. Then we have

$$d_L(G'_n, \text{Sch}(\Gamma, \Gamma_n, S)) \leq d$$

and so G'_n is a rewiring of $\text{Sch}(\Gamma, \Gamma_n, S)$. This implies $\text{cc}(\text{Sch}(\Gamma, \Gamma_n, S)) \leq R$ and proves the reverse inequality.

The Theorem holds. □

Remark. We do not have an example for a strict inequality in Theorem 8. Maybe equality holds in general, but it is not clear how to get a cheap rewiring from a cheap generating set for the measured groupoid. A priori, such cheap generating sets (or any rewirings of them) may lie quite deep in the group structure, in which case generation may happen very slowly.

Theorem 24. *Let Γ be a right angled group and let (Γ_n) be a Farber sequence of subgroups of finite index in Γ. Then*

$$\mathrm{RG}(\Gamma, (\Gamma_n)) = 0.$$

Proof. By passing to a subsequence we can assume that $\mathrm{r}(\Gamma, (\Gamma_n))$ converges. Let S be a finite generating set of Γ and let $\Phi : F_S \to \Gamma$ be the canonical homomorphism with kernel $N = \ker \Phi$. Let $H_n = \Phi^{-1}(\Gamma_n)$ $(n \geq 1)$. Since H_n is Farber, by Lemma 16 the coset actions σ_n of Γ on Γ/Γ_n give a sofic approximation for Γ. By Theorem 19 we have $\mathrm{cc}(\mathrm{Sch}(\Gamma, \Gamma_n, S)) = 1$. Theorem 8 yields

$$\mathrm{RG}(\Gamma, (\Gamma_n)) \leq \mathrm{cc}(\mathrm{Sch}(\Gamma, \Gamma_n, S)) - 1 = 0.$$

The Theorem holds. $\square$

Proof of Theorem 2. By Theorem 13, the sequence (Γ_n) is Farber. Applying Theorem 24 gives $\mathrm{RG}(\Gamma, (\Gamma_n)) = 0.$ $\square$

5. Growth of homology torsion of right angled groups

In this section we give upper estimates on the torsion homology and prove Theorem 9.

For a finite CW-complex M let $\pi_1 M$ denote its fundamental groupoid. This is a measured groupoid defined as follows. Let us put the uniform probability measure on the vertices of M. Let $P(M)$ denote the set of paths in M. This is a groupoid under concatenation. Then $\pi_1 M$ is the quotient of $P(M)$ modulo homotopy. Since the set of directed edges $E(M)$ in M generate $P(M)$, their homotopy classes generate $\pi_1 M$.

We now describe how to rewire a complex to another generating set of its fundamental groupoid.

Let M be a finite CW-complex, let $\mathcal{G} = \pi_1 M$ and let $E = E(M)$. Let $F \subseteq \mathcal{G}$ be another generating set. We define the *rewired complex N with respect to F* as follows. Let $N_0 = M_0$ and $N_1 = F$. For every $e \in E$ choose a path $\varphi(e) = (f_1, \ldots, f_k)$ in N such that $e = f_1 \cdots f_k$. Also, for every $f \in F$ choose a path $\phi(f) = (e_1, \ldots, e_s)$ in M such that $f = e_1 \cdots e_s$ (where k and s may depend on e and f respectively). The maps φ and ϕ extend to $P(M)$ and $P(N)$ by concatenation. For every disc bounded by the path p in M add the disc bounded by $\varphi(p)$ to N (type I). For every $f \in F$ add the disc bounded by $\varphi\phi(f)f^{-1}$ to N (type II).

Note that the rewired complex N does depend on the choices φ and ϕ.

Proposition 25. *Using the above notations, we have $\pi_1 M \cong \pi_1 N$.*

Proof. The maps $\varphi : P(M) \to P(N)$ and $\phi : P(N) \to P(M)$ are well defined groupoid homomorphisms that preserve the starting and endpoints of paths.

For a path $p \in P(N)$ or $p \in P(M)$ let $\Delta(p)$ be the product of its edges (as elements of $\mathcal{G}^1$). Then Δ is a groupoid homomorphism from each of $P(N)$ and $P(M)$ to $\mathcal{G}^1$, and by the definition of φ and ϕ, we have

$$\Delta(\varphi(p)) = \Delta(p) \ (p \in P(M)) \text{ and } \Delta(\phi(p)) = \Delta(p) \ (p \in P(N)).$$

Since $\mathcal{G} = \pi_1 M$, two paths $p_1, p_2 \in P(M)$ are homotopic in M if and only if $\Delta(p_1) = \Delta(p_2)$.

We claim that φ and ϕ respect the homotopy defined by the 2-cells of M and N and so induce well-defined homomorphisms $\varphi : \pi_1 N \to \pi_1 M$ and $\phi : \pi_1 M \to \pi_1 N$.

Indeed, if $p \in P(M)$ bounds a disc, then $\varphi(p)$ is a type I disc in N. So φ respects homotopy. If the loop $p \in P(N)$ starting at x bounds a type I disc in N, then $p = \varphi(p')$ for some disc $p' \in M$, which implies

$$\Delta(\phi(p)) = \Delta(p) = \Delta(\varphi(p')) = \Delta(p') = 1_x$$

that is, $\phi(p)$ is nullhomotopic. If the loop $p \in P(N)$ starting at x bounds a type II disc in N, then $p = \varphi\phi(f)f^{-1}$ for some edge f in M. By the above, we have

$$\Delta(\phi(p)) = \Delta(\varphi\phi(f)f^{-1}) = \Delta(\varphi\phi(f))\Delta(f^{-1}) = 1_x$$

that is, $\phi(p)$ is nullhomotopic. So ϕ respects homotopy. The claim holds and implies that $\varphi : \pi_1 N \to \pi_1 M$ and $\phi : \pi_1 M \to \pi_1 N$ are well defined homomorphisms.

