# A Study in the Use of Maniplatives to Teach Topics in Differential and Integral Calculus

**The Harvard community has made this article openly available. Please share how this access benefits you. Your story matters**

| | |
|---|---|
| Citation | Rivero, Maria de Lourdes. 2016. A Study in the Use of Maniplatives to Teach Topics in Differential and Integral Calculus. Master's thesis, Harvard Extension School. |
| Citable link | http://nrs.harvard.edu/urn-3:HUL.InstRepos:33797357 |
| Terms of Use | This article was downloaded from Harvard University's DASH repository, and is made available under the terms and conditions applicable to Other Posted Material, as set forth at http://nrs.harvard.edu/urn-3:HUL.InstRepos:dash.current.terms-of-use#LAA |

A Study in the Use of Maniplatives to Teach Topics in Differential and Integral Calculus

Maria de Lourdes Rivero

A Thesis in the Field of Mathematics for Teaching

for the Degree of Master of Liberal Arts in Extension Studies

Harvard University

May 2016

# Abstract

The use of mathematics manipulatives for the elementary grades is well-studied, but little research exists on their value for teaching calculus students. This project studied the role of physical manipulatives on student learning in two high school calculus classes. It explored the effect of two lessons taught with manipulatives, and compared two lessons on the same topic, one taught in the traditional way and the other incorporating the use of manipulatives. In evaluating the teaching method and process for the four lessons, quantitative measures involved statistical testing of mean pretest and posttest scores. Qualitative factors considered student feedback on a questionnaire, and the evaluation of the experience by the instructor. Overall, this research found that physical manipulatives improved student understanding and the students reported a positive experience with the visual and hands-on approach of the research study lessons. It is suggested that manipulatives be included among other good teaching practices in calculus, especially in classes taught at the regular and honors level.

# Dedication

I dedicate this work to my family; first to my husband, Juan E. Enjamio, without whose love and support I would not have been able to complete this program. And then to my children, Victoria Maria and Julian Enrique, whose presence in my life renewed a passion for the education and formation of a new generation.

# Acknowledgements

I would like to thank Alexa Kapor-Mater, my thesis director, for her guidance, suggestions and feedback, both on the writing and on the manipulatives lessons themselves. Her Montessori background and experience with hands-on teaching and learning were invaluable to my work.

I would also like to thank Dr. Andrew Engelward, my thesis advisor, professor for several of my program courses and director of the Mathematics for Teaching program. His advice, support, and patience were instrumental in my progression through the Mathematics for Teaching program, especially in the development of this project.

With sincerest appreciation, I acknowledge the kind assistance of Dr. Lourdes Rovira, personal friend, and her colleague Dr. Francisco Vital, both of the Abraham S. Fischler College of Education at Nova Southeastern University, for their guidance and critique of the literature review chapter of this thesis.

I am deeply grateful to my colleague and friend, Josefina Ochoa, for sharing her time and photographic expertise to produce many of the manipulative images documented in Chapter 3.

Finally, I acknowledge the administration and students of Riviera Preparatory School for their support and participation in this project. Especially helpful was Dr. Linda Grant, without whose support and assistance this project would not have been possible.

# Table of Contents

# List of Figures

# List of Tables

# Chapter 1  Introduction

Manipulatives are concrete objects that a student can grasp in their hands, and by being touched, and thus manipulated, can help the student understand a specific educational concept. There are many definitions of manipulatives, some quite wordy, but MIT mathematician, computer scientist, and educator Seymour Papert sums it up quite nicely and succinctly when he states that a manipulative is an "object-to-think-with" (Papert, 1993, p. 11).

Manipulatives primarily used for mathematics concepts made their appearance in the late modern period when Friedrich Froebel provided educational play materials to his kindergarten students and Maria Montessori used and later expanded on materials designed by Édouard Séguin to help young children learn the basic concepts of math (Boggan, Harper & Whitmire, 2010). They have been used in math education since, and, especially in the last twenty years, manipulatives have been discussed and promoted in many articles, research papers and other publications. A review of pertinent selections is included in this thesis (Chapter 2).

## Why Manipulatives Have Been a Common Topic in the Math Education Literature

There are views, often repeated in various communications media reports, that mathematics education in this country requires improvement. Points have been made that students are graduating from high school lacking vital skills in mathematics, that

employers, including the military, have difficulty finding workers with math skills necessary for job performance, that too few students are acquiring degrees in science, technology, engineering and math (STEM) fields, and that Program for International Student Assessment test results show American students lagging well behind students from many other countries (Council on Foreign Relations, 2012, p. 23).

The mathematics education profession, partly in response to this general theme, but also out of its own desire to improve teaching methods and advance the profession, has identified manipulatives as one of many good teaching practices recommended to improve students' learning of math as recommended in the National Council of Teachers of Mathematics (NCTM) Standards, 2000. "One practical route for bringing experience to bear on students' mathematical understanding . . . is the use of manipulatives" (Hartshorn & Boren, 1990, p. 2).

## Area of Interest

Common student difficulties in learning abstract calculus concepts are documented in the literature. For example, students have trouble recognizing differences between a function and its derivative (Nemirovsky & Rubin, 1992). Another problem lies with the concept of accumulation and the definite integral (Bhatia, Premadasa & Martin, 2014). In addition, Ferrini-Mundy and Lauten describe problems in "learning about calculus learning", where students have difficulty assimilating new uses and meanings for concepts they have previously been exposed to, such as functions, slope, tangent, and limit (1994).

This researcher is a practicing high school mathematics teacher who has developed a specific interest in investigating the use of manipulatives to help students understand abstract calculus concepts such as mentioned above. A preliminary review of the literature found that there are few publications that address the use of manipulatives in teaching calculus at the high school level. The idea for the research project, as presented in this thesis, came from a desire to further investigate the use of these promising tools in the teaching of high school calculus, a relatively unexplored area of mathematics education, and to contribute to the body of knowledge in the use of manipulatives to teach mathematics.

## What Other Researchers Have Found

In a broader review of the literature, it was found that there have been numerous studies over the years regarding the impact of manipulatives on student learning. The general consensus is that students benefit when instruction incorporates the use of manipulatives. One recent study by Swan and Marshall (2010) followed up on research done twelve years earlier by Howard, Perry and Tracey (1997), confirming that their "conclusions . . . are still relevant." Among other things, both studies decided that "manipulatives benefit the learning and teaching of mathematics." Indeed, all of the studies included in the literature reviewed for this thesis showed that the use of manipulatives was at least as effective on student learning as a traditional approach. In addition, manipulatives have benefits beyond simply helping students learn the material better. Students typically enjoy working with these items, and once they master the concept, they might have a better attitude about math, and about themselves (McClung, 1998). However, many studies emphasized that factors such as teacher training and

development, using a manipulative appropriate to the task, and teacher guidance are important to the successful use of manipulatives (Carbonneau, Marley & Selig, 2013; Morin & Samelson, 2015; Stein & Bovalino, 2001).

Manipulative use at the elementary and middle school levels has been well studied and documented. In a meta-analysis of 55 studies that compared teaching with manipulatives to teaching with abstract symbols, 93% of the studies dealt with elementary and middle school students and math topics (Carbonneau et al., 2013).

But as early as 1990, Hartshorn and Boren noted that the use of manipulatives at the secondary level was progressing slowly, and that there was very little research on their efficacy. They urged that "forthcoming research should . . . seek to study the use and effects of manipulatives at the secondary level" (Hartshorn & Boren, 1990).

Since then, there have been a number of studies of manipulative use at the secondary level, albeit less than the number of studies at the lower and middle grades. In reviewing the literature, this researcher located six publications that involved the use of manipulatives in high school, in the subjects of Algebra 1, Geometry, and Algebra 2. The populations were general education students (4) and special needs students (2): (Bruins, 2014; Curtain-Phillips, 2015; Dobbins, Gagnon & Ulrich, 2014; Howard, Perry & Lindsay, 1996; Jimenez, 2011; Satsangi & Block, 2015).

One possible reason cited for a decrease in manipulative use at the higher school levels is that older students and their teachers have a perception of manipulatives as being for young children (Moyer, Bolyard & Spikell, 2002; Swan & Marshall, 2010). Another cause described is the classroom structure in secondary schools: timed classes, students and teachers moving from room to room (Howard et al., 1997).

4

This researcher's investigation was focused not only on studies at the secondary math levels, but also on publications related to the use of manipulatives to teach calculus. While several calculus manipulatives were described in the literature, studies involving calculus students were not found.

## The Research Problem

The researcher was interested in developing manipulatives and lessons to teach topics in differential and integral calculus, with the purpose of investigating whether manipulatives can help students better understand calculus concepts and to obtain feedback from the students regarding their experience in using the manipulatives.

The participants in this study were high school students taking Honors Calculus or Advanced Placement Calculus. Two calculus concepts were taught with the use of manipulatives to two groups of students. Another calculus concept was presented in two ways, once with manipulatives, and once in the traditional lecture format, respectively to two groups of students. Data was collected with the use of pretests, posttests and questionnaires.

Findings indicated student learning gains in all lessons, with and without manipulatives, with a difference in favor of manipulatives in the comparison lessons. Additionally, student reception of the manipulative lessons was overwhelmingly positive, and the remaining responses neutral. Constructive feedback was received on two identified concerns, which should be considered in planning calculus lessons with manipulatives. One of these is time, both for the teacher in planning and executing the lesson, and for the students in working with the manipulatives. The other is using

manipulatives appropriate to the concept being taught and to the developmental level of the students.

In the next chapter, many of the publications mentioned above, and others will be presented and reviewed.

# Chapter 2  Review of the Literature

In this project, the researcher intended to use and investigate the effectiveness of physical manipulatives in the teaching of high school calculus. Before embarking on such an endeavor, it was both useful and necessary to understand the history and the research conducted to-date on the topic, and for these reasons, a review of the literature was undertaken. The literature was examined first to find research studies and articles related to the use of manipulatives in the mathematics classroom and their effectiveness with respect to student learning and achievement. Second, the researcher reviewed publications related to the use of manipulatives to teach advanced mathematics, specifically calculus.

The literature was reviewed first for a historical perspective on the development of manipulatives. Second, research and information on the use and effectiveness of manipulatives in general was reviewed. Next, the use of manipulatives at the primary and middle grade levels was studied. This was followed by consideration of virtual manipulatives. Next the use of manipulatives for students with special needs was investigated. The review then focused on manipulative use in high school mathematics classes. During this process, a list and description of published physical and virtual manipulatives for precalculus and calculus concepts was compiled.

It was quickly discovered that physical manipulatives have been used to support the teaching and learning of mathematics for more than a century. During this time, their use and effectiveness have been the subject of numerous research studies, where

educators have proposed innovative ways to incorporate manipulatives into their classrooms. More recently, advances in technology and the advent of the Internet and the World Wide Web have given rise to virtual manipulatives. This, in turn, has resulted in the need for studies examining their effectiveness, both in their own right, and in comparison to their physical counterparts.

## History and Background

In the nineteenth century, educational reformer Johann Pestalozzi proposed the idea that learning starts with impressions made upon the mind by experiences and things outside the brain (Holman, 1908). Pestalozzi, whose first published work was in 1774, stated, "There are two ways of instructing: either we go from words to things, or from things to words. Mine is the second method" (Holman, 1908, p. 198). These "things" that Pestalozzi refers to are what today we call manipulatives. Friedreich Fröbel, a student of Pestalozzi, advanced the idea, using items such as balls, blocks, sticks and string to help children learn to read, write and understand geometric concepts (Froebel, 1895). Maria Montessori followed in the early twentieth century, designing manipulative materials to engage the children in her tutelage to learn reading, writing, history, science, arithmetic and geometry (Montessori, 1917).

The use of manipulatives in mathematics education continued as the twentieth century progressed. Hartshorn and Boren (1990) state that "in every decade since 1940, the NCTM has encouraged the use of manipulatives at all grade levels" (p. 2). In their book, *Multi-Sensory Techniques in Mathematics Teaching*, Krulik and Kaufman (1963) list manipulatives as one of the multi-sensory techniques. They describe a basic toolkit

for each classroom, how teachers can make their own manipulatives, and items such as Cuisenaire rods and expanding and contracting dynamic geometrical figures, available for purchase on the market. In the late 1980's most, if not all, *Arithmetic Teacher* periodicals included an article describing uses of manipulatives, and the February, 1986 issue was entirely devoted to "the practical questions of why, when, what, how and with whom manipulative materials should be used" (Hartshorn & Boren, 1990, p. 2).

Owen (1988) reported on the forthcoming 1989 Curriculum and Evaluation Standards for School Mathematics (Standards), which emphasized problem solving activities at all grade levels, with the use of manipulatives at the elementary and middle school levels. The Standards assumed the availability of "ample sets of manipulative materials . . . for student use" (Owen, 1988, p. 16) and listed recommended materials for each K-8 classroom (Hartshorn & Boren, 1990, p. 4). This resulted in increased attention to the use of manipulatives as states and school districts made plans to reform to the new Standards. One example, drafted in the researcher's home state soon after publication of the Standards, is *A Comprehensive Plan: Improving Mathematics, Science, and Computer Education in Florida*. The plan called for the use of cooperative learning, problem-centered activities, manipulatives, the acquisition of manipulative materials, and staff development on the use of manipulatives (Dana and Shaw, 1992).

In 2000, the Standards once again encouraged the use of manipulatives and problem solving in mathematics instruction (NCTM, 2000), but another development also resulted in an increased emphasis on the use of manipulatives: the availability and enhancement of virtual manipulatives. With resources being directed towards implementing the use of manipulatives in the classroom and the accessibility of virtual

manipulatives, research into the efficacy of this teaching strategy, using either physical manipulatives, virtual manipulatives, or both, seemed to become even more important.

## Publications on the Use and Effectiveness of Manipulatives

In 2013, Carbonneau et al. published the results of their meta-analysis on 55 studies that compared the use of manipulatives in teaching to a control group where only abstract symbols were used in teaching. The studies dated from 1976 to 2010 and encompassed students in Kindergarten through college. Results indicated a small- to medium-sized effect in favor of using manipulatives, and this range was affected by other variables, such as the type of manipulative, the amount of direction from the teacher, and the maturity of the student. These factors are significant enough that they can determine the failure or success of the use of any specific manipulatives and result in the variation of the conclusions reached by different authors. The researchers therefore concluded that these factors should be considered when planning lessons using manipulatives. Of the 55 studies included in this meta-analysis, only four dealt with high-school level mathematics, algebra and geometry. None of the studies considered the use of manipulatives in the teaching of high school calculus

In addition to Carbonneau et al. (2013) article described above, this researcher reviewed several other publications that discuss the efficacy of, and factors to be considered with the use of manipulatives. These are divided into grade levels and specialties, and are described below.

Elementary Grades

The six publications that are summarized below range in dates from 2002 to 2015 and include teacher and teacher educator experiences, a review of several studies, a follow-up to a previous survey study, and an explanatory article.

Moch (2002). This paper describes the author's experience working with a group of 16 fifth grade students who scored an average of 49% on a practice Florida Comprehensive Assessment Test (FCAT). Moch devised innovative lessons using manipulatives and other good teaching practices, and met with the students two times each week for six weeks. At the end of this time the students took a posttest in the same format and covering the same material as the practice FCAT. The class average improved from 49% correct responses to 59% correct responses. Moch also relates that the students enjoyed using the manipulatives, their attitude toward learning mathematics improved, and that they looked forward to learning new mathematical ideas.

Kamina & Iyer (2009). As educators of pre-service elementary teachers in a mathematics methods class, the authors use manipulatives in their training sessions, both to model their use and to teach required mathematical concepts to the future educators. They present a sample geometry lesson and detailed information on how the teacher students can bridge the gap between the concrete and the abstract in using manipulatives with their own students. They concluded that the use of manipulatives can be an effective teaching tool.

Boggan et al. (2010). In a review of five studies at the elementary level ranging from 1996 to 2008 and other published research, the authors found that "the majority of the studies indicate that mathematics achievement increases when manipulatives are put to

good use" (p. 4), and that "many studies also suggest that manipulatives improve children's long-term and short-term retention of math" (p. 4).

Swan and Marshall (2010). These researchers followed up on a 1997 Australian study that surveyed primary educators regarding their use of manipulatives in the mathematics classroom. Findings of the original study were based on responses from 249 primary teachers in New South Wales. Swan and Marshall received responses from 820 teachers and conducted interviews with a sample of the responders. This follow-up study confirmed the results found by Howard et al. in 1997, that "manipulatives benefit the learning and teaching of mathematics" (p. 18).

Nelson (2012). This paper is authored by an elementary mathematics methods teacher whose focus is the use of children's literature and manipulatives in math lessons. Nelson states that she incorporates the Standards into her lessons, and as a result, her students "use manipulatives, engage in problem-solving activities, listen to children's literature, and use technology to experience firsthand how these instructional strategies might benefit their own students" (p. 419). Nelson reports on a component of her class that she calls the Math Box Project, in which her students prepare a math lesson incorporating the elements of literature and manipulatives for their field experience 4th and 5th grade students. The pre-service teachers report that the students thoroughly enjoy and learn from these lessons. This author believes that the experience will positively affect the instructional style of the future teachers as well as educate and inspire the teachers of the classes they visit.

Morin & Samelson (2015). This paper discusses the appropriate use of manipulatives at the elementary level, focusing specifically on the students being able to connect the

concrete manipulative with the concepts being taught. The authors describe manipulative scenarios and how to achieve what they refer to as "conceptual congruence" between the manipulative and the concepts, so as to facilitate "the transition from concrete to formal abstract mathematical knowledge" (p. 369). They go on to say that incongruence between the manipulative and the concepts "can confuse children -- render them less prepared to learn at a level that is commensurate with their potential" (p. 369).

Morin and Samelson give suggestions on how to avoid these problems and achieve conceptual congruence. They state that students must progress from concrete to abstract in steps. For example, initially using blocks, concrete objects, to represent quantity, and moving on to tally marks on paper, semi-concrete items.

Choosing manipulatives appropriate to the task is also important. For example, if asking students to identify the group with more objects among a group that contains 4 discs somewhat larger than the cubes contained in a group of 7, the students will likely be confused, resulting in conceptual incongruence.

Another factor in conceptual congruence is proper instruction and supervision by the teacher. Morin and Samelson suggest the teacher model think-alouds of their manipulative counts and appropriate solution strategies for counting and solving problems, especially when transitioning to independent work.

## Middle Grades

Two of the following three selections discuss issues to think about when planning for the use of manipulatives. The third is a dissertation that compared the achievement of students taught with and without the use of manipulatives.

Stein and Bovalino (2001). With the belief that the use of manipulatives does not necessarily result in a good lesson, the authors set out to identify factors that made for a successful mathematics lesson using manipulatives. These authors observed several teachers in a rural middle school each teach a mathematics lesson with manipulatives. Factors that contributed to an effective lesson were teacher training and preparation. The training included a professional development session devoted solely to the use of manipulatives. The preparation involved designing or adapting a lesson, trying the manipulatives out themselves, and preparing the classroom: organizing and setting up the materials, the desks and the student groups. Once the activity started, the students had ample time to work with the manipulatives, and the teacher helped them to construct their own understanding. In manipulative lessons that did not go well, teachers insisted on showing students how to do things step-by-step and corrected any deviations from this. At the other extreme, things did not go well when teachers did not give a proper introduction and students were left with no clear sense of where the lesson was headed. The article closed by giving a detailed description of the planning and execution of the most successful lesson that they observed.

Moyer & Jones (2004). This study examined whether giving middle school students free access to manipulatives would promote autonomous thinking in the students. Ten teachers who had attended a professional development summer institute session on using manipulatives were selected for the study. After a variety of manipulatives were introduced in teacher-directed activities in the classroom, students were progressively given periods of free access to the manipulative materials to use as needed to do their mathematics work. The researchers concluded that "students began to see these materials

as one of many tools in their mathematics environment and spontaneously and selectively used the materials effectively to mediate their learning." Moyer and Jones propose that it may be important to include the element of choice in developing teaching environments where students have an active role in their own learning.

White (2012). This study compared the performance of seventh grade students in a middle school in rural Georgia on a pretest and posttest. The experimental group was taught a unit on data analysis and probability in a hands-on way, with the use of manipulatives. The control group was taught the same unit in the traditional way, without the use of manipulatives. For further comparison, the two groups were subdivided (on paper) into low-, average-, and high-achieving. The results showed no significant differences in performance between the experimental group and the control group. There were also no differences between the groups of varying achieving levels. There were, however, significant learning gains for a number of individual students in both the experimental and control groups. The researcher recommended that until further research is done, multiple instructional approaches should be considered in planning lessons, so as to reach every student.

Virtual Manipulatives

Virtual manipulatives are discussed at this point, because much of the research and information available deals with their use in lower and middle school populations. Moyer, Bolyard and Spikell (2002) define a virtual manipulative as "an interactive, Web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge" and further stipulate that the manipulatives must

15

have the ability to be used interactively, allowing the user to engage and control their physical actions (p. 1).

Several of the studies discussed below involve head-to-head comparison of physical manipulatives to virtual manipulatives in different areas. Other articles involving physical and virtual manipulatives were not direct comparisons, but rather information and experience on using the two types of manipulatives.

Burris (2010). This researcher focused on how third grade students *think* about place value using both types of manipulatives. He found no differences between how the students thought about place value when using concrete manipulatives than when using virtual manipulatives, and concluded "that virtual manipulatives are a viable instructional tool for the instruction and learning of place value" (p. 175).

Mendiburo & Hasselbring (2011). These researchers believe that practical and pedagogical challenges often prevent teachers from using manipulatives during instruction, and that this results in their students receiving less exposure to manipulatives than the level of exposure recommended by the NCTM Standards. Mendiburo & Hasselbring examined whether virtual manipulatives would be as effective as, and easier to use than, physical manipulatives in teaching fractions to fifth grade students. Sixty-seven students were randomly placed into four groups; two were the control group and the other two were the treatment group. During a 10-day unit on basic fraction concepts, the control groups received instruction using a commercial fraction curriculum and physical fraction strips. The treatment group was instructed using the same curriculum and virtual fraction strips on a laptop computer. Pretest, interim, posttest results, and comparison between the two groups showed that the virtual manipulative groups scored

essentially the same or slightly better than the physical manipulative groups. The researchers concluded that "computers can provide students with virtual representations of mathematical concepts that are just as meaningful as physical manipulatives" (p. 4). However, it was found that the virtual manipulatives were easier to set up and use, and saved considerable class time. As a result, the students in the treatment group were able to complete more practice exercises and games than the other group.

Özgün-Koca & Edwards (2011). This paper describes a lesson activity presented to two eighth grade algebra classes, with the purpose of observing and describing the teacher's and the students' experiences with the activity. The concept taught was that of linear regression with residuals. The activity involved both physical (graph and spaghetti strand) and virtual manipulatives (a virtual graph and spaghetti strand). The virtual manipulative was in the form of a TI-Nspire provided to each student for use in the activity. When the students were asked about their experiences, 15% preferred the physical spaghetti strand, and 85% preferred the virtual spaghetti strand. However, most students said in their comments that they liked being able to touch the physical object. Supporters of the virtual version stated that it was easier to move the strand on the graph, and that the tedious correlations and calculations were done by the calculator, rather than by hand, as with the paper method. The researchers concluded that by using both methods, they were able to reach both visual and kinesthetic learners. They also pointed out that with the availability of tablet technology, virtual manipulatives are a "touch" away, but additional research is needed to determine how much of type of manipulative to use, and to the proper design of virtual manipulatives, so that they are conducive to student learning.

