F-WORM colorings: Results for 2-connected graphs *

Csilla Bujtás and Zsolt Tuza

Department of Computer Science and Systems Technology
University of Pannonia, Veszprém, Hungary
and
Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences, Budapest, Hungary

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Abstract

Given two graphs F and G, an F-WORM coloring of G is an assignment of colors to its vertices in such a way that no F-subgraph of G is monochromatic or rainbow. If G has at least one such coloring, then it is called F-WORM colorable and $W^-(G,F)$ denotes the minimum possible number of colors. Here, we consider F-WORM colorings with a fixed 2-connected graph F and prove the following three main results: (1) For every natural number k, there exists a graph G which is F-WORM colorable and $W^-(G,F) = k$; (2) It is NP-complete to decide whether a graph is F-WORM colorable; (3) For each $k \geq |V(F)| - 1$, it is NP-complete to decide whether a graph G satisfies $W^-(G,F) \leq k$. This remains valid on the class of F-WORM colorable graphs of bounded maximum degree. For complete graphs $F = K_n$ with $n \geq 3$ we also prove: (4) For each $n \geq 3$ there exists a graph G and integers r and s such that $s \geq r + 2$, G has K_n-WORM colorings with exactly r and also with s colors, but it admits no K_n-WORM colorings with exactly $r + 1, \ldots, s - 1$ colors. Moreover, the difference $s - r$ can be arbitrarily large.

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1 Introduction

Given a graph G and a color assignment to its vertices, a subgraph is *monochromatic* if its vertices have the same color; and it is *rainbow* if the vertices have pairwise different colors. For graphs F and G, an *F-WORM coloring* of G is an assignment of colors to the vertices of G such that no subgraph isomorphic to F is monochromatic or rainbow. This concept was introduced recently by Goddard, Wash, and Xu [8].

If G has at least one F-WORM coloring, we say that it is *F-WORM colorable*. In this case, $W^-(G, F)$ denotes the minimum number of colors and $W^+(G, F)$ denotes the maximum number of colors used in an F-WORM coloring of G; they are called the F-WORM *lower* and *upper chromatic number*, respectively. The F-WORM *feasible set* $\Phi_W(G, F)$ of G is the set of those integers s for which G admits an F-WORM coloring with exactly s colors. Moreover, we say that G has a *gap* at k in its F-WORM chromatic spectrum, if $W^-(G, F) < k < W^+(G, F)$ but G has no F-WORM coloring with precisely k colors. The size of a gap is the number of consecutive integers missing from $\Phi_W(G, F)$. If $\Phi_W(G, F)$ has no gap—that is, if it contains all integers from the interval $[W^-(G, F), W^+(G, F)]$—we say that the F-WORM feasible set (or the F-WORM chromatic spectrum) of G is *gap-free*.

The invariants $W^-(G, F)$ and $W^+(G, F)$ are not defined if G is not F-WORM colorable. Hence, wherever W^- or W^+ appears in this paper, we assume without further mention that the graph under consideration is F-WORM colorable.

In the earlier works [7, 8, 4], F-WORM colorings were considered for particular graphs F — cycles, complete graphs, and complete bipartite graphs; but mainly the cases of $F = P_3$ and $F = K_3$ were studied. In this paper we make the first attempt towards a general theory; we study F-WORM colorings for all 2-connected graphs F. Our results presented here concern colorability, lower chromatic number, and gaps in the chromatic spectrum.

1.1 Related coloring concepts

A general structure class within which F-worm colorings can naturally be represented is called *mixed hypergraphs*. In our context its subclass called *bi-hypergraphs* is most relevant. It means a pair $\mathcal{H} = (X, \mathcal{E})$, where $\mathcal{E}$ is a set system (the 'edge set') over the underlying set X (the 'vertex set'), whose feasible colorings are those mappings $\varphi : X \to \mathbb{N}$ in which the set $\varphi(e)$ of colors in every $e \in \mathcal{E}$ satisfies $1 < |\varphi(e)| < |e|$; in other words, the hyperedges are neither monochromatic nor rainbow. In case of F-WORM colorings of a graph $G = (V, E)$ we have $X = V$, and a subset $e \subset V$ is a member of $\mathcal{E}$ if and only if the subgraph induced by e in G contains a subgraph isomorphic to F. For more information on mixed (and bi-) hypergraphs we recommend the monograph [12], the book chapter [5], and the regularly updated list of references [13].

The exclusion of monochromatic or rainbow subgraphs has extensively been studied also separately. Monochromatic subgraphs are the major issue of Ramsey theory, moreover minimal colorings fit naturally in the context of generalized chromatic number with respect to hereditary graph properties [1], since the property of not containing any subgraph isomorphic to F is hereditary.