Let $p \in P(N)$. Then we have $\Delta(\phi\varphi(p)) = \Delta(p)$, thus $\phi\varphi(p)$ and p are homotopic. Let $p \in P(M)$. Then by the type II relations, $\varphi\phi(p)$ is homotopic to p. We get that $\phi\varphi$ and $\varphi\phi$ are the identity maps on $\pi_1 M$ and $\pi_1 N$ and so $\varphi : \pi_1 N \to \pi_1 M$ is an isomorphism. $\qquad\square$

In a more algebraic way, we assert above that the homotopy classes and the Δ-classes on $P(N)$ are equal, which establishes the isomorphism between $\pi_1 N$ and $\pi_1 M$.

The following is a stardard lemma in linear algebra.

Lemma 26. *Let $m, d, l \in \mathbb{N}$ and let $v_1, \ldots, v_l \in \mathbb{Z}^m$. Suppose that each of the vectors v_i is such that the sum of the absolute values of its coordinates is at most b. Let $A = \mathbb{Z}^m / \sum_{i=1}^{l} \mathbb{Z}v_i$. Let $\mathrm{trs(A)}$ denote the size of the torsion subgroup of A. Then $\mathrm{trs}(A) \le b^m$.*

Proof. Let X be the matrix whose rows are the vectors $v_1, \ldots, v_l$. By [Ja] Theorem 3.9, $\mathrm{trs(A)}$ is the greatest common divisor of all non-zero minors of A. So to prove the lemma we need to bind any such minor. Therefore we may assume that X is non-singular and $m = l$. From geometric consideration in the Euclidean space $\mathbb{R}^m$ we know that $|\det X|$ is at most the product of the lengths of the row vectors of X. So $|\det X| \le b^m$ and the lemma follows. $\square$

In the following Lemma, we show that for a finitely presented group, the torsion homology growth stays bounded over finite index subgroups. Note that by [KKN] this is false already in the realm of finitely generated solvable groups.

Lemma 27. *Let $\Gamma = \langle X; R \rangle$ be a finitely presented group. Let $|X| = d$ and let b be the maximal length of the relations in R. Let H be a subgroup of index n in Γ. Then*

$$\frac{\ln \mathrm{trs}H}{n} \le d \ln b.$$

Proof. Consider the Schreier presentation $\langle X'; R' \rangle$ for H obtained from the presentation of Γ. We have $|X'| = n(d-1) + 1 < nd$ and in addition all relations in R' have length at most b considered as words in X'. By considering the Abelianization of the presentation $\langle X'; R' \rangle$ and using Lemma 26, we get

$$\mathrm{trs}(H) \le b^{|X'|} < b^{dn}$$

and the result follows. $\square$

We are ready to prove the main theorem of this section. We will make use of the following definition:

Definition 28. *A graph sequence (G_n) has subexponential distortion if there are d_r and ε_r such that $\epsilon_r \to 0$ and*

$$\lim_{r \to \infty} \varepsilon_r \ln d_r = 0$$

and for every r there is a rewiring $G_n^{(r)}$ of G_n such that $d_L(G_n, G_n^{(r)}) \leq d_r$ and

$$\limsup_{n \to \infty} \frac{|E(G_n^{(r)})|}{|V(G_n^r)|} < 1 + \varepsilon_r.$$

Remark 29. *Note that here we used $\limsup$ instead of $\liminf$ as in the original definition of combinatorial cost for obvious purposes.*

Proof of Theorem 9. Let (S, R) be a finite presentation of Γ. Let M be the corresponding presentation 2-complex, with one vertex, $|S|$ edges labeled by S and the relations glued on as disks. Let b be the maximal size of a disc in M. Let

$$M_n = \widetilde{M}/\Gamma_n$$

be the lift of M with respect to Γ_n. The 1-skeleton of M_n equals the Schreier graph $G_n = \mathrm{Sch}(\Gamma, \Gamma_n, S)$. Since Γ_n is a Farber sequence, G_n is a sofic approximation for Γ. Let $\mathcal{G}_n$ be the measured groupoid of the coset action of Γ on Γ/Γ_n with underlying measure μ_n and let E_n be the standard generating set of $\mathcal{G}_n$.

By the subexponential distortion assumption, there are sequences d_r and ε_r such that $\lim_{r \to \infty} \varepsilon_r \ln d_r = 0$ and for every r there is a rewiring $G_n^{(r)}$ of G_n such that $d_L(G_n, G_n^{(r)}) \leq d_r$ and

$$\frac{|E(G_n^{(r)})|}{|V(G_n^r)|} < 1 + \varepsilon_r + \delta_n^{(r)},$$

for some $\delta_n^{(r)} > 0$ which tends to 0 with n for any given r.

Let K_n be the uniform random conjugate of Γ_n in Γ and let

$$\gamma_n^{(r)} = \sum_{g \in S^{d_r^2+1} \setminus \{1\}} \mathcal{P}(g \text{ is in } K_n).$$

For a fixed r let $H_n = G_n^{(r)}$. By Lemma 23 there exist generating sets F_n of $\mathcal{G}_n$ of measure

$$\widetilde{\mu}_n(F_n) \leq 1 + \varepsilon_r + \delta_n^{(r)} + \gamma_n^{(r)}$$

such that

$$d_L(E_n, H_n) \leq d_r^2 + 1.$$

Let N_n be the rewiring of the complex M_n with respect to F_n, such that the rewiring functions φ and ϕ always take the edges of the respective complex to paths of length at most $d_r^2 + 1$. Taking into account both type I and II 2-discs in N_n, the maximal size of a 2-disc in N_n is at most

$$\max \left\{ b(d_r^2 + 1), (d_r^2 + 1)^2 + 1 \right\} \leq 4bd_r^4.$$

By Proposition 25 the measured groupoid $\pi_1 N_n \cong \pi_1 M_n = \mathcal{G}_n$. This means that for any vertex x of N_n, the fundamental group $\pi_1(N_n, x) \cong \Gamma_n$. That is, N_n is a presentation complex for Γ_n. Choosing a spanning tree in the 1-skeleton of N_n

and contracting it to a point gives that Γ_n admits a presentation with number of generators at most

$$(\widetilde{\mu}_n(F_n) - 1)\,|\Gamma : \Gamma_n| \leq \left(\varepsilon_r + \delta_n^{(r)} + \gamma_n^{(r)}\right)|\Gamma : \Gamma_n|$$

where all the relations have size at most $4bd_r^4$. This yields

$$\frac{\log \text{trs}\Gamma_n}{[\Gamma : \Gamma_n]} \leq \left(\varepsilon_r + \delta_n^{(r)} + \gamma_n^{(r)}\right)\log(4bd_r^4).$$