This study appears to have a weakness in design, which could have skewed the results in favor of the virtual manipulative. The physical manipulative required extensive calculator computations, and at least some students reacted negatively to the "tediousness" of it. A more appropriate study would have removed the aspect of extensive calculator computations for the physical manipulative, resulting in a more accurate comparison of the two methods.

Magruder (2012). This researcher conducted a three-way study, comparing a control group taught with no manipulatives, a treatment group taught with concrete manipulatives, and a treatment group taught with virtual manipulatives. Three sixth grade classes, one class in each respective group above, were given instruction on solving linear equations in a 10-day unit, and results of a pretest and posttest were compared. Results were statistically significant in favor of the control group, both when compared to the concrete manipulative group and compared to the virtual manipulative group. Magruder suggested several possible explanations for this, one of which was that the treatment groups had less time for leaning and practicing due to learning how to use their respective manipulatives. She listed considerations to be included in further research "such as the use of manipulatives in middle and high school" (p. 3).

In addition to the above studies, this researcher reviewed two articles involving physical and virtual manipulatives that were not direct comparisons, but rather information and experience on using the two types of manipulatives.

Moyer-Packenham, Baker, Westenskow, Anderson, Shumway, Rodzon & Jordan. (2013). This study used rigorous research design, and involved 350 third and fourth grade students in 17 classrooms, spread among 7 public elementary schools in 3 school

districts. The students were randomly assigned to a group that would use textbooks and physical manipulatives to learn fractions in a regular classroom, or a group that would use virtual manipulatives to learn fractions in a computer lab. Results showed that "using either physical or virtual manipulatives produce a similar student achievement for third- and fourth-grade students learning fraction concepts, [and that] experienced instructors can use different instructional modalities for mathematics instruction and produce similar achievement results" (p. 37).

Loong (2014). This article gives advice on how to select and use manipulatives, considering mathematical, cognitive, and pedagogical fidelity. The author provides specific, detailed information on using manipulatives with the following concepts: addition and subtraction requiring regrouping, understanding place value, naming fractions, addition of fractions, multiplication and division of fractions, and misconceptions in area and perimeter. Loong's purpose is to provide information that teachers can use to help students better understand these important concepts or to use as a remedial measure for middle school students whose struggle with these concepts is impeding mathematical progress.

Peppers, Wan & Phillips (2014). An action research intervention with eighth grade pre-algebra students working on a conceptual understanding of fractions, a teacher used both physical and virtual manipulatives in remediation. The teacher related the following:

> Because of the positive results from the project, we continued incorporating concrete and virtual manipulatives into regular instruction throughout the year and focusing on critical thinking skills as we covered our standards. Students' attitudes toward math continued to improve; their ability to communicate their understanding improved; and, subsequently, they were able to build on the knowledge they gained during this unit. Students had

19

> positive reactions to both types of manipulatives, so I would use
> both again. (p. 172)

The researchers and teacher felt that the remediation project prepared the students to begin learning the curriculum standards for their eighth grade level. Furthermore, once they started, the impact of the project showed in the minimal time spent revisiting procedures involving fractions.

## Special Needs

Physical manipulatives, and, more recently, virtual manipulatives, have been shown to be effective for students with special needs. According to Boggan et al. (2010), "research . . . indicates that using manipulatives is especially useful for teaching . . . students with learning disabilities" (p. 5). This researcher lists and briefly describes studies and other sources that document this claim.

McNichols (1985). This author, a Mathematics Specialist at a center for educational therapy, writes about her experience in assisting a fifth grade boy diagnosed with "dyscalculia, a specific number disability" (p. 13). She used manipulative materials to work with the student on the basic operations of addition, multiplication, subtraction, and division. The boy progressed from a first grade math level to a fourth grade math level between September and March of the school year.

Maccini & Gagnon (2000). This article describes best practices for teaching math to students with special needs at the secondary level. Maccini and Gagnon state that "use of these concrete aids has been determined to be an effective medium for students across grade and developmental levels, including students with disabilities" (p. 9).

Moyer and Suh (2012). In the literature review for their article on learning mathematics

with technology, the researchers compiled sets of studies involving the use of virtual

manipulatives with students of varying achievement levels. Moyer and Suh report their

findings on the use of virtual manipulatives with special needs students:

> Hitchcock & Noonan (2000) reported that preschool special
> education children using virtual manipulatives made more progress
> than when they used paper and pencil. Suh and Moyer-Packenham
> (2008) reported that fourth grade special needs students were
> supported by the use of the virtual manipulatives because the tools
> allowed students to offload findings to the computer thereby
> reducing their cognitive load. Two studies reported that virtual
> manipulatives improved test scores for ninth- through twelfth-
> grade learning disabled students (Guevara, 2009) and university
> remedial students (Demir, 2009). (p. 42).

Bouck, Satsangi, Doughty and Courtney (2014). The researchers used concrete and

virtual manipulatives to teach subtraction concepts to three students diagnosed with

autism spectrum disorder (ASD). They determined that "Both concrete and virtual

manipulatives resulted in increases in the percentage of accurate and independent

performance for students with ASD when solving subtraction problems" (p. 187).

Dobbins, Gagnon & Ulrich (2014). This article focuses on students diagnosed with

mathematical difficulties, defined as a standardized test score that falls below the $35^{th}$

percentile. The authors describe graduated instruction as "using a graduated sequence that

includes hands on manipulatives to teach difficult mathematical concepts in a concrete

and progressively more abstract manner," and that this "allows students to understand

abstract concepts more easily." The Concrete-Representational-Abstract Sequence of

Instruction (CRA) is an example of graduated instruction. In CRA, instruction starts at

the concrete stage, with physical objects, then moves to pictorial representations before

moving on to the abstract, symbols, numbers and other notations. The students must "graduate" from one level before moving to the next one.

The researchers designed a model for teachers using graduated instruction with a CRA approach and peer-mediated instruction for teachers to use in teaching geometric concepts, and include a sample lesson plan that uses a manipulative for finding the area of a trapezoid. Dobbins et al. state that "there is great potential for these instructional approaches to promote conceptual understanding, provide for interactive learning, and provide students with disabilities the greatest opportunity for success within geometry coursework" (p. 22).

Satsangi & Bouck (2015). The researchers used virtual manipulatives to teach the concepts of area and perimeter to three secondary students identified with a learning disability in mathematics. Satsangi and Bouck reported that the performance of the three students showed statistically significant improvement and stating that the virtual manipulatives were shown to be highly effective.

The student populations in the two studies just reviewed mentioned in this section were secondary school students. The following section will discuss research involving general education high school students.

High School

As the title implies, the focus of this thesis is the use of manipulatives to teach advanced mathematics, specifically calculus. But, as Hartshorn and Boren (1990) stated, "Manipulatives have, unfortunately, been implemented more slowly at the secondary

level. As a result, research on their effectiveness at this level is minimal" (p. 3). This was reiterated by Howard et al. (1996), who stated:

> This paper has presented some initial baseline data on the use of manipulatives in secondary school mathematics classrooms. It would appear that the use of manipulatives in these classrooms is low, particularly compared to such use in primary school mathematics lessons. (p. 9)

Three secondary school studies located by this researcher are described below.

Golafshani (2013). In an attempt to encourage the use of manipulatives in the teaching of mathematics, this study focused on comparing the beliefs of four ninth grade teachers regarding the use of manipulatives in their classroom to the teachers' actual use of manipulatives. The study involved providing the teachers with manipulatives, training, and assistance in developing lesson plans. The teachers also completed periodic questionnaires on their beliefs and experiences. Results showed that the teachers' attitudes toward the use of manipulatives became more positive as the study progressed. Factors that could have affected this were the teachers' increased comfort level with the materials and assistance with overcoming difficulties with use of the materials.

Jimenez, 2011. This study investigated the use of a mathematics program called Hands-On Equations to help ninth and tenth grade students in an Algebra 1 class learn abstract algebra concepts through the use of manipulatives. The performance of these students was compared to students who were taught without the use of manipulatives. The results showed an increase in posttest and 3-week retention test scores, and was judged to be effective for these assessments. However, the program was not as effective on a 6-week retention test when compared to the benchmark test. Jimenez also compared the results of

benchmark tests on students that participated in the Hands-On Program and those that did not, and no significant differences were found.

Bruins, 2014. Six Algebra II classes participated in this study, where the students in three classes where taught two lessons incorporating the use of manipulatives using the Concrete-Representational-Abstract (CRA) model, and the students in the other three classes were taught the same lessons with explicit instruction, where no manipulatives were used. The researcher compared the performance of both sets of students on a pretest, posttest and a follow-up test. The results showed a minor difference in favor of explicit instruction in one lesson (domain and range of quadratic functions) and a minor difference in favor of CRA instruction in the other lesson (transformations of quadratic functions). Bruins concluded successful learning took place for both groups, and that the statistical analysis did not strongly favor either method of teaching in this case. However, she goes on to relate positive observations regarding active student engagement, meaningful mathematics discussions, and display of problem-solving skills by the students in the CRA classes that were not observed with classes taught using traditional explicit instruction

Manipulatives Used in Calculus

This researcher found no research studies in which manipulatives were used in the teaching of calculus topics. The highest mathematics level identified in a research study was Algebra 2. (Bruins, 2014) However, 14 publications were located where the authors described manipulatives that they designed and used in their own teaching. These are listed and described in this document (Appendix 1).

Discussion and Summary

In this researcher's review of the literature, a background and brief history of the use of manipulatives to support the teaching and learning of mathematics from the eighteenth century to the late twentieth century were presented. Factors that resulted in increased attention and research on manipulative use were revisions to the NCTM standards in 1989 and 2000, and the development of virtual manipulatives.

Publications, including research studies and informational articles, that dealt with the use and efficacy of manipulatives were presented. First considered was information regarding the elementary and middle school grades, where the studies showed that the use of manipulatives was more effective or as effective on student learning as a traditional approach: Moch (2002), Boggan et al. (2010). Swan (2010), and White (2012). Teachers and researchers involved in the studies typically reported that students were engaged in learning and enjoyed using the manipulatives. Two articles related innovative ways in which the teacher educator authors used manipulatives in their development of pre-service teachers: Kamina & Iyer (2009) and Nelson (2012). Other papers informed on various aspects of manipulative use, such as how teachers can appropriately and successfully incorporate them into their teaching and other factors to be considered: Morin (2015), Stein (2001), and Moyer (2004).

When virtual manipulatives became available, researchers compared their effectiveness to that of physical manipulatives. In three direct comparison studies, the research indicates no significant differences in achievement between the two types of manipulatives: Burris (2010), Mendiburo (2011) and Moyer-Packenham (2013). In one direct comparison study, the results were in favor of the control group that received

instruction with no manipulatives: Magruder (2012). Two papers described student and teacher experiences in using both types of manipulatives: Özgün-Koca (2011) & Peppers (2014). One article gave detailed information for using virtual manipulatives to teach several concepts traditionally difficult for students: Loong (2014).

Seven publications documented that manipulatives are effective tools for teaching students with the following special needs: dyscalculia, non-specific special needs, general learning disabilities, remedial, ASD, diagnosed mathematical difficulties, and learning disability in mathematics: McNichols (1985), Maccini (2000), Moyer (2012), Bouck (2013), Dobbins et al. (2014), and Satsangi (2015).

In addition to the two studies discussed in the special needs section, three publications involving secondary students were identified. Two studies compared teaching algebra concepts with the use of manipulatives and without the use of manipulatives, with statistical analysis showing no significant differences in the achievement levels: Jimenez (2011) and Bruins (2014). The third publication examined teacher beliefs and action with respect to the use of manipulatives in their classrooms: Golafshani (2013).

The research described did not find any studies involving calculus students, but did find a number of calculus manipulatives described in the literature by practitioners, and these are discussed in Appendix 1. Based on this researcher's findings, the study of manipulatives in teaching advanced mathematics, and specifically calculus, is essentially an unexplored field of study.

# Chapter 3  Study Methodology

Taking into consideration the information on the calculus manipulatives located in the literature, and reflecting on her own ideas, this researcher designed or adapted physical manipulatives to use in teaching several calculus topics. A total of four lessons, three lessons with the use of manipulatives, and one without the use of manipulatives, were taught to four groups of students. Both quantitative and qualitative data was gathered, through the use of pretests, posttests, and questionnaires. The data was analyzed using two types of $t$ tests for the pretest and posttest scores, and survey-interpretation techniques as described in Sauro (2011) for the questionnaire statements and comments.

## The Population

It is important to note that the author/researcher of this study is also the calculus teacher of the students in the two classes participating in the study. In order to safeguard each student's freedom of choice and privacy, strict policies regarding recruitment, consent/assent, and documentation procedures were designed and implemented.

A total of 18 students in two calculus classes at Riviera Preparatory School, an independent private school in Miami, Florida, participated in this study. One of the classes combined 4 Honors and 7 Advanced Placement AB (H/AB) calculus students, and the other class was made up of 8 Advanced Placement BC (BC) calculus students.

With respect to ethnicity, the majority of the school students are categorized as white or Hispanic, with many being both white and Hispanic. There are a minority each

of African American and Asian students. The socioeconomic status of the majority of students is middle or upper middle class.

## Lesson 1

Optimization involves finding extrema of real world problems such as maximum profit or volume and minimum cost or surface area. The mathematical procedure for solving these problems involves defining a function and a domain for the quantity to be maximized or minimized, finding the derivative of the function, setting the derivative equal to zero, and solving for the independent variable.

At this point in the calculus course, most students are very comfortable with finding values of the independent variable where the derivative of the given function is equal to zero. They are also proficient at determining whether these values result in a maximum, minimum, or neither. The problem for many students is in defining the required function, as well as in determining the appropriate domain for the real world situation.

An optimization problem typically found in calculus textbooks is as follows: Given a rectangular piece of cardboard of specified dimensions, a box can be created by cutting out equally-sized squares from each corner of the cardboard. The pieces remaining would be folded up to form an open rectangular prism (box). Determine the length of the side of each square such that the resulting box will have the largest possible volume.

To introduce the concept of optimization to the H/AB class, the researcher designed a multi-day project lesson plan incorporating the use of manipulatives in

investigating a variation of the box in the above problem (Appendix 2). In class, before the beginning of the project, students were given a pretest on optimization of functions (Appendix 3). At the end of class, they were shown how to construct a "pizza" style box. The box was similar to the one described above, except that, in addition to the corner squares, squares were also cut out at the midpoint of each long side. This would form a box with a fold-over top, similar to a pizza box. Students were given a prelesson assignment and a color poster board of length 22 inches and width 14 inches (Appendix 4). The assignment gave each student (or pair of students) a specific value of $h$, a diagram, and additional instructions for building the box and calculating its volume and surface area (Figure 1). Decorating the box was optional, but encouraged.
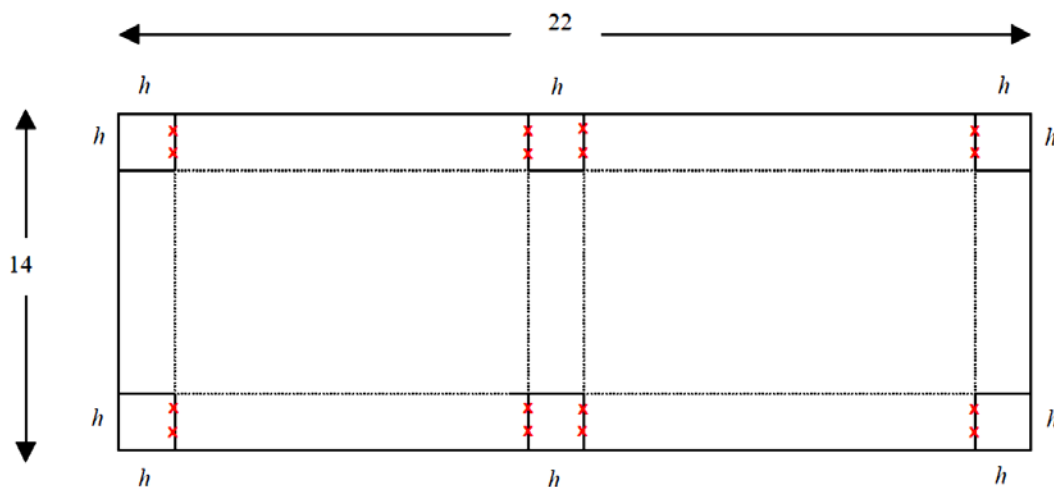


**Figure 1 Diagram for box manipulatives**

On the next class day students brought in their boxes and provided data to complete a class spreadsheet (Figure 2; Appendix 5). They saw that as the values of h increased, the volume of the boxes increased, to a point, and then began to decease. In

29

contrast, the surface area of the boxes continually decreased. The data showed the students how the same rectangular area produced boxes of different volume and surface area, depending on the value of h, and that there was a specific value of h that produced a maximum volume. In addition, the students were able to better understand a previously covered concept, the Extreme Value Theorem (EVT), using a physical object to see how extrema can occur in the middle of an interval, as well as at the endpoints of the interval.
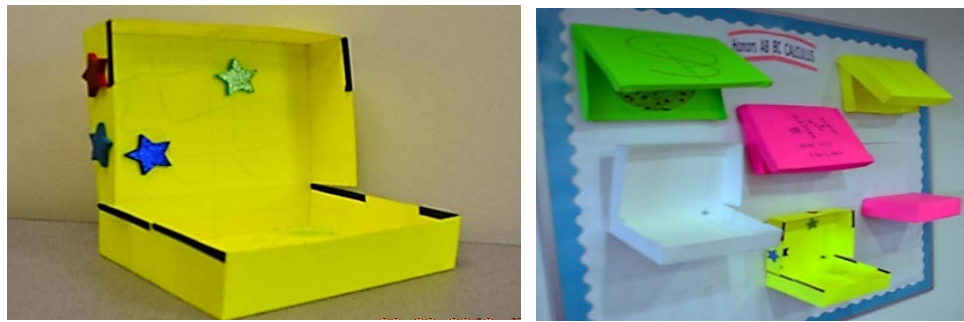


**Figure 2 Student-made boxes used to introduce the concept of optimization**

The exploration continued with the students being given a piece of ½-scale cardstock paper 22 units long by 14 units wide, and asked to make cuts of length $h$, starting with $h = 1$ unit and fold to make a box. Then they unfolded the box and cut an additional 1 unit for each $h$, and folded a new box. They were to repeat this procedure until a natural stopping value occurred (Figure 3). The students discovered that this natural stopping point was half the length of the shorter side, because once they reached that point, there was nothing left to cut on that side. This led the students to see that the interval defined by $h = 0$ and the natural stopping point, $h = 7$ units, represented the

domain of this real world problem. They calculated the volume and surface area at these extremes and identified the range of these two functions of *h*.
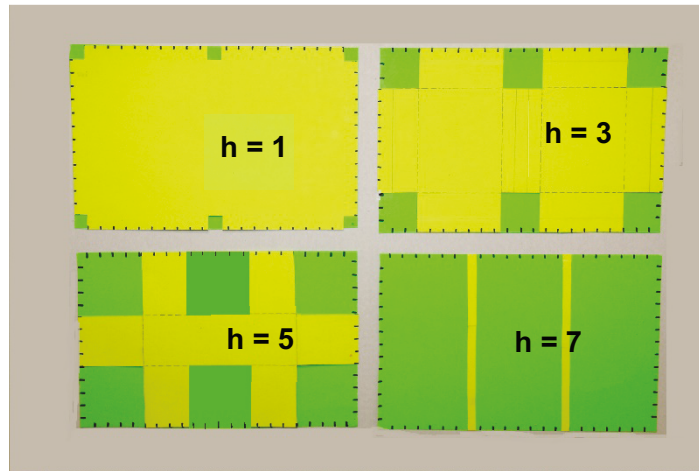


**Figure 3 Fold-over boxes made by varying the height from 1 to 7 units. Green areas are sections cut away. Yellow areas represent the 2-dimensional geometric net of the 3-dimensional boxes.**

The last item for this lesson was for the students to find the extrema for volume and surface area using previously learned calculus concepts. For each function, the students differentiated and found the value(s) of *h* for which the derivative was equal to zero in the domain of the functions. They determined the function values for the above value(s) of *h* and at the endpoints, and applied the EVT to find the extrema.

The students compared the values found on the class spreadsheet to those on graphs of the functions made using Desmos Graphing Calculator (Desmos) and those found using calculus (Appendix 6; https://www.desmos.com/). In this way, they were able to connect the work they did with the physical boxes to previous calculus concepts

and use them in a new way: to describe a physical situation as a function and find the values of the independent variable that optimize the function value.

The students were given optimization problems to practice at home, and were able to check their work and ask questions the next day. On the following day, the students were given a posttest with optimization problems similar to the pretest and practice problems, along with experience ranking statements and the opportunity for open-ended comments (Appendix 7; Appendix 8).

There was no control group for this lesson The goal of the postlesson information collected was to determine if the students showed significant learning of the material (quantitative data), and how they felt about their experience working with manipulatives in this lesson (qualitative data).

## Lesson 2

Finding the volume of solids of revolution is an abstract concept that, in this educator's experience working with calculus students over the years, is challenging for many students. Some will quickly memorize the formula and go about their work, but their lack of conceptual understanding becomes obvious when variations, such as if the area of revolution is between two curves (as opposed to between a curve and x-axis), or if the area is to be revolved about anything other than the x-axis, are introduced.

A concept new to the students is that the typical slice of a solid resulting from an area revolved around an axis is a circle. Understanding this is essential for solids of revolution problems. With this in mind, the researcher designed a multi-day lesson for the class, using manipulatives that would allow the student to physically see that this is the

32

case (Appendix 9). To assess their knowledge before the lesson, the students took a pretest on solids of revolution (Appendix 10).