Also, forbidden polychromatic subgraphs arise in various contexts, most notably in a branch of Ramsey theory. Namely, the maximum number of colors in an edge coloring of G without a rainbow copy of F is termed anti-Ramsey number, and the number one larger — which is the minimum number of colors guaranteeing a rainbow copy of F in G in every coloring with that many colors — is the rainbow number of G with respect to F. We recommend [6] for a survey of results and numerous references. In particular, vertex colorings of graphs without rainbow star $K_{1,s}$ subgraphs were studied in [3, 2].

1.2 Results

Goddard, Wash, and Xu proved in [8] that if G is P_3-WORM colorable, then $W^-(G, P_3) \leq 2$. Motivated by this, in [7] they conjectured that $W^-(G, K_3) \leq 2$ holds for every K_3-WORM colorable graph G. Moreover, they asked whether there is a constant $c(F)$ for every graph F such that $W^-(G, F) \leq c(F)$ for every F-WORM colorable G. It is proved in [4] that the conjecture is false for $F = K_3$, and a finite $c(K_3)$ does not exist. Now, we extend this result from K_3 to every 2-connected graph.

Theorem 1 *For every 2-connected graph F and positive integer k, there exists a graph G with $W^-(G, F) = k$.*

What is more, the structure of those graphs is rich enough to imply that they are hard to recognize. We proved in [4] that for every $k \geq 2$ it is NP-complete to decide whether the K_3-WORM lower chromatic number is at most k; moreover it remains hard on the graphs whose maximum degree is at most a suitably chosen constant d_k, whenever $k \geq 3$. It is left open whether the same is true for $k = 2$. The following general result is stronger also in the sense that for 2-connected graphs F of order $n \geq 4$ the bounded-degree version is available starting already from $k = n - 1$ instead of $k = n$.

Theorem 2 *For every 2-connected graph F of order $n \geq 4$ and for every integer $k \geq n - 1$, it is NP-complete to decide whether $W^-(G, F) \leq k$. This is true already on the class of F-WORM colorable graphs with bounded maximum degree $\Delta(G) < 2n^2$.*

The decision problem of F-WORM colorability is proved to be NP-complete for $F = P_3$ and $F = K_3$ in [8] and [7], respectively. We prove the same complexity for every 2-connected F.

Theorem 3 *For every 2-connected graph F, the decision problem F-WORM colorability is NP-complete.*

Finally, we deal with the case where F is a complete graph. We have proved in [4] that there exist graphs with large gaps in their K_3-WORM chromatic spectrum. Here we show that this remains valid for the K_n-WORM spectrum with each $n \geq 4$. For the sake of completeness we also include the previously known case of $n = 3$ in the formulation.

Theorem 4 *For every $n \geq 3$ and $\ell \geq 1$ there exist K_n-WORM colorable graphs whose K_n-WORM chromatic spectrum contains a gap of size ℓ.*

In Section 2 we present some preliminary results and define a basic construction. Using those lemmas, we prove Theorems 1, 2, and 3 in Section 3. In Section 4, we consider the case $F \cong K_n$ and prove Theorem 4.

1.3 Standard notation

As usual, for any graph G we use the notation $\omega(G)$ for clique number, $\chi(G)$ for chromatic number, $\delta(G)$ for minimum degree, and $\Delta(G)$ for maximum degree.

2 Preliminaries

Here we prove a proposition on the F-WORM colorability and lower chromatic number of complete graphs; for some extremal cases we also consider the possible sizes of color classes. Then, we give a basic construction which will be referred to in proofs of Section 3.

Proposition 5 *For every graph F of order n, with $n \geq 2$, the following hold:*

(i) *For every integer $s > (n-1)^2$, the complete graph K_s is not F-WORM colorable.*

(ii) *For every integer s satisfying $1 \leq s \leq (n-1)^2$, K_s is F-WORM colorable and $W^-(K_s, F) = \lceil \frac{s}{n-1} \rceil$.*

(iii) *In every F-WORM coloring of the complete graph $K_{(n-1)^2}$, there are exactly $n-1$ color classes each of size $n-1$.*

(iv) *In every F-WORM coloring of the complete graph $K_{(n-1)^2-1}$, there are exactly $n-1$ color classes such that one of them contains $n-2$ vertices while the other $n-2$ color classes are of size $n-1$ each.*