Since (Γ_n) is a Farber sequence, we have $\lim_{n\to\infty} \gamma_n^{(r)} = 0$ and we get

$$\limsup_{n\to\infty} \frac{\log \text{trs}\Gamma_n}{[\Gamma : \Gamma_n]} \leq \varepsilon_r \log(4bd_r^4).$$

Letting r tend to infinity yields

$$\lim_{n\to\infty} \frac{\log \text{trs}\Gamma_n}{[\Gamma : \Gamma_n]} = 0.$$

The Theorem follows. $\square$

5.1. **Explicit estimates with and without the CSP.** In this section we will focus on the case when Γ is an arithmetic group. We show that when Γ has the generalized congruence subgroup property (CSP) there is a polynomial bound on the size of the torsion in homology.

Proposition 30. *Let G be a semi-simple algebraic group defined over a number field K and let Γ be an arithmetic subgroup of G. Suppose that Γ has CSP. Then there is a number $c > 0$ (depending on Γ) such that if $H \leq \Gamma$ has index n then $|H^{ab}| < n^c$.*

Proof. By [LuMa] the arithmetic group Γ has CSP if and only if Γ has polynomial representation growth, that is there exists a number $c_0 > 0$ such that Γ has at most n^{c_0} irreducible representations of degree at most n. Suppose now that H is a subgroup of index n. We claim that $|H^{ab}| \leq n^{c_0+1}$. This is provided by Lemma 2.2 of [LuMa] but for completeness we give a proof. Suppose that $|H^{ab}| > n^{c_0+1}$, this means that H has more that n^{c_0+1} one dimensional characters ρ_i. Let ϕ_i be the induced character $Ind_H^G(\rho_i)$, this is a character of G of degree n. We claim that each ϕ_i can share an irreducible constituent with at most $n-1$ other characters ϕ_j. Indeed if ϕ_i and ϕ_j share an irreducible character of G then by Frobenius reciprocity ρ_j is a summand of the restricted character $(\phi_i)|_N$. However the degree of $(\phi_i)|_N$ is n and so it cannot have more that n irreducible constitutents. The claim follows. This means that the total number of different irreducible characters of G appearing as constituents of all ϕ_i is more than $n^{c_0+1}/n = n^{c_0}$, contradiction. $\square$

When CSP fails or is not known to hold the proof of Theorem 9 gives a much weaker, but still explicit bound.

Theorem 31. *Let $\mathcal{G}$ be a simple algebraic group defined over a number field K and let Γ be a cocompact right angled arithmetic subgroup of $\mathcal{G}$. There are constants $\alpha > 0$ and $A > 0$ such that for any congruence subgroup H of index n in Γ*

$$\log \text{trs}H \leq \frac{A \cdot n}{(\log n)^\alpha}.$$

Proof. Fix a right angled generating set S of Γ of size k. For a graph L, $r > 0$ and vertex v of L, by $B_L(R, v)$ we denote the ball of radius r centered at v in the edge metric of L.

We need to estimate the contribution to the edge cost from the bad vertices in the sofic approximation in the proof of Theorem 9.

By the discrete version of Theorem 5.2 of [7Sa] there are constants c and $\mu > 0$ (depending on Γ) such that if $r > 0$ and H is a congruence subgroup of Γ with $n = [\Gamma : H]$ then

$$(1) \qquad |\{v \in \Gamma/H \mid B_{\mathrm{Sch}(\Gamma,\mathrm{H},\mathrm{S})}(r, v) \not\simeq B_{Cay(\Gamma,S)}(r, v)\}| \leq e^{Cr} n^{1-\mu}.$$

Recall that k is the size of the right angled generating set S of Γ. Let R be the even integer closest to $\beta(\log n)^{1/2k}$ for some small constant $\beta > 0$ such that $\max\{(3\beta)^{2k}, \beta\} < \mu/2C$. Proposition 20 provides a rewiring of $\mathrm{Sch}(\Gamma, \mathrm{H}, \mathrm{S})$ of distance at most $d = (2R + 1)^k$ and edge cost at most $1 + (k-1)/R + \delta_n$ where $\delta_n \leq 2k|E_n|/n$ and E_n is the set of vertices of $\mathrm{Sch}(\Gamma, \mathrm{H}, \mathrm{S})$ where the injectivity radius is less than $R + 1$. From (1) we obtain $|E_n| \leq e^{C(R+1)} n^{1-\mu} = O(n^{1-\mu/2})$ since $R \leq \beta \log n$ and $e^{CR} \leq n^{C\beta} \leq n^{\mu/2}$. Similarly (1) shows that the number of bad vertices of $\mathrm{Sch}(\Gamma, \mathrm{H}, \mathrm{S})$ with injectivity radius less than $d^2 + 1 < (3R)^{2k}$ is at most $O(n^{1-\mu/2})$. So the contribution from the bad vertices

$$\gamma_n = \sum_{g \in S^{d^2+1} \setminus \{1\}} \mathcal{P}(g \text{ is in } K)$$

in the proof of Theorem 9 is at most $O(n^{-\mu/2})$.