The manipulatives used in this lesson are referred to as "accordion figures" commonly found in party supply stores. Each consists of two cardboard bases connected by accordion-folded tissue paper. The first such figure used in the lesson had triangular bases, which, when opened and fully revolved, met to form a 2-cone solid (Figure 4). As part of the lesson, the students first measured and sketched the triangle on a coordinate plane, before opening to reveal the solid. They then calculated the volume of the solid using the measured height and radius and the geometric formula for the volume of a right cylindrical cone.
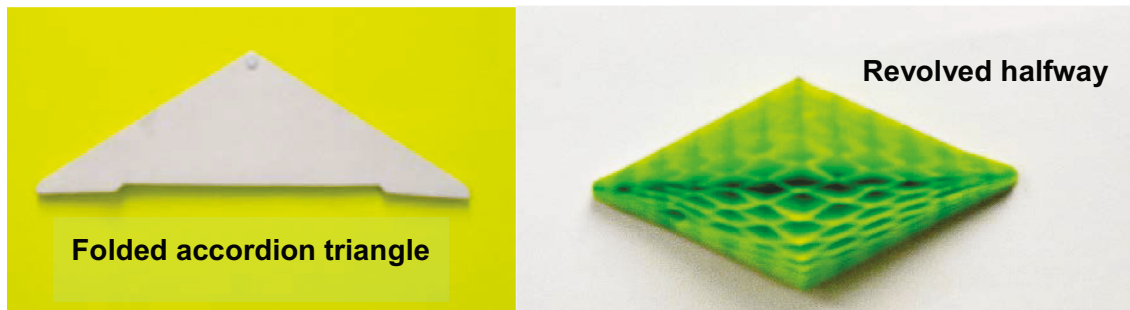


**Figure 4 Manipulatives used to introduce the solids lesson**

The concept of a typical slice of the solid as a circle was demonstrated well by means of a different type of accordion figure. The bases of this manipulative were rectangles of about 1cm x 7cm. Each student was given a manipulative figure and asked to overlay it on the triangle they had drawn, in a location such that the top of the thin rectangle touched the triangle and the bottom touched the x-axis. They were then asked to

trace the rectangle onto the paper and remove the still-flat accordion figure. Now the

students were asked to imagine what type of a figure would result if the thin rectangle

they had traced were to be revolved around the x-axis. This was followed by a thinking

pause, which elicited a few incorrect guesses of "triangle". The students were asked to

"revolve" their rectangle by opening and fastening the two flat rectangles together,

revealing a circle (Figure 5). Thus the fully open manipulative served to model a circular

slice of the 2-cone solid.
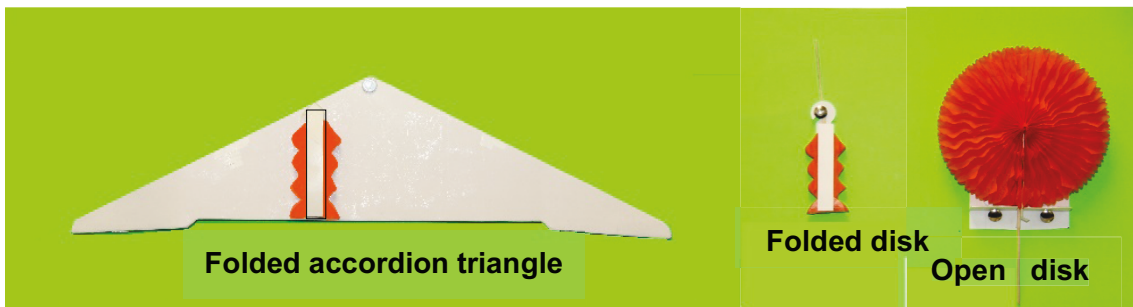


**Figure 5 Accordion figure manipulatives used in volumes of solids lesson**

Understanding that the typical slice of the solid, now redefined as a disk, is a

circle, the students determined the radius of the circle to be the function value and wrote

an expression for its area. Drawing on their knowledge of integration, the students wrote

an integral expression that represented the sum of all such disks over the interval

represented by the length of the base. They then evaluated the integral to calculate the

volume of the 2-cone solid. The last step was to compare this value for the volume with

the value found geometrically at the beginning of the lesson. This step was instrumental

in convincing the students that the integral they used did, indeed, find the volume of the solid.

After paper and pencil practice calculating volumes, this next part of the lesson, inspired by Scherger and Tuerk (2012), went on to show that volumes of solids generated by areas not defined by known functions can be computed as well. Pairs of students were given a closed accordion figure and asked to think about what function might define the curve and what the solid might look like when fully opened. After some thinking and discussing time, the students opened the figure to show a bell (Figure 6).



**Figure 6 Accordion manipulative used to introduce solids generated by unknown functions**

Having planted a seed of curiosity, the students were given a different accordion figure in the shape of a small ornament. They traced the flat side of the figure onto a first-quadrant graph, with the straight edge of the figure on the x-axis, starting at the origin. The students measured the base of the figure at 11 cm and marked the x-axis by whole numbers. They measured the function value at each interval endpoint and recorded the results in an x-y table, in preparation for finding a curve-of-good-fit (Figure 7).
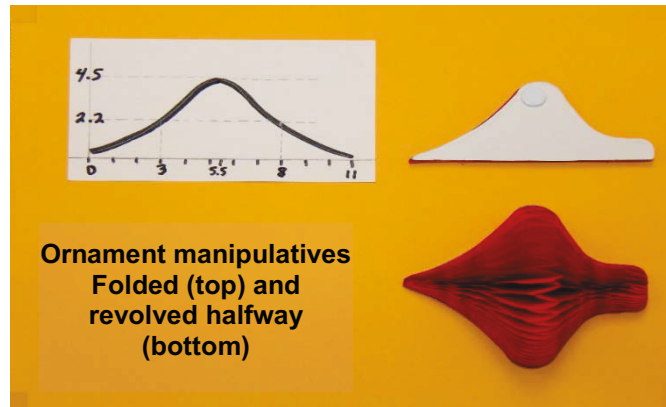
**Figure 7 Manipulatives, measurements and sketch used in finding the area and volume of a solid generated by an unknown function**

Since each student or pair had their own figure, a set of representative points were chosen and graphed on the interactive white board (IWB) using Desmos. Because fitting a function to this complex shape was an unreasonable task, a function of the form

$$f(x) = ab^{-(cx-d)^2}$$

was suggested to the students and the Desmos slider functionality was used to assist in finding values for a, b, c and d that resulted in a curve-of-good-fit (Figure 8). For homework, the students were asked to calculate an approximation for the volume of the ornament using the curve-of-good-fit function, and for four other practice problems, using the technique learned in this lesson (Appendix 11).

The next day, the students had the opportunity to compare answers and check their work. To reinforce the concept the students had learned, they were given a new manipulative accordion figure in the shape of a semicircle, with directions to use integration to derive the volume of a sphere (Figure 9).

www.desmos.com

$$f(x) = 4.5 \cdot 6^{-(.25x - 1.375)^2}$$
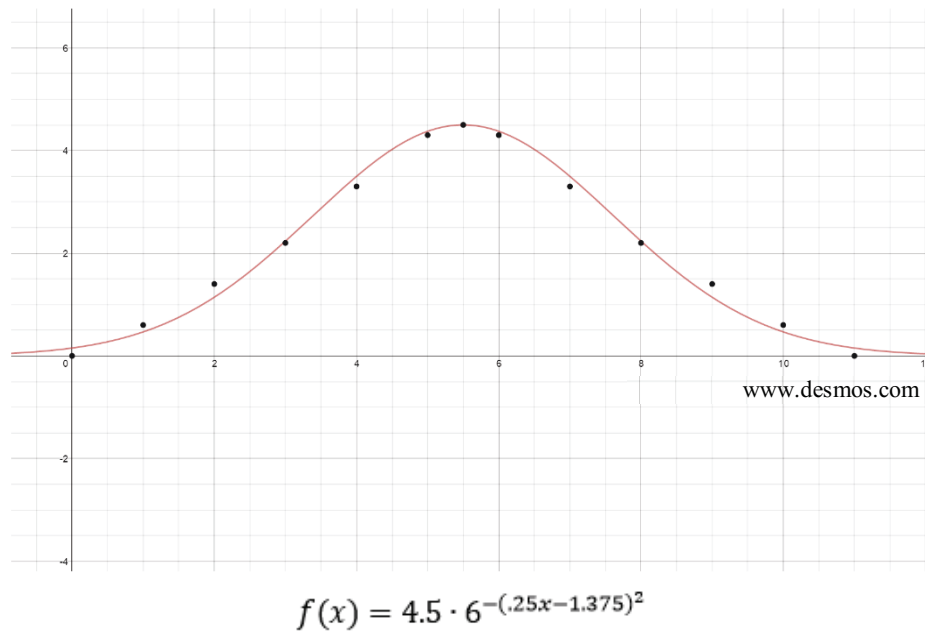
**Figure 8 Graph of function fitted to measured ornament data points**
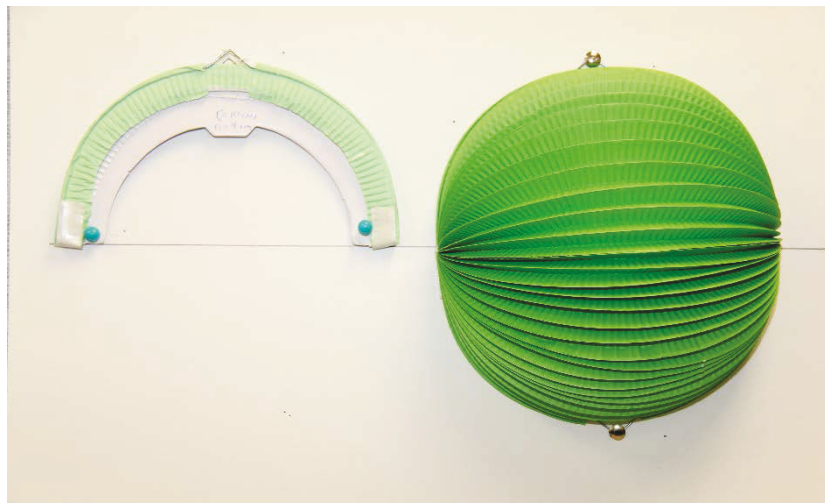


**Figure 9 Manipulative used to derive the formula for the volume of a sphere**

Finally, the students were given a posttest with volume of solids problems similar to the pretest and practice problems, along with experience ranking questions and the

opportunity for open-ended comments, and the end-of-study questionnaire (Appendix 12; Appendix 8).

As with the optimization lesson, no control group was available for this lesson. The goal of the postlesson information collected was to determine if the students showed significant learning of the material (quantitative data), and how they felt about their experience working with manipulatives in this lesson (qualitative data).

## Lessons 3 and 4

When calculus students learn integration, they seem to understand the concept of an indefinite integral well, once they recognize it as the inverse operation of differentiation. They are quickly proficient at working backwards to find the anti-derivatives of simple polynomial, trigonometric, exponential and logarithmic functions. The students are then exposed to the definite integral, which can be evaluated both as the area under a curve and with the Fundamental Theorem of Calculus, Part I (FTOC I).

The next topic discusses the properties of definite integrals, and transitions into the concept of the area between two curves. It is this material, which can be difficult for students to visualize, that the researcher chose to address in this third manipulative lesson. The manipulative for this concept was designed exactly as the first manipulative described in Thirey and Wooster (2015), with the lesson itself adapted to suit the teacher's presentation style, while developing the same ideas and concepts (Appendix 13).

The concept was taught separately to two groups of students in H/AB calculus. The groups were determined by opportunity; a group of four students were in school

while the other students were out of town on a field trip. These four students were taught the third research study lesson described below, with manipulatives. A few days after they returned from the trip, six students were taught the same concepts, in the fourth research study lesson, in the traditional way, using printed notes, practice, and homework problems from the textbook. Even though the groups were not randomly assigned, it turned out that the experimental group was made of two honors students and two AB students, and the control group was made up of three honors students and three AB students. Both groups were given pre-lesson ranking questions and a pretest before the lesson, and post-lesson ranking questions and a posttest after the lesson. The manipulatives lesson given to the experimental group will now be described.

Using masking tape, the students marked the *x* and *y* axes with the bottom left corner as the origin on their respective desks. They chose a colored paper and laid it on the desk with the long side on the x-axis, somewhere to the right of the origin. The students drew a function of their choice, extending from the left side of the paper to the right side of the paper, with the stipulation that the function be always positive. They marked the letters $a$ and $b$ on the bottom left and right corners, respectively, to represent the interval $[a, b]$, and the function was labeled as $f(t)$. The students cut along the upper boundary of the curve line and were asked to identify what was represented by what they held in their hands. Various answers were offered, until everyone agreed that they held a representation of the definite integral $\int_a^b f(t)dt$. The students were asked to choose a paper of contrasting color and repeat the procedure with a function $g(t)$ and the stipulation that $g(t)$ be less than $f(t)$ at every point (Figure 10).
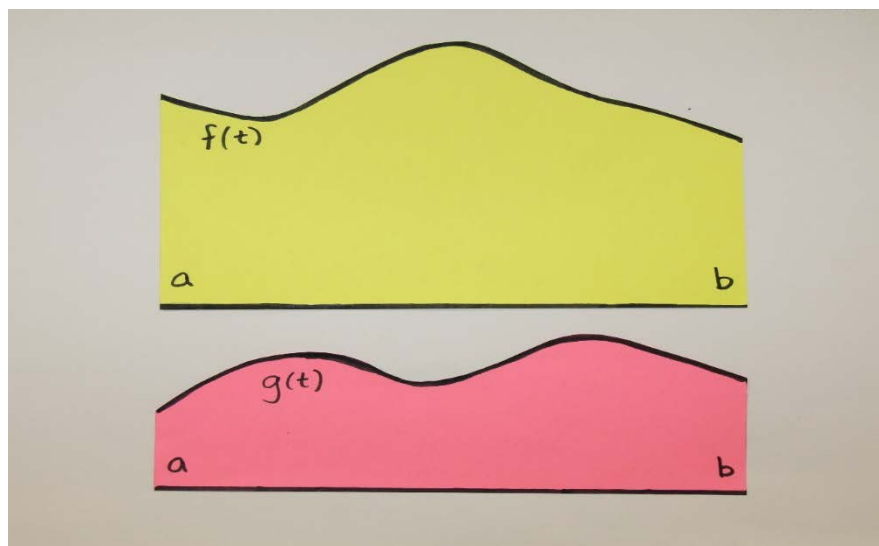
**Figure 10 Student-made manipulatives to introduce the properties of definite integrals**

Subtraction. With Thirey's (2015) manipulatives created, a circle completely inside of a square was drawn on the board, with the region between the circle and square shaded in. The students were asked to think about how they would determine the area of the shaded region, a problem they had likely solved in prior math classes and standardized tests. The immediate response was that they would subtract the area of the circle from the area of the square. They were then asked to place the $g(t)$ cutout over the $f(t)$ cutout, being careful to align the bottom and sides of each. The students were then asked to express the visible area of $f(t)$ in terms of $f(t)$ and $g(t)$. Albeit not quite as quickly as with the previous question, the students responded that the visible area was represented by $\int_a^b [f(t) - g(t)]dt$. Thus the students demonstrated the subtraction property of definite integrals (Figure 11). The result of the subtractions was described as "the area between the two curves."
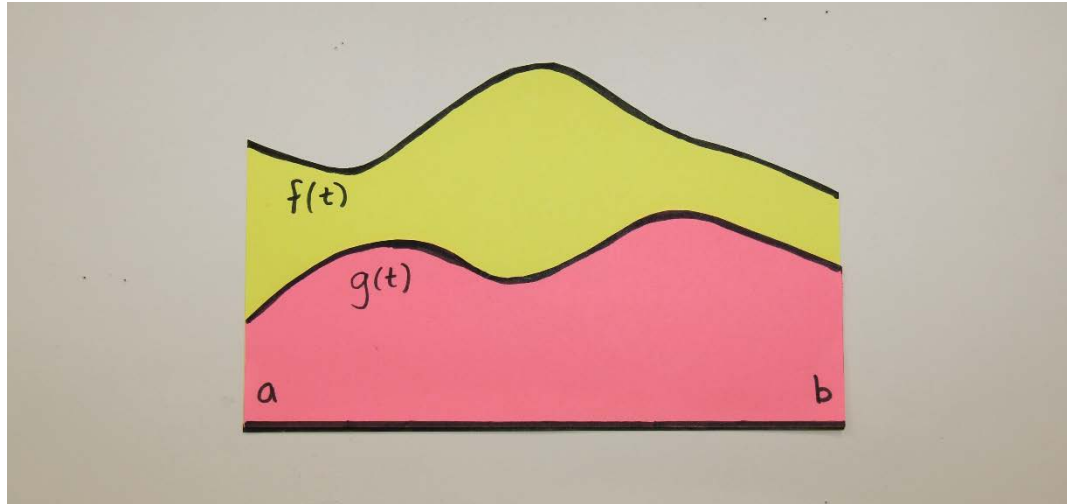
**Figure 11 Subtraction property demonstrated using the concept of compound areas**

Using these manipulatives, other properties were demonstrated:

Addition. Now the students reflected the $g(t)$ manipulative about the x-axis, labeled $a$ and $b$ in the upper left and right corners, respectively, and were asked to describe the resulting function based on their knowledge of transformations. With a bit of prompting, the students saw that it was $-g(t)$. They then lined up the horizontal edges of the two manipulatives, and used the newly-discovered subtraction property to describe the visible area as $\int_a^b [f(t) - (g(t))] dt$. The integral simplifies to $\int_a^b [f(t) + g(t)] dt$, demonstrating the addition property (Figure 12).

41
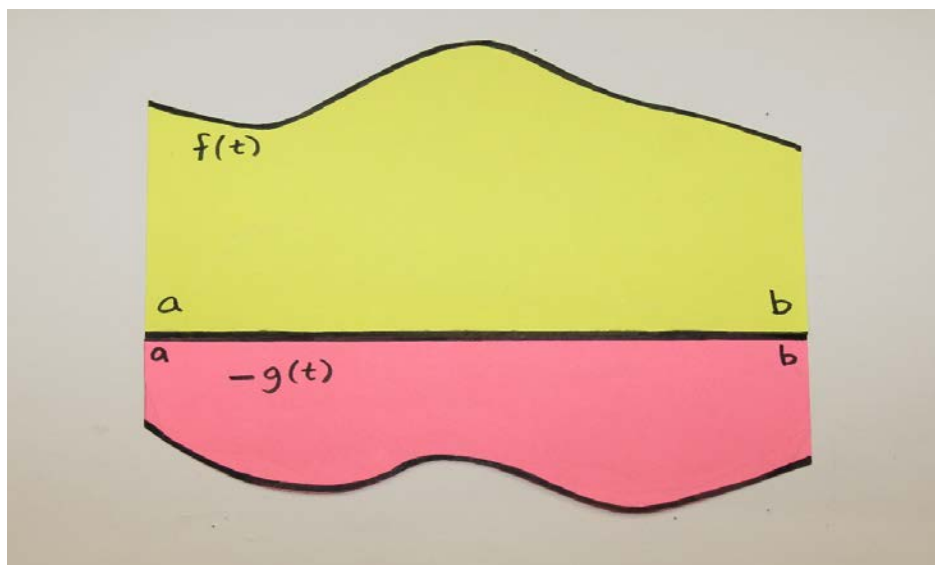
**Figure 12 Addition property demonstrated using the subtraction property**

<u>Split-up.</u> The students were asked to cut the $g(t)$ manipulative at any point between $a$ and $b$, and label it as point $c$. Then they were asked to describe each of the two pieces as a definite integral with new limits and equate the sum of the two integrals to the known expression for the whole, $\int_a^c g(t)\, dt + \int_c^b g(t)\, dt = \int_a^b g(t)\, dt$ (Figure 13).
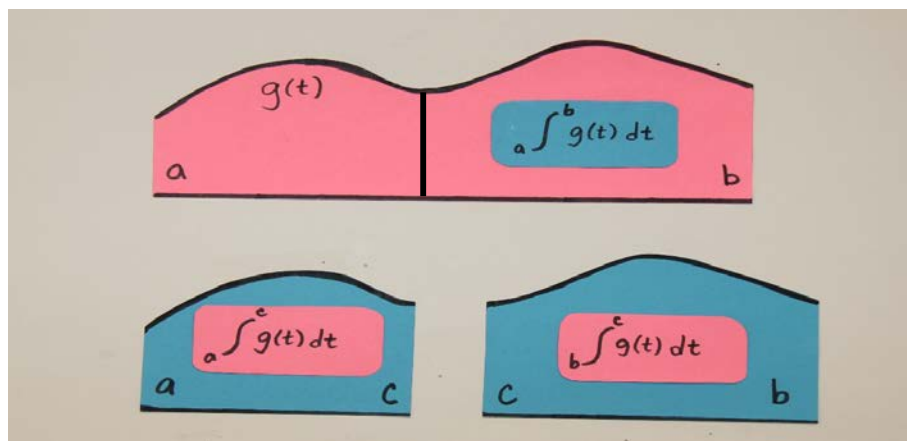


**Figure 13 Split-up property illustrated using manipulatives**

Inverse Area. To demonstrate that reversing the limits of integration results in an area of the same magnitude, but with the opposite sign, a directional approach was used. While demonstrating with the $f(t)$ cutout, the students were asked to consider integration to the right, i.e. integrating from $t = a$ to $t = b$, where $a < b$, as a positive direction. Just as walking from point $A$ to point $B$ would result in a positive distance, the area would be positive for the integration. If one reversed direction and walked from point $B$ to point $A$, one would have walked the same magnitude in the opposite direction, and this would result in a negative distance. Integration to the left, i.e. integration from $t = b$ to $t = a$ where $b > a$, would likewise result in a negative area. $\int_b^a g(t)\, dt = -\int_a^b g(t)\, dt$

Zero Area. To elicit this property, the students were asked how far they traveled if they started at point $D$ and stayed at point $D$ (i.e. they did not move). The answer was unanimous, zero. The students were asked to choose a point on the x-axis of the $g(t)$ graph, label it point $d$, and draw a line straight up to the $g(t)$ curve. They were then asked for the area spanned by the integral $\int_d^d g(t)\, dt$, represented by the line they had just drawn. The answer was again unanimous, zero. So $\int_d^d g(t)\, dt = 0$ (Figure 14).
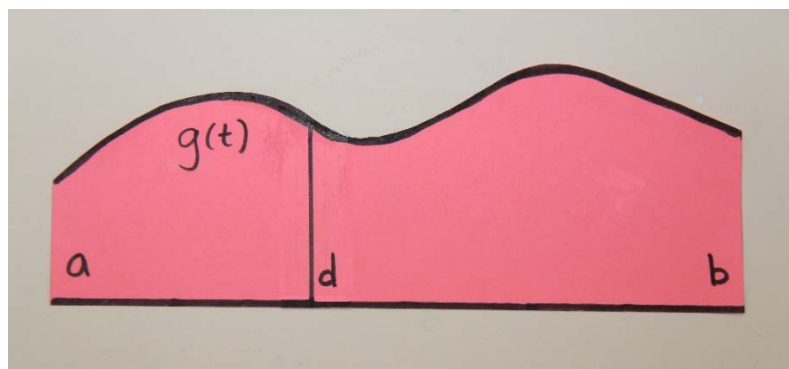


**Figure 14 Zero area property illustrated with a manipulative**

After demonstrating the properties, it was explained that since definite integrals obey the above properties, just as real numbers and variables do, they can be manipulated algebraically in equations. Several examples were worked out together, and the students were given time to practice with problems similar to those on the pretest. They were able to ask questions of the teacher and of their classmates. At the end of the practice time, the students were given ranking questions and a posttest with problems similar to the pretest and practice problems.

This lesson had both a control group and an experimental group. The goal of the pretest data collected was to compare the skill level of the two groups before the lesson, and the goal of the posttest data collected was to compare the skill levels of the two groups after their respective lessons.

## Methodology Summary

This study consisted of four lessons to four groups of students. Two lessons, one on optimization, and one on finding the volume of solids of revolution, both incorporating the use of manipulatives were delivered to two different groups of students. Data collected for these lessons came from ranking pre-lesson questions, a pretest, ranking post-lesson questions, a posttest, and both ranking and open-ended experiential questions. For these two manipulative lessons, the goal of data analysis was to determine whether significant learning took place, and to review the students' experiences in learning a calculus topic using manipulatives.

The third and fourth lessons, on the properties of definite integrals, were delivered to two groups. One group received the lesson with the use of manipulatives, and the other

group received the lesson in the traditional way. Data collected for these lessons was obtained from ranking pre-lesson questions, a pretest, ranking post-lesson questions, and a posttest (Appendix 14; Appendix 15). The goal of data analysis for this lesson was to compare the skills levels of the two groups, both before and after the different types of lessons.

The results of these analyses are discussed in Chapter 4.

# Chapter 4  Analysis and Results

This study examined the role of manipulatives in teaching calculus concepts to high school students. It explored the effect of lessons taught with manipulatives on both student learning and on the students' experiences with the use of manipulatives. For one concept, a lesson was taught to a group of students in the traditional way, without the use of manipulatives, and compared to the same topic taught to another group in a lesson that incorporated the use of manipulatives. A lesson for each of two other concepts was developed and taught using manipulatives as the principal teaching strategy.

The first of these two lessons was on optimization of functions using the derivative (Optimization) and the second was on the volume of solids of revolution about the x-axis (Solids). For each lesson, the use of a pretest and posttest provided numerical data to measure students' change in understanding of the respective concepts between the pretest and the posttest. Qualitative data was gathered by way of pre- and post-lesson 5-point rating scale questions, and two open-ended post-lesson questions.

