FORBIDDEN FAMILIES OF MINIMAL QUADRATIC AND CUBIC CONFIGURATIONS

ATTILA SALI AND SAM SPIRO

Abstract. A matrix is *simple* if it is a (0,1)-matrix and there are no repeated columns. Given a (0,1)-matrix F, we say a matrix A has F as a *configuration*, denoted $F \prec A$, if there is a submatrix of A which is a row and column permutation of F. Let $|A|$ denote the number of columns of A. Let $\mathcal{F}$ be a family of matrices. We define the extremal function $\text{forb}(m, \mathcal{F}) = \max\{|A| \colon A$ is an $m-$rowed simple matrix and has no configuration $F \in \mathcal{F}\}$. We consider pairs $\mathcal{F} = \{F_1, F_2\}$ such that F_1 and F_2 have no common extremal construction and derive that individually each $\text{forb}(m, F_i)$ has greater asymptotic growth than $\text{forb}(m, \mathcal{F})$, extending research started by Anstee and Koch [7].

1. Introduction

The investigations into the extremal problem of the maximum number of edges in an n vertex graph with no subgraph H originated with Erdős and Stone [13] and Erdős and Simonovits [12] . There is a large and illustrious literature. A natural extension to general hypergraphs is to forbid a given *trace*. This latter problem in the language of matrices is our focus. We say a matrix is *simple* if it is a (0,1)-matrix and there are no repeated columns. Given a (0,1)-matrix F, we say a matrix A has F as a *configuration*, denoted $F \prec A$, if there is a submatrix of A which is a row and column permutation of F. Let $|A|$ denote the number of columns in A. We define

$$\text{Avoid}(m, F) = \{A \,:\, A \text{ is } m\text{-rowed simple}, F \nprec A\}\,,$$
$$\text{forb}(m, F) = \max_A\{|A| : A \in \text{Avoid}(m, F)\}.$$

A simple (0,1)-matrix A can be considered as vertex-edge incidence matrix of a hypergraph without repeated edges. A configuration is a trace of a subhypergraph of this hypergraph.

Let A^c denote the 0-1-complement of a (0,1)-matrix A. It is easy to see that $\text{forb}(m, F) = \text{forb}(m, F^c)$.

We recall an important conjecture from [10]. Let I_k denote the $k \times k$ identity matrix, let I_k^c denote the (0,1)-complement of I_k, and let T_k denote the $k \times k$ upper triangular matrix whose ith column has 1's in rows $1, 2, \ldots, i$ and 0's in the remaining rows. For p matrices $m_1 \times n_1$ matrix A_1, an $m_2 \times n_2$ matrix $A_2, \ldots$, an $m_p \times n_p$ matrix A_p we define $A_1 \times A_2 \times \cdots \times A_p$ as the $(m_1 + \cdots + m_p) \times n_1 n_2 \cdots n_p$ matrix whose columns consist of all possible combinations obtained from placing

Research was partially supported by Hungarian National Research, Development and Innovation Office NKFIH, K116769.

Research was done while the second author took the *Research Opportunties* course at Budapest Semesters in Mathematics under the supervision of the first author.

a column of A_1 on top of a column of A_2 on top of a column of A_3 etc. For example, the vertex-edge incidence matrix of the complete bipartite graph $K_{m/2,m/2}$ is $I_{m/2} \times I_{m/2}$. Define 1_k to be the $k \times 1$ column of 1's and 0_ℓ to be the $\ell \times 1$ column of 0's.

Conjecture 1.1. [10] Let F be a $k \times \ell$ matrix with $F \neq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Let $X(F)$ denote the largest p such that there are choices $A_1, A_2, \ldots, A_p \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$ so that $F \not\prec A_1 \times A_2 \times \cdots \times A_p$. Then $\text{forb}(m, F) = \Theta(m^{X(F)})$.

We are assuming p divides m which does not affect asymptotic bounds.

It is natural to extend the concepts of $\text{Avoid}(m, F)$ and $\text{forb}(m, F)$ to the case when not just a single configuration, but a family $\mathcal{F} = \{F_1, F_2, \ldots, F_r\}$ of configurations is forbidden.

$$\text{Avoid}(m, \mathcal{F}) = \{A : A \text{ is } m\text{-rowed simple}, F \not\prec A \text{ for all } F \in \mathcal{F}\},$$
$$\text{forb}(m, \mathcal{F}) = \max_A \{|A| : A \in \text{Avoid}(m, \mathcal{F})\}.$$

One important result in this area is the following theorem of Balogh and Bollobás [11].

Theorem 1.2 (Balogh and Bollobás, 2005)**.** *For a given k, there is a constant $BB(k)$ such that* $\text{forb}(m, \{I_k, T_k, I^c_k\}) = BB(k)$.

The best current estimate for $BB(k)$ is due to Anstee and Lu [8], $BB(k) \leq 2^{ck^2}$ where c is absolute constant, independent of k. It could be tempting to extend Conjecture 1.1 to the case of forbidden families, as well. However, as it was shown in [5] $\text{forb}(m, \{I_2 \times I_2, T_2 \times T_2\})$ is $\Theta(m^{3/2})$ despite the only products missing both $I_2 \times I_2$ and $T_2 \times T_2$.are one-fold products. An even stronger observation is made in Remark 5.10.

In the present paper we continue the investigations started in [7]. Anstee and Koch determined $\text{forb}(m, \{F, G\})$ for all pairs $\{F, G\}$, where both members are *minimal quadratics*, that is both $\text{forb}(m, F) = \Theta(m^2)$ and $\text{forb}(m, G) = \Theta(m^2)$, but no proper subconfiguration of F or G is quadratic. We take this one step further. That is, we consider cases when one of F or G is a simple minimal cubic configuration and the other one is a minimal quadratic or minimal simple cubic. Our results are summarized in Table 3. We solve all cases when the minimal simple cubic configuration has four rows. If Conjecture 8.1 of [3] is true, then there are no minimal simple cubic configurations on 5 rows. The six-rowed ones are discussed in Section 8. The remaining case is $\text{forb}(m, Q_8, F_{14})$, where we believe that non-existence of common quadratic product construction indicates that the order of magnitude is $o(m^2)$.

The structure of the paper is as follows. In Section 2 product constructions and bounds implied by them are treated. Then in Section 3 upper bounds implied by the *standard induction* technique ([3], Section 11) are given. These combined with product constructions give asymptotically sharp bounds for many pairs of configurations. Sections 4, 5, 6 and 7 deal with specific configurations. In Section 4 a stability theorem is proven for matrices avoiding the configuration $Q_3(t)$, which is a generalization of the configuration Q_3 (see Table 1), and this theorem is applied to prove forbidden pairs results involving $Q_3(t)$. Section 5 contains cases when one member of the forbidden pairs is a block of 1's. This naturally involves extremal graph and hypergraph results, as forbidding $1_{k,1}$ restricts the hypergraph corresponding to our simple (0,1)-matrix to be of *rank*-$(k-1)$, that is edges are of size at most $k-1$. Interestingly enough, in one case we use a very recent theorem of Alon and Shikhelman [1] combined with an old fundamental result of Füredi [14]. Section 6 considers F_9 (see Table 2). Interestingly, some exact results are also obtained. Section 7 deals with Q_9 of Table 1 based on the characterization of Q_9 avoiding matrices of [4]. Finally, in Section 8 we observe that $\mathrm{forb}(m, \{F, G\})$ is quadratic if F is a minimal quadratic and G is a 6-rowed minimal cubic in all but one case.

Throughout the paper we use standard extremal graph and hypergraph notations, such as $ex(m, G)$ to denote the largest number of edges a graph on m vertices can have without containing a subgraph isomorphic to G, or $ex^{(k)}(m, \mathcal{H})$ for the largest number of edges a k-uniform hypergraph can have without containing a subhypergraph $\mathcal{H}$. The complete k-partite k-uniform hypergraph on partite sets of sizes $s_1, \ldots, s_k$, respectively is denoted by $K(s_1, \ldots, s_k)$. Also, when forbidden pairs of configurations are considered, we use the notational simplification $\mathrm{forb}(m, \{F, G\}) = \mathrm{forb}(m, F, G)$ for typesetting convenience. We allow ourselves the ambiguity of writing $I \times I^c$ instead of the technically precise $I_{m/2} \times I^c_{m/2}$ in product constructions.

2. Product Constructions

What follows are tables of all minimal quadratic configurations and simple minimal cubic configurations with 4 rows. In addition to the configurations, we have included a list of all 2-fold and 3-fold products of I, I^c and T that avoid these configurations. The list of constructions avoiding quadratic configurations comes from [7], and the lists for cubic configurations are proved in Section 2, with the statement that proves the result listed under "Proposition."

Table 1. Minimal Quadratic Configurations

	Configuration Q_i	Construction(s)
$1_{3,1}$	$\begin{bmatrix}1\\1\\1\end{bmatrix}$	$I \times I$
$1_{2,2}$	$\begin{bmatrix}1&1\\1&1\end{bmatrix}$	$I \times I$
I_3	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$	$I^c \times I^c$ $I^c \times T$ $T \times T$
Q_3	$\begin{bmatrix}0&0&0&1&1&1\\0&1&1&0&0&1\end{bmatrix}$	$I \times I^c$
Q_8	$\begin{bmatrix}0&0&1&1\\1&0&1&0\\0&1&0&1\end{bmatrix}$	$T \times T$
Q_9	$\begin{bmatrix}1&0\\1&0\\0&1\\0&1\end{bmatrix}$	$I \times T$ $I^c \times T$

Note that we have not included the complements of $1_{3,1}$, $1_{2,2}$, and I_3 in this table, even though these are also minimal quadratic configurations. This is because if Q denotes any of these configurations then $\text{forb}(m, Q, F) = \text{forb}(m, Q^c, F^c)$, which is already included in Table 3.

Table 2. Minimal Simple Cubic Configurations with 4 Rows

	Configuration F_i	Quadratic Const.(s)	Cubic Const.(s)	Proposition
$1_{4,1}$	$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$	$I \times I$	$I \times I \times I$	Prop. 2.2
F_9	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$	$I^c \times I^c$ $I^c \times T$ $T \times T$	$I^c \times I^c \times T$	Prop. 2.4
F_{10}	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$	$I^c \times I^c$ $I^c \times T$ $T \times T$	$I^c \times I^c \times T$	Prop. 2.4
F_{11}	$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$	$I \times T$ $I^c \times T$ $T \times T$	$T \times T \times T$	Prop. 2.6
F_{12}	$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$	All	All	Lem. 2.7
F_{13}	$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$	All	$T \times T \times T$	Lem. 2.7 Prop. 2.8

In addition to this, the compliment of $1_{4,1}$ (which we denote by $0_{4,1}$), F_9^c, F_{10}^c, and F_{12}^c are minimal simple cubic configurations, and the products avoiding these configurations are the complements of the products avoiding their complements.

Table 3 contains the asymptotic values for all pairings of the configurations mentioned above when at least one of the configurations is cubic. We note that all exact results stated below hold for m sufficiently large.

Table 3. Results

	$1_{4,1}$	F_9	F_{10}	F_{11}	F_{12}	F_{13}	$0_{4,1}$	F_9^c	F_{10}^c	F_{12}^c
$1_{3,1}$	$\Theta(m^2)$ Rm 2.1	$m+2$ Cr 6.16	$\Theta(1)$ Cr 5.1	$\Theta(m^{3/2})$ Cr 5.3	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(1)$ Cr 5.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1
$1_{2,2}$	$\Theta(m^2)$ Rm 2.1	$m+3$ Cr 6.16	$\Theta(1)$ Cr 5.1	$\Theta(m^{3/2})$ Cr 5.5	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(1)$ Cr 5.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1
I_3	$\Theta(1)$ Cr 5.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1
Q_3	$\Theta(m)$ Cr 4.2	$\Theta(m)$ Th 6.1	$\Theta(m)$ Cr 4.2	$\Theta(m^{3/2})$ Cr 4.13	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m)$ Cr 4.2	$\Theta(m)$ Th 6.1	$\Theta(m)$ Cr 4.2	$\Theta(m^2)$ Rm 2.1
Q_8	$\Theta(m)$ Pr 3.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m)$ Pr 3.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1
Q_9	$3m-2$ Cr 7.3	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$3m-2$ Cr 7.3	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1
$1_{4,1}$		$m+5$ Cr 6.16	$\Theta(1)$ Cr 5.1	$\Theta(m^{3/2})$ Pr 5.7	$\Theta(m^3)$ Rm 2.1	$\Theta(m^2)$ Pr 3.3	$\Theta(1)$ Cr 5.1	$\Theta(m^3)$ Rm 2.1	$\Theta(m^3)$ Rm 2.1	$\Theta(m^3)$ Rm 2.1
F_9			$\Theta(m^3)$ Rm 2.1	$\Theta(m^2)$ Pr 3.3	$\Theta(m^3)$ Rm 2.1	$\Theta(m^2)$ Pr 3.3	$\Theta(m^3)$ Rm 2.1	$\Theta(m^2)$ Pr 3.4	$\Theta(m^2)$ Pr 3.4	$\Theta(m^3)$ Rm 2.1
F_{10}				$\Theta(m^2)$ Pr 3.3	$\Theta(m^3)$ Rm 2.1	$\Theta(m^2)$ Pr 3.3	$\Theta(m^3)$ Rm 2.1	$\Theta(m^2)$ Pr 3.4	$\Theta(m^2)$ Pr 3.4	$\Theta(m^3)$ Rm 2.1
F_{11}					$\Theta(m^3)$ Rm 2.1	$\Theta(m^3)$ Rm 2.1	$\Theta(m^{3/2})$ Pr 5.7	$\Theta(m^2)$ Pr 3.3	$\Theta(m^2)$ Pr 3.3	$\Theta(m^3)$ Rm 2.1
F_{12}						$\Theta(m^3)$ Rm 2.1	$\Theta(m^3)$ Rm 2.1	$\Theta(m^3)$ Rm 2.1	$\Theta(m^3)$ Rm 2.1	$\Theta(m^3)$ Rm 2.1
F_{13}							$\Theta(m^2)$ Pr 3.3	$\Theta(m^2)$ Pr 3.3	$\Theta(m^2)$ Pr 3.3	$\Theta(m^3)$ Rm 2.1

In this section we determine all product constructions that avoid the minimal cubic configurations mentioned above, where we note that if a configuration A is avoided by the product B then A^c is avoided by the product B^c. We will then be able to obtain most of our lower bound results from the following observation:

Remark 2.1. If F and G are both avoided by the same p-fold product construction then $\text{forb}(m, F, G) = \Omega(m^p)$.