The proof of Theorem 9 now gives that H can be presented by x elements, where

$$x = n((k-1)/R + \delta_n + \gamma_n) = O(n/(\log n)^{1/2k})$$

and relations of length at most $(3R)^{4k} = O((\log n)^2)$. Therefore by Lemma 26 we obtain

$$\log |\mathrm{trs}(H^{ab})| \leq x \log((3R)^{4k})) = O\left(\frac{n}{(\log n)^{1/2k}}\right) O(\log \log n) = O\left(\frac{n \log \log n}{(\log n)^{1/2k}}\right).$$

The Theorem follows with any $\alpha < 1/2k$. $\qquad \square$

We note that in case Γ has property FAb, (e.g. when Γ has property T) we can simplify the above argument because H contains a principal congruence subgroup N of index at most n^b for some constant b. Then it is sufficient to bind $|N^{ab}|$ and the above argument is simpler: For normal subgroups the avoidance of bad vertices in the sofic approximation is equivalent to the requirement that N intersects the ball of radius $(2R+1)^{2k}$ in $\mathrm{Cay}(\Gamma, S)$ trivially. This holds when $R = O((\log n)^{1/2k})$ and we reach essentially the same bound as above.

6. Right angled lattices in higher rank Lie groups

In this section we construct the lattices Γ from Theorems 5 and 6. The main idea is to make sure that Γ has a semisimple element $g \neq 1$ which is not regular (i.e. has repeated eigenvalues). The (arithmetic) Borel density theorem then can be applied to ensure that $C_\Gamma(g)$ contains elements of infinite order and in this way obtain a right angled generating set for some finite index subgroup of Γ.

6.1. **Co-compact Right angled lattices in orthogonal groups.** Let $d \in \mathbb{N}$ be a square-free natural number, set $K = \mathbb{Q}(\sqrt{d})$ and denote its ring of integers $\mathcal{O}_K$. Let $n \geq 7$ and define

$$f(x_1, \ldots, x_{n+2}) = x_1^2 + \ldots + x_n^2 - \sqrt{d}(x_{n+1}^2 + x_{n+2}^2)$$

be a quadratic form on $V = K^{n+2}$. Let $\mathbb{G} = SO(f)$ and $\Gamma = \mathbb{G}(\mathcal{O}_K)$. We will construct a finite set $\{b_1, \ldots, b_m\}$ such that $\langle b_i, b_{i+1} \rangle \simeq \mathbb{Z}^2$ for each $i = 1, \ldots, m-1$, and such that $\langle b_1, \ldots, b_m \rangle$ is of finite index in Γ.

Denote by V the quadratic module (K^{n+2}, f). As K is a subfield of $\mathbb{R}$ the sign $\pm$ is defined for elements in K. For subspaces A, B of V we will write $A \perp B$ to indicate that they are orthogonal with respect to f and denote by $A \oplus B$ their orthogonal direct sum. Let $\mathcal{L}$ be the set of 3-dimensional subspaces W of V of signature (2,1), i.e. $W = W_+ \oplus W_-$ with $\dim W_- = 1$ and $\dim W_+ = 2$, and f is positive (resp. negative) definite on W_+ (resp. on W_-). Define an equivalence relation $\sim$ on $\mathcal{L}$ by taking the reflexive and transitive closure of the relation $W_1 \perp W_2$.

Proposition 32. *The relation $\sim$ has a single equivalence class on $\mathcal{L}$.*

Denote $Q(v) = f(v, v)$ and $\langle v, u \rangle = \mathrm{span}_K\{v, u\}$ for $v, u \in V$. We say that a vector v or a subspace W is positive (resp. negative) if Q is positive (resp. negative) definite on $\langle v \rangle$ or W.

Lemma 33. *Let $V' \leq V$ be a 5 dimensional subspace and v_1, v_2 two negative vectors in V'. There is a negative vector $x \in V'$ such that $\langle x, v_1 \rangle$ and $\langle x, v_2 \rangle$ are both non-degenerate and isotropic.*

Proof. By multiplying v_2 by -1 if necessary, we may suppose that

$$Q(v_1 + v_2) < \min \{Q(v_1), Q(v_2)\} < 0.$$

Since $K = \mathbb{Q}[\sqrt{d}]$ is dense in $\mathbb{R}$ and $\langle v_1, v_2 \rangle^\perp \cap V'$ is at least 3-dimensional and so cannot be negative, we may find a vector $s \in V'$ orthogonal to $\langle v_1, v_2 \rangle$ such that

$$-Q(v_1 + v_2) > Q(s) > \max \{-Q(v_1), -Q(v_2)\} > 0.$$

Set $x = v_1 + v_2 + s$. It is easy to see that x is negative and that $s + v_2, s + v_1$ are positive and lie in $\langle x, v_1 \rangle$ and $\langle x, v_2 \rangle$ respectively. $\square$

Proof of Proposition 32. Let $W_1, W_2 \in \mathcal{L}$. Note that $(W_1+W_2)^\perp$, having dimension at least 3 and rank at most 1, contains a two dimensional positive subspace U.

Case 1: Suppose first that $W_1 + W_2$ has rank 1. Then $(W_1 + W_2)^\perp$ also has rank one and since its dimension is at least 3 it contains a subspace $W_3 \in \mathcal{L}$. Therefore $W_1 \sim W_3 \sim W_2$.

Case 2: Suppose that W_1 and W_2 share a common negative vector v. Then taking $W_4 = \langle v \rangle \oplus U \in \mathcal{L}$ we have that $W_1 + W_4$ and $W_2 + W_4$ have rank 1 and therefore $W_1 \sim W_4 \sim W_2$ by Case 1.

Case 3: Now suppose that W_1 and W_2 share a common 2-dimensional positive subspace Y. We can write $W_1 = Y \oplus \langle v_1 \rangle, W_2 = Y \oplus \langle v_2 \rangle$ where the negative vectors v_1, v_2 are in $Y^\perp$. By Lemma 33 (applied to a subspace of the space $Y^\perp$) we can find a negative vector $x \in Y^\perp$ such that both $\langle v_1, x \rangle$ and $\langle v_2, x \rangle$ are of signature $(1, 1)$.

Put $W_5 = Y \oplus \langle x \rangle$. We have that $W_1 + W_5 = Y \oplus \langle v_1, x \rangle$ and $W_2 + W_5 = Y \oplus \langle v_2, x \rangle$ are both of rank 1 and therefore $W_1 \sim W_5 \sim W_2$ by Case 1.