The comparison lessons were on understanding and using the properties of definite integrals (Definite Integrals). The topic was taught without manipulatives to one group of students, and with manipulatives to a different group of students. Qualitative data was collected in the same manner as it was for the Optimization and Solids lessons, and a pretest and posttest were also administered to measure change in comprehension of the concepts. However, for these lessons, additional data analysis was performed to compare results between the two groups.

In addition to the above, all students were given an end-of-study questionnaire consisting of 5-point ranking scale statements relating to their feedback on working with manipulatives in this study and in prior math experience.

## Quantitative Data Results

For each of the four study lessons, the researcher sought to determine whether there was a difference between the students' understanding of the concept before the lesson and after the lesson, as measured by the pretest and posttest scores. With the assumption of a normal distribution for the underlying population and two test scores for each student, a paired t-test was appropriate for this purpose (Kokoska, 2015, p. 492).

In addition, for the concept where there was a control lesson and an experimental lesson, respective pretest scores and posttest scores were compared between the two groups. The purpose of the comparison for the pretest scores was to determine whether the two groups were at approximately the same skill level before proceeding with the lessons. After the lessons, comparison of the posttest scores between the groups was to ascertain whether there was a significant difference between the mean scores of the two groups, presumably due to the difference in method of instruction. It should be noted that while the two groups could not be randomized using a randomization procedure, and this may limit the reliability of the results, there is no reason to suspect any systematic difference between the two groups. Assuming a normal distribution and independent samples of unequal size and variance, a two-sample $t$-test with unequal variances was the appropriate tool (Kokoska, 2015, p. 482).

It is noted that all statistical tests were conducted with a confidence level of 95% ($\alpha = 0.05$). The information in Kokoska (2015) was used to compute the test statistic ($t$) (p. 481 and p. 491) and critical value (CV) for all statistical tests described in this thesis. The test result tables were generated by running the "t-Test: Paired Two Sample for Means" or the "t-Test: Two-Sample Assuming Unequal Variances" tests in Excel 2013.

## Lessons 1 and 2

For Lesson 1 (Optimization) and Lesson 2 (Solids), there were no control groups, i.e. all students were taught with manipulatives. The comparison for these lessons was based on the group's pretest and posttest scores. A paired t-test was conducted on the Lesson 1 sample data (Table 1).

$$\text{Null Hypothesis: } H_0\text{: } X_D = X_2 - X_1 \leq 0$$

There is no difference (or a negative difference) between the mean scores on the pretest and posttest for Lesson 1.

$$\text{Alternate Hypothesis: } H_A\text{: } X_D = X_2 - X_1 > 0$$

The mean score on the posttest is higher than the mean score on the pretest for Lesson 1.

Comparing the test statistic to the critical value,

$$\text{t Stat} = 11.057 > \text{t Critical one-tail} = 1.860,$$

the test statistic is in the rejection region and $p < 0.05$. The null hypothesis is rejected for the Lesson 1 sample.

**Table 1 Pretest to Posttest Comparison – Lesson 1**

| t-Test: Paired Two Sample for Means | Pretest | Posttest |
|---|---:|---:|
| Mean | 0.222222222 | 4.777777778 |
| Variance | 0.444444444 | 1.694444444 |
| Observations | 9 | 9 |
| Hypothesized Mean Difference | 0 | |
| df | 8 | |
| t Stat | 11.05687774 | |
| P(T<=t) one-tail | 0.000001995 | |
| t Critical one-tail | 1.859548038 | |

The paired t-test was also run on the Lesson 2 pretest and posttest sample data (Table 2).

$$\text{Null Hypothesis: } H_0\text{: } X_D = X_2 - X_1 \leq 0$$

There is no difference (or a negative difference) between the mean scores on the pretest and posttest for Lesson 2.

$$\text{Alternate Hypothesis: } H_A\text{: } X_D = X_2 - X_1 > 0$$

The mean score on the posttest is higher than the mean score on the pretest for Lesson 2.

Comparing the test statistic to the critical value,

$$\text{t Stat} = 10.304 > \text{t Critical one-tail} = 2.015,$$

the test statistic is in the rejection region. The null hypothesis is rejected for the Lesson 2 sample as well.

**Table 2 Pretest to Posttest Comparison – Lesson 2**

| t-Test: Paired Two Sample for Means | *Pretest* | *Posttest* |
|---|---|---|
| Mean | 0.333333333 | 5.083333333 |
| Variance | 0.666666667 | 0.541666667 |
| Observations | 6 | 6 |
| Hypothesized Mean Difference | 0 | |
| df | 5 | |
| t Stat | 10.30419675 | |
| P(T<=t) one-tail | 0.000074020 | |
| t Critical one-tail | 2.015048373 | |

The significant P-values are $P < 0.00001$ and $P = 0.00074$ for the Lesson 1 and Lesson 2 data, respectively. The results are significant for $p < 0.05$, so it appears, then, that the manipulatives-based study lessons on optimization of functions (Lesson 1) and on finding the volumes of solids of revolution (Lesson 2) increased the students' conceptual and practical understanding of these concepts.

## Lessons 3 and 4 –Control and Experimental

In the last two lessons of the research study, a concept (Properties of Definite Integrals) was taught in the traditional way, without the use of manipulatives, to a control group. The same lesson topic was taught, with the use of manipulatives, to a second group of students, the experimental group. Each group completed a pretest on the lesson material before the lessons were administered to either group. A two-sample t-test, for unequal variances, was run on the data (Table 3), with the purpose of determining whether both sets of students were at approximately the same skill level before their respective lessons.

Null Hypothesis: $H_0: X_2 - X_1 = 0$

There is no difference between the mean pretest scores of the control group and the experimental group.

$$\text{Alternate Hypothesis: } H_A: X_2 - X_1 \neq 0$$

There is a difference between the mean posttest scores of the control group and the experimental group.

**Table 3 Control to Experimental Comparison – Lessons 3 and 4 Pretest**

| t-Test: Two-Sample Assuming Unequal Variances | Control | Experimental |
|---|---|---|
| Mean | 0.33333333 | 1.75000000 |
| Variance | 0.66666667 | 1.58333333 |
| Observations | 6 | 4 |
| Hypothesized Mean Difference | 0 | |
| df | 5 | |
| t Stat | 1.98969950 | |
| P(T<=t) two-tail | 0.103289653 | |
| t Critical two-tail | 2.570581836 | |

Comparing the test statistic to the critical value,

$$\text{t Stat} = 1.990 < \text{t Critical two-tail} = 2.57,$$

the null hypothesis failed to be rejected because the test statistic was not in the rejection region and $p > 1.0$ for a two-tailed test.

It seemed to be the case, then, that there was no significant difference between the skill levels of the two groups of students before the lesson. The researcher was satisfied that a comparison of the two groups would be valid, and proceeded with the lessons.

After Lesson 3 and Lesson 4 were taught to the control and the experimental groups, respectively, the researcher first wanted to determine, separately, whether each group had improved their understanding of the concept, based on pretest and posttest scores.

A paired *t-test* was conducted on the control group's pretest and posttest sample data (Table 4).

$$\text{Null Hypothesis: } H_0: X_D = X_2 - X_1 \le 0$$

There is no difference (or a negative difference) between the mean scores on the pretest and posttest, for Lesson 3, the control group.

$$\text{Alternate Hypothesis: } H_A: X_D = X_2 - X_1 > 0$$

The mean score on the posttest is higher than the mean score on the pretest, for Lesson 3, the control group.

**Table 4 Pretest to Posttest Comparison – Lesson 3 – Control**

| t-Test: Paired Two Sample for Means | Pretest | Posttest |
|---|---|---|
| Mean | 0.333333333 | 3.333333333 |
| Variance | 0.666666667 | 3.466666667 |
| Observations | 6 | 6 |
| Hypothesized Mean Difference | 0 | |
| df | 5 | |
| t Stat | 3.503245249 | |
| P(T<=t) one-tail | 0.008566496 | |
| t Critical one-tail | 2.015048373 | |

Comparing the test statistic to the critical value,

$$t \text{ Stat} = 3.503 > t \text{ Critical one-tail} = 2.015,$$

the test statistic is in the rejection region, so the null hypothesis is rejected for comparison of pretest to posttest for the control group.

The paired t-test was also conducted on the experimental group's pretest and posttest sample data (Table 5).

$$\text{Null Hypothesis: } H_0: \ X_D = X_2 - X_1 \le 0$$

There is no difference (or a negative difference) between the mean scores on the pretest and posttest for Lesson 4, the experimental group.

$$\text{Alternate Hypothesis: } H_A: \ X_D = X_2 - X_1 > 0$$

The mean score on the posttest is higher than the mean score on the pretest for Lesson 4, the experimental group.

**Table 5 Pretest to Posttest Comparison – Lesson 4 – Experimental**

| t-Test: Paired Two Sample for Means | Pretest | Posttest |
|---|---|---|
| Mean | 1.75 | 6 |
| Variance | 1.583333333 | 0 |
| Observations | 4 | 4 |
| Hypothesized Mean Difference | 0 | |
| df | 3 | |
| t Stat | 6.755115021 | |
| P(T<=t) one-tail | 0.003313589 | |
| t Critical one-tail | 2.353363435 | |

Comparing the test statistic to the critical value,

$$t \text{ Stat} = 6.755 > t \text{ Critical one-tail} = 2.353,$$

the test statistic is in the rejection region, so the null hypothesis is rejected for comparison of pretest to posttest for the experimental group. It appears, then, that both the lesson without the use of manipulatives (Lesson 3) and the lesson that incorporated the use of manipulatives (Lesson 4) increased the students' conceptual and practical understanding of the properties of definite integrals.

The remaining research question for Lessons 3 and 4 was whether there was a significant difference in the gain of conceptual understanding between the control and experimental groups, as measured by a comparison of the posttest scores between the respective groups. To this end, a two-sample $t$-test for unequal variance was conducted on the data (Table 6).

$$\text{Null Hypothesis: } H_0: X_2 - X_1 \leq 0$$

There is no difference (or a negative difference) between the mean posttest scores of the control group and the experimental group.

$$\text{Alternate Hypothesis: } H_A: X_2 - X_1 > 0$$

The mean posttest scores of the experimental group are significantly greater than those of the control group.

Comparing the test statistic to the critical value,

$$\text{t Stat=3.508} > \text{t Critical one-tail =2.015,}$$

the null hypothesis was rejected, because the test statistic was in the rejection region. The results of this test seem to indicate a statistically significant difference in the mean posttest scores in favor of the group taught with manipulatives.

**Table 6 Control to Experimental Comparison – Lessons 3 and 4 Posttest**

| t-Test: Two-Sample Assuming Unequal Variances | Control | Experimental |
|---|---|---|
| Mean | 3.33333333 | 6.00000000 |
| Variance | 3.46666667 | 0.00000000 |
| Observations | 6 | 4 |
| Hypothesized Mean Difference | 0 | |
| df | 5 | |
| t Stat | 3.50823208 | |
| P(T<=t) one-tail | 0.008566496 | |
| t Critical one-tail | 2.01504837 | |

## Summary of Quantitative Results

Several comparisons were done using quantitative methods on pretest and posttest sample data generated in this research study. First, for the control and experimental group lessons, a test was run to determine if the two groups were at approximately the same level of content knowledge of the lesson material, as measured by a statistically significant difference (or no difference) in the pretest means of the two groups. The test results indicated no significant difference between the two groups.

The next comparison, for all four study lessons, was to conduct tests on the sample data to determine if there was a gain in conceptual understanding of the lesson material, as measured by a statistically significant increase in the mean pretest and posttest scores. In the four cases, results indicated a difference in favor of the posttest. This is interpreted to mean that learning took place with the use of manipulatives in instruction as well as in traditional instruction, without the use of manipulatives.

Finally, a two-sample *t*-test assuming unequal variances was run on sample posttest mean scores to determine whether there was a significant difference in

understanding between the control and experimental groups. Results indicated a difference in favor of the experimental group.

In conclusion, with respect to quantitative results, while all four groups of students exhibited significant learning in their respective lessons, in a direct comparison between the control group and the experimental group, the experimental (manipulative) group exhibited a higher learning gain. The next section of the data analysis looks at the students' and teacher's experiences in using manipulatives with the calculus concepts taught in the research study lessons.

## Qualitative Data Results

With the goal of expounding on the quantitative results of the study, this section examines the students' observations and comments on working with manipulatives, prior use of math manipulatives, and statements related to attributes of manipulatives.

After the posttest, each student completed a questionnaire that included ten 5-point ranking statements and two open-ended questions (Appendix 7), summarized below (Table 7). The students were asked to rank their level of agreement, ranging from agree "not at all" to agree "a great deal", for each statement, and to answer the two open-ended questions.

### 5-point ranking statements

To summarize the ranking statement data, the researcher used survey-interpretation techniques described in Sauro (2011). First, numbers were assigned to the levels of agreement, with 1 corresponding to "not at all", and 5 corresponding to "a great

deal" (Table 8). The number of responses at the 4 and 5 level, considered to be "agree" responses, were counted, and these numbers were divided by the total number of responses to get the *Percent Agree* figure, also called *Top Two Box* scoring. The mean and standard deviation were calculated for the data in each statement (Table 9).

**Table 7 End-of-Study Questionnaire 5-point Ranking Scores and Statements**

| Item | Statement |
|------|-----------|
| A | I now understand the concept taught in the research study lesson. |
| B | I would not have learned the concept as well without any manipulatives (transposed). |
| C | In past math classes I have had one or more math lessons where manipulatives were used. |
| D | It would have taken longer for me to understand the concept without the use of the manipulatives in the study lesson. |
| E | I understand the concept presented to me in the study lesson. |
| F | The use of manipulatives in the lesson were helpful to me in understanding the concept. |
| G | I am a tactile learner; I learn better when I can touch things. |
| H | I want to know where formulas come from, not just memorize them (transposed). |
| I | It is easier for me to learn when I can see a diagram, sketch or picture. |
| J | I do not learn mathematics well in a traditional lecture format (transposed). |
| Q1 | Please describe how the manipulatives used in the lesson enhanced or detracted from your learning of the concept taught in the lesson. |
| Q2 | Please provide any other feedback, suggestions or comments with respect to your experience in learning a calculus concept with the use of a manipulative. |

**Table 8 Numerical Scores for 5-point Ranking Statements**

| Level of Agreement | Not at all | Just a little | Somewhat | A lot | A great deal |
|---|---|---|---|---|---|
| Numerical score | 1 | 2 | 3 | 4 | 5 |

**Table 9 Statistical Tests on 5-point Ranking Statement Data (n = 18)**

| Statement | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| Mean | 3.88 | 2.82 | 2.76 | 3.29 | 4.06 | 3.94 | 3.00 | 3.47 | 3.65 | 3.24 |
| Standard Dev | 1.11 | 1.19 | 1.20 | 1.21 | 1.09 | 0.90 | 1.22 | 1.23 | 1.11 | 1.25 |
| Top 2 Box | 13.00 | 3.00 | 5.00 | 8.00 | 13.00 | 12.00 | 6.00 | 10.00 | 9.00 | 6.00 |
| Percent Agree | 0.76 | 0.18 | 0.29 | 0.47 | 0.76 | 0.71 | 0.35 | 0.59 | 0.53 | 0.35 |

Item A and Item E in the questionnaire were designed to gauge students'
perception of their understanding of the lesson. The *Percent Agree* for both of these items
was 76%, demonstrating consistency in the responses. The researcher was interested in
comparing the students' perception of understanding to their scores on the posttest. The
data for responses to the statement "I understand the concept presented to me in the study
lesson" (Item E) was chosen for comparison with the posttest scores, which were each
converted from a 6-point score to an equivalent 5-point score for this purpose. The
assumption of normality of the underlying population for the test scores was stated in the
previous section, and "rating scale means often follow a normal or close to normal
distribution" (Sauro, 2011, para. 4). With two scores for each student, a paired *t* test was
appropriate, and was run on the Item E and posttest data for all students (Table 10).

$$\text{Null Hypothesis: } H_0: X_D = X_2 - X_1 = 0$$

There is no difference between mean scores on Item E on the end-of-study

questionnaire and mean posttest scores.

$$\text{Alternate Hypothesis: } H_A\text{: } X_D = X_2 - X_1 \neq 0$$

There is a significant difference between mean scores on Item E on the end-of-

study questionnaire and mean posttest scores.

**Table 10 Comparison of Posttest Scores to Perception of Learning (Item E) All Students**

| t-Test: Paired Two Sample for Means | Posttest | Item E |
|---|---|---|
| Mean | 4.191176471 | 4.058823529 |
| Variance | 0.813163807 | 1.183823529 |
| Observations | 17 | 17 |
| Hypothesized Mean Difference | 0 | |
| df | 16 | |
| t Stat | 0.494301178 | |
| P(T<=t) two-tail | 0.627811216 | |
| t Critical two-tail | 2.119905299 | |

Comparing the test statistic to the critical value,

$$\text{t Critical two-tail} = 0.494 < \text{CV} = 2.120,$$

the null hypothesis failed to be rejected because the test statistic was not in the rejection

region. It appears then, that students' perception of their understanding of the lesson was

confirmed by the objective posttest scores.

Regarding the *Percent Agree* entries in Table 7, items B, C, D and F are

statements about the use of manipulatives. Less than one-third of the students, 29%, had

prior experience with math manipulatives (Item C), so it seems that the research study

lesson was the first exposure to manipulatives for many students. Regarding the use of

manipulatives in the lesson, 29% felt that they would not have learned the concept as well without them (Item B), while 47% expressed that it would have taken them longer to understand the concept without the manipulatives (Item D). Finally, when asked their level of agreement to the statement that the use of manipulatives was helpful in understanding the lesson concept (Item F), 71% of the students agreed that they were. Based on this feedback, it seems that for the majority of students the manipulatives were helpful in learning the concept more quickly or in learning the concept overall.

In Item H, students were asked their level of agreement to the statement that they want to know where formulas come from, not just memorize them. The intent of this item was to gauge students' interest in knowing concepts in depth, rather than memorizing formulas (or information) to pass a test, and the response to this was 59% agreement. Because "students who use concrete materials develop more comprehensive mental representations . . . and exhibit a deeper understanding of mathematical concepts" Dobbins et al. (2014), it appears that manipulative use in the study lessons met the needs of these students.

Because the manipulatives used in this study gave students the opportunity to visualize and physicalize abstract concepts, a much different presentation means than that which is usually used in a calculus class, the researcher was interested in the students' perception of how they learn. 35% of the students agreed that they learn better when they can touch things (Item G), 53% agreed that it is easier for them to learn when they can see a diagram, sketch or picture (Item I), 29% agreed to both Items G and I, and 35% agreed that they do not learn mathematics as well in a traditional lecture format (Item J). It is noted here that learning style theory has been debated, with at least one recent study

citing lack of evidence for its support (Cuevas, 2015). Statements G, I and J on the questionnaire were directed toward the ability of manipulatives to provide multiple pathways to learning, rather than to assess the students' preferred learning styles.

Open-Ended Responses

In the end-of-study questionnaire, the students answered two open-ended questions regarding manipulative use in the calculus research study lesson (Table 7). In analyzing responses to these questions, the researcher reviewed each response, assigned it to one or more categories, and tabulated the results (Table 11).

An overall positive experience with using manipulatives in the lesson was reported by 82% of the students, and 59% stated the manipulatives made the lesson more fun, interesting, or enjoyable. Examples of these comments are shown below (Table 12).

It was reported by 76% of the students that manipulatives helped them understand the lesson, and of these, 69% used several forms of the word "visualize" in describing how the manipulatives were of help to them. Abbreviated versions of these comments are shown below (Table 13). This report by the students is significant, because as they progress to higher levels of mathematics, abstract concepts become harder for some students to visualize (Bhatia et al., 2014; Cruse, 2012; Doias, 2013; Kaplan, 2009, Koss, 2015; Thirey &Wooster, 2015; Uhl et al. 2006).

**Table 11 Analysis of Open-Ended Responses**

| Description of Comment Category | Number Reported | Out of | Percent | |
|---|---|---|---|---|
| Reported overall positive experience | 14 | 17 | 82% | 100% |
| Neutral experience | 3 | 17 | 18% | |
| Did not enhance or detract from learning the concept | 3 | 17 | 18% | 100% |
| No comment on helpfulness | 1 | 17 | 6% | |
| Helped them to understand the concept . . . | 13 | 17 | 76% | |
| *by visualization** | 9 | 13 | 69% | 100% |
| *but time factor *** | 2 | 13 | 15% | |
| *but was distracting**** | 1 | 13 | 8% | |
| *no reason given* | 1 | 13 | 8% | |
| Made lesson more fun, interesting, and/or enjoyable . . . | 10 | 17 | 59% | |

**Table 12 Sample Student Responses – Overall Positive Experience**

This experiment was awesome; very fun, also made it much quicker to learn.

I really enjoyed the lessons because it made learning revolutions fun.

It was hands-on experience, which I liked.

Use more manipulatives - they are more interesting and fun.

They made the lesson a lot more interesting and easy to learn.

Manipulatives should be used more in everyday teaching of higher math classes.

**Table 13 Sample Student Responses – Manipulatives Helped with Visualization**

| |
|---|
| The manipulatives helped me visualize what I was learning. |
| I am a visual learner, so manipulatives helped a great deal. |
| It is a really good help for visual learners. |
| It showed me visually what I was solving for. |
| The use of manipulatives allowed me to clearly visualize the concept of revolutions. |
| I am better able to VISUALIZE what modifying an aspect of an object does to the whole. |
| The manipulatives in this lesson helped me visualize a 3D object that we are evaluating in terms of a 2D graph. |
| The manipulatives gave me an interactive visual representation of what I was learning. It really took the stress off. |
| I think manipulatives help a great deal by providing a fun, interactive, visual way to learn concepts. |

In their statements that manipulatives were helpful to them, several students gave reasons other than visualization and others did not give a reason. Abbreviated versions of these comments are shown below (Table 14).

There were no negative comments, however, three students that found the manipulatives helpful also reported some difficulties. Two students, 15% of the sample, stated that they could have used more time to work with the manipulatives. One student, 8% of the sample, said that while very helpful, the manipulatives were also distracting, as many of the students in the class found them fun to play with. These comments corroborate study recommendations in the literature that discuss the need for effective teacher control, additional time, and guidance in the use of manipulatives (Doias, 2013; Golafshani, 2013; Hartshorn & Boren, 1990; Magruder, 2012; Mendiburo & Hasselbring, 2011; Morin & Samelson, 2015; Stein & Bovalino, 2001; Swan & Marshall, 2010).

**Table 14 Sample Student Responses – Manipulatives Helped – Other Reasons**

| |
|---|
| It helps you break down something extremely complicated into something simple. |
| You can observe easier how the concepts and formulas work with your own hands. |
| Without the manipulatives, I would not have fully understood the lesson. |
| It was essential to learning the concepts. |
| They helped a lot. |
| It was refreshing and helpful. |
| It helped me learn. |
| This lesson worked and I feel that the manipulative method helped me understand the concept. |

Finally, several responses were neutral. Three students, 18% of the sample, stated that the manipulative neither enhanced nor detracted from their learning of the concept, although one of these students stated that the manipulatives "helped keep the concept interesting and made it more entertaining to do the math."

## Teacher Impressions

From the perspective of the researcher, also the regular teacher of the students that participated in this study, all of the manipulative lessons were very well received by the students. In general, the students were enthusiastic and cooperative during the four research study lessons. Specific observations regarding student interest, learning, and behavior were made. Due to differences in the concept, manipulative used, and students, there will be observations that pertain only to one lesson, and others apply to all lessons.