We note that proving $\text{forb}(m, F, G) = \Omega(m^2)$ when either F or G is a minimal quadratic configuration implies that $\text{forb}(m, F, G) = \Theta(m^2)$, and similarly if $\text{forb}(m, F, G) = \Omega(m^3)$ for F or G a minimal cubic configuration then $\text{forb}(m, F, G) = \Theta(m^3)$.

Proposition 2.2. *The only 2-fold product avoiding $1_{4,1}$ is $I \times I$. The only 3-fold product avoiding $1_{4,1}$ is $I \times I \times I$.*

Proof. Note that $1_{4,1} \prec I_5^c, T_5$, so any product using I^c or T will contain $1_{4,1}$. There are only three 1's in each column of $I \times I \times I$, so $1_{4,1} \nprec I \times I \times I$, and it follows that $1_{4,1} \nprec I \times I$ as well. □

Lemma 2.3. $F_9, F_{10}, F_9^c, F_{10}^c \prec [01] \times [01] \times T_4$.

Proof. The last two rows of $F_9, F_{10}, F_9^c, F_{10}^c$ are contained in T_4, and hence the last three rows of these configurations will be contained in $[01] \times T_4$ and all of the configurations will be contained in $[01] \times [01] \times T_4$. □

Proposition 2.4. *F_9 and F_{10} are avoided by every 2-fold product not involving I, and they are contained in every 2-fold product involving I. The only 3-fold product avoiding F_9 and F_{10} is $I^c \times I^c \times I^c$.*

Proof. Note that I_3 is avoided by every 2-fold product not involving I by [7], and because $I_3 \prec F_9, F_{10}$ it follows that these products must also avoid F_9 and F_{10}. Observe that $F_9, F_{10} \prec [01] \times I_3$, and hence F_9 and F_{10} will be contained in any 2-fold product involving I. It follows from Lemma 2.3 that F_9, F_{10} will be contained in any 3-fold product involving T, so the only 3-fold product that can avoid these configurations is $I^c \times I^c \times I^c$, and [3] notes that this is indeed the case. □

Lemma 2.5. *$F_{11}, F_{13} \prec [01] \times [01] \times I_2 = [01] \times [01] \times I_2^c$.*

Proof. $F_{11} = I_2 \times I_2 \prec [01] \times [01] \times I_2$. The second and third rows of F_{13} are equal to $[01] \times [01]$, and the remaining rows consist of columns of I_2. We thus have $F_{13} \prec [01] \times [01] \times I_2$. □

Proposition 2.6. *$F_{11} \not\prec I \times T, I^c \times T, T \times T$ and it is contained in all other 2-fold products. The only 3-fold product that avoids F_{11} is $T \times T \times T$.*

Proof. Note that $Q_9 \prec F_{11}$ and that $Q_9 \not\prec I \times T, I^c \times T$, so it follows that this is also the case for F_{11}. Because $F_{11} = I_2 \times I_2$ and $I_2 \prec I, I^c$, it follows that every 2-fold product consisting only of I's and I^c's contains F_{11}. [3] notes that $F_{11} \not\prec T \times T \times T$, so it also follows that $F_{11} \not\prec T \times T$. It follows from Lemma 2.5 that every 3-fold product involving an I or I^c contains F_{11}, so the only 3-fold product that can avoid F_{11} is $T \times T \times T$. □

Lemma 2.7. *All 2-fold products of I, I^c and T avoid F_{13}. All 3-fold products avoid F_{12} and F_{12}^c*

Proof. Every two rows of the first three rows of F_{13} contains $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, and as no two rows of I, I^c, or T contains this configuration, the first three rows of F_{13} can not be found in any 2-fold product of these matrices. Any two rows of F_{12} contains $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, which again is contained in no two rows of I, I^c or T, so this can not be found in any 3-fold product of these matrices. Similar logic holds for F_{12}^c. □

Proposition 2.8. *The only 3-fold product that avoids F_{13} is $T \times T \times T$.*

Proof. By Lemma 2.5 every 3-product involving I or I^c contains F_{13}, and [3] notes that $F_{13} \not\prec T \times T \times T$. □

3. Inductive Results

In this section we prove a variety of upper bounds by using two standard techniques: Theorem 1.2 and the following standard induction method. Let F be a k-rowed matrix. Suppose we have $A \in \text{Avoid}(m, F)$ such that $|A| = \text{forb}(m, F)$. Consider deleting a row r. Let $C_r(A)$ be the matrix that consists of the repeated columns of the matrix that is obtained when deleting row r from A. If we permute the rows of A so that r becomes the first row, then after some column permutations, A looks like this:

$$A = {}^{r}\begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B_r(A) & & C_r(A) & C_r(A) & & D_r(A) \end{bmatrix}.$$

where $B_r(A)$ are the columns that appear with a 0 on row r, but don't appear with a 1, and $D_r(A)$ are the columns that appear with a 1 but not a 0. We have that

$$\text{forb}(m, F) \le |C_r(A)| + \text{forb}(m-1, F),$$

as $[B_r(A)C_r(A)D_r(A)] \in \text{Avoid}(m-1, F)$. This is used usually in the form that if $F \prec [01] \times F'$, then

$$\text{forb}(m, F) \le \text{forb}(m-1, F') + \text{forb}(m-1, F).$$

We let $1_{k,\ell}$ denote the $k \times \ell$ matrix where every entry is 1. Similarly, we define $0_{k,\ell}$ to be the $k \times \ell$ matrix where every entry is 0. We use the notation $C_r := C_r(A)$ when it is clear from context what the underlying matrix A is.

Proposition 3.1. $\text{forb}(m, Q_8, 1_{k,\ell}) = \text{forb}(m, Q_8, 0_{k,\ell}) = \Theta(m)$.

Proof. As $Q_8^c = Q_8$ we see that these two values are equal, so we only address the $1_{k,\ell}$ case. Note that I_m gives the lower bound. For the upper bound, note that $Q_8 = [01] \times I_2$. It follows that when we apply the standard induction that C_r can not contain $I_2 = I_2^c$. But by Theorem 1.2 if $|C_r| > BB(k+\ell)$ we must have $T_{k+\ell} \prec C_r$, which would contradict $1_{k,\ell} \not\prec A$. Thus we must have $|C_r| \le BB(k+\ell)$, so we can inductively assume a linear bound for $\text{forb}(m, Q_8, 1_{k,\ell})$. □

Lemma 3.2. $\text{forb}(m, [01] \times [01] \times I_r, [01] \times [01] \times I_r^c, [01] \times [01] \times T_r) = O(m^2)$.

Proof. By using the standard induction and Theorem 1.2 one gets that $\text{forb}(m, [01] \times I_r, [01] \times I_r^c, [01] \times T_r) = O(m)$. Given this, when we apply the standard induction for $\text{forb}(m, [01] \times [01] \times I_r, [01] \times [01] \times I_r^c, [01] \times [01] \times T_r)$ we get a quadratic upper bound. □

Proposition 3.3. $\text{forb}(m, F, G) = O(m^2)$ *for* $F = 1_{4,1}, F_9, F_{10}, F_9^c,$ *or* F_{10}^c *and* $G = F_{11}$ *or* F_{13}.

In Table 3 Proposition 3.3 is frequently quoted to prove Θ bounds. This is done so when common quadratic lower bound exists for F and G by product constructions listed in Table 2.

Proof. This follows from Lemma 3.2, along with the observations that $1_{4,1} \prec [01] \times [01] \times T_4$, $F_9, F_{10} \prec [01] \times [01] \times T_4$ by Lemma 2.3, and $F_{11}, F_{13} \prec [01] \times [01] \times I_2$ by Lemma 2.5. □

Proposition 3.4. $\text{forb}(m, F, G) = \Theta(m^2)$ *where* $F = F_9$ *or* F_{10} *and* $G = F_9^c$ *or* F_{10}^c.

Proof. The lower bound follows from the construction $T \times T$, and the upper bound is a consequence of Lemma 3.2 and the observations that $F_9, F_{10}, F_9^c, F_{10}^c \prec [01] \times [01] \times T_4$, $F_9, F_{10} \prec [01] \times I_3 \prec [01] \times [01] \times I_3$ and $F_9^c, F_{10}^c \prec [01] \times I_3^c \prec [01] \times [01] \times I_3^c$. □

4. Avoiding $Q_3(t)$

We consider a slight generalization of Q_3

$$Q_3(t) = \begin{bmatrix} 0 & \overbrace{1\cdots 1}^{t} & \overbrace{0\cdots 0}^{t} & 1 \\ 0 & 0\cdots 0 & 1\cdots 1 & 1 \end{bmatrix},$$

where we always assume $t \geq 2$ when we write $Q_3(t)$. We have the following result from [7].

Theorem 4.1. *$\text{forb}(m, Q_3(t), t \cdot I_k) = \text{forb}(m, Q_3(t), t \cdot I_k^c) = \Theta(m)$ for any fixed k.*

Corollary 4.2. *$\text{forb}(m, Q_3, F) = \Theta(m)$ for $F = 1_{4,1},\ F_{10},\ 0_{4,1},\ F_{10}^c$.*

Proof. Each of these F is contained in either I_k or I_k^c for sufficiently large k, so Theorem 4.1 gives the upper bound, and either I_m or I_m^c gives the lower bound. □

Our main result for this section will be a stability theorem which says that large $Q_3(t)$ avoiding matrices "look like" $I \times I^c$, and from this we will be able to prove an upper bound for $\text{forb}(m, Q_3, F_{11})$, and more generally for $\text{forb}(m, Q_3(t), I_r \times I_s)$. We first introduce some terminology for the proof.

We will say that a row r is *sparse* when restricted to a set of columns C if, restricted to C, r has at least one 0 but fewer than t 0's (i.e. r has few 0's but is not identically 1), and we will say that a row r is *dense* when restricted to a set of columns C if r has at least one 1 and at least t 0's within the columns of C (i.e. r has many 0's but is not identically 0). We will say that a column $c \in C$ is *identified* by a sparse row r if r has a 0 in column c.

If A is a matrix and C is a set of columns (not necessarily a subset of the columns of A), then $A \setminus C$ will denote the set of columns in A that are not in C. We define the matrix $Q_3(t;0)$ to be $Q_3(t)$ without its column of 1's. Lastly, we restate Theorem 4.1 as follows: for any fixed k and t there exists a constant $c_{k,t}$ such that if A is an m-rowed simple matrix with $|A| > c_{k,t}m$ and $Q_3(t) \not\prec A$, then $t \cdot I_k \prec A$.

Theorem 4.3. *Let $A \in \text{Avoid}(m, Q_3(t))$ with $|A| = \omega(m \log m)$. There exists a set of integers $\{k_1, \ldots, k_y\}$ and a set $A' = \{A'_1, \ldots, A'_y\}$, of configurations $A'_j \prec A$ such that:*

1. *(1) $k_{j+1} \leq \frac{1}{2}k_j$ for all j, and $y \leq \log m$.*
2. *(2) There exists k_j rows of A such that the columns of A'_j restricted to these rows are columns of I_{k_j}.*
3. *(3) If i is a column of I_{k_j} and C_i^j is the set of columns in A'_j that are an i column in the rows mentioned above, then no row restricted to C_i^j is dense, and every column of C_i^j is identified by some sparse row.*
4. *(4) $|A| = \Theta(\sum |A'_j|)$.*

We first present an outline of the proof before going into the details. We are given a large $Q_3(t)$ avoiding matrix A_0, and as a first step we remove all rows from A_0 that have few 1's (for technical reasons) to get a new matrix A_1. We then find the largest $t \cdot I_k$ in A_1, and our goal is to use this as the I_{k_1} base for A'_1. To do so, we trim A_1 by getting rid of all columns of C_i^1 that are not identified by a sparse row, as well as all rows that are dense restricted to some C_i^1. This gives us A'_1, and we repeat the process on the remaining columns of A_1, A_2 (after again removing rows with few 1's). It turns out that the largest $t \cdot I$ in A_2, I_{k_2}, will satisfy $k_2 \leq \frac{1}{2}k_1$, and thus we can repeat this process at most $\log m$ times. At each step we remove only $O(m)$ columns, so in total only $O(m \log m)$ columns of A_0 were removed. As $|A_0| = \omega(m \log m)$, the columns that remain (those of A') must be asymptotically as large as our original A_0.

Proof. Let $A_0 \in \text{Avoid}(m, Q_3(t))$ with $|A_0| = \omega(m \log m)$. Let R_1 denote the set of rows of A_0 that have fewer than $3t-2$ 1's, and let A_1 denote A_0 with these rows removed. Note that A_1 need not be a simple matrix, but if C_{R_1} denotes the set of columns that have a 1 in some row of R_1, then $A_1 \setminus C_{R_1}$ will be simple. As $|C_{R_1}| \leq (3t-2)m = O(m)$, $|A_1 \setminus C_{R_1}| = \Theta(|A_0|)$. Note that we will be working with the matrix A_1, *not* its simplification $A_1 \setminus C_{R_1}$, in order to use the fact that every row has at least $3t-2$ 1's.