The general case: We may always choose a positive two dimensional subspace $Y \leq W_1$ and a negative 1-dimensional subspace $Z \leq W_2$. We have $Y + Z \in \mathcal{L}$ and $W_1 \sim Y + Z \sim W_2$ by Cases 3 and 2. This completes the proof of Proposition 32. $\qquad\square$

For an integral lattice M in the K-vector space V (i.e. a free $\mathcal{O}_K$-submodule of V of maximal rank) let us write $GL(M)$ for the arithmetic subgroup of $GL(V)$ which stabilizes M. The following is elementary:

Proposition 34. *Let M_1, M_2 be two integral lattices in V. The groups $GL(M_1)$ and $GL(M_2)$ are commensurable.*

In fact more is true: The class of arithmetic subgroups is preserved under any rational isomorphism of their ambient algebraic groups, (see [PR], Propostion 4.1). The next result we need is the following (well known) "arithmetic" variant of the Borel density theorem:

Theorem 35. *Let k be a number field and let G be a k-defined simple algebraic group of non-compact type, i.e. such that $G(k_v)$ is noncompact for at least one archimedian valuation v of k. Then any arithmetic subgroup Γ of G is Zariski-dense in G.*

To prove Theorem 35 one can argue as follow. It can be easily deduced from Proposition 34 that the commensurator of Γ in G_{k_v} is dense with respect to the local topology. In addition, commensurable algebraic groups must share a common identity connected component. Therefore the Zariski connected component of the Zariski closure of Γ is normal in the simple algebraic group G, and hence is either finite or equals G. However, by [PR, Corollary 2, page 210] Γ is infinite, and hence it is Zariski dense. An immediate consequence is that Γ admits elements of infinite order.

For any subspace $W \in \mathcal{L}$ choose a basis B_W containing a basis for W and a basis for $W^\perp$ and let M_W be the integral lattice it generates. Let G_W be the subgroup of G which stabilizes W and fixes $W^\perp$ pointwise. We have that G_W is a K-form of $SO(2,1)$ of non-compact type. The group $\Delta = G_W \cap GL(M_W)$ is an arithmetic subgroup of G_W hence admits a non-torsion element, say γ. By Proposition 34, for some $n \in \mathbb{N}$, the nontrivial element $h_W = \gamma^n$ lies in $\Gamma \cap G_W$. If $W_1, W_2 \in \mathcal{L}$ and $W_1 \perp W_2$ then by definition the element h_{W_1} commutes with any element of G_{W_2} in particular with h_{W_2}.

We also invoke the celebrated:

Theorem 36 (Margulis normal subgroup theorem [Ma], Ch. VIII). *A lattice in a higher rank simple Lie group has the normal subgroup property, i.e. all infinite normal subgroups are of finite index.*

We are now in a position to complete the:

Proof of Theorem 6. Choose any subspace $W \in \mathcal{L}$ and let $a = h_W \in \Gamma$ be the element defined above. Let Γ_0 be the normal closure of a in Γ. By Theorem 36,

Γ_0 is of finite index in Γ. In particular it is finitely generated, and hence we can find n distinct conjugates $a_i = a^{\gamma_i}$ of a $(i = 1, \ldots, n$ with $\gamma_i \in \Gamma)$ which together generate Γ_0. By Proposition 32 the sequence $\gamma_1 W, \ldots, \gamma_n W$ can be refined to a sequence $W_1, \ldots, W_m \in \mathcal{L}$, $m \geq n$, so that $W_j \perp W_{j+1}$ for $j = 1, 2, \ldots, m-1$. If $W_j = \gamma_i W$ for some $i = 1, \ldots, n$, set $b_j = a^{\gamma_i}$, and otherwise set $b_j = h_{W_j}$. Then the elements b_i, $i = 1, \ldots m$ form an ordered set of generators to Γ_0 with the desired property. $\qquad\square$

6.2. **Cocompact right angled lattices in** $SL(n, \mathbb{R})$. Let K be a totally real number field of degree $s > 1$ over $\mathbb{Q}$ and let L be a quadratic extension of K. Let σ be the involution automorphism of $\mathrm{Gal}(L/K)$. Suppose that L has 2 real embeddings $i_1, i_2 = i_1 \circ \sigma$ and $2s - 2$ complex embeddings $j_1, \bar{j}_1 = j_1 \circ \sigma, \cdots j_{s-1}, \bar{j}_{s-1} = j_{s-1} \circ \sigma$ into $\mathbb{C}$. Note that the restrictions of $i_1, j_1, \ldots, j_{s-1}$ to K are all the embeddings of K into $\mathbb{R}$. Let $\mathcal{O}$ be the ring of integers of K and $\mathcal{V}$ the ring of integers of L.

For example we can take $K = \mathbb{Q}(\sqrt{2})$, $\mathcal{O} = \mathbb{Z}[\sqrt{2}]$ and L to be the extension $K(\alpha)$ where $\alpha^2 = \sqrt{2}$.

For $n \geq 3$ let $V = L^n$ and let $f : V \to K$ be the unitary form defined by

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i x_i^{\sigma}$$

for any $(x_1, \ldots, x_n) \in V = L^n$.

For a subring R of L such that $R^{\sigma} = R$ let us define

$$H(R) = \{x \in SL_n(R) \mid x(x^{\sigma})^t = \mathrm{Id}\}$$

We are interested in the group $H(L) = SU(f, L)$ and its integral points $H(\mathcal{V})$. Note however that the equation $x(x^{\sigma})^t = Id$ does not define an algebraic group over L because σ is not a linear map over L. Following [PR, Section 2.3.3] we realise $H(L)$ as the K-rational points of a related K-defined algebraic group $\mathcal{G}$ using the restriction of scalars of L.

Choose a basis u_1, u_2 for L over K such that $\mathcal{O}u_1 + \mathcal{O}u_2 \subset \mathcal{V}$. The left regular representation of $\rho : L \to End_K(L)$ embeds it as a K-subalgebra $\rho(L)$ of $M_2(K)$. This is just the restriction of scalars $R_{L/K}(L)$. Moreover $\rho(L) \cap M_2(\mathcal{O})$ is isomorphic to $\mathcal{O}u_1 + \mathcal{O}u_2$.