In the Optimization lesson, the students were cooperative and motivated for the first part of the project, in which they built the box at home and reported data and

analyzed the results in class the next day. The finished boxes, along with the online graphs of the associated functions, were effective in showing the different volumes and surface areas that resulted from distinct values of the cuts used to make the boxes.

In the second part of the lesson, the students used cardstock paper to make boxes with cuts "starting" at zero and increasing until there was nothing left to cut. The purpose of this was for them to discover a restricted domain and to "see" the Extreme Value Theorem. This part caused some confusion, however, with one student exhibiting signs of frustration. In reflecting on this, the researcher now understands that this particular type of box was more complex than was necessary for the students to realize the objectives. This underscores findings in the literature that mention the "great deal of prior planning and organization" required to effectively use manipulatives (Curtain-Phillips, 2015) and the importance of selecting an appropriate manipulative for the concept (Morin & Samelson, 2015). In relation to this, longtime Montessori educator A. Kapor-Mater explains that "one of the reasons manipulatives work so well in Montessori schools is that the manipulatives themselves and the lessons used to give them have been carefully refined over the last hundred years" (personal communication, March 31, 2016).

Manipulative use in the Solids lesson was successful in terms of the conceptual understanding of the objectives, and of student interest and motivation. When the students opened the rectangular accordion figure to reveal that a "slice" of the solid was the figure of a circle, the teacher heard the following verbal comments: "It's a circle!" "I get it!" "Oh, wow!" and "Thank you, Ms. Rivero, for teaching us this way." This level of enthusiasm continued, and the students mastered the objective of the lesson and progressed well in the topic. Based on years of experience, the topic of solids of

revolution is very difficult for students to understand, and in the opinion of this researcher, the manipulatives in this lesson were instrumental in the students' success with the concept.

There were some problems with this lesson, however. Many of the students in the class were fascinated with the accordion figures, both as objects and as manipulatives. As objects, they were fun to open and close, and were colorful and pretty. Considering them as manipulatives, the students were amazed at what they had just learned, the way in which they had learned it, and were very excited about it. Both of these factors contributed to lively interactions and a noisier-than-normal atmosphere in the classroom. This was distracting to at least one student, enough so as to mention it in the questionnaire comment item. This researcher included, teachers must be mindful to keep the students focused on the activity and the class under control.

The experimental group for the Definite Integrals lesson also responded eagerly to the manipulative used in their lesson, but with no disruptions to the class. The control group had their lesson in the traditional lecture format, but they were exposed to the manipulatives when the topic was revisited in preparation for the AP Calculus AB exam.

In addition to the lesson-specific observations, there are two additional notes. First, among the two calculus classes that participated in this study, there are several students that are doing very well in their respective class, with an A+ average each quarter. From observations during the manipulative lessons, these students, while by no means indifferent, were not particularly enthusiastic. One of these is the student mentioned earlier that became frustrated with the box-cutting in the optimization lesson.

This researcher considers the possibility that students such as these, that can assimilate abstract concepts quickly, may be impatient to move on to new concepts.

This leads to the second point, time. It seemed to this researcher that when manipulatives were used in the lesson, developing the concepts took more time. The concern that it "takes more class time to introduce a concept using manipulatives," is mentioned in the literature as well (Golafshani, 2013). This issue of time has additional implications with respect to the use of manipulatives, and will be discussed further in the next chapter.

## Summary of Qualitative Results

Qualitative data on the research study was collected by means of a questionnaire administered at the end of the students' participation in the study. The students indicated their level of agreement to ten statements using a 5-point ranking scale, ranging from agreeing "not at all" to agreeing "a great deal." The ranking levels were assigned numerical scores ranging from 1 to 5, respectively. One comparison was made using a paired *t*-test, and the other items were interpreted using *Top Two Box* scoring, where the top two scores of 4 and 5 were considered as "agree" scores.

The data indicated that the students' perception of their understanding of the lesson material was consistent with their scores on the posttest. A majority of the students reported an overall positive experience with the use of manipulatives, and both in the ranking statements and in the open-ended questions, more than two-thirds of the students indicated that the manipulatives were helpful to them in learning the lesson concept.

A small number of students stated that the manipulatives neither helped nor hindered their learning experience, while two students mentioned concerns with not

enough time to work with the manipulatives and one was distracted by other students using the manipulatives quasi-inappropriately.

The teacher's impressions were also taken into account as qualitative information. Although the lessons went very well, each one begot concerns. In one of the lessons, a manipulative was not well-suited for the objective, resulting in frustration for one student. In another lesson, the students were fascinated and distracted by some of the manipulatives.

Regarding the teacher's observations of the students, most were motivated and cooperative, and behaved well, in spite of the distractions just mentioned. The teacher was concerned with the seemingly additional time required to present and develop the concept properly with the manipulatives, and with the possibility of the more advanced being ready to move before the rest.

Implications of these results and conclusions and recommendations are discussed in the next chapter.

# Chapter 5  Summary and Conclusions

This research study investigated the use of manipulatives in teaching calculus concepts at the secondary school level. The objective of the study was to learn whether manipulatives can facilitate student learning of abstract calculus concepts. With this is mind, the literature was reviewed for publications related to the use of manipulatives in teaching advanced mathematics, specifically calculus. Manipulatives were developed or adapted to support the teaching of calculus concepts. Lessons incorporating the use of these manipulatives were taught to students in two high school calculus classes. One topic was taught with manipulatives to one group of students and in a traditional lecture format, without the use of manipulatives, to a different group of students.

## Findings

A search of the literature for information involving the use of manipulatives in teaching mathematics found numerous research studies and articles involving the elementary and middle school math levels. However, only four publications were found that addressed the use of manipulatives at the secondary school math level. Three of these were in Geometry and one was in Algebra 2.

While no research studies involving calculus students were found, several calculus manipulatives were described in the literature by educators, mostly for use at the college level. One of these manipulatives was adapted for use in this project. The other two sets of manipulatives used in this project were developed by the researcher

Regarding the research study lessons, a comparison of mean pretest and posttest scores for each of the groups indicated that all groups experienced a statistically significant positive learning outcome, indicating an understanding of the lesson concepts, regardless of method of instruction. With respect to the control and experimental groups, although a comparison of mean pretest scores showed no significant difference between the groups, a comparison of mean posttest scores indicated a statistically significant difference in favor of the experimental (manipulative) group.

In their responses to an end-of-study questionnaire, a large majority of the students reported overall positive experiences in the manipulative lessons, using words such as "fun", "interesting", "enjoyable", and "helpful." More than half of the students also reported that the manipulatives were helpful to them by allowing them to visualize the concepts. Other students stated that manipulatives helped them to learn for other reasons, such as that manipulatives "break down something extremely complicated into something simple", or that "you can observe . . . with your own hands."

A few of the students reporting that the manipulatives were conducive to learning also mentioned that they needed more time to work with the manipulatives, and one mentioned being distracted by noise level due to the novelty of the manipulative objects. In addition, the researcher observed a usually high achieving student frustrated in working with one of the manipulatives. A few students reported that the manipulatives neither helped nor hindered their learning.

## Implications

In reflecting on the quantitative results, it seems to be the case that for the group of students that participated in the project, instruction with and without manipulatives was

effective as measured by posttest results. There is also evidence to suggest that instruction was more effective with the use of manipulatives when directly compared to instruction in the traditional lecture format.

Impressions related by the students and teacher were primarily positive, which combined with the learning gains observed form the quantitative results, suggest that the use of manipulatives should be considered in teaching calculus concepts at the high school level. However, the concerns discussed below should be taken into account when considering the use of manipulatives in teaching advanced math concepts at the high school.

## Time Factor

Based on many years of teaching at all levels, and corroborated in the literature, the researcher believes it takes more classroom time to introduce and develop concepts with manipulatives. This increased time may be acceptable in elementary and middle school math classes when teaching Algebra or Geometry, because manipulatives help to develop a conceptual understanding, i.e. a solid foundation of the concepts needed to progress to the next level, as described in the literature. But due to the large amount of material to be covered in an AP Calculus class in a limited amount of time, this researcher/teacher believes that manipulatives should be used judiciously in AP Calculus. For example, manipulatives would be especially beneficial with concepts such as the chain rule and the limits and properties of definite integrals. Time invested in these fundamentals would be well spent. For concepts that are difficult to visualize, manipulatives such as the accordion figures used in the solids of revolution lesson in this study can be used for demonstration purposes without taking much out of the class time for an AP Calculus class.

The researcher believes that manipulatives can be used successfully for a number of topics in a regular calculus class, where the fast pace required to cover the curriculum for a standardized test, the AP Calculus exam, is not a factor. This recommendation is aimed at students who are not ready for an AP Calculus class in high school, but who would need to take a calculus class for their intended college major. For these students, the researcher recommends a thorough treatment of the basic calculus concepts, using manipulatives for many of them. The goal would be provide the students with a solid foundation in calculus so that they may succeed in an introductory calculus class at the college level. To effect this at her own school, the researcher plans to thoroughly investigate the manipulatives compiled in Appendix 1 for use in the school's Honors Calculus class next school year.

## Appropriate Use of Manipulatives

The idea of using manipulatives appropriately was discussed in the literature, and its importance was encountered in the classroom lessons in this study.

Because younger children and adolescents are at different developmental stages, they use manipulatives differently. Older students are able to abstract and can generally comprehend information given to them in a traditional lecture format. But if presented to them properly, manipulatives can be used to deepen their understanding of abstract concepts (Waski, 2012). For example, at the elementary level, many manipulatives, such as fraction bars and abacus, are used for calculations. These types of manipulatives would not be necessary at advanced math levels, as students are adept at calculations at this stage. However, demonstrative manipulatives, which illustrate a concept, but are not

precise enough to be used as a tool for calculation, would work for many concepts at all levels (A. Kapor-Mater, personal communication, April 5, 2016).

Several demonstrative manipulatives were used in the lessons for this study. For example, in the solids of revolutions lesson, the instructor used ready-made materials, the accordion figures, to show the students how a 3-dimensional solid was generated from a 2-dimensional area. This allowed the students to see the shape of the cross-section and derive the integral for the volume. In the definite integrals lesson, the students constructed the manipulatives themselves and used them to discover and demonstrate the properties of definite integrals.

It should also be noted that higher level math students do not necessarily require more complex manipulatives. For example, the box manipulative designed for the optimization lesson turned out to be more complicated than was needed to get the point across, where a simpler box would have likely made the activity progress more smoothly. This illustrates the idea of "conceptual congruence" espoused by Morin and Samelson (2015).

Another consideration regarding appropriate use of manipulatives is the school and classroom environment. Manipulatives in and of themselves will not necessarily benefit the students without other elements in place. Teachers must be trained and developed, both for their own knowledge of the manipulatives and to enable them to design constructive learning activities for their students. They must also provide the students with guidance and support, and maintain effective control of the class.

## Study Limitations

Although this study resulted in mostly positive results for the prospect of using manipulatives in teaching calculus students, the researcher acknowledges that caution be exercised in generalizing the results.

The total number of students that participated in this study was 19. Of this total, 9 students participated in one manipulative lesson and 6 participated in the other manipulative lesson. In the experimental/control lessons, 4 students participated in the experimental group, and 6 students participated in the control group. While quantitative results were generated, it is understood that they may not transfer to other situations due to the small number of students in the study. However, the results give an indication that it is possible to teach calculus with manipulatives, and that learning took place. Further research with a greater number of students can give generate additional information and more robust results.

The student participants in this study attend a private school of about 400 students in grades 6-12. Schools with a greater population may provide diverse pool of students for subsequent research.

## Conclusion

In general, the findings of this study were consistent with studies in the literature that have shown manipulatives to be effective at the lower levels. This research showed learning gains for all groups, and a slight edge to manipulatives in a direct comparison. Also, as widely discussed by other authors, the use of manipulatives raises concerns about the need for teacher training, selection of the right manipulatives, and the required

level of guidance. Overall, this researcher believes that both the qualitative and quantitative results of this study are promising and indicate that further research would be appropriate. More importantly, it is believed that manipulatives can help calculus students with abstract concepts, and pending additional research, teachers can and should consider the manipulatives presented here, and manipulatives that they may develop, in the instruction of high school level calculus.

# Appendix 1  Compilation of Calculus Manipulatives

In a review of the literature for the use of manipulatives used in high school calculus classes, none designed for use at the high school level were located. The manipulatives described in this appendix were designed by university professors, but can be used in a regular high school or Advanced Placement calculus class.

## Teaching Integration Applications Using Manipulatives (Bhatia, Premadasa and Martin, 2014)

The authors focused designing manipulatives and activities to address students' difficulty with integration as accumulation of pieces of a whole. The approach was to reinforce the concept of the definite integral as a limit of Riemann sums, including the concept of the sum of small quantities producing a larger whole and the concept of a larger number of smaller pieces producing a better approximation.

The manipulatives developed by Bhatia et al. included 7 wooden discs of uniform thickness and radii varying from smaller to larger. When laid one on top of the other, the discs fit into a circular glass bowl (Figure 15). In the activities, the students calculated the volume of the bowl using geometry and approximated the volume of the bowl by adding the volume of the seven discs. The approximation was repeated using a second set of 14 discs of a smaller thickness. The second approximation was closer to the actual volume, demonstrating that the larger number of discs resulted in a better approximation.

**Figure 15 Manipulatives developed by Bhatia et al. (2014) for integration applications**

Additional volume activities included measuring data points and fitting a curve to generate a solid of revolution. The authors also described an activity to demonstrate the concept of work, which was not discussed in this document.

It is noted that the manipulatives described above serve a similar purpose as those this author developed for one of the research study lessons in this thesis. However, the two sets differ in that the accordion figure manipulatives used in Lesson 2 (Solids) focused on the definite integral as an area revolved to generate a solid. The circular discs developed by Bhatia et al. are 3-dimensional objects that were used to form a model of the solid and whose volumes were added to approximate the volume of the solid.

## Manipulatives for 3-Dimensional Coordinate Systems (Koss, 2011)

These manipulatives were designed for a college level multivariable calculus class in response to the author's observation of the frustration experienced by some students in understanding three-dimensional coordinate systems represented by computer animation or drawn on a board. In addition to Cartesian three-dimensional coordinate systems,

which are not used in high school calculus, Koss describes manipulatives for use with polar coordinates, which are part of the AP Calculus BC curriculum. The manipulatives consisted of pipe cleaners used to represent the x and y axes, and a ray representing the angle θ. Beads attached to the pipe cleaners with fine gauge wire were used to represent the origin and the radius (Figure 16). The article includes instructions for constructing the materials and a copy of the student handout for the activities.
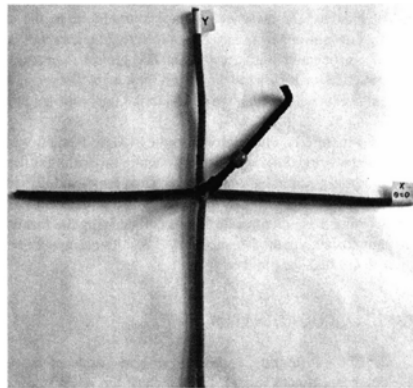


**Figure 16 Manipulative model of polar coordinates (Koss, 2011)**

Koss relates that, having used manipulatives in learning alternate coordinate systems, including polar coordinates, the students demonstrated better mastery of the material as evidenced by their responses to questions on the homework, regular exams, and on the final exam. In addition, the students reported positive comments on their experiences in using the manipulatives in their end-of-course evaluations.

The Touchy-Feely Integral: Using Manipulatives to Teach the Basic Properties of Integration (Thirey and Wooster, 2013)

This paper describes two sets of manipulatives used in relation to definite integrals. The first set was used in this thesis to illustrate many of the properties of definite integrals in Lesson 3, and is fully described in Chapter 3.

The second manipulative helps students understand the first part of the Fundamental Theorem of Calculus I (FTOC I). As Thirey and Wooster point out, students quickly learn and use the second part of FTOC I to evaluate definite integrals:

$$\int_a^b f'(x)dx = f(b) - f(a)$$

However, they often have difficulty in understanding and applying the first part of theorem, which deals with functions defined as integrals:

$$F(x) = \int_a^x f(x)dx, \text{ where } F'(x) = f(x)$$

In their description of the manipulative and how to use it, the authors refer to $F(x)$, as defined above, as the *area-so-far-function*. They use the cutout for the $f(t)$ function from the first set of manipulatives, cover it with a paper with the left edges and bottom horizontal edges aligned, and move the paper to the right to expose the *area-so-far* (Figure 17).

At this point, Thirey and Wooster apply the limit definition of the derivative to $F(x)$ as defined above, and use additional manipulative cutouts to follow the logic of a proof of the FTOC I (Figure 18).
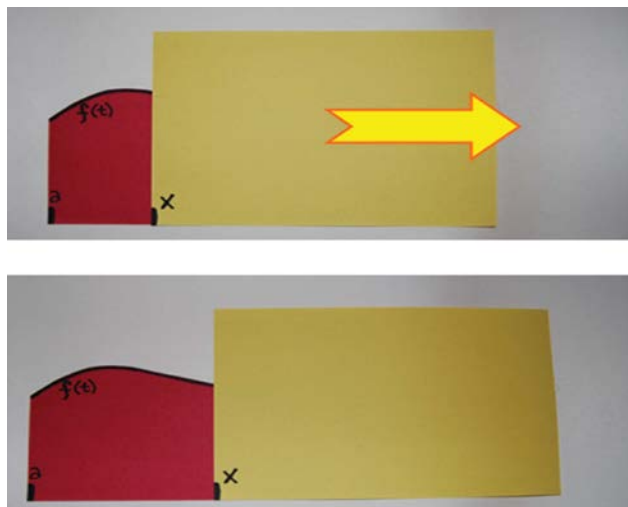
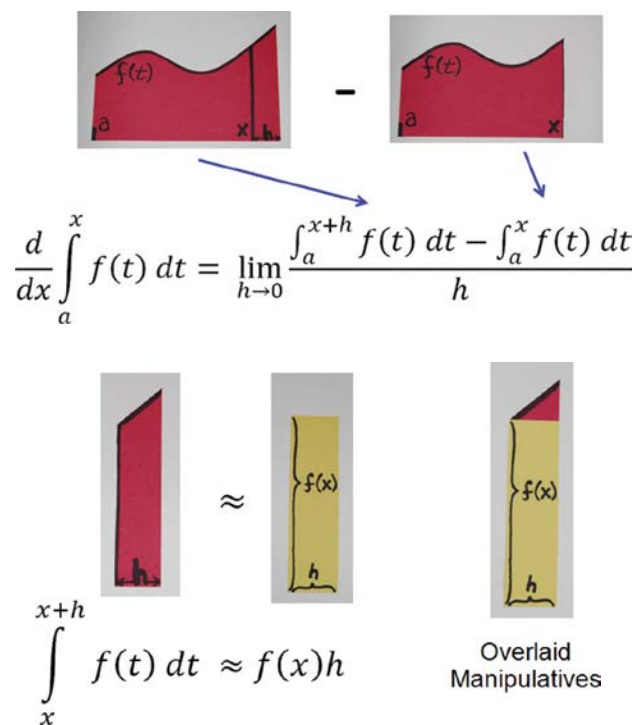**Figure 17 Manipulative used to demonstrate a function defined by an integral (Thirey & Wooster, 2013)**



$$\frac{d}{dx}\int_a^x f(t)\,dt = \lim_{h\to 0}\frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}$$

$$\int_x^{x+h} f(t)\,dt \approx f(x)h$$

**Figure 18 Intuitive steps toward a proof of the FTOC I using manipulatives (Thirey & Wooster, 2013)**

## Tactile Reinforcement for Early Calculus Concepts: Grocery Bags and Laser Beams (Cone, 2013)

In a continual search to engage his students and help them to become independent learners, this professor of an undergraduate calculus class describes his use of manipulatives to help students understand the chain rule and the trapezoidal rule.

Cone addresses confusion between function multiplication and function composition as the source of student problems with the chain rule. He addresses these in an innovative way using brown paper grocery bags and lunch bags. In the activity described in this paper, the author uses grocery bags labeled with the functions $x^2, 2x, \sin(x)$ and $\cos(x)$, and two smaller lunch bags labeled with the symbols 2 and +. Function multiplication is represented by placing the bags side-by-side, or concatenated, and function composition is represented by placing one bag inside of the other. However, the author uses an Inquiry-based Learning approach and has the students figure this out for themselves. He does this by asking them to use the bags to represent the product rule for the derivative of the function $x^2 \sin(x)$ (Figure 19). This is followed by using the bags to represent the derivative of the function $\sin(x^2)$ using the chain rule. (Figure 20).



**Figure 19 Manipulatives used to represent the product rule for derivatives**

**Figure 20 Manipulatives used to represent the product rule for derivative of a function (Cone, 2013)**

The second topic the author discusses is using the trapezoidal rule to approximate area under a curve, or in this case, area under an arch. He acknowledges that, although it does not use manipulatives in the true sense of the word, the activity serves to make the concept tangible.

In the activity described, the area under an architectural arch on campus grounds is approximated, but any large non-polygonal structure can be used. The students use a standard tape measure and chalk to measure and mark the ground length and subintervals, a laser measuring device for the heights, and a worksheet to record the data. The approximate area under the arch is then calculated using the trapezoidal rule.

The author relates positive feedback from the students both in the classroom and in end-of-course evaluations. He concludes by offering these the activities as examples and encourages teachers to rework and expand on them to actively engage students in their own classrooms.

Visualizing the Method of Finding Volumes by Cross Sections – An
Eggsperiment (Uhl, Humphrey and Braselton, 2006)

This paper described an interdisciplinary activity involving mathematics, engineering, and statistics, with hard-boiled eggs serving as manipulatives. The eggs were assumed to be ellipsoids, with circular cross-sections. The authors state that this setup will work to illustrate the volume of a solid of revolution as well as the volume of a solid with known cross-section, but the activity described here employs the latter.

Groups of students carefully measure the major and minor axis of their egg using a Vernier caliper, and write the equation of the ellipse that forms the base of the solid (half of the egg). The known cross-sections perpendicular to the base are semi-circles. The integral is written and doubled to derive a formula for the volume of the entire egg (Figure 21).
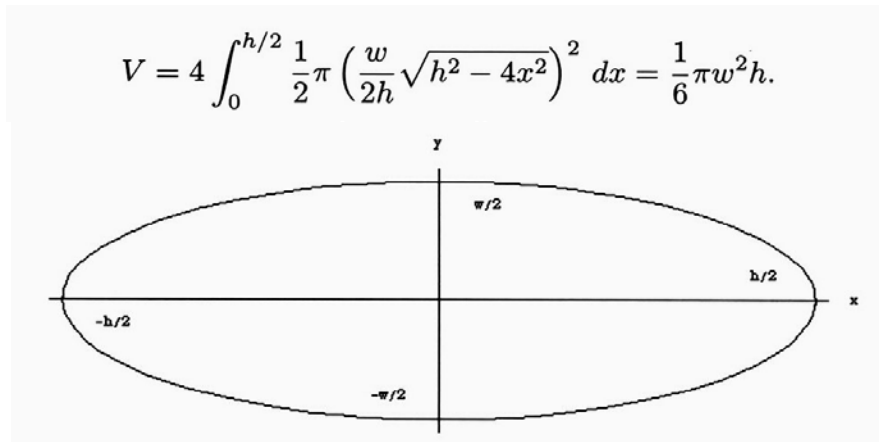
$$V = 4 \int_0^{h/2} \frac{1}{2}\pi \left( \frac{w}{2h} \sqrt{h^2 - 4x^2} \right)^2 dx = \frac{1}{6}\pi w^2 h.$$



**Figure 21 Manipulatives used to represent the derivative of a function using the chain rule (Cone, 2013)**

Each group calculates the volume of the egg using calculus with the derived formula, and next the engineering and statistics parts come into play. The volume of each egg is estimated by measuring the amount of water displaced by placing the egg into a graduated cylinder. The results for the class are compiled and analyzed statistically to determine the soundness of the derived formula.