Proof. First, observe that if $s < n$, K_s contains no subgraphs isomorphic to F and therefore, $W^-(K_s, F) = 1 = \lceil \frac{s}{n-1} \rceil$. If $s \geq n$, a subgraph isomorphic to F occurs on any n vertices of K_s. Hence, in an F-WORM coloring of K_s, no n vertices have the same color and no n vertices are polychromatic; on the other hand, this is also a sufficient condition for F-WORM colorability. By the pigeonhole principle, if $s > (n-1)^2$, the complete graph K_s does not have such a color partition, while $K_{(n-1)^2}$ and $K_{(n-1)^2-1}$ can be F-WORM colored only with color classes of sizes as stated in (iii) and (iv), respectively. It also follows that for each $s \leq (n-1)^2$, a vertex coloring of K_s with $\lceil s/(n-1) \rceil$ color classes of size at most $n-1$ each determines an F-WORM coloring with the smallest possible number of colors. $\square$

Construction of the gadget $\mathbf{G_1(F)}$. For a given graph F whose order is n and has minimum degree $\delta \geq 2$, let $G_1(F)$ be the following graph. The vertex set is $V(G_1(F)) = S \cup S' \cup \{x, y\}$ where the three sets are vertex-disjoint and $|S'| = n - \delta - 1$, $|S' \cup S| = (n-1)^2 - 1$. Moreover, $S' \cup S$ induces

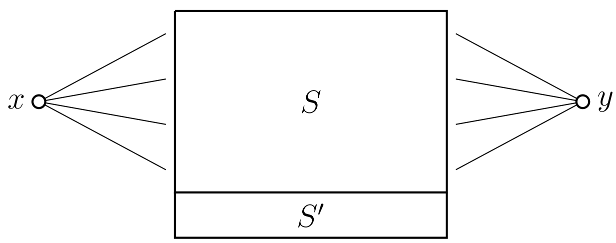

Figure 1: Gadget $G_1(F)$

a complete graph and the vertices x and y are adjacent to all vertices of S, but not to each other, neither to any vertex in S'. The vertices x and y will be called *outer vertices*, while the elements of $S \cup S'$ are called *inner vertices*. For illustration see Fig. 1.

Lemma 6 *For every graph F of order n and with minimum degree $\delta \geq 2$, the graph $G_1(F)$ is F-WORM colorable. Moreover, in any F-WORM coloring of $G_1(F)$, the outer vertices x and y get the same color which is repeated on exactly $(n-2)$ inner vertices.*

Proof. Assume that φ is an F-WORM coloring of $G_1(F)$. By Proposition 5(iv), $S' \cup S$ is partitioned into $n-1$ color classes and one of them is of size $n-2$, while each further class contains exactly $n-1$ vertices. The color of the $(n-2)$-element color class will be denoted by c^*.

First assume that $F \cong K_n$. Then, $S' = \emptyset$ and both $S \cup \{x\}$ and $S \cup \{y\}$ induce a complete subgraph on $(n-1)^2$ vertices. By Proposition 5(iii), $\varphi(x) = \varphi(y) = c^*$ follows.

If $F \not\cong K_n$, then $\delta \leq n-2$ and we can take the following observations on φ.

- Since S contains at least $n-2-|S'| = \delta - 1 \geq 1$ vertices from each color class, we can choose an $(n-1)$-element polychromatic subset S'' of S. Then, on the vertex set $S'' \cup \{x\}$, which induces a complete graph, we consider a subgraph isomorphic to F. This subgraph cannot be polychromatic, hence the color $\varphi(x)$ (and similarly, $\varphi(y)$) must be assigned to at least one vertex of S.

- Now assume that $\varphi(x) \neq c^*$. Then, we have $n-1$ vertices in $S' \cup S$ colored with $\varphi(x)$, and at least $(n-1)-|S'| = \delta$ of them are adjacent to

x. Hence, we can identify a copy of F monochromatic in c^*, in which x is a vertex of degree δ. This cannot be the case in an F-WORM coloring. Thus, $\varphi(x) = c^*$ and similarly $\varphi(y) = c^*$ that proves the second part of the lemma.