Consider the K-algebra $M_n(\rho(L))$ as the subalgebra of $M_{2n}(K)$ consisting of the $n \times n$ matrices with entries in $\rho(L) \subset M_2(K)$. The involution σ acts as a K-linear automorhism of $M_n(\rho(L))$ by acting on the matrix entries from $\rho(L)$.

Let $\mathbb{G}$ be the K-defined algebraic group

$$SU(n, f) = \{X \in M_n(\rho(L)) \subset M_{2n}(K) \mid X(X^{\sigma})^t = Id, \det X = 1\},$$

Observe that $\mathbb{G}(K) = H(L)$. Let $\Gamma = \mathbb{G}(\mathcal{O})$ be the corresponding arithmetic subgroup of integral points, and note that $\mathbb{G}(\mathcal{O})$ embeds as a subgroup of finite index in $H(\mathcal{V})$.

For each $c = 1, \ldots, s - 1$ the embedding $j_c : L \to \mathbb{C}$ extends to an injection $J_c : \mathbb{G}(K) \to SL(n, \mathbb{C})$ under which f becomes a positive definite unitary form and therefore J_c induces a K-defined homomorphism $J_c : \mathbb{G}(K) \to SU(n)$. On the other hand the map $I : \mathbb{G}(K) \to SL(n, \mathbb{R}) \times SL(n, \mathbb{R})$ given by $g \mapsto (i_1(g), i_2(g))$ has image $\{(X, (X^t)^{-1}) \mid X \in SL(n, K)\}$ which is isomorphic to $SL(n, K)$.

By the Borel–Harish-Chandra theorem, the image of the embedding of Γ into $D := SU(n)^{s-1} \times SL(n, \mathbb{R})$ given by $g \mapsto (J_1(g), \ldots, J_{s-1}(g), I(g))$ is a lattice in

D and since the first $s-1$ factors are compact, it projects to a lattice in $SL(n, \mathbb{R})$. Moreover Γ is cocompact because the form f is anisotropic, i.e. $f(v) = 0$ for $v \in V$ implies $v = 0$. This can be seen from the fact that under the embedding $J_1 : \mathbb{G}(K) \to SU(n)$, f becomes a positive definite form.

Lemma 37. *There exists an element u which is a unit of infinite order in $\mathcal{V}^*$ and such that $uu^\sigma = 1$.*

Proof. The map $x \mapsto xx^\sigma$ is a homomorphism π from $\mathcal{V}^*$ to $\mathcal{O}^*$. By the Dirichlet unit theorem $\mathcal{O}^*$ has torsion free rank $s-1$ while $\mathcal{V}^*$ has rank s. Therefore $\ker \pi$ contains some element of infinite order in $\mathcal{V}^*$. $\square$

Theorem 38. *The group Γ is virtually right-angled.*

Let $B = \{e_1, \ldots, e_n\}$ be any f-orthogonal basis of V over L, i.e. $f(e_i, e_j) = 0$ for all $i \neq j$. Define the following diagonal subgroup of $H(L)$ with respect to the basis B:

$$A_B = \{diag(u^{d_1}, u^{d_2}, \ldots, u^{d_n}) \mid d_i \in \mathbb{Z}, \ d_1 + \cdots + d_n = 0\}.$$

Note that, under the identification of $H(L)$ with $\mathbb{G}(K)$ we have that $A_B < \mathbb{G}(K)$ since $uu^\sigma = 1$. The group A_B preserves the $\mathcal{V}$-integral lattice of V with respect to the basis B. In view of Proposition 34, $|A_B : A_B \cap H(\mathcal{V})|$ is finite and therefore the group $T_B := A_B \cap \Gamma$ is of finite index in A_B.

Define a relation on the set X of orthogonal bases of V over L by declaring $B_1 \sim B_2$ if B_1 and B_2 share a vector.

Suppose that this is the case, specifically $B_1 = \{e_1, e_2, \cdots e_n\}$, $B_2 = \{e_1, e_2', \cdots, e_n'\}$. Then $Le_2 + \cdots + Le_n = (Le_1)^\perp = Le_2' + \cdots + Le_n'$ and so $A_{B_1} \cap A_{B_2}$ contains the infinite cyclic subgroup

$$\{diag(u^{-nd+d}, u^d, \ldots, u^d) \mid d \in \mathbb{Z}\}$$

(the definition gives the same group with respect to either basis B_1 or basis B_2) We can deduce that $A_{B_1} \cap A_{B_2}$ is infinite and hence $T_{B_1} \cap T_{B_2}$ is infinite.

Finally we need to check

Lemma 39. *The transitive closure of $\sim$ is all of X, all orthonormal bases.*

Proof. Consider two arbitrary elements $B_1, B_2 \in X$. Let $e_1 \in B_1$ and $e_1' \in B_2$.

If e_1 and e_1' are scalar multiples of each other we just rescale one to the other without changing the rest. Otherwise the subspace $W = Le_1 + Le_2$ is two dimensional, in which case take an orthogonal basis C for $W^\perp$. Choose a vector $h \in W$ orthogonal to e_1 and a vector $d \in W$ orthogonal to e_1'. Now

$$B_1 \sim \{h, e_1\} \cup C \sim \{d, e_1'\} \cup C \sim B_2.$$

$\square$

We can now complete the proof of Theorem 38. Let $B = B_1$ be any orthonormal basis for V and choose $g_1 \in T_{B_1} \backslash \{1\}$. The normal closure of g_1 in Γ has finite index in Γ by Theorem 36. Therefore there exist conjugates $g_2, \ldots, g_k \in g^\Gamma$ such that $\langle g_1 \ldots, g_k \rangle$ has finite index in Γ. Now each g_i lies in some T_{B_i} for an appropriate basis $B_i \in X$. By Lemma 39 we can find some sequence of bases $C_1, \ldots, C_m$ in X which contains all B_j and such that $C_i \sim C_{i+1}$ for each $i = 1, \ldots, m-1$. This means that $T_{C_1}, \ldots, T_{C_n}$ is a sequence of abelian groups with infinite consecutive

intersections whose union contains all of g_j and therefore generates a subgroup of finite index in Γ. □

7. Non-archimedean local fields

Recently many of the results of [7Sb] have been extended by A. Levit and the second named author to IRS in simple Lie groups over non-archimedean local fields [GL] (this work is under preparation and is expected to be on the Arxiv soon). Below we shall use them in combination with Theorem 24 to prove Theorem 11.