In addition to discovering the variability in the volumes of Grade A eggs, the authors found this exercise to be pedagogically meaningful and thoroughly enjoyable for their students.

## Playing with Dominoes: Proof by Induction (Kaplan, 2009)

This activity was designed to provide an additional way for students to practice the concepts of reasoning and proof, as stated in the NCTM Standards. The author provides groups of students with dominoes and has them design and construct a pattern of dominoes that will all fall when the first domino is pushed. The designs are tested one by one. Next the students are challenged to design patterns where the all of the dominoes do not fall. These exercises force the students to think about what causes patterns to work or not work. The students continue their discussion and analysis, listing conditions necessary for the pattern to result in all of the dominoes collapsing (Figure 22).
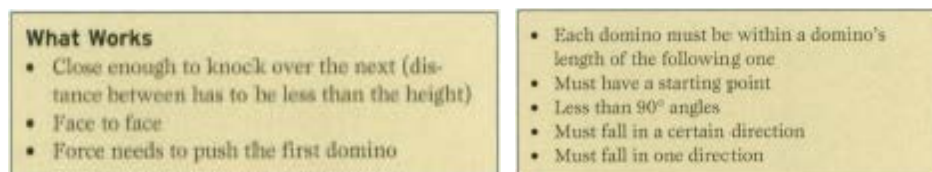


**What Works**
- Close enough to knock over the next (distance between has to be less than the height)
- Face to face
- Force needs to push the first domino

- Each domino must be within a domino's length of the following one
- Must have a starting point
- Less than 90° angles
- Must fall in a certain direction
- Must fall in one direction

**Figure 22 Samples of student-generated lists of conditions**

Eventually, it is determined that there are two conditions necessary for successful collapse of all of the dominoes. First, a force must push and cause the first domino to fall. Second, it is necessary that the collapse of each domino will cause the next domino to collapse as well. Once the students understand this, the teacher moves forward with the mathematical explanation of proof by induction. The beginning and assumption steps correspond to "the first domino must fall", and the induction step corresponds to "each domino collapsing must cause the next domino to collapse."

The author uses proof by induction to review previous concepts in precalculus, such as trigonometric identities, and the differentiation power rule in calculus. She also believes that proofs by mathematical induction are a good tool to help students improve their algebraic manipulation skills.

## Hands-On Calculus (Sutherland, 2006)

In this paper, the author describes assorted items that she has successfully used in her college level Calculus I and Calculus II classes. Here is a summary of the items she described.

- A loaf of sliced, hard-crusted bread can be used to help teaching how to compute the volume of solids with known cross-sections. Students will see the need to calculate the area of the face of each slice to compute the volume. Assuming that the loaf is larger in the middle and tapers off at the ends, the students will also understand that the area of a slice varies depending on its position in the loaf.

- Holiday decorations made out of tissue paper and cardboard used to demonstrate the idea of solids formed by revolving an area around an axis.

- Washers of varying radii held together by a nut and bolt used to show that to calculate the area of the face of each washer, both the outer and inner radii are used.

- Cans of different heights and radii, with the tops cut off, are nested to illustrate the cylindrical shell method of finding the volume of a solid of revolution. A pipe cleaner is then used to simulate the curve that was revolved to form the solid.

- Paper plates with a piece of chalk (or marker, for a white board) attached to the top rim used to demonstrate the path of a cycloid.

- Prior to using a sphere to show students how to find the area of a surface obtained by revolving a parameterized curve about an axis, the author has the students derive the surface area of a sphere using a hands-on activity with oranges of various sizes.

Each student (or group) cuts an orange in half, and after covering it with plastic wrap, draws five circles on a piece of paper by tracing the circle formed by the cut. Then they peel the orange, tear up the pieces and flatten them out. The student then uses the pieces to fill the circles they drew. They will see that four of the circles can be filled, thus the surface area of the sphere is $4\pi r^2$. This is further emphasized by the fact that each orange was of a different diameter, and still four circles were filled.

The author reports positive feedback from both regular college students and prospective math teachers taking the calculus courses for certification.

## Cookies and Pi (Dempsey, 2009)

In this article, circular cookies covered with vanilla and chocolate icing provide a challenge to students that have learned to evaluate definite integrals. The problem posed to the students is to use calculus to determine where to make the cuts so that the cookie is divided into four equal pieces, each containing an equal amount of vanilla and chocolate icing (Figure 23).

The students are first asked to estimate the answer using iterations starting at the halfway point between the center of the circle and the length of the radius. The teacher then moves on to show the students how to compute the answer, teaching (or reviewing) integration using trigonometric substitution.
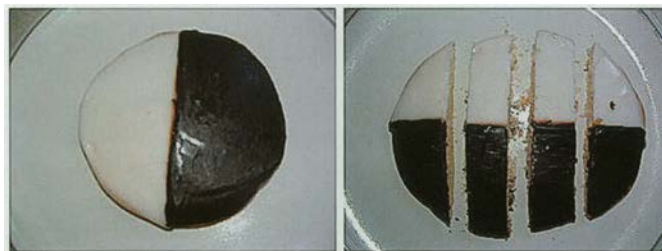


**Figure 23 Black and white cookie divided in the preferred manner**

## The Calculus of a Vase (Scherger & Tuerk, 2012)

The authors created an activity where students use calculus to find the volume, surface area and arc length of a vase that they are given in class. The students were to devise a technique to measure the dimensions of the vase, translate these into ordered pair coordinates, and use technology to find a curve to fit the data. The next part of the project was to use calculus to calculate the volume, surface area and arc length.

To compare to their calculations, the students were asked to measure the actual volume and arc length of the vase, and compute the percent error.

## Using Origami Boxes to Explore Concepts of Geometry and Calculus (Wares, 2010)

The authors introduce the concept of using the art of origami, or paper folding, to create a manipulative. The students used a square sheet of paper to make a box (Figure 24). Once the box was created the students were asked to find the internal surface and volume of the box. This was followed by an exploration of how to fold the box to maximize the volume that could be achieved with a sheet paper of fixed size. In this process the students were introduced to concepts of geometry and calculus.
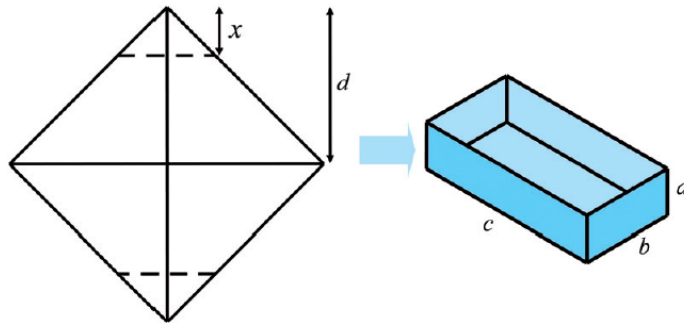


**Figure 24 Original square sheet and the constructed box (Wares, 2010)**

# Appendix 2  Optimization Lesson Plan

**TOPIC:** Solving Optimization Problems

**OBJECTIVES:**
At the end of this lesson, students will be able to:
o   Define a function to represent a real world quantity to be optimized.
o   Identify the domain and range of the function.
o   Find the optimal value and where it occurs on the domain interval.

**MATERIALS:**
o   Marked scaled down sample of poster board (for demonstration purposes)
o   Poster board, 14 in. x 22 in., one per student/pair
o   Student-made boxes (they will have used the above poster board to make these as homework)
o   Interactive white board
o   DESMOS graphing software (online)
o   Spreadsheet of volume & surface area for assigned dimensions
o   Cardstock paper cut to 7 in. by 11 in.
o   Teacher-made items: 1 box, h $\neq$ 0; suggested h = 2

                                   1 "box", h=0 & 1 "box", h=7

**PRE-LESSON ACTIVITIES:**
o   Demonstrate the cuts and folding required to make box.
o   Distribute poster board and project description sheets.
o   Students will make boxes of specified heights for homework.

**LESSON ACTIVITIES:**
o   Students will complete a questionnaire and take a pretest on solving optimization.
o   Fill in the spreadsheet using student results and discuss/elicit the following points:
    - Point out that everyone started out with the same size poster board.
    - But what happens to the volume values as h increases?
    - Class observes that for their data points, volume increases until h=2.375, then decreases.

- How do surface area values behave?
- Class observes that the surface area always decreases.
- Distribute the ½-scale cardstock paper and have students make a box using h=0:
- What is the volume of this "box"? Answer: Zero.
- What is the surface area of this "box"? Answer: 308 in.$^2$

o Guide students through increasing the h value until its natural limit.
o What is the volume of this box? Answer: Zero.
o What is the surface area of this box? Answer: 7 in.$^2$
o Discuss what the implications of these h=0 and h=7 and relate to domain.
- If we consider volume/surface to be functions of h, what is their domain?
- Answer: [0, 7]

o Lead class into defining the length, width and height of the box in terms of h.
o Describe the volume & surface area functions in terms of h.
- Answers: $V(h) = \frac{1}{2}(22 - 3h)(14 - 2h)h$

$$S(h) = 2[\frac{1}{2}(22 - 3h)(14 - 2h) + (14 - 2h)h + h \cdot \frac{1}{2}(22 - 3h)]$$

o Use procedure learned in previous unit to find extreme values:
- Differentiate, set derivative equal to zero, and solve for h.
- Calculate the function value for this h and each endpoint.
- Identify extrema (EVT) and where they occur.
- Answers: Maximum volume = 163.4 in$^3$, at h = 2.387 in.
  - Minimum volume = 0 in.$^3$, at h = 0 in. and h = 7 in. *
  - Maximum surface area = 308 in.$^2$, at h = 0 in. *
  - Minimum surface area = 7 in.$^2$, at h = 7 in. *
- Point out that for these measurements, the object is not 3-dimensional, and thus not useful in the real world, but serve to illustrate the "extreme" values.

o Graph the two functions; point out the extrema, which should match those just found.
o Reinforce EVT previously learned by showing the different types of extrema (inside the interval and endpoints of interval).
o Compare to extreme values and locations from class data spreadsheet
o Continue with examples to be solved with calculus:
- Open box made cutting four corners.
- Product of 2 numbers with constraints.
- For given value of surface area, find x to maximize volume.

o For practice at home, solve selected problems from the Optimization Lesson pages.

**POST-LESSON ACTIVITIES:**
o Check student's work and assist them with corrections and questions.
o Students will complete a questionnaire and take a posttest on optimization problems.

# Appendix 3  Optimization Pretest

Participant Code: _____

**OPTIMIZATION PROBLEMS**
**PRETEST: PRE-LESSON QUESTIONS AND PROBLEMS**

**PART I: QUESTIONS - UNDERSTANDING OF PREREQUISITE MATERIAL**
There are certain concepts, listed in the questions below, which students should know before they learn to solve optimization problems. Please mark the box or the phrase that best describes your understanding of these concepts.

**NOTE:** It is OK to respond "just a little"or "somewhat" if you just recently covered it in class.

*Presently I understand . . .*

    a.   Given an equation with more than variable, how to solve for one of the variables in terms of the others.

| Not at all | Just a little | Somewhat | A lot | A great deal |
|:---:|:---:|:---:|:---:|:---:|
| ☐ | ☐ | ☐ | ☐ | ☐ |

    b.   How to find the derivative of a given function (polynomials, sin(x), cos(x), ln(x), etc.

| Not at all | Just a little | Somewhat | A lot | A great deal |
|:---:|:---:|:---:|:---:|:---:|
| ☐ | ☐ | ☐ | ☐ | ☐ |

    c.   How to use the product rule, quotient rule and chain rule to find the derivative of a function.

| Not at all | Just a little | Somewhat | A lot | A great deal |
|:---:|:---:|:---:|:---:|:---:|
| ☐ | ☐ | ☐ | ☐ | ☐ |

    d.   How to find the relative and absolute extrema of a function.

| Not at all | Just a little | Somewhat | A lot | A great deal |
|:---:|:---:|:---:|:---:|:---:|
| ☐ | ☐ | ☐ | ☐ | ☐ |

**PART II: OPTIMIZATION PROBLEMS- PRETEST PROBLEMS**

You will now have a lesson on solving optimization problems. Below is a pretest on that topic. Attempting these problems will give you a preview of what you will learn in the lesson.

1) A supermarket employee wants to construct an open-top box from a 16 by 30 in. piece of cardboard. To do this, the employee plans to cut out squares of equal size from the four

corners so that the four sides can be bent upwards. What size should the squares be in order to create a box with the largest possible volume?

Illustration of problem

16
in

30
in
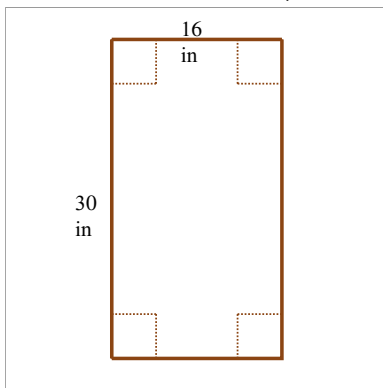
2) A company has started selling a new type of smartphone at the price of $150.0 − 0.05x$ where $x$ is the number of smartphones manufactured per day. The parts for each smartphone cost $70 and the labor and overhead for running the plant cost $7000 per day. How many smartphones should the company manufacture and sell per day to maximize profit?

3) A farmer wants to construct a rectangular pigpen using 400 ft of fencing. The pen will be built next to an existing stone wall, so only three sides of fencing need to be constructed to enclose the pen. What dimensions should the farmer use to construct the pen with the largest possible area?

Illustration of problem
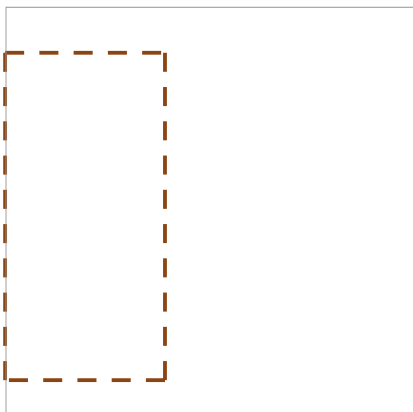
# Appendix 4  Optimization Prelesson Assignment

Participant 1: _____          Optimization Project Part 1 Honors          Participant 2: _____
                                        & AB Calculus - Ms. Rivero               (if applicable)

### Making a Box and Calculating Its Volume & Surface Area

**Background:** Many calculus problems involve finding the greatest or smallest value (or values) that a function assumes over an appropriate domain. These greatest or smallest value points are called global or absolute extrema. The purpose of this project is to gather data and prepare to study this topic.

**The Box:** There are many ways to make boxes, but usually the box starts out as a 2-dimensional rectangular piece of cardboard. Cuts and folds are made in strategic places, glue is used to keep pieces together, and the result is a 3-dimensional box.

**Instructions:** For the first part of this project, you will make a "pizza-style" box. The box is to be made from a piece of poster board of size 14 inches by 22 inches. (Poster board was given to you, or to you and your partner, today in class.) Six squares of size $h$ are to be marked on the poster board, as indicated on the diagram below.

**NOTE: See the list at the end of these instructions for which specific value of $h$ you, or you and your partner, are to use!**

Refer to the diagram below and cut only on the edges of the squares marked with $\overset{\mathbf{x}}{\mathbf{x}}$ . Make crisp folds on the long dashed lines and on the "flaps". Glue the small square flaps to the edges as demonstrated in class today so that the box will stay together.

NOTE: You may decorate the box any way you wish. Be creative!

Now that the box is together, calculate its volume and surface area. HINT: The height of the box is the $h$ that you used. You can calculate the length and width by figuring out what is left of the original dimensions after the flaps are folded. You should measure with a ruler to confirm these measurements; they should be close, but may not be exact due to cutting and folding. Use the CALCULATED values of length and width for the volume/surface area computations. ROUND TO ONE DECIMAL PLACE.
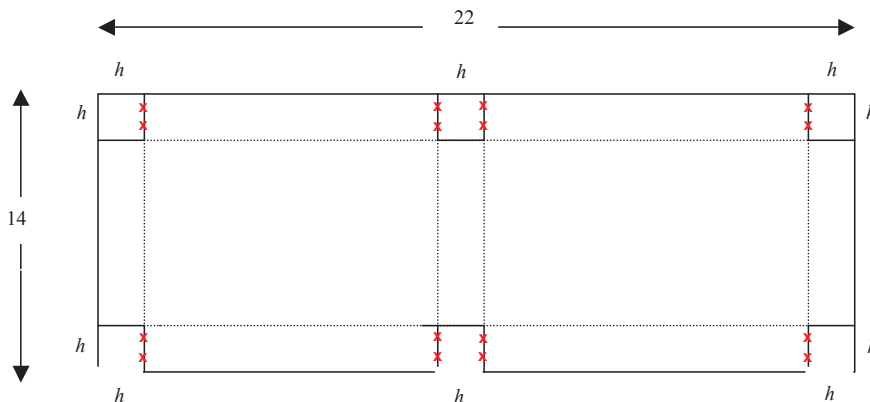
$$\text{Volume} = L \times W \times H \qquad\qquad \text{Surface Area} = 2 ( L \times W + W \times H + H \times L )$$

Write your name(s) and V = _____ in.$^3$   and SA = _____ in.$^2$ on the bottom of the inside of the box and on this paper.

LIST OF $h$ VALUES:   Group A  2-7/8 in. = 0.875 in.     Group B  1-1/4 in. = 1.25 in.     Group C  1-5/8 in. = 1.625 in.

      Group D  2 in.     Group E  2-3/8 in. = 2.375 in.     Group F  2-3/4 in. 2.75 in.     Group G  3-1/8 = 3.125 in.

# Appendix 5  Student-Constructed Box Data

**Table 15 Student Box Dimensions, Volume and Surface Area**

| Student(s) | Value of h (in.) | Volume (in.$^3$) | Surface Area (in.$^2$) |
|---|---|---|---|
| Lower Endpoint | 0.000 | 0.0 | 308.0 |
| Group A | 0.875 | 103.8 | 275.7 |
| Group B | 1.250 | 131.2 | 261.4 |
| Group C | 1.625 | 149.6 | 246.9 |
| Group D | 2.000 | 160.0 | 232.0 |
| Group E | 2.375 | 163.4 | 216.9 |
| Group F | 2.750 | 160.7 | 201.4 |
| Group G | 3.125 | 152.9 | 185.7 |
| Higher Endpoint | 7.000 | 0.0 | 7.0 |

# Appendix 6  Function Graphs and Optimized Values



$$V(h) = h\left(\frac{22-3h}{2}\right)(14-2h)$$

$$S(h) = 2\left(h\left(\frac{22-3h}{2}\right) + \left(\frac{22-3h}{2}\right)(14-2h) + (14-2h)\,h\right)$$
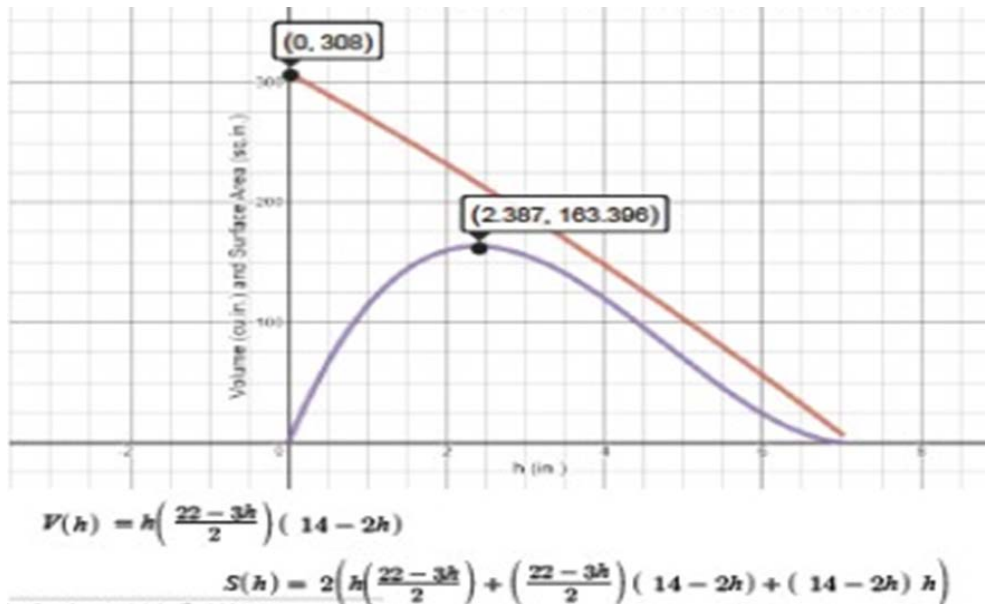
**Figure 25 Volume and surface area as functions of h**

**Table 16 Volume and surface area function equations and optimized values**

| Actual Optimized Values | 163.4 | 308.0 |
|---|---|---|
| | at x = 2.387 | at x = 0 |
| Volume Function | $V(h) = \frac{1}{2}h(14-2h)(22-3h)$ | |
| Surface Area Function | $A(h) = 2[h(14-2h) + \frac{1}{2}(22-3h)(14-2h) + \frac{1}{2}h(22-3h)$ | |

# Appendix 7  Optimization Posttest

Participant Code: _____          **OPTIMIZATION**
**POSTTEST: POST-LESSON QUESTIONS AND PROBLEMS**

**PART I: QUESTIONS - UNDERSTANDING OF LESSON MATERIAL**
Now that you have completed the lesson on optimization, please mark the box for the phrase that best describes your understanding of this concept. To help you connect the words with the math, an example is given below each concept.

*I now understand . . .*

**a. How to translate optimization word problems into mathematical functions and equations.**

**Example:**  The volume of a rectangular open box formed by cutting four squares of side length $x$ from each corner of a 10 in. by 15 in. rectangular piece of cardboard and folding up the sides is represented by the volume function:

$$V(x) = (10 - 2x)(15 - 2x)x$$

| Not at all | Just a little | Somewhat | A lot | A great deal |
|---|---|---|---|---|
| ☐ | ☐ | ☐ | ☐ | ☐ |

**b.  How to determine the absolute maximum and minimum values of a function over an interval [a,b]**

**Example:**  The absolute maximum and/or minimum of a function $A(x) = x(20 - x)$ on an interval [a, b] can be determined by finding values of $x$ where $A'(x) = 0$ or is undefined. Then these values as well as the endpoints would be tested to see where the maximum and/or minimum value(s) occur.

| Not at all | Just a little | Somewhat | A lot | A great deal |
|---|---|---|---|---|
| ☐ | ☐ | ☐ | ☐ | ☐ |

**c.  How to solve optimization problems involving volume, area, revenue, profits, cost, distance and time.**

**Example:**

If the profit for a company spending an amount $x$ thousands of dollars on advertising is represented by the function $P(x) = -\frac{1}{10}x^3 + 6x^2 + 400$, the amount to be spent on advertising that will maximize profit  can be determined by finding $P'(x)$, setting it equal to zero, and solving for $x$.