Define k_1 to be the largest integer such that $t \cdot I_{k_1} \prec A_1$. As $|A_1 \setminus C_1| = \omega(m)$, Theorem 4.1 tells us that we have $t \cdot I_k \prec A_1 \setminus C_1 \prec A_1$ for any fixed k (so in particular we can assume that $k_1 \geq 3$). Rearrange rows so that this $t \cdot I_{k_1}$ appears in the first k_1 rows of A_1.

Note that no column of A_1 can have two 1's in the first k_1 rows. Indeed, any two rows of $t \cdot I_{k_1}$ for $k_1 \geq 3$ induce a $Q_3(t; 0)$, and hence if a column had 1's in two of these rows we would have $Q_3(t) \prec A_1$. We can thus partition the columns of A_1 as follows. We will say that a column c belongs to the set C_i^1 for $1 \leq i \leq k_1$ if c has a 1 in row i, and we will say that $c \in C^2$ if c has no 1's in these rows. We will make the additional assumption that the $t \cdot I_{k_1}$ we placed in the first k_1 rows was such that $|C^2|$ is minimal. Note that $|C_i^1| \geq 3t-2$ for all i, as otherwise the ith row would belong to R_1 and hence not be in A_1.

We now examine the rows that are dense in some C_i^1.

Lemma 4.4. *If a row r restricted to C_i^1 is dense, then restricted to $A_1 \setminus C_i^1$, r has at most $t-1$ 1's or r is identically 1.*

Proof. Assume r is dense restricted to C_i^1, i.e. it has at least t 0's and one 1 restricted to C_i^1. If r had t 1's and a 0 in $A \setminus C_i^1$, then by looking at the ith row, row r, and the relevant columns, we would find a $Q_3(t)$. □

We would like to strengthen the above lemma to say that dense rows are either identically 0 or identically 1 outside of their C_i^1, and to do so we'll have to ignore a small number of columns of A_1. We will say that a column c is "bad" if there exists a row r and integer i such that r is dense restricted to C_i^1, r is not identically 1 in $A \setminus C_i^1$, and c has a 1 in row r. Let $\overline{C^1}$ denote the set of bad columns.

Lemma 4.5. $|\overline{C^1}| = O(m)$.

Proof. Each dense row r contributes at most $t-1$ columns to $\overline{C^1}$ by Lemma 4.4, and hence $|\overline{C^1}| \leq (t-1)m = O(m)$. □

We now wish to ignore the dense rows of A_1, as well as any rows of $\bigcup C_i^1$ that are not identified by a sparse row. Rearrange rows so that the bottom ℓ rows of A_1 consist of all rows that when restricted to some C_i^1 are dense. Let $\widehat{C_i^1}$ denote the columns of C_i^1 that are not identified by a sparse row and that are not in C_{R_1} or $\overline{C^1}$. Let $\widehat{A_1}$ denote A_1 restricted to the top k_1 rows, the bottom ℓ rows, and the columns of $\bigcup \widehat{C_i^1}$.

Lemma 4.6. $\widehat{A_1}$ *is a simple matrix.*

Proof. Let $\hat{c}$ and $\hat{d}$ be columns of $\widehat{A_1}$ with corresponding columns c, d in $A_1 \setminus C_{R_1}$ (as no $\widehat{C_i^1}$ columns are in C_{R_1}). If $\hat{c} = \hat{d}$, then clearly we must have $c, d \in C_i^1$ for some i. As $c \neq d$ (because $A_1 \setminus C_{R_1}$ is a simple matrix), we must have c and d differing in some row r above the bottom ℓ rows, say c has a 0 in row r and d has a 1. But this means that r must be sparse (as every row between the top k_1 rows and bottom ℓ rows is either identically 0, identically 1, or sparse), and hence c is identified by a sparse row, contradicting $\hat{c}$ belonging to $\widehat{A_1}$. □

Lemma 4.7. $|\widehat{A_1}| = O(m)$.

Proof. By Lemma 4.4 (and the fact that $\widehat{A_1}$ contains no columns of $\overline{C^1}$), we know that each row r restricted to $\widehat{C_i^1}$ can be one of four types: r can be identically 0 restricted to $A_1 \setminus C_i^1$ (in which case we will say it is a row of $B_{i,0}$), r can be identically 1 restricted to $A_1 \setminus C_i^1$ (in which case we will say it is a row of $B_{i,1}$), or r can itself be either identically 0 or identically 1. We thus have that the matrix B_i formed by restricting $\widehat{A_1}$ to the columns $\widehat{C_i^1}$ and to the rows of $B_{i,0}$ and $B_{i,1}$ is simple with $|\widehat{C_i^1}|$ columns. Let b_i denote the number of rows in B_i.

If $|B_i| > c_{3,t}b_i$, then we must have $t \cdot I_3 \prec B_i$, and hence either $B_{i,0}$ or $B_{i,1}$ must contain a $Q_3(t;0)$. If $B_{i,1}$ contains a $Q_3(t;0)$, then these rows and columns together with any column of $A_1 \setminus C_i^1$ gives a $Q_3(t)$. If $B_{i,0}$ contains a $Q_3(t;0)$, then one can find a $t \cdot I_{k_1+1}$ in A_1. Indeed, in A_1 (note that we are no longer ignoring the columns of $\overline{C^1}$ and C_{R_1}), take the two rows from $B_{i,0}$ that contain a $Q_3(t;0)$, ignore the at most $2t-2$ columns that have 1's in these rows outside of C_i^1, and swap these rows with rows i and k_1+1. After performing these steps, no column of A_1 has two 1's in any of the first k_1+1 rows (since we removed the at most $2t-2$ columns that could pose a problem), rows i and k_1+1 by assumption have at least t 1's, and as every other row had at least $3t-2$ 1's before ignoring the at most $2t-2$ columns, they all still have at least t 1's. Hence we have $t \cdot I_{k_1+1} \prec A_1$, contradicting our definition of k_1. Thus we must have $|B_i| = |\widehat{C_i}| \le c_{3,t}b_i$, and in total we have

$$|\widehat{A_1}| = \sum |\widehat{C_i^1}| \le \sum c_t b_i \le c_t \ell \le c_t m,$$

proving the statement. □

We now let A_1' be $\bigcup C_i^1$ after removing the columns of $\widehat{A_1}$, C_{R_1}, and $\overline{C^1}$ (which in total are only of size $O(m)$), along with the bottom ℓ rows. If $|C^2| = O(m \log m)$, then $A' = \{A_1'\}$ meets all of the conditions of the theorem. Otherwise we can repeat our argument.

Let R_2 denote the set of rows below the first k_1 rows such that if $r \in R_2$ then r has fewer than $3t-2$ 1's when restricted to C^2, and let C_{R_2} be the set of columns where one of these rows has a 1 in C^2. Let A_2 be A_1 restricted to C^2 after ignoring the rows of R_2 and let k_2 be the largest integer such that $t \cdot I_{k_2} \prec A_2$. Note that we can assume $k_2 \ge 3$.

Lemma 4.8. $k_2 \le \frac{1}{2}k_1$.

Proof. Note that any row r that is part of this $t \cdot I_{k_2}$ must appear above the bottom ℓ rows (as restricted to C^2 the bottom ℓ rows either have fewer than t 1's or they are identically 1). Thus restricted to any C_i^1, r is either identically 0, identically 1 or sparse. We will say that a row r is "mostly 1" restricted to C_i^1 if r is identically 1 or sparse restricted to C_i^1 (i.e. r has fewer than t 0's restricted to these columns). Rearrange rows so that this $t \cdot I_{k_2}$ appears in the first k_2 rows.

Note that because $k_2 \geq 3$, no column can have two 1's in the first k_2 rows. As $|C_i^1| \geq 3t - 2 \geq 2t - 1$ for all i, any two rows that are mostly 1 restricted to any C_i^1 must contain a column with 1's in both of these rows. Hence restricted to any C_i^1 and the first k_2 rows, there can be at most one mostly 1 row.

If row $1 \leq j \leq k_2$ is not mostly 1 when restricted to any C_i^1, then we could use row j to create a $t \cdot I_{k_1+1} \prec A_1$ by swapping it with our original $k_1 + 1$th row, contradicting the definition of k_1. If there is precisely one i such that j restricted to C_i^1 is mostly 1, then swapping row j with the original ith row gives a $t \cdot I_{k_1}$ that would have given us a smaller value for $|C^2|$ (as at least $3t - 2$ 1's get added from C^2 and at most $t-1$ 1's are replaced by 0's of the mostly 1 row), which contradicts our choice of $t \cdot I_{k_1} \prec A_1$. Hence every row $1 \leq j \leq k_2$ must be mostly 1 restricted to at least two different C_i^1, but as each C_i^1 can only contribute at most one mostly 1 row we must have $k_2 \leq \frac{1}{2}k_1$. □

We then perform identical arguments for the corresponding C_i^2 columns as we did with the C_i^1 columns to get an A_2'. If C^3 is defined analogous to C^2 and if $C^3 = O(m \log m)$, then we can take $A' = \{A_1', A_2'\}$ which satisfies all the conditions of the theorems. If not, we repeat the same argument. But by Lemma 4.8 this process can continue at most $\log m$ times, and when the process terminates A' excludes only $O(m \log m)$ columns of A_0 (as it ignores $O(m)$ columns at each of the potentially $\log m$ steps), so it meets all of the criteria of the theorem. □

Theorem 4.3 allows us to reduce computing upper bounds of matrices in $\text{Avoid}(m, \mathcal{F})$ where $Q_3(t) \in \mathcal{F}$ to computing upper bounds of matrices that are of the same form as the A_j' matrices.

Corollary 4.9. *For $\mathcal{F}$ with $Q_3(t) \in \mathcal{F}$, let $\tilde{A}$ be the largest matrix such that $\tilde{A} \in \text{Avoid}(m, \mathcal{F})$ and such that it meets all the requirements of the A_j' matrices in the statement of Theorem 4.3. Then* $\text{forb}(m, \mathcal{F}) = O(\max\left\{|\tilde{A}|, m\right\} \log m)$.

Proof. The statement certainly holds if $\text{forb}(m, \mathcal{F}) = O(m \log m)$. Assume $\text{forb}(m, \mathcal{F}) = \omega(m \log m)$. Then if A is a maximum sized matrix in $\text{Avoid}(m, \mathcal{F})$ we can apply Theorem 4.3 to get a set of configurations $A' = \left\{A_j'\right\}$ with $|A_j'| \leq |\tilde{A}|$ for all j (as necessarily $A_j' \in \text{Avoid}(m, \mathcal{F})$ since $A_j' \prec A \in \text{Avoid}(m, \mathcal{F})$), and we have $|A| = O(\sum |A_j'|)$ or $|A| = O(|\tilde{A}| \log m)$. □

We suspect that the statement of Corollary 4.9 can be strengthened to $O(\max\left\{|\tilde{A}|, m\right\})$, but as stated the Corollary can still be used to prove near optimal results. It is possible to get tighter upper bounds for certain configurations by using some of the additional structure provided by Theorem 4.3.

Theorem 4.10. *If $s \leq r$ then* $\text{forb}(m, Q_3(t), I_r \times I_s^c) = O(m^{2-1/s})$.

Proof. We first prove this for the case $t = 2$. Let $A \in \text{Avoid}(m, Q_3(2), I_r \times I_s^c)$ with $|A| = \omega(m \log m)$ and let A' be the corresponding set obtained from Theorem 4.3. We focus our attention on bounding $|A_1'|$. Note that restricted to C_i^1, there must exist $|C_i^1|$ rows that are distinct rows of $I^c_{|C_i^1|}$ (one to identify each column of

C_i^1). Denote a set of such rows by R_i. If there exists a set of integers $\{i_1, \ldots, i_r\}$ such that $|R_{i_1} \cap \cdots \cap R_{i_r}| \geq s$, then by taking these s rows, the rows $i_1, \ldots, i_r$ and the relevant columns we can find an $I_r \times I_s^c$ in A_1' (since we have an I_s^c occurring simultaneously under r different I_{k_1} columns). How large can $|A_1'| = \sum |C_i^1|$ be given this restriction?

We rephrase this problem in terms of graph theory. We form a bipartite graph $G(C, R)$ where $v_i \in C$ for $1 \leq i \leq k_1$ corresponding to the C_i^1 columns, and $r \in R$ corresponding to each row below the first k_1 rows. G will contain the edge $v_i r$ iff $r \in R_i$. Our restriction of no set $\{i_1, \ldots, i_r\}$ such that $|R_{i_1} \cap \cdots \cap R_{i_r}| \geq s$ means that G does not contain a $K_{r,s}$, the complete bipartite graph with vertex sets of size r and s, with the r vertices coming from C and the s vertices coming from R. Using standard arguments from extremal graph theory, this graph can have at most $c|R||C|^{1-1/s} + d|C| \leq cmk_1^{1/s} + dk_1$ edges for some constants c and d. Hence in total we have that

$$\sum |A_i'| \leq \sum (cmk_i^{1-1/s} + dk_i) \leq cmk_1^{1-1/s} \sum \left(\frac{1}{2}\right)^{i(1-1/s)} + dk_1 \sum \left(\frac{1}{2}\right)^i = O(m^{2-1/s}),$$

and thus this is an asymptotic upper bound for $|A| = \Theta(\sum |A_i'|)$.

We wish to generalize this argument for arbitrary t. The key idea is that for each set C_i^j we must find a set of rows R_i^j with $|R_i^j| = \Theta_t(|C_i^j|)$ and such that R_i^j contains an $I^c_{|R_i^j|}$. Once we have this, we can perform the same graph argument on these R_i^j rows as we did for the R_i rows above and get the same asymptotic results. The following lemma accomplishes this goal by taking $B = C_i^j$ after ignoring rows that are identically 0.

□

Lemma 4.11. *Given an integer t, let B be a matrix consisting of rows with fewer than t 0's such that every column of B has a 0 in some row. Then there exists a set of rows R of B such that:*

(1) R contains an $I^c_{|R|}$.
(2) $|R| \geq 2^{2-t}|B|$.