- Consider the following coloring ϕ of $G_1(F)$. The color c^* is assigned to x, y, to all vertices in S', and to exactly $\delta - 1$ vertices from S. The remaining $(n-2)(n-1)$ vertices in S are partitioned equally among $n-2$ further colors. As we used only $n-1$ colors, no subgraph isomorphic to F can be polychromatic. Further, each color different from c^* is assigned to only $n-1$ vertices, so no copy of F can be monochromatic in those colors. The only color occurring on n vertices is c^*. But x (and also y) shares this color with only $\delta - 1$ of its neighbors. Therefore, we cannot have a subgraph isomorphic to F and monochromatic in c^*. These facts prove that ϕ is an F-WORM coloring. □

Construction of $\mathbf{C^1(G, F, N_0)}$ Given an integer N_0, a 2-connected graph F of order n, and a graph G, construct the following graph $C^1(G, F, N_0)$. If $V(G) = \{v_1, v_2, \ldots, v_\ell\}$, take $N_0 + 1$ copies for each vertex v_i; these vertices are denoted by $v_i^0, v_i^1, \ldots, v_i^{N_0}$. For each $1 \le i \le \ell$ and $0 \le j \le N_0 - 1$ take a copy of the gadget $G_1(F)$ such that its two outer vertices are identified with v_i^j and v_i^{j+1}, respectively. The edges contained in these copies of $G_1(F)$ are referred to as *gadget-edges*.

When we define the further edges of the construction, only the copy vertices of the form $v_i^{k\lceil n/2\rceil}$ ($k \in \mathbb{N}_0$) will be used, each of them at most once. The sequence

$$v_i^0, v_i^{\lceil \frac{n}{2} \rceil}, v_i^{2\lceil \frac{n}{2} \rceil}, \ldots, v_i^{\left\lfloor \frac{N_0}{\lceil \frac{n}{2} \rceil} \right\rfloor \lceil \frac{n}{2} \rceil}$$

is called V_i*-sequence*.

To finalize the construction of $C^1(G, F, N_0)$, assume $N_0 \ge (n+1)^2 \Delta(G)/4$ and consider the edges of G one by one in an arbitrarily fixed order. When an edge $v_i v_j$ (with $i < j$) is treated, take the next $\lceil n/2 \rceil$ vertices from the V_i-sequence and the next $\lfloor n/2 \rfloor$ ones from the V_j-sequence, and connect them with edges to obtain an induced subgraph isomorphic to F. These edges are called *supplementary edges*. For an illustration with $F = C_4$ see Fig. 2.

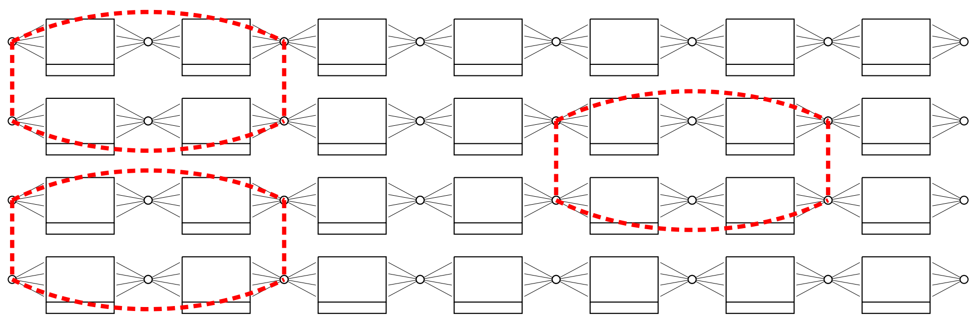

Figure 2: The graph $C^1(P_4, C_4, 7)$. Supplementary edges are drawn with dashed lines.

Lemma 7 *Assume that F is a 2-connected graph of order n, G is a graph, and $N_0 \geq \frac{(n+1)^2\,\Delta(G)}{4}$. Then the graph $C^1(G, F, N_0)$ satisfies the following properties.*

(i) In each F-WORM coloring φ of $C^1(G, F, N_0)$, the vertices $v_i^0, v_i^1, \ldots, v_i^{N_0}$ are monochromatic for each i with $1 \leq i \leq |V(G)|$. Moreover, if v_iv_j is an edge in G then $\varphi(v_i^0) \neq \varphi(v_j^0)$.

(ii) For every integer k with $n-1 \leq k \leq |V(G)|$ the graph $C^1(G, F, N_0)$ is F-WORM colorable with exactly k colors if and only if G is k-colorable.

(iii) For every integer $k \leq |V(G)|$, there exists an F-WORM coloring φ of $C^1(G, F, N_0)$ which uses exactly k different colors on the set of outer vertices of gadgets, if and only if G is k-colorable.

Proof. To simplify notation, let us write $G^* = C^1(G, F, N_0)$. First, consider an F-WORM coloring φ of G^*. By Lemma 6, in each gadget $G_1(F)$ the two outer vertices have the same color. Thus, for each i, the vertices $v_i^0, v_i^1, \ldots, v_i^{N_0}$, and particularly the vertices contained in the V_i-sequence, share their color. We denote this color by $\varphi(V_i)$. By construction, if v_iv_j is an edge in G, we have an F-subgraph in G^* such that every vertex of the subgraph belongs to the V_i- or V_j-sequence. Since F is not monochromatic in φ, we infer that $\varphi(V_i) \neq \varphi(V_j)$. These prove (i).