In this case, $P'(x) = -\frac{3}{10}x^2 + 12x = 0$ is solved by the value $x = 40$, and the amount to be spent on advertising is $40,000.

| Not at all | Just a little | Somewhat | A lot | A great deal |
|---|---|---|---|---|
| ☐ | ☐ | ☐ | ☐ | ☐ |

d.  **How to utilize a graphing calculator to represent and solve optimization problems.**

**Example:**  If the function to be minimized is the distance from a line to a point, say (1, 2), and this distance is represented by a function D(x), the function description would be entered into the "Y=" area of the calculator. There are several ways that this can then be solved using the CALC or MATH menus on the calculator.

| Not at all | Just a little | Somewhat | A lot | A great deal |
|---|---|---|---|---|
| ☐ | ☐ | ☐ | ☐ | ☐ |

**PART II: OPTIMIZATION – POSTTEST PROBLEMS**

1.  A supermarket employee wants to construct an open-top box from a 16 by 30 in. piece of cardboard. To do this, the employee plans to cut out squares of equal size from the four corners so that the four sides can be bent upwards. What size should the squares be in order to create a box with the largest possible volume?

Illustration of problem

16 in

30 in
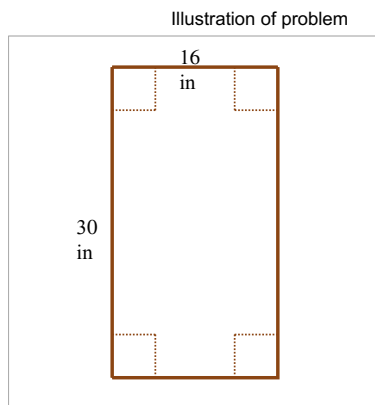
2.  An employee's monthly production M is found to be a function of the number of years of service, t. The monthly productivity function is given by:

$$M(t) = -2t^2 + 100t + 180$$

Where $t$ is in the interval [0, 40].

Find the maximum monthly productivity and the years of employee service at which it occurs.

# Appendix 8  End-of-Study Questionnaire

**END-OF-STUDY QUESTIONNAIRE** - Part I

In the recent research study lesson in your class, I used manipulatives for a hands-on approach to help you understand the concept of optimization of functions, finding volumes of solids of revolution, or the properties of definite integrals.
*Keeping in mind your present and past learning experiences, please mark the box for the phrase that best describes your level of agreement with the statement.*

a. I now understand the concepts taught in the research study lesson.

| Not at all | Just a little | Somewhat | A lot | A great deal |
|---|---|---|---|---|
| ☐ | ☐ | ☐ | ☐ | ☐ |

b. I would have learned the concept just as well without any manipulatives.

| Not at all | Just a little | Somewhat | A lot | A great deal |
|---|---|---|---|---|
| ☐ | ☐ | ☐ | ☐ | ☐ |

c. In past math classes I have had one or more math lessons where manipulatives were used. Examples: algebra tiles, blocks, and geometric solids.

| Never | Just a little | Somewhat | A lot | A great deal |
|---|---|---|---|---|
| ☐ | ☐ | ☐ | ☐ | ☐ |

d. It would have taken longer for me to understand the concept without the use of the manipulatives in the research study lesson.

| Do not agree | Agree a little | Somewhat | Agree a lot | Agree a great deal |
|---|---|---|---|---|
| ☐ | ☐ | ☐ | ☐ | ☐ |

e. I understand the concept presented to me in the research study lesson.

| Not at all | Just a little | Somewhat | A lot | A great deal |
|---|---|---|---|---|
| ☐ | ☐ | ☐ | ☐ | ☐ |

f. The use of manipulatives in the lesson were helpful to me in understanding the concept.

| Not at all | Just a little | Somewhat | A lot | A great deal |
|---|---|---|---|---|
| ☐ | ☐ | ☐ | ☐ | ☐ |

g. I am a tactile learner; I learn better when I can touch things.

| Not at all | Just a little | Somewhat | A lot | A great deal |
|---|---|---|---|---|
| ☐ | ☐ | ☐ | ☐ | ☐ |

h. I don't care where formulas come from; I just want to memorize them.

| Do not agree | Agree a little | Somewhat | Agree a lot | A great deal |
|---|---|---|---|---|
| ☐ | ☐ | ☐ | ☐ | ☐ |

i.  It is easier for me to learn when I can see a diagram, sketch or picture.

| Not at all | Just a little | Somewhat | A lot | A great deal |
|:---:|:---:|:---:|:---:|:---:|
| ☐ | ☐ | ☐ | ☐ | ☐ |

j.  I learn mathematics well in a traditional lecture format.

| Not at all | Just a little | Somewhat | A lot | A great deal |
|:---:|:---:|:---:|:---:|:---:|
| ☐ | ☐ | ☐ | ☐ | ☐ |

**END-OF-STUDY QUESTIONNAIRE - Part 2**

**Please elaborate on the following aspects of the lesson.**

Q1.  Please describe how the manipulatives used in the lesson enhanced or detracted from your learning of the concept taught in the lesson.

_____

_____

_____

_____

_____

_____

Q2.  Please provide any other feedback, suggestions or comments with respect to your experience in learning a calculus concept with the use of a manipulative.

_____

_____

_____

_____

_____

_____

# Appendix 9  Solids of Revolution Lesson Plan

**TOPIC:** Finding the Volume of Solids Revolved About the x-Axis

**DAY 1 & 2 LESSON (Combined if done on 2-hour block day, otherwise separated into 2 days)**

**Day 1 Objective:** Students will be able to find the volume of solids of revolution about the x-axis using the disk method.

**Materials:**
o   For teacher demonstration and student manipulation:
    Small, medium and large accordion rectangles (open into disks)
o   Large accordion triangles (open into two right circular cones joined at the base)
o   Accordion 2-dimensional figures (open into bells)
o   Interactive white board
o   DESMOS graphing software (online)

For student manipulation and projects:
o   Small accordion rectangular strips (open into disks)
o   Accordion concentric semi-circles (open into concentric spheres)
o   Small accordion 2-dimensional figures (open into ornaments)
o   Paper and ruler

**ACTIVITIES:**
o   Students will complete pre-lesson ranking questions and pretest problems.
o   Distribute unopened accordion triangles to pairs of students and have each trace the triangles on paper.
o   Students measure the base and height of the triangle to the nearest half-inch and label.
o   Ask the students to think about what 3-dimesional solid would result if the triangle were rotated about the x-axis.
o   After allowing them discuss briefly, have them open the triangle accordion and secure with a clip.
o   Students observe that the solid consists of two right circular cones joined at the base.

- Have the students calculate the solid's volume on paper using geometry. (Formula for volume of a cone x 2.)
- Display the functions previously saved on DESMOS that form the triangle the students worked with and review finding the area under the curve.
- Draw a thin rectangle (the same height and width as an unopened small accordion rectangular strip) from the curve to the x-axis to represent a typical rectangle.
- Place one of the small accordion rectangular strips over the thin rectangle and ask the students to think about what figure would result if the strip were revolved about the x-axis.
- Students should conclude that the figure formed is a circle, but if they don't see it yet, they will see it in the next step.
- Tell the students that you are "removing" the small rectangle from the solid and giving them each a copy of it.
- Then distribute a strip to each student, and ask them to use the strip to help them determine what type of figure results from "revolving" the strip about the x-axis.
- Allow a few minutes for the students to "play" with the strips, closing and revolving again so that they see what is happening when the thin rectangle is revolved.
- The students should have seen that the resulting figure is a circle.
- Ask the student to think about the area of this circle, and how they can calculate it.
- Elicit that to calculate the area of the circle, the radius is needed, and ask them if we know what the radius of the circle is.
- Give time for the students to see that the radius of this circle is the function value at the particular x-value where we drew the rectangle, and ask them to write a formula for the area of the circle in terms of $\pi$ and the radius (function value).
- Explain to the students that this thin rectangle that they revolved around the x-axis to form a circle is typical of a thin rectangle that they take for any x value on the graph.
- Refer the students to the triangle-turned-double-cone that they have on their desks, and tell them to imagine slicing the solid at any point and see that they would have a circle.
- NOTE: For students that still do not see this, show them the cone solids from a geometry solids kit. (First have the cones together, base-to-base, and then "slice" them apart to show the circles.)
- Once they have accepted that the "slices" are all circles, define the term "disk" to replace the term "slice".
- Then ask them to use their knowledge of integrals to set up an integral that will find the sum of all of the disks. (The integral should look like this $\int_0^l \pi f(x)^2 dx$ , where $l$ is the length of the base of the triangle that was revolved.)
- Elicit that the definite integral represents the volume of the solid.
- Now ask the students to evaluate the definite integral and compare it to the answer previously found using geometry. (They should be the same!)

o  Explain that the volume of any solid formed by revolving a function around the x-axis can be calculated using an integral such as the one above, and that in general we factor out $\pi$ so that the integral has this form:

- $\pi \int_a^b f(x)^2 dx$

o  Have the students practice finding volumes of solids of revolution about the x-axis with several functions selected from pages 640-641 of their student packets.

**Day 2 Objective:** Students will be able to find the volume of a solid generated by a function whose area-under-the-curve is not a standard geometric figure (i.e. not a triangle, rectangle, circle, etc.).
**Materials:**

o  For teacher demonstration and student manipulation:
   Interactive white bard
o  DESMOS graphing software (online)
o  Accordion 2-dimensional figures (open into bells, will call these "accordion bells")
o  Small accordion 2-dimensional figures (open into ornaments, will call the "accordion ornaments")

For student manipulation and projects:
o  Small accordion rectangular strips (open into disks)
o  Accordion concentric semi-circles (open into concentric spheres)
o  Accordion 2-dimensional figures (open into bells, will call these "accordion bells")
o  Small accordion 2-dimensional figures (open into ornaments, will call the "accordion ornaments")
o  Paper and ruler

**ACTIVITIES:**
o  Distribute the unopened accordion bells and have the students think about revolving this figure around the x-axis and try to guess what the resulting solid would look like.
o  Then have the students open and revolve the figure to see the answer.
o  Then have the students think of a type of function that may approximate the upper boundary of the plane figure before the revolution. (Answers will vary, but there will not be an obvious function as there was with the triangle in the previous lesson.)
o  Explain that this shows that not all solids of revolution have known areas-under-the-curve that are geometric figures such as triangles and semi-circles.
o  Explain that if we had a function that modeled the upper boundary of the figure, we could calculate the volume of the bell using the techniques learned in Day 1 Lesson.
o  Now distribute an accordion ornament to each student, and explain that we will find a curve-of-best-fit for the figure and find an approximation of its volume.
o  Have the students:

102

- Place the ornament on graph paper with the left bottom corner of the ornament at the origin. NOTE: Take all x-value measurements to the nearest whole of half-centimeter.
- Measure the base of the ornament and record this number on the x-axis
- Mark centimeters (cm) for units on the x-axis of the graph and label from x=0 to the end of the ornament (should be approximately 11cm).
- Mark the x-value of the maximum point.
- Make a 2-column x-y table with the x-values in order from 0 to the end, making sure to include the x-value of the maximum in its proper place.
- Measure and write the function value (y) for each x-value on the table. NOTE: For these measurements, measure to the nearest tenth; 2.1, 3.6, and so forth
- Graph the points from the table.
- Now the students will think about what type of function might fit the curve. Answers will vary, and several forms will be suggested, tried using the graphing calculator and discarded.

From past experience provide an equation of this form:
$$f(x) = ab^{-(cx-d)^2}$$
As a class the teacher and students will:
o Enter the data points and the above equation into DESMOS and create sliders for a, b, c & d.
o Have students take turns at the IWB using the sliders to come find an equation that best fits the data. (They will quickly discover that each constant controls some aspect of the graph, and they should find the best value for each letter while leaving the others constant.) NOTE: They may also do this work on their tablets.
o A function will be agreed upon and will be used for the evening's assignment.

**HOMEWORK:**
o The assignment will be posted on the class website with the function-of-best-fit and graph worked on in class.
o The students will find the volume of the solid formed by revolving the area over the x-axis.

**DAY 3 - ASSESSMENT**

**Objective:** Students will demonstrate their understanding of the integration techniques for calculating the volume of solids of revolution by deriving the general formula for the volume of a sphere.

**Materials:**
For student manipulation and work:
o   Accordion semi-circles
o   Ruler and calculator as needed
o   Paper

**ACTIVITIES:**
o   Distribute the accordion semi-circles.
o   Using integration, students will derive the general formula for the volume of a sphere.

**POSTTEST & QUESTIONNAIRE:**
Students will complete post-lesson ranking questions and posttest problems.

# Appendix 10  Solids of Revolution Pretest

Participant Code: _____

**SOLIDS OF REVOLUTION**
**PRETEST: PRE-LESSON QUESTIONS AND PROBLEMS**

**PART I: QUESTIONS - UNDERSTANDING OF PREREQUISITE MATERIAL**
There are certain concepts, listed in the questions below, which students should know before they learn to solve optimization problems. Please mark the box or the phrase that best describes your understanding of these concepts.

**NOTE:** It is OK to respond "just a little"or "somewhat" if you just recently covered it in class.

## *Presently I understand . . .*

a.  How to evaluate a definite integral.

| Not at all | Just a little | Somewhat | A lot | A great deal |
|------------|---------------|----------|-------|--------------|
| ☐ | ☐ | ☐ | ☐ | ☐ |

b.  How to find the area under a curve.

| Not at all | Just a little | Somewhat | A lot | A great deal |
|------------|---------------|----------|-------|--------------|
| ☐ | ☐ | ☐ | ☐ | ☐ |

c.  How to find the point(s) of intersection of two curves.

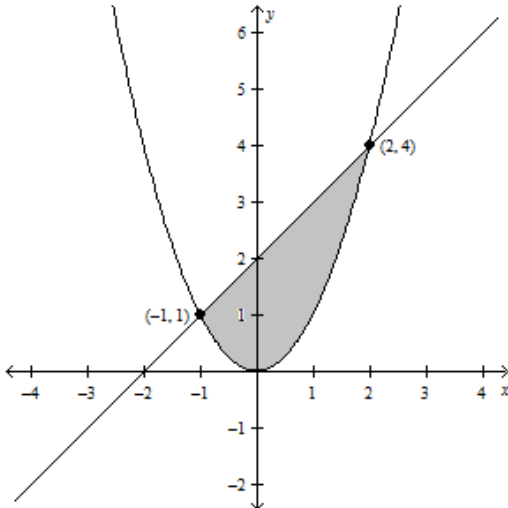| Not at all | Just a little | Somewhat | A lot | A great deal |
|------------|---------------|----------|-------|--------------|
| ☐ | ☐ | ☐ | ☐ | ☐ |

d.  How to find the area between two curves.

| Not at all | Just a little | Somewhat | A lot | A great deal |
|------------|---------------|----------|-------|--------------|
| ☐ | ☐ | ☐ | ☐ | ☐ |

**PART II: SOLIDS OF REVOLUTION- PRETEST PROBLEMS**

You will now have a lesson on finding the volume of solids of revolution. Below is a pretest on that topic. Attempting these problems will give you a preview of what you will learn in the lesson.

Consider the region pictured to the right that is bounded by the graphs of $y = x^2$ and $y = x + 2$.

| Find the volume of the solid formed when the region is rotated about the $x$ – axis. |  |
|---|---|
| Find the volume when the region is rotated about the line $y = 4$. | Find the volume when the region is rotated about the line $y = -2$. |

# Appendix 11  Ornament Assignment

ORNAMENT ASSIGNMENT            Participant Code: _____

The purpose of this assignment is to review and reinforce the work we did today in class on finding the volume of solids of revolution.

Below is a scatter plot and the curve of best fit for a cross-section of the tissue paper ornament we worked with. (These were the small ones; some were white and some were red.)

The equation for the curve that best represents the shape of the ornament is $f(x) = 4.5 \, (6)^{-(0.25x - 1.375)^2}$ .

For your reference I also included the data table of the actual measurements, in centimeters.

**Instructions:**

1. Print this paper.

2. On the graph, draw a small rectangle from the x-axis to the curve, to represent the radius of a generic circular "slice" of the ornament.

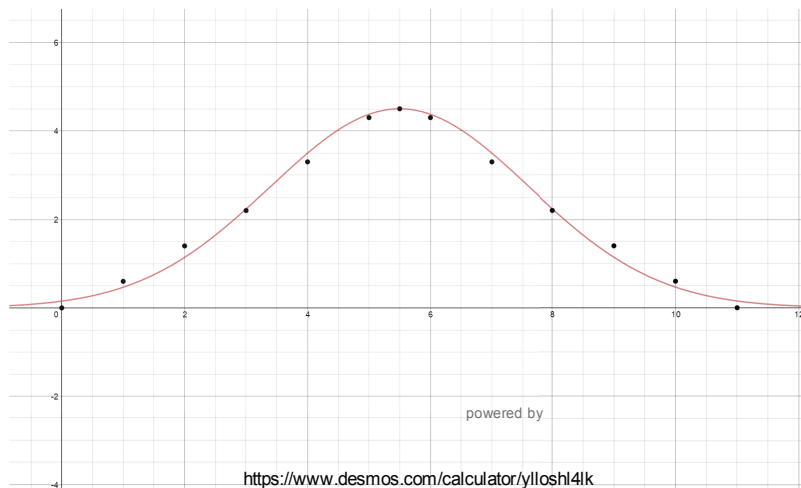3. Write the integral that represents the volume of the ornament.

        V(x) = _____

4. Evaluate the integral on your graphing calculator and write the answer below.
   NOTE: Round to 3 decimal places, and use the appropriate units in your answer.

        The volume of the ornament is: _____

| $x_1$ | $y_1$ |
|-------|-------|
| 0 | 0 |
| 1 | .6 |
| 2 | 1.4 |
| 3 | 2.2 |
| 4 | 3.3 |
| 5 | 4.3 |
| 5.5 | 4.5 |
| 6 | 4.3 |
| 7 | 3.3 |
| 8 | 2.2 |
| 9 | 1.4 |
| 10 | .6 |
| 11 | 0 |

$$f(x) = 4.5 \cdot 6^{-\left( .25x - 1.375 \right)^2}$$



powered by

https://www.desmos.com/calculator/ylloshl4lk

# Appendix 12  Solids of Revolution Posttest

Participant Code: _____     **SOLIDS OF REVOLUTION - POSTTEST**

**PART I: QUESTIONS - UNDERSTANDING OF LESSON MATERIAL**

Now that you have completed the lesson on solids of revolution, please mark the box for the phrase that best describes your understanding of this concept. To help you connect the words with the math, an example is given below each concept.



*I now understand . . .*

**a. How to find the are between two curves.**

**Example:** To find the area between the two curves on this graph, the integral would be the difference between the top function and the bottom function from $x = 0$ to $x = 5$.
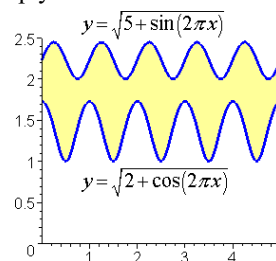
| Not applicable | Not at all | Just a little | Somewhat | A lot | A great deal |
|:---:|:---:|:---:|:---:|:---:|:---:|
| ☐ | ☐ | ☐ | ☐ | ☐ | ☐ |

**b. That a solid of revolution is formed by revolving a flat, two-dimensional area about the x-axis.**

**Example:** Revolving a semi-circle with its diameter on the x-axis would generate a sphere. Revolving a rectangle about the x-axis would generate a cylinder.

| Not applicable | Not at all | Just a little | Somewhat | A lot | A great deal |
|:---:|:---:|:---:|:---:|:---:|:---:|
| ☐ | ☐ | ☐ | ☐ | ☐ | ☐ |

**c. That a "slice" of such a solid has the shape of a circle whose radius at any given *x*-value is f(x). Example:** Think about the "slices" of the cones used in the class demonstration.

| Not applicable | Not at all | Just a little | Somewhat | A lot | A great deal |
|:---:|:---:|:---:|:---:|:---:|:---:|
| ☐ | ☐ | ☐ | ☐ | ☐ | ☐ |

**d. How to find the volume of a solid of revolution generated by revolving a function about the *x*-axis.          Example:**

The general formula for finding the volume of a solid formed by revolving a function about the x-axis is:

$$V(x) = \pi \int_a^b [f(x)]^2 \, dx$$

| Not applicable | Not at all | Just a little | Somewhat | A lot | A great deal |
|:---:|:---:|:---:|:---:|:---:|:---:|
| ☐ | ☐ | ☐ | ☐ | ☐ | ☐ |

**e. How to use the graphing calculator to evaluate the integrals used in finding these volumes.**

| Not applicable | Not at all | Just a little | Somewhat | A lot | A great deal |
|:---:|:---:|:---:|:---:|:---:|:---:|
| ☐ | ☐ | ☐ | ☐ | ☐ | ☐ |

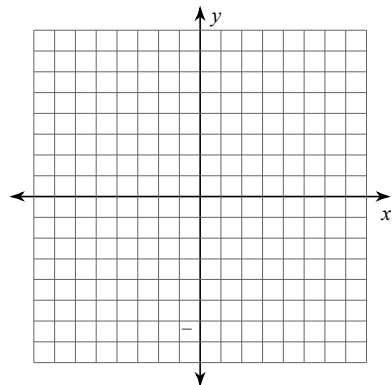**PART II: SOLIDS OF REVOLUTION – POSTTEST PROBLEMS**

**For the problems below, find the volume of the solid generated revolving the given function about the x-axis using given limits of integration.**

1. $f(x) = \sqrt{1 - x^2}$     from $x = -1$ to $x = 1$     no calculator; leave answer in terms of pi
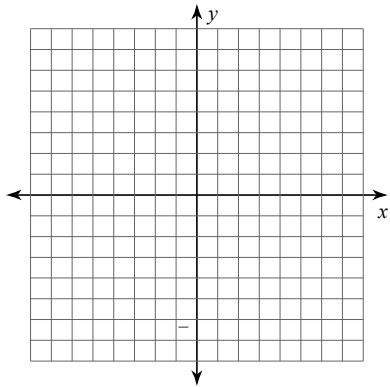
2. $f(x) = \cos x$   from x = 0 to x = $1.5\pi$  calculator allowed; round to 3 decimals places.

**For the funcions below, find the volume of the solid generated by revolving the area between the graphs of the two functions about the given lines.**

**You may use the graph provided to sketch the graphs. NOTE: If you do not show the graph, you must show your work for finding the limits of integration.**

$$x = -y^2 + 2$$
$$x = y$$

3. About the x-axis.  NO CALCULATOR FOR THIS PART.
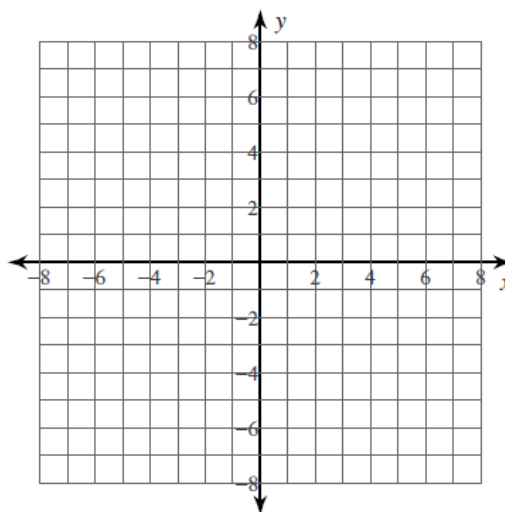


4. About the line y = 3

CALCULATOR ALLOWED, BUT YOU MUST SHOW THE SETUP OF THE INTEGRAL

# Appendix 13  Definite Integrals Lesson Plan

**TOPIC:** Properties of Definite Integrals

**MATERIALS:**
Cardstock paper, various colors
Markers, rulers, scissors, masking tape

- o **ACTIVITIES:**
- o NOTE 1: Students will have completed a questionnaire and pretest problems.
- o Orient the first quadrant on the desk (origin at lower left corner) or mark with tape.
- o Distribute materials.
- o Align bottom of paper to the x-axis (not at the origin).
- o Sketch a curve freehand from the left edge of the paper to the right edge and label as $f(t)$.
- o Label the bottom left and right corners as $a$ and $b$, respectively.
- o Cut along the curve (demonstrate mine).
- o On the back of the paper, write the notation of what this represents $\int_a^b \int f(t)dt$.
- o Tell students that what they are holding in their hands represents:
    - • "the integral of a positive function from $a$ to $b$;"
    - • "the area between a positive function and the horizontal axis;"
- o Do the same thing in steps 3-7, but . . .
    - • The curve must be LOWER i.e., LESS THAN the first one
    - • Label it as g(x)
    - • Write the integral notation on the back

- o **The Properties of Definite Integrals**

    SUBTRACTION:  $\int_a^b [f(t) - g(t)]dt = \int_a^b f(t)dt - \int_a^b g(t)dt.$

- Sketch on the board a circle inside of a square (not touching the edges) and shade the area between the square and the circle.
- How do we calculate the area of the shaded region?
  - Answer: Subtract area of circle from area of square.
- Lining up the bottoms, place the cutout of the $g(t)$ curve on top of the $f(t)$ curve.
- The amount of paper visible of the $f(t)$ function represents "the area between the curves", or the difference of the two definite integrals.
- We have just demonstrated the Subtraction Property of Definite Integrals
- Does that make sense?