Proof. The $t = 2$ case is obvious (for every column take a row that has a 0 in the column), so inductively assume the statement holds up to $t - 1$. We wish to partition the columns of B into two sets, B_1 and B_2. Remove the leftmost column c of B and add it to B_1, and remove all columns c' of B where there exists a row r such that r has a 0 in both column c and column c' and add these columns to B_2. Repeat this process until every column of B is in one of these sets, and note that $B_i \geq \frac{1}{2}|B|$ for some i. Note that as every column of B was identified, every column of B_1 and B_2 is also identified.

If $B_1 \geq \frac{1}{2}|B|$, then note that no row r has more than one 0 in B_1 (if r had 0's in $c, c' \in B_1$ with c to the left of c', then c' should have been added to B_2), so by the $t = 2$ case we can find a set R with $|R| = |B_1| \geq \frac{1}{2}|B|$ that contains an $I^c_{|R|}$.

If $|B_2| \geq \frac{1}{2}|B|$, then note that B_2's rows all have at most $t - 2$ 0's (as every row with a 0 in some c' originally had a 0 in the corresponding c column from B_1), so by the inductive hypothesis we can find a set R with $|R| \geq 2^{2-(t-1)}|B_2| \geq 2^{2-t}|B|$ that contains an $I^c_{|R|}$. □

We can use the graph idea from the proof of Theorem 4.10 to achieve lower bounds as well.

Theorem 4.12. $\text{forb}(m, Q_3(t), I_r \times I_s^c) = \Omega(ex(m, K_{r,s}))$.

Proof. We define a generalized product operation for matrices. Let A and B be simple matrices with m_1 and m_2 rows respectively and $G = G(C_A, C_B)$ a bipartite graph with the vertex set C_A corresponding to the set of columns of A and C_B to the set of columns of B. We define $A \times_G B$ to be the simple matrix on $m_1 + m_2$ rows such that it contains the column defined by placing the column $a \in C_A$ on the column $b \in C_B$ iff $ab \in E(G)$. Thus $|A \times_G B| = |E(G)|$.

Let $G(V, W)$ be a bipartite graph on m vertices such that G avoids $K_{r,s}$ and such that G has the maximum number of edges. Note that using the probabilistic method it is easy to show that $|E(G)| \geq \frac{1}{2}ex(m, K_{r,s})$. We claim that $A = I_{|V|} \times_G I^c_{|W|} \in \text{Avoid}(m, Q_3(t), I_r \times I_s^c)$, and hence $\text{forb}(m, Q_3(t), I_r \times I_s^c) \geq \frac{1}{2}ex(m, K_{r,s})$. We certainly have $Q_3(t) \nprec A$ as A is a sub-matrix of $I_a \times I_a^c$ for $a = \max\{|V|, |W|\}$, which avoids $Q_3(t)$. Note that if $I_r \times I_s^c \prec A$ Then we must have all of the I_r rows coming entirely from either the $I_{|V|}$ rows of A or the $I^c_{|W|}$ rows and the I_s^c rows coming entirely from the other. Indeed, no two rows of the $I_{|V|}$ block of A contains a column of two 1's, but every row of I_r in $I_r \times I_s^c$ together with a row of I_s^c contains a column of two 1's, so the $I_{|V|}$ rows can contribute to at most one of these blocks. Further note that if $s \geq 3$ then the I_s^c must come from the $I^c_{|W|}$ block (as it needs a column with two 1's), and similarly if $r \geq 3$ then I_r must come from the $I_{|V|}$ block (and hence again the I_s^c must come from the $I^c_{|W|}$ block).

Now consider $B = I_{|V|} \times_G I_{|W|}$. If $I_r \times I_s^c \prec A$ then we certainly have $I_r \times I_s \prec B$ (if s or r were at least 3 then the I_s^c must have been in $I^c_{|W|}$ and then complimented to become an I_s, and if $s = r = 2$ complimenting either block would still leave you with an $I_2 \times I_2$). But $I_{|V|} \times_G I_{|W|}$ is the incidence matrix of G, a graph that avoids $K_{r,s}$, and hence it must avoid $I_r \times I_s$, the incidence matrix of $K_{r,s}$. Thus we could not have had $I_r \times I_s^c \prec A$.

□

It is known that $ex(m, K_{r,s}) = \Theta(m^{2-1/s})$ for $(s-1)! \leq r$, so for these values of s and r our bounds from Theorems 4.10 and 4.12 are sharp. In particular, because $F_{11} = I_2 \times I_2 = I_2 \times I_2^c$, we have the following result.

Corollary 4.13. $\text{forb}(m, Q_3, F_{11}) = \Theta(m^{3/2})$.

5. Avoiding $1_{k,\ell}$

In this section we study the identically 1 matrices $1_{k,\ell}$. We first note an immediate consequence of Theorem 1.2.

Corollary 5.1. $\text{forb}(m, 1_{k,\ell}, F) = \Theta(1)$ *for* $F = I_3,\ F_{10},$ *or* $0_{k,\ell}$.

Proof. Note that $1_{k,\ell} \prec T_{k+\ell}, I^c_{k+\ell}$ and that $I_3, F_{10} \prec I_4$ and $0_{k,\ell} \prec I_{k+\ell}$. We thus have an upper bound of $BB(k+\ell)$ by Theorem 1.2. □

We next consider a slight generalization of a result from [7].

Theorem 5.2. *Let F be the incidence matrix of a $(k-1)$-uniform hypergraph $\mathcal{H}$. Then*

$$\text{forb}(m, 1_{k,1}, F) = \binom{m}{0} + \binom{m}{1} + \cdots + \binom{m}{k-2} + ex^{(k-1)}(m, \mathcal{H})$$

Proof. As a lower bound one can take all columns with fewer than $k-1$ 1's, along with the incidence matrix of a maximum $(k-1)$-uniform $\mathcal{H}$ avoiding hypergraph. For an upper bound, note that one can have at most $\binom{m}{0}+\cdots+\binom{m}{k-2}$ columns with fewer than $k-1$ 1's, and the columns with weight $k-1$ define the incidence matrix of a $(k-1)$-uniform hypergraph that avoids $\mathcal{H}$, and hence can be no larger than $ex^{(k-1)}(m,H)$. □

Corollary 5.3.

$$\text{forb}(m,1_{k,1},I_{s_1}\times\cdots I_{s_{k-1}})=\binom{m}{0}+\cdots+\binom{m}{k-2}+ex(m,K^{(k-1)}(s_1,\ldots,s_{k-1})).$$

In particular, $\text{forb}(m,1_{3,1},F_{11})=1+m+ex(m,K_{2,2})=\Theta(m^{3/2})$.

We can get similar results when considering configurations of the form $1_{k,2}$.

Theorem 5.4. *Let F be the incidence matrix of a k-uniform complete r-partite hypergraph $\mathcal{H}$ with $r\geq k$. Then*

$$\text{forb}(m,1_{k,2},F)=\binom{m}{0}+\binom{m}{1}+\cdots+\binom{m}{k-1}+ex^{(k)}(m,\mathcal{H})$$

Proof. For a lower bound, again take all columns with fewer than k 1's along with the incidence matrix of a maximum $\mathcal{H}$ avoiding k-uniform hypergraph. Let A be a maximum matrix of $\text{Avoid}(m,1_{k,2},F)$ and let A' be a matrix obtained from A by taking every column with more than k 1's and removing 1's until these columns have k 1's. We claim that $A'\in\text{Avoid}(m,1_{k,2},F)$. Clearly $1_{k,2}\not\prec A'$ (if $1_{k,2}\not\prec A$ then removing 1's from A can't induce this configuration) and A' is simple (the columns with fewer than k 1's were already distinct, and if any columns with k 1's were identical we would have a $1_{k,2}$), so all that remains is to show that $F\not\prec A'$.

To see this, we claim that if F' is the matrix obtained by changing any 0 of F to a 1 then F' contains a $1_{k,2}$. This claim is equivalent to saying that if one extends any $e\in E(\mathcal{H})$ to $e'=e\cup\{v\}$ for some $v\in V(\mathcal{H})$, $v\notin e$, then there exists an $f\in E(\mathcal{H})$ such that $|e'\cap f|=k$. If e contains no vertices that are in the same partition class as v, then if f is any k-subset of e' that includes v then $f\in E(\mathcal{H})$ and $|e'\cap f|=k$. If e contains a vertex v' that belongs to the same partition class as v, then $f=e'\setminus\{v'\}\in E(\mathcal{H})$ with $|e'\cap f|=k$, and thus we've proven the claim. This means that A can not contain any configuration that is obtained by taking 0's of F and changing them to 1's (since A avoids $1_{k,2}$), and hence the procedure of deleting 1's from A can not induce an F if $F\not\prec A$, so we have $F\not\prec A'$.

Thus for an upper bound of $\text{forb}(m,1_{k,2},F)$, one only needs to consider matrices where each column has at most k 1's, and this clearly gives the above upper bound. □

Corollary 5.5.

$$\text{forb}(m,1_{k,2},I_{s_1}\times\cdots I_{s_k})=\binom{m}{0}+\cdots+\binom{m}{k-1}+ex(m,K^{(k)}(s_1,\ldots,s_k)).$$

In particular, $\text{forb}(m,1_{2,2},F_{11})=1+m+ex(m,K_{2,2})=\Theta(m^{3/2})$.

We note that in general $\text{forb}(m,1_{k+1,1},F)\neq\text{forb}(m,1_{k,2},F)$ when F is the incidence matrix of a k-uniform hypergraph. That is, the statement of Theorem 5.4 can not be strengthened to include all hypergraphs as in Theorem 5.2. For example, Q_9 is the incidence matrix of two disjoint edges. It isn't difficult to see that the extremal number for this graph is $m-1$, and hence $\text{forb}(m, 1_{3,1}, Q_9) = 2m$. However, the following matrix A satisfies $|A| = 2m+1$ and $A \in \text{Avoid}(1, 1_{2,2}, Q_9)$:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0\cdots 0 & 1 & 1\cdots 1 & 0 \\ 0 & 0 & 1 & 0\cdots 0 & 1 & 0\cdots 0 & 1 \\ 0 & 0 & 0 & 1\cdots 0 & 0 & 1\cdots 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0\cdots 1 & 0 & 0\cdots 1 & 1 \end{bmatrix}$$

It should also be noted that the statement of Theorem 5.4 is not as strong as possible. For example, the theorem statement and general proof also applies to the configuration F stated below, despite it not being the incidence matrix of a complete r-partite 3-uniform hypergraph. It would be interesting to know of a complete characterization of k-uniform hypergraphs that satisfy Theorem 5.4.

$$F = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Unfortunately for $1_{k,\ell}$ with $\ell > 2$, this "downgrading" technique no longer works. We are, however, able to obtain some partial results.

Theorem 5.6. *For $\ell > 2$,*

$$\text{forb}(m, 1_{k,\ell}, I_{s_1} \times \cdots \times I_{s_k}) = \Omega(ex^{(k)}(m, K(s_1, \ldots, s_k)))$$

$$\text{forb}(m, 1_{k,\ell}, I_{s_1} \times \cdots \times I_{s_k}) = O(ex^{(k)}(m, K(s_1 + c_1, \ldots, s_k + c_k))),$$

where $c_i = (\ell - 1) \max_{j \neq i} \left\{ \frac{s_j - 1}{2} \right\} \prod_{j \neq i} s_j$.

We believe that this can be improved to $\text{forb}(m, 1_{k,\ell}, I_{s_1} \times \cdots \times I_{s_k}) = \Theta(ex^{(k)}(m, K(s_1, \ldots, s_k)))$, though we are unable to do so here. Nevertheless, $ex^{(k)}(m, K(s_1+c_1, \ldots, s_k+c_k)) = o(m^k)$, so this bound is non-trivial.

Proof. The lower bound is simply the incidence matrix of the extremal hypergraph. We first prove the upper bound for $k = 2$ to demonstrate the general idea of the proof. Let A be a maximum matrix in $\text{Avoid}(m, 1_{2,\ell}, I_r \times I_s)$ that has no columns with fewer than two 1's (and hence the forb function will be at most $O(m)$ larger than $|A|$). Let C_i denote the set of columns of A whose first 1 is in row i. Note that any row $j \neq i$ restricted to C_i has at most $\ell - 1$ 1's (otherwise the row together with the ith would induce a $1_{2,\ell}$), and further note that each column of C_i has a 1 in some row other than the ith (since every column has at least two 1's), i.e. every column of C_i is identified by a 1. We can thus use Lemma 4.11 (after switching 0's and 1's in the lemma statement) to find a set of rows R_i such that restricted to C_i these rows contain a $I_{|R_i|}$ and such that $|R_i| \geq 2^{2-\ell}|C_i|$. We then define a bipartite graph with one vertex set corresponding to the C_i column sets and the other vertex set corresponding to the rows of A, and we draw an edge between C_i and r if $r \in R_i$. We would like to say that if this graph contains a $K_{r,s}$ (say the r vertices coming from the C_i vertex set and the s vertices coming from the R_i vertex set, which is a non-trivial assumption we will deal with later), then A contains an

$I_r \times I_s$. Unfortunately, this is not true. For example, if

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

then A does not contain a $I_2 \times I_2$, despite the corresponding graph being $K_{2,2}$. The problem is that if we want to use columns from C_i and $C_{i'}$ with $i < i'$, it's possible that there are 1's in the i'th row of C_i, and if these 1 columns correspond with the I_s under C_i then we can't actually use these columns. Fortunately, each row below the ith row of C_i contains fewer than ℓ 1's, so this problem can't happen too many times. We claim that if instead of having an I_s simultaneously under r different C_i we had an I_{s+c_2}, where $c_2 = (\ell-1)\frac{r(r-1)}{2}$, simultaneously under r different C_i, then we could find an $I_r \times I_s$.