Now, assume again that φ is an F-WORM coloring of G^*. Then, the coloring ϕ which assigns the color $\varphi(V_i)$ to every vertex $v_i \in V(G)$ is a proper vertex coloring of G and it uses precisely $|\{\varphi(V_i) : 1 \leq i \leq |V(G)|\}|$

colors. This proves the "only if" direction of (iii). Further, we infer that $W^-(G^*, F) \geq \chi(G)$, and if $W^-(G^*, F) \leq k \leq |V(G)|$ then G has a proper coloring with exactly k colors. Since $n-1 \leq W^-(G^*, F)$, this proves the "only if" direction of the statement (ii).

To prove the other direction, we consider an integer k in the range $\chi(G) \leq k \leq |V(G)|$. Let ϕ be a proper coloring of G which uses the colors $1, \dots, k$. We define a vertex coloring φ of G^* as follows. For every i and s, with $1 \leq i \leq |V(G)|$ and $0 \leq s \leq N_0$, let $\varphi(v_i^s) := \phi(v_i)$. Moreover, for each copy of gadget $G_1(F)$ whose outer vertices are v_i^s and v_i^{s+1}, let its inner vertices be assigned with $n-1$ different colors from $1, \dots, \max\{k, n-1\}$ without creating rainbow or monochromatic copies of F inside the gadget. We can specify this assignment corresponding to Lemma 6. That is, $\varphi(v_i^s)$ is repeated on all inner vertices nonadjacent to v_i^s and on further $\delta - 1$ inner vertices; each of the further $n-2$ colors is assigned to exactly $n-1$ inner vertices.

It is clear from the definition that any F-subgraph which is contained entirely in one gadget or contains only supplementary edges is neither monochromatic nor rainbow under ϕ. Next, we prove that there are no further F-subgraphs in G^*. First, assume that a subgraph isomorphic to F contains only gadget edges but from at least two different gadgets. Then, this subgraph meets two consecutive gadgets and contains their common outer vertex v_i^s. As $s \neq 0$ and $s \neq N_0$, this outer vertex is a cut vertex in the subgraph determined by the gadget edges. Thus, v_i^s would also be a cut vertex in the F-subgraph, what contradicts the 2-connectivity of F. Therefore, such an F-subgraph does not occur in G^*. The only case that remains to be excluded is an F-subgraph which contains both gadget edges and supplementary edges. In such a subgraph F^*, we would have a vertex which is incident to gadget edges and supplementary edges as well. This vertex, say v_i^r, belongs to the V_i-sequence. If only the gadget edges are considered, any further vertex of V_i is at distance at least n apart from v_i^r, while F^* has only n vertices and at least one of them belongs to a different V_j-sequence. Hence, by deleting v_i^r from F^* we obtain a disconnected graph, one component of which is contained entirely in the sequence of gadgets between v_i^r and $v_i^{r+\lceil n/2 \rceil}$, or between v_i^r and $v_i^{r-\lceil n/2 \rceil}$. Again, this contradicts the 2-connectivity of F. Therefore, we have only non-monochromatic and non-rainbow F-subgraphs, and φ is an F-WORM coloring of G^* with exactly k colors. This completes the proof of the lemma. $\square$

3 Lower chromatic number and WORM-colorability

Having Lemma 7 in hand, we are now in a position to prove Theorems 1, 2, and 3. Before the proofs, we will recall the statements of the theorems.

Theorem 1. *For every 2-connected graph F and positive integer k, there exists a graph G with $W^-(G,F)=k$.*

Proof. Let F be a 2-connected graph of order $n \geq 3$. By Proposition 5(ii), if $1 \leq k \leq n-1$ and $(k-1)(n-1) < s \leq k(n-1)$, then $W^-(K_s)=k$. Hence, we may assume $k \geq n$. We consider the graph $G^* = C^1(G,F,N_0)$ where G is a graph of chromatic number k, and $N_0 = \left\lceil \frac{(n+1)^2\,\Delta(G)}{4} \right\rceil$. By Lemma 7, for every integer $k' \in [n-1, |V(G)|]$, G^* has an F-WORM coloring using exactly k' colors if and only if $k' \geq \chi(G)$. Since $\chi(G)=k$ by assumption, this implies $W^-(G^*,F)=\chi(G)=k$, as desired. □

Theorem 2. *For every 2-connected graph F of order $n \geq 4$ and for every integer $k \geq n-1$, it is NP-complete to decide whether $W^-(G,F) \leq k$. This is true already on the class of F-WORM colorable graphs with bounded maximum degree $\Delta(G) < 2n^2$.*

Proof. Let a 2-connected graph F of order $n \geq 4$ and an integer $k \geq n-1$ be given. Clearly, the decision problem '*Is $W^-(G,F) \leq k$?*' belongs to NP. To prove that it is NP-hard (also under the assumption of bounded maximum degree), we apply reduction from the classical problem of graph k-colorability, which is NP-complete for every $k \geq 3$.