We shall make use of the following definition from [GL]:

Definition 40. *Let G be a locally compact group. A family of IRS, $\mathcal{F} \subset IRS(G)$ is weakly uniformly discrete if for every $\epsilon > 0$ there is an identity neighbourhood $U \subset G$ such that for all $\mu \in \mathcal{F}$ we have*

$$\mu\{\Gamma : \Gamma \cap U \neq id_G)\} < \epsilon.$$

One advantage of this notion of uniform discreteness is that it also applies to non-uniform lattices. In particular, the following proposition is straightforward:

Proposition 41. *Let $\Gamma \leq G$ be a lattice and let $\mathcal{F}$ be the family consisting of all μ_Δ where Δ runs over the finite index subgroups of Γ. Then $\mathcal{F}$ is weakly uniformly discrete.* □

The following theorem is proved in [GL]:

Theorem 42. *Let k be a non-Archimedean local field (of positive characteristic) and let $G = \mathbb{G}(k)$ be the group of k rational points of a simple algebraic group $\mathbb{G}$ of k-rank at least 2. Let μ_n be a weakly uniformly discrete sequence of distinct non-atomic IRS in G. Then $\mu_n \to \mu_{id_G}$.*

Note that Theorem 42 is an analogue of Theorem 13 for higher rank simple groups over general local fields. The 0 characteristic (non-Archimedean) case is more general and easier (see [GL]). The proof of Theorem 42 makes use of ideas originated in [7Sb] and relies on a version of the Nevo–Stuck–Zimmer theorem which was recently obtained by Arie Levit (see [Levit]).

The following is an immidiate consequence of Lemma 14, Theorem 42 and Proposition 41:

Corollary 43. *Let G and Γ be as in Theorem 42, and let $\Gamma_n \leq \Gamma$ be a sequence of subgroups of finite index, such that $|\Gamma : \Gamma_n| \to \infty$. Then (Γ_n) is Farber.*

Theorem 11 now follows from Theorem 42 and Theorem 24 by observing that Γ is generated by its elementary subgroups $E_{i,j}$ for $1 \leq i \neq j \leq n$ which can be enumerated as $\{A_n\}_{n=1}^{n^2-n}$ such that A_k commutes with A_{k+1} for every $1 \leq k \leq n^2 - 2$.

Remark 44. *It is shown in [Ge2] that if G is a semisimple Lie group over a local field of characteristic 0, then the set of all IRS in G which have no atom at G is weakly uniformly discrete. The analog statement for local field of positive characteristic is still unknown.*

8. Open problems

In this section we present some open problems and suggest further directions for research.

8.1. **The growth of rank for lattices in a fixed Lie group.** We already pointed out in the introduction that Conjecture 3 does not hold for rank 1 simple Lie groups in general. In particular, lattices surjecting to a non-Abelian free group provide easy counterexamples. However, all these give sequences of lattices that do not approximate the ambient Lie group in the following sense.

Definition 45. *Let G be a Lie group. A sequence of lattices Γ_n is Farber, if the invariant random subgroups $\mu_{\Gamma_n} \to \mu_1$.*

We suggest that the asymptotic growth of the rank of a lattice only depends on its covolume and ambient Lie group.

Problem 46. *Let G be a connected Lie group and let Γ_n be a Farber sequence of lattices in G. Then*
$$\frac{d(\Gamma_n) - 1}{\mathrm{Vol}(\Gamma_n)}$$
converges.

We further suspect that the above limit (which only depends on G) should be 0 except when G satisfies the following:

- The amenable radical $A \lhd G$ is compact;
- $G/A \cong \mathrm{PSL}(2, \mathbb{R})$.

Note that by [Ge1] there is a constant $C = C(G)$ such that
$$d(\Gamma) \leq C \cdot \mathrm{Vol}(\Gamma)$$
for every lattice $\Gamma \leq G$. In particular, this implies that the limsup of the sequence in Problem 46 exists and is at most C.

The corresponding question about discrete groups is as follows.

Problem 47. *Let Γ be a finitely generated group and let Γ_n be a Farber sequence of subgroups of finite index in Γ. Then*
$$\frac{d(\Gamma_n) - 1}{|\Gamma : \Gamma_n|}$$
converges.

In [AbN] the weaker question is investigated: is the rank gradient of a Farber chain independent of the chain? That question is related to the fixed price problem of Gaboriau, that is, whether every free action of a countable group must have the same cost. In particular, when the ambient group has fixed price, the rank gradient is independent of the chain (assuming it is Farber). To the best of our knowledge a cost correspondence of this nature has not been worked out in the realm of Lie groups.

8.2. **The distortion function of a p.m.p. action.** Let Γ be generated by the finite symmetric set S and let Φ be a p.m.p. action of Γ on the Borel space (X, μ). One can define the bi-Lipshitz distortion function of Φ as follows. Let $\mathcal{G}$ be the measured groupoid associated to the action. Let the standard generating set
$$R_S = \left\{ (x, s) \in \mathcal{G}^1 \mid x \in X, s \in S \right\}$$
and let
$$N = \left\{ (x, e) \in \mathcal{G}^1 \mid x \in X \right\}.$$

Recall that the cost of $\mathcal{G}$ is defined as

$$c(\mathcal{G}) = \inf_{\substack{T \subseteq \mathcal{G}^1 \text{ Borel} \\ T \text{ generates } \mathcal{G}}} \widetilde{\mu}(T)$$

Recall that in Definition 22 we defined the distance of two generating sets $A, B \subseteq \mathcal{G}^1$ as

$$d_L(A, B) = \inf \left\{ k \in \mathbb{N} \mid A \subseteq (B \cup B^{-1} \cup N)^k \text{ and } B \subseteq (A \cup A^{-1} \cup N)^k \right\}.$$

A priori, the distance of two generating sets can be infinite. However, using a compactness argument, it is proved in [AbN] that for any $\varepsilon > 0$ there exists a generating set $T \subseteq \mathcal{G}^1$ with $\widetilde{\mu}(T) \leq c(\mathcal{G}) + \varepsilon$ such that $d_L(T, R_S)$ is finite.