Assume that we have this situation with the i's of our C_i's belonging to the set $\{i_1, \ldots, i_r\}_<$, and let R_0' denotes the set of rows that contain the simultaneous I_{s+c_2} under these C_i, noting that $|R_0'| = s + (\ell-1)\frac{r(r-1)}{2}$. For $r \in R_0'$, we will say that its corresponding column restricted to C_{i_j} is the column where r contains the 1 it contributes to the $I_{|R_0'|}$ in C_{i_j}. Note that restricted to the $r-1$ rows $\{i_2, \ldots, i_r\}$, C_{i_1} contains at most $(\ell-1)(r-1)$ 1's (as each row has at most $\ell-1$ 1's). Thus if B_1 is the set of columns of C_{i_1} with 1's in these rows we have $|B_1| \le (\ell-1)(r-1)$. Define $R_1' \subseteq R_0'$ to be the set of rows that have corresponding columns in C_{i_1} that are not in B_1, and hence $|R_1'| \ge |R_0'| - (\ell-1)(r-1) = s + (\ell-1)\frac{(r-1)(r-2)}{2}$. Note that restricted to the corresponding columns of R_1' and the rows $\{i_2, \ldots, i_r\}$, C_{i_1} is identically 0. We can similarly define the subset $R_2' \subseteq R_1'$ consisting of the rows whose corresponding columns in C_{i_2} are 0 in the rows $\{i_3, \ldots, i_r\}$ (row i_1 is automatically identically 0 restricted to C_{i_2} since $i_1 < i_2$) with $|R_2'| \ge |R_1'| - (\ell-1)(r-2) \ge s + (\ell-1)\frac{(r-2)(r-3)}{2}$. We repeat this process until we reach the set R_r' which satisfies $|R_r'| \ge s$ and under each C_{i_j}, the corresponding columns of R_r' are identically 0 in the other $i_{j'}$ rows. This gives an $I_r \times I_s$.

However, to guarantee an $I_r \times I_s$ in A it is insufficient to simply guarantee the existence of a $K_{r,s+c_2}$ in the graph we constructed, since we could have the $s + c_2$ vertices coming from the C_i vertex set instead of the row vertex set. To remedy this, we must increase r by a suitable amount as well, namely by $c_1 = (\ell-1)\frac{s(s-1)}{2}$, as in this case a symmetric argument will guarantee our result. Thus the existence of a $K_{r+c_1,s+c_2}$ in this graph guarantees an $I_r \times I_s$, so the graph must have $O(ex(m, K_{r+c_1,s+c_2}))$ edges, and hence $|A| = O(ex(m, K_{r+c_1,s+c_2}))$ as well.

For the general problem, again consider a maximum A with every column having at least k 1's and define the set $C(i_1, \ldots, i_{k-1})$ to be the columns which have their first $k-1$ 1's in rows $i_1, \ldots, i_{k-1}$ and with $i_j > i_{j-1}$. Again we can find rows $R(i_1, \ldots, i_{k-1})$ such that the number of rows is proportional to the number of columns of $C(i_1, \ldots, i_{k-1})$, and restricted to these rows and columns there is a large identity matrix. We can then define a k-uniform k-partite hypergraph with vertex sets V_j for $1 \le j < k$ corresponding to all possible choices of i_j, and vertex set V_k corresponding to all rows of A. We then add the hyperedge $\{i_1, \ldots, i_{k-1}, r\}$ to our hypergraph iff $r \in R(i_1, \ldots, i_{k-1})$. If this hypergraph contains a $K^{(k)}(s_1 + c_1, \ldots, s_k + c_k)$ where $c_i = (\ell-1)\max_{j \ne i}\left\{\frac{s_j-1}{2}\right\}\prod_{j \ne i} s_j$, then we claim that A contains an $I_{s_1} \times \cdots \times I_{s_k}$.

Assume that this hypergraph contains a $K^{(k)}(s_1+c_1,\dots,s_k+c_k)$, say on the vertex sets $V_1',\dots,V_k'$ with $V_j'\subseteq V_j$ and $|V_i'|=s_i+c_i$ (again, an assumption we'll have to address later). First note that if $i_j\in V_j'$ and $i_{j'}\in V_{j'}'$ with $j<j'$, then $i_j<i_{j'}$. Indeed, because we have a complete k-partite hypergraph, $i_j\in V_j'$ and $i_{j'}\in V_{j'}'$ means that there exists an edge containing both i_j and $i_{j'}$ from these vertex sets. If $j'<k$ then this edge corresponds to a column whose jth 1 is in row i_j and j'th 1 is in row $i_{j'}$,and if $j<j'$ this only makes sense if $i_j<i_{j'}$. If $j'=k$ then the $i_{j'}$th row must come after the rows where this column has its first $k-1$ 1's by definition, and hence again $i_j<i_{j'}$. This means that for any $C(i_1,\dots,i_{k-1})$, $i\in V_j$ with $i\neq i_j$ and $j<k-1$, the ith row of $C(i_1,\dots,i_{k-1})$ is identically 0 (since its $(j+1)$th row with a 1 in it comes from row $i_{j+1}>i$ and its $(j-1)$th comes from $i_{j-1}<i$ if $j\neq 1$), and hence when choosing corresponding rows from V_k' the only potential pitfall will be the rows from V_{k-1}' (as it is possible for $C(i_1,\dots,i_{k-1})$ to have 1's in row $i\neq i_{k-1}$ even if $i\in V_{k-1}'$).

For $j<k$ let $V_j''\subseteq V_j'$ be any subset with $|V_j''|=s_j$ and let R_0' be the set of rows corresponding to the $I_{s_k+c_k}$ simultaneously under all of the $C(i_1,\dots,i_{k-1})$ columns with $i_j\in V_j''$, and we emphasize that our observations in the preceding paragraph shows us that the rows of R_0' lie entirely below the rows of every V_j'' for $1\le j<k-1$. Let $i_1,\dots,i_{k-2}$ be any fixed elements from the V_j'''s. Restricted to the columns $C(i_1,\dots,i_{k-1})$, where i_{k-1} varies amongst all V_{k-1}'', we perform the same procedure that we used for the $k=2$ case to obtain a set of rows R_1', after removing at most $(\ell-1)\frac{s_{k-1}(s_{k-1}-1)}{2}$ rows from R_0', such that that for any $i_{k-1}\in V_{k-1}''$ and any corresponding column of R_1' restricted to the rows $V_{k-1}''\setminus\{i_{k-1}\}$, $C(i_1,\dots,i_{k-1})$ is identically 0. We then repeat this process for all possible sequences of $i_1,\dots,i_{k-2}$, in total removing at most $\frac{s_{k-1}(s_{k-1}-1)}{2}\prod_{j<k-1}s_j$ rows (which in the worst case scenario is $(\ell-1)\max_{j\neq k}\left\{\frac{s_j-1}{2}\right\}\prod_{j\neq k}s_j$). In the end we are left with a set $R'\subseteq R_0'$ with $|R'|\ge s_k$ and in the corresponding columns of any $C(i_1,\dots,i_{k-1})$ for $i_j\in V_j''$ and restricted to the rows $V_{k-1}''\setminus\{i_{k-1}\}$ the matrix is identically 0. This gives an $I_{s_1}\times\cdots\times I_{s_k}$ in A. Hence the hypergraph can have at most $ex^{(k)}(m,K^{(k)}(s_1+c_1,\dots,s_k+c_k))$ edges, which means that overall $|A|=O(ex^{(k)}(m,K^{(k)}(s_1+c_1,\dots,s_k+c_k)))$. □

Next we consider $\text{forb}(m,1_{k,1},F_{11})$ The following was proven by Gyárfás et. al. [15].

Proposition 5.7. $\text{forb}(m,1_{4,1},F_{11})=\Theta(m^{3/2})$.

Proposition 5.7 is a corollary of the following theorem that was first proven by Füredi and Sali [16]

Theorem 5.8. $r\ge s\ge k-2\ge 1$ *be fixed integers. Then* $\text{forb}(m,1_{k,1},I_r\times I_s)=O(m^{k-1-\frac{1}{s}\binom{k-1}{2}})$. *Furthermore, if* $r\ge(s-1)!+1$ *and* $s\ge 2k-4$*, then* $\text{forb}(m,1_{k,1},I_r\times I_s)=\Theta(m^{k-1-\frac{1}{s}\binom{k-1}{2}})$

For the sake of completeness we give a simpler proof extending ideas of [15] We need the following theorem of Alon and Shikhelman. Let $ex(m,G,H)$ mean the largest possible number of subgraphs isomorphic to G in an m-vertex graph that does not have H as subgraph. Alon and Shikhelman prove

Theorem 5.9 (Alon and Shikhelman)**.** *Let $r \geq s \geq k-1$ be fixed integers. Then $ex(m, K_k, K_{r,s}) = O(m^{k-\frac{1}{s}\binom{k}{2}})$, furthermore, if $r \geq (s-1)!+1$ and $s \geq 2k-2$, then $ex(m, K_k, K_{r,s}) = \Theta(m^{k-\frac{1}{s}\binom{k}{2}})$.*

Simpler Proof of Theorem 5.8. Let $A \in \text{Avoid}(m, 1_{k+1,1}, I_r \times I_s)$. We can inductively conclude that $\text{forb}(m, 1_{k+1,1}, I_r \times I_s)^{<k} = O(m^{k-1-\frac{1}{s}\binom{k-1}{2}})$, base case being $k=3$. let A' be obtained by deleting columns of sum less than k from A. Consider columns of A' as characteristic vectors of a k-uniform hypergraph $\mathcal{F}$. Let $\mathcal{F}_1'$ be a largest size k-partite subhypergraph of $\mathcal{F}$, with partite classes $V_1, V_2, \dots, V_k$. It is well know that $|\mathcal{F}| \leq c_k|\mathcal{F}_1'|$ for some constant c_k. Let $\mathcal{H}_i$ be the $(k-1)$-partite graph induced by $\mathcal{F}_1'$ after ignoring V_i. Observe that no $\mathcal{H}_i$ contains $K_{r,s}$ as a trace. Call a hyperedge $F \in \mathcal{F}'_1$ 1-thick if restricted to each $\mathcal{H}_i$, F is contained in at least $r+s-2$ other hyperedges of $\mathcal{F}'_1$, and call F 0-thick otherwise. There are at most $(r+s-2)|E(\mathcal{H}_i)|$ 0-thick edges. Recursively define $\mathcal{F}_i'$ to consist of all $F \in \mathcal{F}_{i-1}'$ that are $i-1$ thick, and call $F \in \mathcal{F}_i'$ i-thick if restricted to each $\mathcal{H}_i$ it is contained in at least $r+s-1$ hyperedges of $\mathcal{F}_i'$. By the same reasoning as before, $|\left\{F | F \in \mathcal{F}_{i-1}', F \notin \mathcal{F}_i'\right\}| \leq (r+s-2)|E(\mathcal{H}_i)|$, and thus the number of $F \in \mathcal{F}_1'$ that are not k-thick is at most $k(r+s-2)|E(\mathcal{H}_i)| = O(m^{k-1-\frac{1}{s}\binom{k-1}{2}})$ by the inductive hypothesis. On the other hand, the 2-shadow of $\mathcal{F}_k'$ can not contain an $K_{r,s}$.

Assume in contrary that this is the case and consider an edge $\{x_1, x_2\}$ used in this $K_{r,s}$ and let F_0 be a k-thick edge with $\{x_1, x_2\} \in F_0$. If F_0 contains no vertex in $(V(K_{r,s}) \setminus \{x_1, x_2\}) \cap V_1$, then define $F_1 = F_0$. Otherwise, by definition of F_0 being a k-thick edge there exists $r+s-1$ hyperedges that are $(k-1)$-thick and that differ with F_0 only in the vertex set V_1. By the pigeonhole principle, one of these hyperedges, call it F_1, does not contain any vertex of $(V(K_{r,s}) \setminus \{x_1, x_2\}) \cap V_1$ and still has $\{x_1, x_2\} \in F_1$. Continue this way, defining F_i to be a $(k-i)$-thick hyperedge that contains $\{x_1, x_2\}$ and no vertices of $(V(K_{r,s}) \setminus \{x_1, x_2\}) \cap \bigcup_{j \leq i} V_j$, and we can do this at each step by the way we defined $(k-i)$-thickness. In the end we obtain a hyperedge F_k that contains $\{x_1, x_2\}$ and no other vertices of the $K_{r,s}$. We can repeat this process for each edge of the $K_{r,s}$, and thus these hyperedges contain $I_r \times I_s$ as a trace. Thus, we inferred that the 2-shadow does not have $K_{r,s}$ as a subgraph. Apply Theorem 5.9 to the graph determined by the 2-shadow of $\mathcal{F}_k'$ and obtain that the number of K_k subgraphs is at most $O(m^{k-\frac{1}{s}\binom{k}{2}})$, which clearly is an upper bound for $|\mathcal{F}_k'|$.

Summarising,

$$|A| = |A \setminus A'| + |A'| \leq |A \setminus A'| + \frac{1}{c_k}(k(r+s-1)|E(\mathcal{H}_i)| + |\mathcal{F}_k'|) = O(m^{k-\frac{1}{s}\binom{k}{2}}).$$

To prove the lower bound take a graph G that gives the lower bound in Alon-Shikhelman' Theorem and let $\mathcal{F}$ consists of those k-subsets of the vertices that induce a complete graph. Since G does not have $K_{r,s}$ subgraph, $\mathcal{F}$ does not have $K_{r,s}$ as *trace*, so if A is the vertex-edge incidence matrix of $\mathcal{F}$, then $A \in \text{Avoid}(m, 1_{k+1,1}, I_r \times I_s)$. □

Note that the upper bound in Proposition 5.7 is obtained by putting $r = s = k-1 = 2$. The lower bound in Theorem 5.8 does not give the lower bound of Proposition 5.7 directly, however the vertex-edge incidence matrix of a maximal C_4-free grah works.