For a generic instance G of the graph k-colorability problem, construct $G^* = C^1(G,F,N_0)$ with $N_0 = \left\lceil \frac{(n+1)^2\,\Delta(G)}{4} \right\rceil$. By Lemma 7, $W^-(G^*,F) \leq k$ if and only if $\chi(G) \leq k$. Concerning the order and maximum degree of G^*, we observe that

$$|V(G^*)| = \left((n-1)^2 N_0 + 1\right)|V(G)|$$

and

$$\Delta(G^*) \leq \max\{(n-1)^2, 2\left((n-1)^2 - 1 - (n-\delta-1)\right) + \Delta(F)\} < 2n^2.$$

Therefore, the order of G^* is polynomially bounded in terms of $|V(G)|$ and its maximum degree satisfies the condition given in the theorem. This completes the proof. □

Theorem 3. *For every 2-connected graph F, the decision problem F-WORM colorability is NP-complete.*

Proof. Let us consider a 2-connected graph F and denote its order by n. The problem is clearly in NP. It is proved in [7] that the decision problem of K_3-WORM colorability is NP-complete. Hence, we may assume that $n \geq 4$. The algorithmic hardness will be reduced from the decision problem of $\chi(G) \leq n-1$ that is NP-complete for each $n \geq 4$.

For a general instance G of the decision problem '$\chi(G) \leq n-1$' we again begin with constructing a graph $C^1(G, F, N_0)$, but now with a much larger N_0, namely

$$N_0 = \left\lceil \frac{(n+1)^2\,\Delta(G)}{4} \right\rceil + \binom{|V(G)|-1}{n-1} \left\lceil \frac{n}{2} \right\rceil .$$

It will be extended with further supplementary edges, as follows.

We consider those n-element subsets $\{i_1, \dots, i_n\}$ of the index set $\{1, \dots, |V(G)|\}$ for which the subgraph induced by $\{v_{i_1}, \dots, v_{i_n}\}$ contains at least one edge. For each such $\{i_1, \dots, i_n\}$ we choose one vertex (the first one which has not been used so far) from each V_i-sequence with indices $i = i_1, \dots, i_n$, and take $|E(F)|$ new supplementary edges in such a way that these n vertices induce a subgraph isomorphic to F. These edges will be called supplementary edges of the second type. As F is 2-connected and the vertices in the V_i-sequences are far enough, this supplementation does not create any further new F-subgraphs different from the ones inserted for the selected n-element subsets.

Let us denote by $C^2(G, F)$ the graph obtained in this way. It has fewer than $|V(G)| \cdot N_0 \cdot n^2$ vertices, which is smaller than $|V(G)|^{n+3}$ if $|V(G)| > n$. Therefore, once the graph F is fixed, the size of $C^2(G, F)$ is bounded above by a polynomial in the size of G. Thus, the proof will be done if we show that $C^2(G, F)$ is F-WORM colorable if and only if G has a proper vertex coloring with at most $n-1$ colors.

Suppose first that G admits a proper $(n-1)$-coloring φ. This yields an F-WORM coloring of $C^1(G, F, N_0)$ by Lemma 7, in which each V_i-sequence is monochromatic, they altogether contain precisely $n-1$ colors, and if $v_i v_j$ is an edge in G then the colors of V_i and V_j are different. Then the F-subgraphs formed by the supplementary edges of the second type cannot be monochromatic, because each selected n-set $\{v_{i_1}, \dots, v_{i_n}\}$ is supposed to induce at least one edge in G; and they cannot be rainbow F-subgraphs either, because only $n-1$ colors occur on the V_i-sequences. Thus, $C^2(G,F)$ is F-WORM colorable in this case.