Definition 48. *Let the distortion function $\delta : \mathbb{R}^+ \to \mathbb{R}$ of Φ be defined as*

$$\delta(x) = \inf_{\substack{T \text{ generates } \mathcal{G} \\ \widetilde{\mu}(T) \leq c(\mathcal{G}) + 1/x}} d_L(T, R_S).$$

The distortion function δ does depend on S, however, it is easy to see that it is well-defined up to a scale factor over all finite generating sets of Γ.

One can show the following.

Theorem 49. *Let Γ be a right angled group and let Φ be a free p.m.p. action of Γ. Then Φ has polynomial distortion.*

The proof is implicitly contained in Gaboriau [Gab]. It follows the same argument as in the proof of Theorem 7 except that one may not be able to choose a maximal R-separated vertex set for the C_i-cycle metric in a measurable way. However, it is possible to do this by paying an arbitrary small extra price using Rokhlin's lemma. Indeed, the powers of each generator g_m provide a free action of $\mathbb{Z}$, and thus one can choose a measurable subset $X_m \subseteq X$ of measure $1/R + \varepsilon$ such that

$$\bigcup_{i=0}^{R-1} X_m g_m^i = X.$$

The above distortion function seems to be interesting already for amenable groups. By the Ornstein-Weiss theorem [OrW], all free actions of such groups are orbit equivalent to $\mathbb{Z}$, and thus have cost 1, but the orbit equivalence may be a quite complicated coding and the theorem does not tell anything about its distortion. Torsion free nilpotent groups are right angled, so they fall under Theorem 49, but already for solvable groups, we do not see any general upper bound for the distortion function. In particular, there may be examples of exponential distortion already in this class. For arbitrary groups, we do not know the answer to the following.

Problem 50. *Is the distortion function of a free p.m.p. action of a finitely generated group always bounded by an exponential function?*

Note that by adapting the Abert-Weiss theorem on weak containment [AbW], one can show that the Bernoulli actions of Γ dominate all free actions of Γ in terms of the distortion function.

8.3. **The growth of the first homology torsion.** The first problem listed here (communicated to us by Marc Lackenby) clearly shows the general lack of understanding on how to estimate the torsion homology from below.

Problem 51. *Is there a finitely presented group* Γ *and a normal chain* Γ_n *in* Γ *with trivial intersection with*

$$\mathrm{Trs}(\Gamma, (\Gamma_n)) > 0?$$

The expected results are much bolder. In particular, it is conjectured by Bergeron and Venkatesh [BV] that for any closed arithmetic hyperbolic 3-manifold group Γ and congruence chain (Γ_n) in Γ, one has

$$\mathrm{Trs}(\Gamma, (\Gamma_n)) = \frac{1}{6\pi}\mathrm{Vol}(\Gamma).$$

This, in particular, would be a rather direct way to express the covolume of Γ using torsion homology. The later paper of Bergeron, Sengun and Venkatesh [BSV] advances towards a 3-manifold group example for Problem 51 but it is still conditional on a conjectured density property of a certain spectral measure.

A very recent result of Bader, Gelander and Sauer [BGS2] states that for every $d \neq 3$ there is a constant $C = C(d)$ such that for every negatively curved analytic d-manifold M with sectional curvature bounded below by -1, we have

$$\log \mathrm{TrsH}_k(M, \mathbb{Z}) \leq C \cdot \mathrm{Vol}(M)$$

for all k. In particular if Γ is a lattice in a rank one Lie group and $\Gamma_n \leq \Gamma$ is as in Problem 51 then $\limsup \log \mathrm{TrsH}_1(\Gamma_n, \mathbb{Z})$ is finite. Conjecturally, the analogue of the result of [BGS2] should hold for every simple Lie group not locally isomorphic to $\mathrm{PSL}(2, \mathbb{C})$.

One could attempt to extend the Bergeron–Venkatesh conjecture to a general problem in the spirit of Problem 46, using normalized log torsion instead of the normalized rank. However, as a recent example of Brock and Dunfield [BrD] shows, already for hyperbolic 3-manifolds, this direct translation would fail. They prove the existence of a Farber sequence of hyperbolic integer homology 3-spheres. This in particular means that the first homology torsion stays trivial. Note that as Brock and Dunfield [BrD, 1.11] point out, this apparent deviation from the Bergeron-Venkatesh conjecture may be because the right invariant to study in this situation also has to involve the so-called regulator. In the other direction, it is shown in the recent preprint by Bader, Gelander and Sauer, [BGS2] that for any function $f : \mathbb{N} \to \mathbb{R}^+$ there is a Farber sequence of (non-arithmetic) compact rational homology spheres hyperbolic 3 manifolds whose normalised torsion grows faster than f. By the main result of [BGS2] this can happen only in dimension 3. The conjecture [BrD, Conjecture 1.13] stated for general Lie groups instead of just $\mathrm{SL}(2, \mathbb{C})$, suggests itself as the right analogue of Problem 46 for torsion.

8.4. **Higher degree torsion.** It is natural to ask whether the right angled condition implies the vanishing of the torsion growth on higher homologies. As we expect the second homology torsion growth of $SL(3, \mathbb{Z})$ to be positive, the direct answer is no. However, note that in a work in progress, Abert, Bergeron and Gaboriau introduce a higher homotopy version of being right angled and prove the vanishing of the torsion growth on some higher homologies for these groups. Note that as of now, their approach do not apply to cocompact lattices. Also note that

the congruence subgroup property does not seem to be relevant for higher torsion. So, as of now, we do not know a cocompact lattice with vanishing second torsion homology growth over Farber sequences.

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