Remark 5.10. Despite the largest product avoiding 1_4 and $I_r \times I_s$ being a 1-fold product, Theorem 5.8 shows that one can make $\text{forb}(m, 1_4, I_r \times I_s) = \Theta(m^{3-\epsilon})$. Thus the best we could hope for as an extension of Conjecture 1.1 for general forbidden families is $\text{forb}(m, F, G) = o(m^p)$ if $\text{forb}(m, F) = \Theta(m^p)$ and there exists no p-fold product avoiding both F and G. However, we do not dare to formulate this as a conjecture.

The following extension of Proposition 5.7 was proven in [16].

Proposition 5.11. *Let $k \geq 3$ be a positive integer. Then* $\text{forb}(m, 1_{k,1}, F_{11}) = \Theta(m^{3/2})$.

An alternate proof of this Proposition could be given using similar ideas as in the simpler proof of Theorem 5.8.

6. Avoiding F_9

$$F_9 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem 6.1. $\text{forb}(m, Q_3(t), F_9) = \Theta(m)$.

Proof. Note that I_m gives the lower bound. For the upper bound, we first take a look at what our preliminary data tells us. We have that $F_9 \prec I_3 \times I_2^c$, so by Theorem 4.10 we know that $\text{forb}(m, Q_3(t), F_9) = O(m^{3/2})$. It also isn't too hard to show (using methods similar to what we'll use below) that $|\tilde{A}| = O(m)$ if $\tilde{A} \in \text{Avoid}(m, Q_3(t), F_9)$ meets all the requirements of the A'_j matrices in the statement of Theorem 4.3, so we have $\text{forb}(m, Q_3(t), F_9) = O(m \log m)$ by Corollary 4.9, and this suggests that $\text{forb}(m, Q_3(t), F_9) = O(m)$. Unfortunately, this is as far as we can get using the results of Theorem 4.3. However, by following the same basic argument of the proof of the theorem, and by using the extra information that we must also avoid F_9, we will be able to show the $O(m)$ result.

Let $A \in \text{Avoid}(m, Q_3(t), F_9)$ such that $|A|$ is maximal and assume $|A| = \omega(m)$. Let k be the largest integer such that $t \cdot I_k \prec A$ (we don't consider the R_1 rows as that technical step will not be required for this proof). Rearrange rows so that this $t \cdot I_k$ appears in the first k rows and let C_i denote the set of columns with a 1 in row i and C^2 the columns with no 1's in the first k rows (and we can assume that $k \geq 3$, thus having no $Q_3(t)$ implies that no column can have two 1's in the first k rows, so all columns belong to precisely one of these sets).

Lemma 6.2. *No row r restricted to $\bigcup C_i$ is identically 0.*

Proof. Assume there is an r such that r is identically 0 restricted to $\bigcup C_i$. Consider how many 1's r has in C^2. If r has fewer than t 1's, then by using the standard induction with row r we see that $|C_r| \leq t - 1 = O(1)$, so we could inductively conclude that $|A| = O(m)$. Otherwise there are at least t 1's, in which case one could use this row to find a $t \cdot I_{k+1}$ in A, a contradiction. □

Lemma 6.3. *If row r with $r > k$ has a 0 restricted to $\bigcup C_i$ then it has 0's in precisely one C_i.*

Proof. Assume r has a 0 in C_i and $C_{i'}$. If there is a 1 in any column of $C_{i''}$, $i'' \neq i, i'$, then by taking these columns and rows r, i, i', and i'' we get an F_9. If every $C_{i''}$ is identically 0 then by Lemma 6.2 one of C_i, $C_{i'}$ must have a 1 in some column, say $c \in C_i$. But then by taking c, the column with a 0 in $C_{i'}$, and any column in any other $C_{i''}$ along with the relevant rows gives an F_9. □

Lemma 6.4. $|C^2| = O(m)$.

Proof. Assume $|C^2| = \omega(m)$, in which case there must exist a $Q_3(t;0)$ in C^2 and it must lie below the top k rows. But as $k \geq 3$, for any two rows $r_1, r_2 \geq k$ one can find a $\begin{bmatrix}1\\1\end{bmatrix}$ in some C_i (if r_1 has 0's in C_1 and r_2 has 0's in C_2 then neither can have 0's in C_3 by Lemma 6.3). Thus whatever rows the $Q_3(t;0)$ lies in one can find a column to give a $Q_3(t)$, a contradiction. □

Lemma 6.5. $|\bigcup C_i| = O(m)$.

Proof. Let R_i denote C_i restricted to its rows that are not identically 1. Note that R_i is a simple matrix, and let r_i denote the number of rows it has. We can't have $|C_i| > c_{3,t} r_i$ (as then we could find a $Q_3(t;0)$ in R_i and take any column of $C_{i'}$, $i' \neq i$ to get a $Q_3(t)$), so we must have $|\bigcup C_i| = \sum |C_i| \leq c_{3,t} r_i \leq c_{3,t} m = O(m)$. □

Thus $|A| = |\bigcup C_i| + |C^2| = O(m)$. □

Theorem 6.6. $\mathrm{forb}(m, 1_{k,\ell}, F_9) = \Theta(m)$ *provided we don't have* $k = \ell = 1$.

Proof. Note that I_m gives the lower bound. Let A be a maximum sized matrix in $\mathrm{Avoid}(m, 1_{k,\ell}, F_9)$ and apply the standard induction on any row r to get the matrix of repeated columns C_r. If $C_r \leq BB(k+\ell+1)$ then we inductively conclude that $|A| = O(m)$. Otherwise, we must have either a $I_3, I^c_{k+\ell+1}$ or $T_{k+\ell+1}$ in C_r. As $1_{k,\ell} \prec I^c_{k+\ell+1}, T_{k+\ell+1}$, we must have $I_3 \prec C_r$ and hence $[01] \times I_3 \prec A$. But $F_9 \prec [01] \times I_3$, which contradicts $F_9 \not\prec A$. □

It is possible to get a finer value for $\mathrm{forb}(m, 1_{k,\ell}, F_9)$, and even an exact value in a few select cases when m is sufficiently large. We say that a column in A is an n-column if its column sum is n. We define $\mathrm{Avoid}(m, F)^{=n}$ to be the set of matrices A that avoid F and whose columns are all n-columns, and analogously we define $\mathrm{forb}(m, F)^{=n}$. We similarly define $\mathrm{Avoid}(m, F)^{\geq n}$ and $\mathrm{forb}(m, F)^{\geq n}$. For columns c, d we will let $c \cap d$ denote the set of rows that c and d both have 1's in, and we similarly define $c \cup d$.

Lemma 6.7. *For any fixed* $t > k$, $\mathrm{forb}(m, 1_{k,\ell}, F_9)^{=t} \leq (BB(k+2) + \ell)2^t$.

Proof. We first consider the $\ell = 2$ case (the $\ell = 1$ case is trivial). Assume the first column c of a matrix $A \in \mathrm{Avoid}(m, 1_{k,2}, F_9)^{=t}$ has all its 1's in the first t rows. For $S \subseteq [t]$ with $|S| \leq k-1$, let C_S denote the set of columns c' of A such that $c \cap c' = S$, and note that every column of A belongs to precisely one such set. But note that $|[t] \setminus S| \geq 2$, which means that for every S there exists two rows such that c has a 1 in these rows and every column of C_S has 0's. Hence, below the first t rows the columns of C_S can not induce an I_2 (as in these rows c is 0, so these together with the 2 rows mentioned above give an F_9). But C_S is a simple matrix so if $|C_S| > BB(k+2)$ it must contain a T_{k+2}, which in particular contains $1_{k,2}$. Thus $|C_S| \leq BB(k+2)$ for all S, and as there are fewer than 2^t such sets (and they partition all of A), we must have $|A| \leq BB(k+2)2^t$.

For $\ell > 2$ one can consider $S \subseteq [t]$ with $|S| \geq k$, but for such S we must have $|C_S| < \ell$ to avoid $1_{k,\ell}$, so we have the bound $|A| \leq (BB(k+2)+\ell)2^t$. □

Lemma 6.8. $\text{forb}(m, 1_{k,\ell}, F_9)^{\geq c_{k,\ell}} = c'_{k,\ell}$ *where* $c_{k,\ell} = 2^{\ell-1}(k+1)-1$ *and* $c'_{k,\ell} = O(1)$.

Proof. We have $c_{k,1} = k$, so the statement is trivially true for $\ell = 1$. Assume for the purpose of induction that this result is true up to $\ell - 1$ and consider a matrix $A \in \text{Avoid}(m, 1_{k,\ell}, F_9)^{\geq c_{k,\ell}}$ and any column d in A. Let R_0 denote the rows where d has 0's and R_1 the rows where d has 1's. We claim that restricted to R_0 there exists no I_z where $z = (\ell-1)(c'_{k,\ell-1}+1)+1$. Indeed, any two columns of such a I_z, say c_1 and c_2, induce an I_2 in R_0, and using column d as well as c_1 and c_2 would give a $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, thus if there exists two rows in R_1 where c_1 and c_2 are both 0 then one could find an F_9. As d has at least $2^{\ell-1}(k+1)-1$ 1's, we must have (restricted to R_1) $|c_1 \cup c_2| \geq 2^{\ell-1}(k+1)-2$ (otherwise there will be at least two rows of R_1 that aren't covered by c_1 and c_2), and hence one of these c_i must have at least $2^{\ell-2}(k+1)-1 = c_{k,\ell-1}$ 1's in R_1. Thus all but at most one of the I_c columns must have at least $c_{k,\ell-1}$ 1's in R_1. Let A' be A restricted to the R_1 rows and the columns of the I_c that have at least $c_{k,\ell-1}$ 1's in these rows. A' need not be simple, but each column can be repeated at most $\ell-1$ times before inducing a $1_{k,\ell}$, so there are at least $c'_{k,\ell-1}+1$ distinct columns in A'. But by the inductive hypothesis this means that there exists either an F_9 (in which case we're done) or a $1_{k,\ell-1}$ in R_1, and using column d in addition to this would give a $1_{k,\ell}$. Thus there can exist no I_c in R_0, but similarly there can't exist sufficiently large I^c's or T's (as these automatically contain $1_{k,\ell}$), so restricted to R_0 there can be at most $BB(c)$ column types.

Any column type restricted to R_0 with at least k 1's can't appear more than $\ell-1$ times (as this would give a $1_{k,\ell}$), and columns restricted to R_0 with fewer than k 1's must have at least $c_{k,\ell}-(k-1) = 2^{\ell-1}(k+1)-1-(k-1) \geq 2^{\ell-2}(k+1)-1 = c_{k,\ell-1}$ 1's in R_1 (since every column of A has at least $c_{k,\ell}$ 1's), and thus can't appear more than $c'_{k,\ell-1}$ times without inducing in R_1 either an F_9 or a $1_{k,\ell-1}$ (and hence a $1_{k,\ell}$ by using column d). Thus each of the constant number of column types appears at most a constant number of times, so we have $\text{forb}(m, 1_{k,\ell}, F_9)^{\geq c_{k,\ell}} \leq BB(c)(\ell-1+c'_{k,\ell-1}) = O(1)$. □

Lemma 6.9. *For any fixed t, if $A \in \text{Avoid}(m, 1_{k,\ell}, F_9)^{=t}$ and if c is any column of A, then there are at most $O(1)$ columns c' of A with $|c \cap c'| < t-1$.*

Proof. The statement is trivially true for $t > k$ (since there can only be at most $O(1)$ such columns by Lemma 6.7) and $t = 1$, so assume $1 < t \leq k$. Rearrange rows so that the 1's of c appear in the first t rows of A, and for any $S \subseteq [t]$ let C_S denote the columns of A with $c \cap c' = S$. If S is a set with $|S| < t-1$, then as argued in Lemma 6.7 the columns of C_S can't contain an I_2 (since there exists at least two of the first t rows with 1's in c and 0's in all of C_S) and it also can't contain a $T_{k+\ell+1}$, so we must have $|C_S| \leq BB(k+\ell+1)$, and since there are fewer than 2^t such sets of A we have $|A| \leq BB(k+\ell+1)2^t = O(1)$. □

Let $A^{\neq t}$ denote the collection of columns of a matrix A that are not t-columns.

Lemma 6.10. *There exists a constant $p \in \mathbb{N}$ such that if $A \in \mathrm{Avoid}(m, 1_{k,\ell}, F_9)$ with $|A| \geq 2pc_{k,\ell} + c'_{k,\ell}$, then there exists a unique $t \leq k$ such that $|A^{\neq t}| \leq (2p - 1)k + p$. Further, there exists $t-1$ rows where every t-column of A has $t-1$ 1's in these rows.*

Note that implicitly this statement requires that m be sufficiently large in order for $|A| \geq 2pc_{k,\ell} + c'_{k,\ell}$.