Next, assume that $\chi(G) \geq n$, and suppose for a contradiction that $C^2(G,F)$ admits an F-WORM coloring ϕ. Since $C^1(G,F,N_0)$ is a subgraph of $C^2(G,F)$, Lemma 7 implies also for the latter graph that each V_i-sequence is monochromatic in every F-WORM coloring, and any k-coloring of the V_i-sequences induced by an F-WORM coloring of $C^2(G,F)$ is a proper k-coloring of G. Such a coloring necessarily uses at least n colors. Selecting an arbitrary edge v_iv_j of G, we can extend $\{v_i, v_j\}$ to an n-element set $\{v_{i_1}, \ldots, v_{i_n}\}$ such that all those vertices have mutually distinct colors. It follows that the F-subgraph formed by the supplementary edges of the second type inserted for $\{v_{i_1}, \ldots, v_{i_n}\}$ is a rainbow copy of F, contradicting the assumption that ϕ is an F-WORM coloring,

Therefore, once F is fixed according to the conditions in the theorem and $n \geq 4$, the decision problem of $\chi(G) \leq n-1$ can be polynomially reduced to the F-WORM colorability problem, and it follows that the latter problem is NP-complete. $\square$

We close this section with a positive result, implying that important graph classes admit efficiently solvable instances of WORM colorability.

Proposition 8 *Let $n \geq 3$ be an integer, and G a graph with $\chi(G) = \omega(G)$. Then G is K_n-WORM colorable if and only if $\omega(G) \leq (n-1)^2$.*

Proof. We know from Proposition 5(i) that $K_{(n-1)^2+1}$ is not K_n-WORM colorable, therefore the condition $\omega(G) \leq (n-1)^2$ is necessary. Conversely, suppose that $\chi(G) \leq (n-1)^2$. Take any proper coloring of G with at most $(n-1)^2$ colors. It is possible to group the color classes into exactly $n-1$ disjoint non-empty parts, say $C^1, \ldots, C^{n-1}$, each of them consisting of at most $n-1$ colors. (We may assume $\omega(G) \geq n$, otherwise G trivially is K_n-WORM colorable.) Assign color i to the vertices in C^i, for $i = 1, \ldots, n-1$. Then no rainbow K_n can occur because there are at most $n-1$ colors are used, and no monochromatic K_n can occur because each K_n-subgraph meets exactly n color classes in the original proper coloring of G, at most $n-1$ of which belong to the same C^i. Thus, G is K_n-WORM colorable. $\square$

Since a proper coloring of a perfect graph with the minimum number of colors can be determined in polynomial time [9], we obtain:

Corollary 9 *For every fixed $n \geq 3$, the problem of K_n-WORM colorability can be solved in polynomial time on perfect graphs.*

4 Gaps in the chromatic spectrum

The following kind of graph product will play an important role in the proof below. Given two graphs G_1 and G_2, the *strong product* denoted by $G_1 \boxtimes G_2$ has vertex set $V(G_1) \times V(G_2)$, and any two edges $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$ give rise to a copy of K_4 in $G_1 \boxtimes G_2$ with the following six edges:

$$\{(u_1,u_2),(u_1,v_2)\}, \quad \{(u_1,u_2),(v_1,v_2)\}, \quad \{(u_1,u_2),(v_1,u_2)\},$$

$$\{(u_1,v_2),(v_1,u_2)\}, \quad \{(u_1,v_2),(v_1,v_2)\}, \quad \{(v_1,u_2),(v_1,v_2)\}.$$

Moreover, we denote by $G_1 \vee G_2$ the join of G_1 and G_2, that is the graph whose vertex set is the disjoint union $V(G_1) \cup V(G_2)$, and has the edge set

$$E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{v_1v_2 : v_1 \in V(G_1), v_2 \in V(G_2)\}.$$

Applying these operations, here we prove Theorem 4; let us recall its assertion.

Theorem 4. *For every $n \geq 3$ and $\ell \geq 1$ there exist K_n-WORM colorable graphs whose K_n-WORM chromatic spectrum contains a gap of size ℓ.*

Proof. As we mentioned in the Introduction, for K_3 the theorem was proved in [4]. Hence, from now on we assume $n \geq 4$.

Consider a triangle-free, connected graph G with $\chi(G) = k \geq 3$, and construct the graph $G^* = (G \boxtimes K_{n-1}) \vee K_{(n-3)(n-1)}$. When G^* is obtained from G, each vertex $v_i \in V(G)$ is replaced with a complete graph on $n-1$ vertices — this vertex set will be denoted by V_i — and each edge $v_iv_j \in E(G)$ is replaced with a complete bipartite graph between V_i and V_j. To complete the construction, we extend the graph with $(n-3)(n-1)$ universal vertices whose set is denoted by V^*. Note that the vertex sets $V_1, \ldots, V_{|V(G)|}, V^*$ are pairwise disjoint.