ADDITION: $$\int_a^b [f(t) + g(t)]dt = \int_a^b f(t)dt + \int_a^b g(t)dt.$$

- Sketch on the board a triangle on top of a square.
- How do we calculate the area of this compound region?
  - Answer: Add the area of the triangle to the area of the square.
- Take the $g(t)$ cutout and rotate it about its horizontal axis.
- Label this side of the cutout $-g(t)$ and the top left and right corners as $a$ and $b$, respectively.
  - NOTE that the area is negative because of its direction, but the magnitude of the area is still $g(t)$.
- Lining up the horizontal axes, place the $f(t)$ and $-g(t)$ cutouts with $f(t)$ above the x-axis and $-g(t)$ below the x-axis.
- In this case, the "area between the two curves" is the sum of the two areas.
  - NOTE: We can also consider it as the difference between $f(t)$ and $-g(t)$:
    - $f(t) - (-g(t)) = f(t) + g(t)$.
- We have just demonstrated the Addition Property of Definite Integrals
- Does that make sense?

SPLIT UP: $$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt.$$

- Take the cutout of $g(t)$ and make a vertical cut anywhere.
- Label the lower right corner of the left piece with the letter $c$.
- Label the lower left corner of the right piece with the letter $c$.
- As you can see, the two pieces added together have the same area as the original piece.
- We have just demonstrated the Split Up Property of Definite Integrals
- Does that make sense?

INVERSE AREA:

$$\int_a^b g(t) = -\int_b^a g(t)$$

o   To demonstrate this property, think about direction. If you are moving in a certain direction when you go from point A to B, then when you turn around and move from point B to point A you are moving in the opposite (or inverse) direction.

o   OR you can think about subtraction. $5 - 3 = 2$. When you reverse the order, $3 - 5 = -2$. When you reverse the order of the subtraction, the answer is the inverse of the original answer.

o   So for definite integrals if you reverse the limits of integration, the sign of the answer will change. NOTE: This means that in integral-land you can have "negative" area.

ZERO AREA:

$$\int_a^a g(t) = 0$$

o   When you are moving from point A to point A, have you gone anywhere?
   •   If you haven't moved, then there is no area.
o   Or you can think of it as a line. Lines have no width, so they have no area.

ALGEBRA WITH DEFINITE INTEGRALS:

o   Since definite integrals are just numbers, we can do algebra with them.

o   For example, if we start out with this equa    $\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt.$

o   We can subtract $\int_a^c f(t)dt$ from both sides and get:    $\int_a^b f(t)dt - \int_a^c f(t)dt = \int_c^b f(t)dt.$

o   We can also multiply definite integrals by constants.

   •   If $\int_a^b f(t) = -3$, then $\int_a^b 4f(t) = 4\int_a^b f(t) = 4(-3) = -12$

o   Now we will work out practice problems.

# Appendix 14  Definite Integrals Pretest

**PART I: QUESTIONS - UNDERSTANDING OF PREREQUISITE MATERIAL**
There are certain concepts, listed in the questions below, which students should know before they learn about the properties of definite integrals. Please mark the box or the phrase that best describes your understanding of these concepts. To help you connect the words with the math, an example is given below each concept.

**NOTE:** It is OK to respond "just a little"or "somewhat" if you are presently covering the topic or just recently finished covering it in your calculus class.

*Presently I understand . . .*

a.  What an anti-derivative/indefinite integral is.

| Not applicable | Not at all | Just a little | Somewhat | A lot | A great deal |
|---|---|---|---|---|---|
| ☐ | ☐ | ☐ | ☐ | ☐ | ☐ |

**Example:**
The phrase *the antiderivative of the function* $3x^2$ is written mathematically as: $\int 3x^2\,dx$

b.  How to find anti-derivatives of polynomial-type and trigonometric functions.

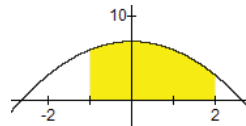| Not applicable | Not at all | Just a little | Somewhat | A lot | A great deal |
|---|---|---|---|---|---|
| ☐ | ☐ | ☐ | ☐ | ☐ | ☐ |

**Examples:**
$\int 3x^2\,dx = x^3 + C$ $\qquad$ $\int 3x^2\,dx = x^3 + C$ $\qquad$ $\int sinx\,dx = -cosx + C$

c.  The concept of the area between a curve and the *x*-axis.

| Not applicable | Not at all | Just a little | Somewhat | A lot | A great deal |
|---|---|---|---|---|---|
| ☐ | ☐ | ☐ | ☐ | ☐ | ☐ |

**Example**:
The area between the curve $y = 7 - x^2$
and the *x*-axis is represented by the yellow
region in the diagram on the right.



d.  How to use the Fundamental Theorem of Calculus Part 1 to find the area between a curve and the *x*-axis.

| Not applicable | Not at all | Just a little | Somewhat | A lot | A great deal |
|---|---|---|---|---|---|
| ☐ | ☐ | ☐ | ☐ | ☐ | ☐ |

**Example**:
For the example shown in letter c above:

$$\text{Area} = \int_{-1}^{2} \left(7 - x^2\right) dx$$
$$= \left(7x - \frac{1}{3}x^3\right)\Big|_{-1}^{2}$$
$$= \left[7\cdot 2 - \frac{1}{3}(8)\right] - \left[7(-1) - \frac{1}{3}(-1)\right]$$
$$= 18$$

**PART II: PROPERTIES OF DEFINITE INTEGRALS - PRETEST PROBLEMS**

Soon you will have a lesson on the properties of definite integrals. Below is a pretest on that topic. Attemping these problems will give you a preview of what you will learn in the lesson.

1.   Suppose that f and g are continuous and that $\int_{7}^{11} f(x)\,dx = -3$   and   $\int_{7}^{11} g(x)\,dx = 9$

   Find   $\int_{7}^{11} [\,g(x) - f(x)\,]\ dx$

   A) 12                    B) −12                    C) 6                    D) −6

2. Suppose that f is continuous and that $\int_{-3}^{3} f(x)\,dx = 10$ and $\int_{-3}^{5} f(x)\,dx = 6$.   Find $\int_{3}^{5} f(x)\,dx$ .

   A) 4                    B) −4                    C) 24                    D) −6

3. Suppose that f and g are continuous and that $\int_{6}^{10} f(x)\,dx = -6$ and $\int_{6}^{10} g(x)\,dx = 9$.

   Find   $\int_{6}^{10} \left[4f(x) + g(x)\right] dx$ .

   A) 12                    B) 13                    C) −15                    D) 30

For problems 4, 5 and 6, suppose that h is continuous and that $\int_{-2}^{2} h(x)\,dx = 6$ and $\int_{2}^{7} h(x)\,dx = -9$.

4.   Find   $\int_{-2}^{7} h(x)\,dx$

   A) -3                    B) 0                    C) -15                    D) 15

5.   Find   $\int_{7}^{-2} h(x)\,dx$

   A) -3                    B) 3                    C) -15                    D) 0

6.   Find   $\int_{7}^{7} h(x)\,dx$

   A) 0                    B) 3                    C) -15                    D) 15

115

# Appendix 15  Definite Integrals Posttest

Participant Code: _____ 

**PROPERTIES OF DEFINITE INTEGRALS - POSTTEST**

**PART I: QUESTIONS - UNDERSTANDING OF LESSON MATERIAL**
Now that you have completed the lesson on the properties of definite integrals, please mark the box for the phrase that best describes your understanding of these concepts. To help you connect the words with the math, an example is given below each concept.

*I now understand . . .*

a.  What a definite integral represents, and that a definite integral is a value, rather than a function.

**Example:**
The definite integral $\int_{-1}^{2} h(x)\, dx$ represents the area

between the curve $h(x) = 7 - x^2$ and the $x$-axis, between the values $x = -1$ and $x = 2$ on the diagram at the right. The value of this definite integral/area is the number 18.
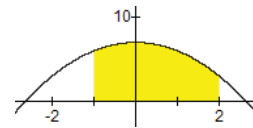


| Not applicable | Not at all | Just a little | Somewhat | A lot | A great deal |
|---|---|---|---|---|---|
| ☐ | ☐ | ☐ | ☐ | ☐ | ☐ |

b.  How to evaluate a linear combination of definite integrals with the same limits of integration.

**Examples:**   If $\int_{6}^{10} f(x)\, dx = -6$   and   $\int_{6}^{10} g(x)\, dx = 9$   then

$$\int_{6}^{10} [f(x) + g(x)]\, dx = -6 + 9 = 3 \qquad \int_{6}^{10} [f(x) - g(x)]\, dx = -6 - 9 = -15$$

$$\int_{6}^{10} [f(x) - g(x)]\, dx = -6 - 9 = -15 \qquad \int_{6}^{10} [4f(x) + g(x)]\, dx = 4(-6) + 9 = -15$$

$$\int_{6}^{10} 2\,[g(x) - f(x)]\, dx = 2\,[9 - (-6)] = 2(15) = 30$$

| Not applicable | Not at all | Just a little | Somewhat | A lot | A great deal |
|---|---|---|---|---|---|
| ☐ | ☐ | ☐ | ☐ | ☐ | ☐ |

c.  A definite integral can be "split up" at any point between the limits of integration, and the sum of the resulting two integrals is equal to the original integral.

**Examples:**     $$\int_{-2}^{7} h(x)\,dx \;=\; \int_{-2}^{2} h(x)\,dx \;+\; \int_{2}^{7} h(x)\,dx$$

and     $$\int_{-2}^{7} h(x)\,dx \;-\; \int_{-2}^{2} h(x)\,dx \;=\; \int_{2}^{7} h(x)\,dx$$

| Not applicable | Not at all | Just a little | Somewhat | A lot | A great deal |
|---|---|---|---|---|---|
| ☐ | ☐ | ☐ | ☐ | ☐ | ☐ |

d.  That reversing the limits of integration results in a value with the opposite sign, making it possible to have "negative" area under a curve.

**Example:** In the example above, if we were to reverse the limits of integration and integrate from $x = 2$ to $x = -1$ instead of from $x = -1$ to $x = 2$, the resulting value of the area would be *-18*, instead of *18*.

So . . . if     $\int_{-1}^{2} h(x)\,dx = 18$         then . . .     $\int_{2}^{-1} h(x)\,dx = -18$

| Not applicable | Not at all | Just a little | Somewhat | A lot | A great deal |
|---|---|---|---|---|---|
| ☐ | ☐ | ☐ | ☐ | ☐ | ☐ |

e.  Using limits of integration that are equal to each other results in an area of zero.

**Examples:**     $\int_{7}^{7} f(x)\,dx = 0$         and         $\int_{-1}^{-1} f(x)\,dx = 0$

| Not applicable | Not at all | Just a little | Somewhat | A lot | A great deal |
|---|---|---|---|---|---|
| ☐ | ☐ | ☐ | ☐ | ☐ | ☐ |

Participant Code: _____

**PART II: PROPERTIES OF DEFINITE INTEGRALS – POSTTEST PROBLEMS**

1.  Suppose that f and g are continuous and that $\int_{4}^{8} f(x)\,dx = -2$ and $\int_{4}^{8} g(x)\,dx = 8$.

    Find $\int_{4}^{8} \left[g(x) - f(x)\right]\,dx$ .

    A) 6            B) –6            C) 10            D) –10

2. Suppose that f is continuous and that $\int_{-2}^{2} f(x)\,dx = 8$ and $\int_{-2}^{7} f(x)\,dx = 6$.  Find $\int_{2}^{7} f(x)\,dx$ .

    A) –2            B) –12            C) –6            D) 12

3. Suppose that f and g are continuous and that $\int_{2}^{6} f(x)\,dx = -5$ and $\int_{2}^{6} g(x)\,dx = 7$.

    Find $\int_{2}^{6} \left[4f(x) + g(x)\right]\,dx$ .

    A) 12            B) –13            C) 15            D) 30

For problems 4, 5 and 6, suppose that h is continuous and that $\int_{-1}^{4} h(x)\,dx = 5$ and $\int_{4}^{7} h(x)\,dx = -12$.  Find

4.  Find $\int_{-1}^{7} h(x)\,dx$

    A) 0            B) –7            C)   –17            D)   7

5.  Find $\int_{7}^{-1} h(x)\,dx$

    A) -7            B) 0            C) –17            D) 7

6.  Find $\int_{-1}^{-1} h(x)\,dx$

    A) 7            B) -7            C) 0            D) 17

# References

Bhatia, K., Premadasa, K., & Martin, P. (2014). Teaching integration applications using manipulatives. *Primus: Problems, Resources & Issues in Mathematics Undergraduate Studies, 24*(4), 335-346. doi:10.1080/10511970.2013.879971

Boggan, M., Harper, S., & Whitmire, A. (2010). Using manipulatives to teach elementary mathematics. *Journal of Instructional Pedagogies, 3*(1), 1-10.

Bouck, E. C., Satsangi, R., Doughty, T. T., & Courtney, W. T. (2014). Virtual and concrete manipulatives: A comparison of approaches for solving mathematics problems for students with autism spectrum disorder. *Journal of Autism and Developmental Disorders, 44*(1), 180-193.

Bruins, B. (2014). *The effectiveness of manipulatives in a high school algebra II class.* Retrieved from Eastern Kentucky University's Encompass Online Theses and Dissertations Digital Archive. (Paper 236)

Burris, J. T. (2010). Third graders' mathematical thinking of place value through the use of concrete and virtual manipulatives.

Carbonneau, K. J., Marley, S. C., & Selig, J. P. (2013). A meta-analysis of the efficacy of teaching mathematics with concrete manipulatives. *Journal of Educational Psychology, 105*(2), 380-400.

Cone, R. E. (2013). Tactile reinforcement for early calculus concepts: Grocery bags and laser beams. *Primus: Problems, Resources & Issues in Mathematics Undergraduate Studies, 23*(7), 617-626.

Council on Foreign Relations. (2012). *U.S. education reform and national security* (Independent Task Force Report No. 68). New York, NY: Council on Foreign Relations.

Cruse, A. (2012). *Using hands-on learning activities in high school mathematics classes to impact student success* (Doctoral dissertation). Retrieved from ProQuest Dissertations and Theses Database. (UMI 3518540)

Cuevas, J. (2015). Is learning styles-based instruction effective? A comprehensive analysis of recent research on learning styles. *Theory and Research in Education, 13*(3), 308-333.

Curtain- Phillips, M. (2015). Manipulatives: The missing link in high school math. Retrieved from http://www.mathgoodies.com/articles/manipulatives.html

Dana, T. M., & Shaw, K. L. (1992). An evaluation of the implementation of Florida's comprehensive plan to improve mathematics, science and computer education. Annual Meeting of the American Educational Research Association.

Dempsey, M. (2009). Cookies & PI. *Mathematics Teacher, 102*(9), 686-690.

Dobbins, A., Gagnon, J. C., & Ulrich, T. (2014). Teaching geometry to students with math difficulties using graduated and peer-mediated instruction in a response-to-intervention model. *Preventing School Failure, 58*(1), 17-25.

Doias, E. D. (2013). *The effect of manipulatives on achievement scores in the middle school mathematics class* (Doctoral dissertation). Retrieved from ProQuest Dissertations and Theses Database. (UMI 3598021)

Ferrini-Mundy, J., & Lauten, D. (1994). Connecting research to teaching: Learning about calculus learning. *Mathematics Teacher, 87*(2), 115-21.

Froebel, F. (1895). *Pedagogics of the kindergarten* (J. Jarvis Trans.). New York: D. Appleton and Company.

Golafshani, N. (2013). Teachers' beliefs and teaching mathematics with manipulatives. *Canadian Journal of Education, 36*(3), 137-159.

Hartshorn, R., Boren, S., (1990). *Experiential learning of mathematics: Using manipulatives.* Charleston, WV: ERIC Clearinghouse on Rural Education and Small Schools. (ED321967)

Holman, H. (1908). *Pestalozzi; an account of his life and work.* London, New York: Longmans, Green, and Co.

Howard, P., Perry, B., & Lindsay, M. (1996). Mathematics and manipulatives: Views from the secondary schools. *Annual Meeting of the Educational Research Association.*

Howard, P., Perry, B., & Tracey, D. (1997). Mathematics and manipulatives: Comparing primary and secondary teachers' views. *Australian Association for Research in Education Annual Conference.*

Jimenez, A. (2011). *Hands-on equations program: An approach to teaching linear equations using manipulatives.* (Doctoral dissertation). Retrieved from ProQuest Dissertations and Theses Database. (UMI 3500447)

Kamina, P., & Iyer, N. (2009). From concrete to abstract: Teaching for transfer of learning when using manipulatives. *NERA Conference Proceedings 2009. Paper 6.*

Kaplan, G. (2009). Playing with dominoes: Proof by induction. *Mathematics Teacher, 102*(6), 426-431.

Kokoska, S. (2015). Chapter 10 confidence intervals and hypothesis tests based on two samples or treatments. Freeman and Company, W. H. (Ed.), *Introductory statistics:*

*A problem-solving approach* (Second ed., pp. 461-481-484, 491-496). New York, NY: Macmillan Education.

Koss, L. (2011). Manipulatives for 3-dimensional coordinate systems. *Primus, 21*(4), 364-376.

Krulik, S., & Kaufman, I. (1963). *Multi-sensory techniques in mathematics teaching.* Englewood Cliffs, N.J.: Teachers Practical Press.

Loong, E. Y. K. (2014). Fostering mathematical understanding through physical and virtual manipulatives. *Australian Mathematics Teacher, 70*(4), 3-10.

Maccini, P., & Gagnon, J. C. (2000). Best practices for teaching mathematics to secondary students with special needs. *Focus on Exceptional Children, 32*(5), 1.

Magruder, R. L. (2012). *Solving linear equations: A comparison of concrete and virtual manipulatives in middle school mathematics.* Retrieved from University of Kentucky UKnowledge Theses and Dissertations (Paper 2)

McClung, L. W. (1998). *A study on the use of manipulatives and their effect on student achievement in a high school algebra I class* (Master's thesis). Retrieved from ERIC database. (ED425077)

McNichols, J. (1985). Manipulative materials make the difference. In Northeast Regional Exchange (Ed.), *Teaching mathematics: Strategies that work* (pp. 13).

Mendiburo, M., Hasselbring, T. (2011). *Technology's impact on fraction learning: An experimental comparison of virtual and physical manipulatives.* 2011 Society for Research on Educational Effectiveness Conference.

Moch, P. L. (2002). Manipulatives work! *Educational Forum, 66*(1), 81-87.

Montessori, M. (1917). *The advanced Montessori method* (A. Livingston Trans.). New York: Frederick A. Stokes Company.

Morin, J., & Samelson, V. M. (2015). Count on it: Congruent manipulative displays. *Teaching Children Mathematics, 21*(6), 362-370.

Moyer, P. S., Bolyard, J. J., & Spikell, M. A. (2002). What are virtual manipulatives? *Teaching Children Mathematics, 8*(6), 372-77.

Moyer, P. S., & Jones, M. G. (2004). Controlling choice: Teachers, students, and manipulatives in mathematics classrooms. *School Science & Mathematics, 104*(1), 16-31. doi:10.1111/j.1949-8594.2004.tb17978.x

Moyer-Packenham, P., Baker, J., Westenskow, A., Anderson, K., Shumway, J., Rodzon, K., & Jordan, K. (2013). A study comparing virtual manipulatives with other instructional treatments in third- and fourth-grade classrooms. *Journal of Education, 193*(2), 25-39.

Moyer-Packenham, P., & Suh, J. (2012). Learning mathematics with technology: The influence of virtual manipulatives on different achievement groups. *Journal of Computers in Mathematics and Science Teaching, 31*(1), 39-59.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Nelson, C. J. (2012). A math-box tale. *Teaching Children Mathematics, 18*(7), 418-425.

Nemirovsky, R., Rubin, A. (1992). *Students' tendency to assume resemblances between a function and its derivative.* Cambridge, MA: TERC Communications.

Owens, J. E. (1988). *Curriculum and evaluation standards for school mathematics: Report of the national council of teachers of mathematics' commission on standards for school mathematics.* Retrieved from ERIC database. (ED302515)

Özgün-Koca, S., & Edwards, T. (2011). Hands-on, minds-on or both? A discussion of the development of a mathematics activity by using virtual and physical manipulatives. *Journal of Computers in Mathematics and Science Teaching, 30*(4), 389-402.

Papert, S. (1993). *Mindstorms*. New York, NY: Basic Books.

Peppers, D. S., Wan, A., & Phillips, H. E. (2014). A closer look at manipulatives in remediation. *Mathematics Teaching in the Middle School, 20*(3), 166-173.

Satsangi, R., & Bouck, E. C. (2015). Using virtual manipulative instruction to teach the concepts of area and perimeter to secondary students with learning disabilities. *Learning Disability Quarterly, 38*(3), 174-186.

Sauro, J. (2011). How to interpret survey responses: 5 techniques. Retrieved from http://www.measuringu.com/blog/interpret-responses.php

Sauro, J., & Lewis, J. R. (2012). *Quantifying the user experience: Practical statistics for user research*. Waltham, MA: Elsevier.

Scherger, N. (2012). The calculus of a vase. *MathAMATYC Educator, 3*(2), 11-13.

Stein, M. K., & Bovalino, J. W. (2001). Manipulatives: One piece of the puzzle. *Mathematics Teaching in the Middle School, 6*(6), 356-59.

Sutherland, M. (2006). Hands-on calculus. *Primus, 16*(4), 289-299.

Swan, P., & Marshall, L. (2010). Revisiting mathematics manipulative materials. *Australian Primary Mathematics Classroom, 15*(2), 13-19.

Thirey, B., & Wooster, R. (2013). The touchy-feely integral: Using manipulatives to teach the basic properties of integration. *Primus: Problems, Resources & Issues in Mathematics Undergraduate Studies, 23*(7), 605-616. doi:10.1080/10511970.2013.796576

Uhl, J., Humphrey, P., & Braselton, J. (2006). Visualizing the method of finding volumes by cross sections: An eggsperiment. *Primus, 16*(2), 138-146.

White, K. M. (2012). *The effect of an instructional model utilizing hands-on learning and manipulatives on math achievement of middle school students in Georgia* (Doctoral dissertation). Retrieved from ProQuest Dissertations and Theses Database. (UMI 3546