Proof. Let p be the smallest (constant) value such that it is larger than $c_{k,\ell}+1$, $c'_{k,\ell}$ and all the $O(1)$ constants obtained from Lemma 6.7 for $k < t \leq c_{k,\ell}$ and Lemma 6.9 for $t \leq k$. Let $t \leq k$ be the smallest t such that A contains at least $2p$ t-columns (and at least one such t must exist by the previous lemmas and the assumption that $|A| \geq 2pc_{k,\ell} + c'_{k,\ell}$). We claim that this is the only such t. Indeed, by Lemma 6.9 at most p of these t-columns don't intersect in the same $t-1$ rows, or in other words, at least p of these t-columns must intersect in the same $t-1$ rows, say the first $t-1$. Their last 1's must all be in separate rows, and this induces an I_p below the first $t-1$ rows. We claim that A contains no t'-column with $t < t' < p-1$. Indeed, such a t' must contain at least two 1's outside of the first $t-1$ rows (since $t' > t$), and it does not have 1's in at least two rows of the I_p (since $t' < p-1$). Take two rows where t' has 1's below the first $t-1$ rows and two rows where t' does not have 1's in rows of the I_p, as well as the t' column and the two columns of the I_p that give an I_2 from the rows chosen. The t' column gives a $\begin{bmatrix}0\\0\\1\\1\end{bmatrix}$ (the first two rows where it doesn't intersect with I_p) and the other columns give a $\begin{bmatrix}1&0\\0&1\\0&0\\0&0\end{bmatrix}$ (since all these rows are after the first $t-1$, and hence every column of the I_p has only one 1 in these columns), and this gives an F_9, so there can be no such t'-columns (the same argument shows that any t-column must have 1's in the first $t-1$ rows). As t was chosen to be the smallest column type with at least $2p$ columns, in addition to the fact that $\mathrm{forb}(m, 1_{k,2}, F_9)^{\geq p} \leq c'_{k,\ell} \leq p$, it is the only such column type with at least this many columns, and thus A can contain at most $(2p-1)t + p \leq (2p-1)k + p$ columns that are not t-columns. □

Corollary 6.11. *For m sufficiently large, $\mathrm{forb}(m, 1_{k,1}, F_9) = m + c_k$, where c_k is some constant depending only on k.*

Proof. Note that I_m gives the lower bound. For any $A \in \mathrm{Avoid}(m, 1_{k,1}, F_9)$ with $|A| \geq 2pc_{k,\ell} + c'_{k,\ell}$ and m sufficiently large, Lemma 6.10 tells us that only one column type appears more than $2p$ times, say the t-columns for some $t \leq k$. But $|A^{=t}| \leq m - t + 1$ (only this many t-columns can intersect in the same $t-1$ rows, and every t-column in A does this) and $|A^{\neq t}| \leq (2p-1)k + p$, and hence $|A| \leq m - t + 1 + (2p-1)k + p \leq m + (2p-1)k + p$, where $(2p-1)k + p$ is a constant depending only on k. □

Corollary 6.12. *For $\ell \geq 2$ and m sufficiently large,*

$$\mathrm{forb}(m, 1_{k,\ell}, F_9) = \mathrm{forb}(m, 1_{k+1,1}, F_9) + \ell - 1 = m + c_{k+1} + \ell - 1.$$

Proof. Let p be the constant defined in Lemma 6.10 and let $A \in \text{Avoid}(m, 1_{k,\ell}, F_9)$ with $|A| \geq 2pc_{k,\ell} + c'_{k,\ell}$. We claim that A contains at most $\ell - 1$ columns with at least k 1's. Indeed, consider the I_p in A and note that any column with at least k 1's must have 1's in all but at most one of the rows that contains the I_p (as otherwise one can find an F_9). As $p > k + \ell$, there can exist at most $\ell - 1$ such columns before the columns induce a $1_{k,\ell}$. Thus we can reduce sufficiently large $A \in \text{Avoid}(m, 1_{k,\ell}, F_9)$ to an $A' \in \text{Avoid}(m, 1_{k+1,1})$ after removing at most $\ell - 1$ columns, so we have $\text{forb}(m, 1_{k,\ell}, F_9) \leq \text{forb}(m, 1_{k+1,1}, F_9) + \ell - 1$.

Take any $A \in \text{forb}(m, 1_{k+1,1}, F_9)$ and let A' be A after adjoining $\ell - 1$ $(m-1)$-columns to A. A' avoids F_9 (since A avoided F_9 and no $(m-1)$-column can contain an F_9 since they don't have two 0's) and it avoids $1_{k,\ell}$ (as there are only $\ell - 1$ columns of A' with at least k 1's). Hence $A' \in \text{Avoid}(m, 1_{k,\ell}, F_9)$ so we have $\text{forb}(m, 1_{k,\ell}, F_9) \geq \text{forb}(m, 1_{k+1,1}, F_9) + \ell - 1$. □

It is somewhat surprising that, despite the extra care needed to deal with $\ell > 1$ in our lemmas, the value of ℓ only contributes linearly to $\text{forb}(m, 1_{k,\ell}, F_9)$. This will also be the case for $\text{forb}(m, 1_{k,\ell}, Q_9)$ in the next section, and this provides some evidence that the upper bound for $\text{forb}(m, 1_{k,\ell}, I_{s_1} \times \cdots I_{s_k})$ should asymptotically be the same as $\text{forb}(m, 1_{k,2}, I_{s_1} \times \cdots I_{s_k})$.

The exact value of c_k seems to be difficult to compute in general, but for specific (small) values of k it is possible to compute.

Proposition 6.13. $c_2 = 1$.

Proof. Take $[0_{m,1}|I_m]$. Clearly this avoids F_9 and this includes every column that avoids $1_{2,1}$. □

Proposition 6.14. $c_3 = 2$.

Proof. To do better than our bound of c_2 we must use 2-columns in our construction (and hence we must use $\Theta(m)$ of them all intersecting in some row, say row 1). In such a construction, there can't be more than two 1-columns (otherwise we'd have an I_2 below row 1, and then taking any 2-column that doesn't intersect with these 1-columns gives an F_9) and we can only have one 0-column. Thus we must have $\text{forb}(m, 1_{3,1}, F_9) \leq 1 + 2 + (m-1) = m + 2$, and this can be achieved by considering A with the 0-column, two 1-columns in rows 1 and 2, and all 2-columns that have 1's in row 1. □

Proposition 6.15. $c_4 = 5$.

Proof. Let A be an extremal matrix in $\text{Avoid}(m, 1_{4,1}, F_9)$ that has a large number of 3-columns that intersect in the first two rows (which again is the only chance of a higher bound than c_3) and let A' denote the matrix of 0, 1, and 2-columns in A. If A' contains an I_2 below the first two rows (say in rows 3 and 4 and columns c_1 and c_2 respectively), then c_1 and c_2 restricted to rows 1 and 2 must look like $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (they can't contain two 1's in these rows without being a 3-column, and if c_1 and c_2 both had 0's in one of these rows, say the first, then we could find an F_9 by considering rows 1, 2 and 3, columns c_1, c_2, a 3-column that has a 1 in row $i \neq 3, 4$ and row i). In this situation one can't have a third column c_3 of A' with a 1 beyond the first two rows, as either c_3 has a 1 in row 3 (in which case it can't be equal to $\begin{bmatrix}1\\0\end{bmatrix}$ in the first two rows since $c_3 \neq c_1$, and hence c_3 and c_1 contain a row of 0's in the first two rows, giving an F_9), row 4 (symmetric argument), or some row other than 3 and 4 (in which case c_3 restricted to the first two rows must be $\begin{bmatrix}1\\0\end{bmatrix}$ to not induce an F_9 with c_2 and $\begin{bmatrix}0\\1\end{bmatrix}$ to not induce an F_9 with c_1, which is impossible). The only other columns that would be allowed are the four columns with no 1's beyond the first two rows, so in this case we have $|A'| \le 6$.

The only other case to consider is when all the 1's beyond the second row lie in the same row (say the third), in which case there can be at most $\binom{3}{2}+\binom{3}{1}+\binom{3}{0}=7$ columns of A', obtained by considering all columns which have fewer than two 1's in the first three rows and no 1's outside these rows. Such an A' avoids F_9 (since F_9 requires four rows with 1's in them), so in total we have that $|A'| \le 7$ and that $|A'| = 7$ can be obtained. Thus in total we have $\mathrm{forb}(m, 1_{4,1}, F_9) \le 7 + (m-2) = m+5$, and this can be achieved by letting A have all 0, 1 and 2-columns with fewer than three 1's in the first three rows and all 3-columns that have 1's in rows 1 and 2. □

Corollary 6.16. *For sufficiently large m:*

$$\mathrm{forb}(m, 1_{3,1}, F_9) = m+2$$
$$\mathrm{forb}(m, 1_{2,2}, F_9) = m+3$$
$$\mathrm{forb}(m, 1_{4,1}, F_9) = m+5.$$

7. Avoiding Q_9

$$Q_9 = \begin{bmatrix}1 & 0\\1 & 0\\0 & 1\\0 & 1\end{bmatrix}$$

It turns out that the problem of avoiding Q_9 and $1_{k,\ell}$ has a very similar flavor to the problem of avoiding F_9 and $1_{k,\ell}$, and because of this we will once again be able to achieve exact results. We maintain all of our notation and terminology from the previous section.

The bound $\mathrm{forb}(m, Q_9) = \binom{m}{2} + 2m - 1$ was proven in [4], where the following classification of Q_9 avoiding matrices was established (following [2]). For each $2 \le t \le m-2$ we can divide the rows into three disjoint sets $A_t, B_t, C_t \subseteq \{1, 2, \dots, m\}$ so that after permuting the rows the t-columns can either be given as

$$\text{type 1:}\ \begin{matrix}A_t\{\\ B_t\{\\ C_t\{\end{matrix}\begin{bmatrix} I_{|A_t|}\\ 1_{|B_t|,|A_t|}\\ 0_{|C_t|,|A_t|}\end{bmatrix}\ \text{or type 2:}\ \begin{matrix}A_t\{\\ B_t\{\\ C_t\{\end{matrix}\begin{bmatrix} I^c_{|A_t|}\\ 1_{|B_t|,|A_t|}\\ 0_{|C_t|,|A_t|}\end{bmatrix}.$$

We will say t *is of type* i ($i = 1$ or $i = 2$) if the t-columns are of type i.

Lemma 7.1. *Let $m \ge 2k$, then* $\mathrm{forb}(m, Q_9)^{=t} = m - (t-1)$ *for* $1 < t \le k$.

Proof. The size of a type 1 matrix of column sum t is at most $m - (t-1)$, while the size of a type 2 matrix of the same column sum is bounded by $t+1$. □

Proposition 7.2. *Let $m \ge 2k$, then* $\mathrm{forb}(m, Q_9, 1_{k,1}) = 1 + (k-1)m - \binom{k-1}{2}$.

Proof. By the previous lemma, $\text{forb}(m, Q_9, 1_{k,1})$ is upper bounded by $1 + m + \sum_{t=2}^{k}(m-(t-1)) = 1+(k-1)m-\binom{k-1}{2}$, and this value can be achieved by having $m-(t-1)$ t-columns intersecting in the first $t-1$ rows, along with all columns of column sum 0 and 1. □

Corollary 7.3. *For $m \geq 8$,*

$$\text{forb}(m, Q_9, 1_{4,1}) = 3m - 2.$$

We can extend these results for $\ell > 1$.

Proposition 7.4. $\text{forb}(m, Q_9, 1_{k,2}) = \text{forb}(m, Q_9, 1_{k+1,1}) + 1$.

Proof. For the lower bound take the lower bound construction for $\text{forb}(m, Q_9, 1_{k+1,1})$ given above and add in the $(m-1)$-column with a 0 in the first row. This new column can't be used to make a Q_9 since it has too few 0's, and it doesn't intersect any other column in k rows so it can't be used to find a $1_{k,2}$. Thus this new matrix is in $\text{Avoid}(m, Q_9, 1_{k,\ell})$. For the upper bound, note that if c, d are columns with at least $k+1$ 1's then either $|c \cap d| \geq k$ (in which case we have $1_{k,2}$) or there exists two rows where c has 1's and d does not and vice versa (in which case we have Q_9), so a matrix in $\text{Avoid}(m, Q_9, 1_{k,2})$ can have at most one column that has more than k 1's. □

Analyzing the $\ell > 2$ case once again turns out to be significantly more difficult than the $\ell \leq 2$ cases, but nonetheless we are able to achieve some nearly tight bounds for this problem.

Lemma 7.5. $\text{forb}(m, Q_9, 1_{k,\ell})^{=t} \leq k + \ell$ *for $k + \ell > t > k$.*

Proof. The size of a type 1 matrix of column sum t can be at most $\ell - 1$ without inducing a $1_{k,\ell}$, and the size of a type 2 matrix of the same column sum is bounded by $t + 1 \leq k + \ell$. □

Lemma 7.6. $\text{forb}(m, Q_9, 1_{k,\ell})^{\geq k+\ell} = \ell - 1$.

Proof. Let c be a column of $A \in \text{Avoid}(m, Q_9, 1_{k,\ell})^{\geq k+\ell}$ with the fewest number of 1's (say t of them). We must have $|c \cap d| \geq t - 1$ for any other d (as if d has two 0's in rows where c has 1's, by virtue of c having the fewest number of 1's d must have at least two 1's where c has 0's, giving a Q_9), and hence for any other $\ell - 1$ columns in A there exists k rows such that c and all of these other columns have 1's in these rows (since each can have at most one 0 in the at least $k+\ell$ rows where c has 1's), so we must have $|A| \leq \ell - 1$. □

Proposition 7.7. *For $k \geq 2$, $\ell \geq 3$ and $m > (\ell+1)(k+\ell) + k$,*

$$\text{forb}(m, Q_9, 1_{k,\ell}) \geq \text{forb}(m, Q_9, 1_{k+1,1}) + 2\ell - 5$$
$$\text{forb}(m, Q_9, 1_{k,\ell}) \leq \text{forb}(m, Q_9, 1_{k+1,1}) + 3\ell - 5.$$

Proof. Take the lower bound construction for $\text{forb}(m, Q_9, 1_{k+1,1})$ and adjoin to this $\ell - 2$ columns with column sum $(k+1)$ such that k of their 1's are in the first k rows and their remaining 1's are in rows $k+1$ through $k+\ell-2$. Additionally adjoin $\ell - 3$ columns with column sum $(k+\ell-2)$ with $k+\ell-3$ of their 1's in the first $k+\ell-2$ rows excluding row k and their remaining 1's anywhere below these rows. One can't use a $(k+\ell-2)$-column to find a Q_9 (only the $(k+1)$-columns and t-columns with a 1 in row $k+1$ have 1's in a row where a $(k+\ell-2)$-column has a 0 in the first $(k+\ell-2)$ rows, but no such row exists beyond that for these columns, and for all other t-columns there exists at most one such row beyond the first $(k+\ell-2)$ and none before this) and one can't use a $(k+1)$-column either (it can't be used with a t-column for $t \le k+1$ as below the first $t-1$ rows of the t-column there aren't enough 1's), so this avoids Q_9. To find a $1_{k,\ell}$, first note that at most one t-column with $t \le k$ could be used (as there exists no k rows where two such t-columns both have 1's). If one uses more than one $(k+1)$-column to find a $1_{k,\ell}$, then one must use the first k rows (since these are the only rows that two distinct $(k+1)$-columns agree); but there are only $\ell-2$ $(k+1)$-columns and one k-column with 1's in the first k rows, and no $(k+\ell-2)$-column can be used as they each have a 0 in row k, so one can't find ℓ such columns. Thus in total one could use at most one t-column with $t \le k$, one $(k+1)$-column and all $\ell-3$ $(k+\ell-2)$-columns, but this can't be used to find a $1_{k,\ell}$ since there are at most $\ell-1$ columns.