If a K_n subgraph of G^* meets both sets V_i and V_j (with $i \neq j$), then there exist some edges between these sets and hence v_i and v_j must be adjacent in G. Moreover, as G is triangle-free, a complete subgraph of G^* cannot meet three different vertex sets V_s. This implies that for each K_n subgraph K of G^*

there exists an edge $v_iv_j \in E(G)$ such that $V(K) \subset V_i \cup V_j \cup V^*$. Therefore, a vertex coloring φ of G^* is K_n-WORM if and only if the complete subgraph of order $(n-1)^2$ induced by $V_i \cup V_j \cup V^*$ in G^* is K_n-WORM colored for each edge v_iv_j of G. By Proposition 5, this gives the following necessary and sufficient condition for φ to be a K_n-WORM coloring:

($\star$) For each $v_iv_j \in E(G)$, φ uses exactly $n-1$ colors on $V_i \cup V_j \cup V^*$, and each color occurs on exactly $n-1$ vertices of this complete subgraph.

Now, we assume that φ is a K_n-WORM coloring of G^*. We make the following observations.

- Since there exist $K_{(n-1)^2}$-subgraphs, φ uses at least $n-1$ colors. On the other hand, by ($\star$) a K_n-WORM coloring is obtained if each of the colors $1, 2, \ldots, n-1$ occurs on exactly $n-3$ vertices from V^*, and on exactly one vertex from each V_i. This proves $W^-(G^*, K_n) = n-1$.

- If φ uses exactly $n-1$ colors on V^*, it follows from ($\star$) that no further colors appear on the sets V_i.

- If $|\varphi(V^*)| = n-2$, then for each $v_iv_j \in E(G)$ the set $\varphi(V_i \cup V_j)$ contains exactly one color different from those in $\varphi(V^*)$. We have two cases. If there exists a monochromatic V_s, its color c^* appears on $n-1$ vertices in V_s. By ($\star$), $c^* \notin \varphi(V^*)$ follows, and also that for every neighbor v_p of v_s, $c^* \notin \varphi(V_p)$. Then, $|\varphi(V^* \cup V_p)| = n-2$ and for each neighbor v_q of v_p, the vertex set V_q in G^* must be monochromatic in a color not included in $\varphi(V^*)$. As G is connected, this property propagates along the edges and for every adjacent vertex pair v_i, v_j, one of the sets V_i and V_j is monochromatic and the other is not. This gives a bipartition of G, which contradicts our assumption $\chi(G) \geq 3$. In the other case, there is no monochromatic V_i, therefore the $n-1$ vertices of the $(n-1)$st color of $V_i \cup V_j \cup V^*$ have to be distributed between V_i and V_j. This implies

$$\varphi(V_i \cup V^*) = \varphi(V_i \cup V_j \cup V^*) = \varphi(V_j \cup V^*)$$

for every pair i, j with $v_iv_j \in E(G)$. By the connectivity of G, we conclude that $|\varphi(G^*)| = n-1$.

- Assume that $|\varphi(V^*)| = n-3$. Then, each of these $n-3$ colors occurs on exactly $n-1$ vertices of V^* and occurs on no further vertices of

G^*. Moreover, for each $v_iv_j \in E(G)$, the vertices in $V_i \cup V_j$ are colored with exactly two colors such that each color is assigned to exactly $n-1$ vertices. If there is a non-monochromatic V_s, then $\varphi(V_s) = \varphi(V_p)$ for every p satisfying $v_sv_p \in E(G)$. Then, since G is connected, this equality will be also valid if $v_sv_p \notin E(G)$. Therefore, we have only $n-1$ different colors on the vertices of $V(G^*)$, again. On the other hand, if every V_i is made monochromatic by φ, the condition $(\star)$ is satisfied if and only if (i) the color of V_i is not in $\varphi(V^*)$; and (ii) for every adjacent vertex pair v_i, v_j of G, the colors $\varphi(V_i)$ and $\varphi(V_j)$ are disjoint. Conditions (i) and (ii) imply that the color assignment ϕ defined as $\phi(v_i) = \varphi(V_i)$ gives a proper vertex coloring of G with $|\varphi(V(G^*))| - n + 3$ colors. Hence, this type of K_n-WORM coloring of G^* can be constructed such that the number of used colors is one from the range $\chi(G) + n - 3, \ldots, |V(G)| + n - 3$.

We have proved that the K_n-WORM feasible set of G^* is

$$\{n-1\} \cup \{k+n-3, \ldots, |V(G)| + n - 3\}.$$

If we choose a triangle-free connected graph G with $\chi(G) = k = \ell + 3$, the gap in the feasible set $\Phi_W(G^*, K_n)$ is of size ℓ. □

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