For the upper bound, take $A \in \text{Avoid}(m, Q_9, 1_{k,\ell})$ with $|A| \ge 1+km-\binom{k}{2}$. Let p denote the number of k-columns that A has. Because $\text{forb}(m, Q_9, 1_{k,\ell})^{\ge k+1} \le \ell(k+\ell)+(\ell-1)$, the only way we can have $|A| \ge 1+km-\binom{k}{2}$ is if $p \ge m-k-\ell(k+\ell)-(\ell-1)$ by Proposition 7.2 and Lemmas 7.5 and 7.6. Now using that $m > (\ell+1)(k+\ell)+k$, this can only happen if columns of sum k are of type 1. We assume that their common 1's are in the first $k-1$ rows, which induces an I_p in the rows below the first $k-1$ rows.

No column with at least $k+1$ 1's can have two 0's in the first $k-1$ rows (as any k-column has two rows where it has 0's and this large column does not, and this large column necessarily has two rows where it has 1's and the k-column does not, since it has at least $k+1$ 1's and two of them aren't in the first $k-1$ rows). If a column with at least $k+1$ 1's has one 0 in the first $k-1$ rows and $k \ge 2$ then this column must cover the entire I_p (otherwise we could find a column that isn't covered by the large column, take these two columns, the rows where the k-column has 1's and the large column has 0's and any rows that the large column has that other doesn't to find a Q_9), but because I_p is large we can have at most $\ell-1$ columns that cover it before inducing a $1_{k,\ell}$. We ignore these covering columns for now and restrict our attention to columns with at least $k+1$ 1's and that are identically 1 in the first $k-1$ rows. Let c be such a column with the fewest number of 1's and assume it has 1's in the first $k+1$ rows. As argued in the second lemma, any other column must have $|c \cap d| \ge k$ and in particular (since all the columns we're considering are identically 1 in the first $k-1$ rows) the only 0's the other columns can have are in the kth and $k+1$st rows. There can be at most $\ell-1$ columns with a 0 in the kth row before inducing a $1_{k,\ell}$, but if there are precisely $\ell-1$ such columns then A can not contain the k-column with 1's in rows 1 through $k-1$ and row $k+1$, decreasing the maximum value p can take by 1, so "effectively" these columns can contribute at most $\ell-2$. Similar results hold for columns with a 0 in the $k+1$st row, so in total we have $|A| \le \text{forb}(m, Q_9, 1_{k+1,1})+2(\ell-2)+\ell-1 = \text{forb}(m, Q_9, 1_{k+1,1})+3\ell-5$ □

We can get a slightly larger lower bound when k is sufficiently large.

Proposition 7.8. *If $\ell = 3$ and $k \ge 3$ or if $k \ge \ell-1 \ge 3$ then*

$$\text{forb}(m, Q_9, 1_{k,\ell}) \ge \text{forb}(m, Q_9, 1_{k+1,1}) + 2\ell - 3.$$

Proof. If $k \ge \ell-1$ then take the lower bound construction for $\text{forb}(m, Q_9, 1_{k+1,1})$ and adjoin to this $\ell-2$ columns with column sum $(k+1)$ with k of their 1's in the first k rows and also adjoin $\ell - 1$ $(m-1)$-columns with their 0's in the first $\ell - 1$ rows (which by assumption is in the first k rows). None of the $(m-1)$-columns can be used to find a Q_9 (as they have too few 0's), and by the same logic as before neither can the $(k+1)$-columns. To find a $1_{k,\ell}$, again note that at most one t-column with $t \leq k$ could be used and if one uses more than one $(k+1)$-column to find a $1_{k,\ell}$, then one must use the first k rows which means no $(m-1)$-column can be used (since each has a 0 in the first k rows), so again we conclude that at most one $(k+1)$-column can be used. One can't use only $(m-1)$-columns since there are at most $\ell - 1$ of them, but if any two $(m-1)$-columns are used then one can't use two of the first k rows (since each has a different 0 in these rows), and hence one can't use any of the t-columns with $t \leq k+1$ (since outside of these rows they have at most $k-1$ 1's). Thus the only way one can find a $1_{k,\ell}$ is to use one $(m-1)$-column, one $(k+1)$-column and one k-column. If $\ell \geq 4$ then we clearly can not find a $1_{k,\ell}$, but if $\ell = 3$ and $k = 2$ one could use the 2-column with 1's in row 1 and row 3, the 3-column with 1's in rows 1 through 3, and the $(m-1)$-column with a 0 in row 2 to find a $1_{2,3}$. If $k \geq \ell = 3$ then each $(m-1)$-column and k-column only share $k-1$ rows with 1's in both columns, so in this case we avoid $1_{k,\ell}$. $\square$

8. Future Directions

A natural extension to this work would be to consider all simple minimal cubic configurations, not just those with 4 rows. [3] does not explicitly list these configurations, but it is possible to determine the complete list (provided a certain conjecture is true).

First, note that there exists no minimal cubic configuration with 7 or more rows. Indeed, each column of a 7 rowed matrix contains $1_{4,1}$ or $0_{4,1}$, meaning the configuration can't be a minimal cubic.

Conjecture 8.1. There exists no 5-rowed minimal cubic configuration.

Proposition 8.2. *Conjecture 8.1 holds provided Conjecture 8.1 of [3] is true.*

Proof. Indeed, if Conjecture 8.1 holds then we need only consider the configurations $F'_{12}, \ldots, F'_{24}$ (where F'_i in our notation corresponds to F_i of [3]). We note that $1_{4,1} \prec F'_{12}$, $0_{4,1} \prec F'_{13}$, $F_9^c \prec F'_{14}, F'_{22}$, $F_9 \prec F'_{15}, F'_{23}$, $F_{10}^c \prec F'_{16}$, $F_{10} \prec F'_{17}$, $F_{11} \prec F'_{21}, F'_{24}$, and thus none of these configurations can be minimal. $\square$

Proposition 8.3. *The configurations F_{14} and F_{15} listed below are minimal cubic configurations. Moreover, they are the only simple 6-rowed minimal cubic configurations.*

Table 4. Minimal Simple Cubic Configurations with 6 Rows

	Configuration F_i	Quadratic Const.(s)	Cubic Const.(s)	Proposition
F_{14}	$\begin{bmatrix}1&0\\1&0\\1&0\\0&1\\0&1\\0&1\end{bmatrix}$	$I\times I$ $I\times I^c$ $I\times T$ $I^c\times I^c$ $I^c\times T$	$I\times I\times T$ $I\times I^c\times T$ $I^c\times I^c\times T$	Prop. 8.4
F_{15}	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&1&1\\1&0&1\\1&1&0\end{bmatrix}$	$I\times I$ $I\times T$ $I^c\times I^c$ $I^c\times T$ $T\times T$	$I\times I\times T$ $I^c\times I^c\times T$	Prop. 8.5

Proof. Note that we need only consider configurations whose column sum's are precisely 3, as otherwise the configuration will not be minimal. It is noted in [9] that the following configurations are the only six-rowed simple matrices with at least a cubic lower bound such that removing any column would make the configuration less than cubic:

$$F_{14},\ F_{15},\ F_{16}=\begin{bmatrix}1&1&1\\1&1&1\\1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix},\ F_{16}^c,\ F_{17}=\begin{bmatrix}1&1&1\\1&1&0\\1&0&0\\0&1&0\\0&0&1\\0&0&1\end{bmatrix},\ F_{17}^c.$$

Note that $F_{10}\prec F_{16}$ and $F_9\prec F_{17}$, and consequently $F_{10}^c\prec F_{16}^c$ and $F_9^c\prec F_{17}^c$. Thus the only configurations that could be minimal cubics are F_{14} and F_{15}.

Anstee and Keevash in [6] note that F_{14} is cubic, and moreover, that it with any row removed is quadratic, so this is a minimal cubic configuration. [3] notes that the following configuration is quadratic:

$$F_7=\begin{bmatrix}1&1&0&1&1&0\\1&0&1&1&1&1\\0&1&0&1&0&1\\0&0&1&0&0&1\\0&0&0&0&1&0\end{bmatrix}$$

If F_7' consists of the 2nd, 3rd and 5th columns of F_7 then we note that F_7' is F_{15} without one of its rows (so if F_{15} is a cubic configuration it must be a minimal cubic). If we apply the standard induction for forb(m,F_{15}), we must have $F_7'\not\prec C_r$ (as otherwise $F_{15}\prec[01]\times F_7'\prec A$), and hence $|C_r|=O(m^2)$, so we conclude that $forb(m,F_{15})=O(m^3)$. □

Proposition 8.4. *$F_{14}\not\prec I\times I, I\times I^c, I\times T, I^c\times I^c, I^c\times T$ and $F_{14}\not\prec I\times I\times T, I\times I^c\times T, I^c\times I^c\times T$. Moreover, these are the only 2 and 3-fold products that avoid F_{14}.*

Proof. Note that any selection of three rows of F_{14} contains $1_{2,1}$ and $0_{2,1}$, but neither I nor I^c contains both of these configurations so any I or I^c in a product could contribute at most 2 rows to find F_{14}. Similarly, any four rows of F_{14} contains

I_2, and hence T can contribute at most 3 rows in finding F_{14} for any product it is involved in. This shows that all 2-fold products except possibly $T \times T$ avoids F_{14}, but it isn't too difficult to see that $F_{14} \prec T_4 \times T_4 \prec T \times T$.

Any 3-fold product involving only I's and I^c's will contain F_{14}, as each of these can contribute an I_2 from two of their rows and three of these put together give F_{14}. Thus the only possible 3-fold product that could avoid F_{14} are products using precisely one T and the rest I's and I^c's. And this does in fact avoid F_{14}, as the most each I and I^c can contribute is two rows that form an I_2, but this still leaves at least one I_2 to be covered by the T, which it can not do. □

Proposition 8.5. *$F_{15} \not\prec I \times I, I \times T, I^c \times I^c, I^c \times T, T \times T$ and $F_{15} \not\prec I \times I \times T, I^c \times I^c \times T$. Moreover, these are the only 2 and 3-fold products that avoid F_{15}.*

Proof. As F_{15} consists of an I_3 on top of an I_3^c, it is clear that $F_{15} \prec I \times I^c$. Note that $I_3^c \not\prec I \times I, I \times T, T \times T$, and hence F_{15} will not be contained in any of these products. Similarly $I_3 \not\prec I^c \times I^c$ implies that $F_{15} \not\prec I^c \times I^c, I^c \times T$.

To see that $F_{15} \not\prec I \times I \times T$, note that any two rows of the I_3^c of F_{15} contains $1_{2,1}$ (so I can contribute to at most one row of I_3^c) and I_2 (so T can contribute to at most one row of I_3^c). Consequently, each of the I's and the T must contribute to precisely one row of the I_3^c. But if an I contributes to the ith row of F_{15} ($i \geq 4$), then the only other row it can contribute to is the $(i-3)$rd row (as using any other row gives a $1_{2,1}$). But if T covers the ith row ($i \geq 4$), it can not also contribute to the $(i-3)$rd row, as these two rows contain an I_2. Thus no matter which rows of the I_3^c the I and T blocks cover, it will be impossible to cover all 6 rows of F_{15}. It is not difficult to show that $F_{15} \prec I \times T \times T$ by finding rows 1 and 4 in I, rows 3 and 5 in the first T and rows 2 and 6 in the second T. Similarly $F_{15} \prec T \times T \times T$ by finding rows 1 and 5 in one T, 2 and 6 in another, and 3 and 4 in the last. □

From these constructions we are able to show that $\text{forb}(m, Q, F) = \Theta(m^2)$ where Q is a minimal quadratic configuration and F is either F_{14} or F_{15} with the exception of the pairing $Q = Q_8$ and $F = F_{14}$ (as the only 2-fold product that avoids Q_8 is $T \times T$, which is the only 2-fold product that contains F_{14}). We would predict based on our previous work that $\text{forb}(m, Q_8, F_{14}) = o(m^2)$, but we are unable to show this.

Question 1. What is $\text{forb}(m, Q_8, F_{14})$?

The problem of pairing F_{14} and F_{15} with other cubics is also a difficult question. Through the constructions we listed, it is possible to show that $\text{forb}(m, F_1, F_2) = \Omega(m^2)$ for F_1 either F_{14} and F_{15} and F_2 any other simple minimal cubic configuration, and that $\text{forb}(m, F_{14}, F_{15}) = \Theta(m^3)$, as well as $\text{forb}(m, F_1, F_2) = \Theta(m^3)$ where F_1 is F_{14} or F_{15} and F_2 is F_{12} or F_{12}^c. Unfortunately, we are unable to prove any tighter bounds.

Question 2. What is $\text{forb}(m, F_1, F_2)$ in general for $F_1 = F_{14}$ or F_{15} and F_2 any simple minimal cubic configuration?

One potential route for proving these results, at least for F_{14}, would be to characterize how matrices in $A \in \text{Avoid}(m, F_{14})^{=t}$ must look like as was done for Q_9 in [4]. However, classifying t-columns of F_{14} seems to be a more difficult problem compared to Q_9.

Question 3. Is there a nice characterization of matrices $A \in \text{Avoid}(m, F_{14})^{=t}$?

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Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences

University of Miami