Algebraicity Criteria and Their Applications

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Yunqing Tang

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Dissertation Advisor: Professor Mark Kisin Yunqing Tang

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Abstract

We use generalizations of the Borel–Dwork criterion to prove variants of the Grothedieck–Katz p-curvature conjecture and the conjecture of Ogus for some classes of abelian varieties over number fields.

The Grothendieck–Katz p-curvature conjecture predicts that an arithmetic differential equation whose reduction modulo p has vanishing p-curvatures for all but finitely many primes p, has finite monodromy. It is known that it suffices to prove the conjecture for differential equations on $\mathbb{P}^1 - \{0, 1, \infty\}$. We prove a variant of this conjecture for $\mathbb{P}^1 - \{0, 1, \infty\}$, which asserts that if the equation satisfies a certain convergence condition for *all* p, then its monodromy is trivial. For those p for which the p-curvature makes sense, its vanishing implies our condition. We deduce from this a description of the differential Galois group of the equation in terms of p-curvatures and certain local monodromy groups. We also prove similar variants of the p-curvature conjecture for a certain elliptic curve with j-invariant 1728 minus its identity and for $\mathbb{P}^1 - \{\pm 1, \pm i, \infty\}$.

Ogus defined a class of cycles in the de Rham cohomology of smooth proper varieties over number fields. This notion is a crystalline analogue of ℓ-adic Tate cycles. In the case of abelian varieties, this class includes all the Hodge cycles by the work of Deligne, Ogus, and Blasius. Ogus predicted that such cycles coincide with Hodge cycles for abelian varieties. We confirm Ogus' conjecture for some classes of abelian varieties, under the assumption that these cycles lie in the Betti cohomology with real coefficients. These classes include abelian varieties of prime dimension that have nontrivial endomorphism ring. The proof uses a crystalline analogue of Faltings' isogeny theorem due to Bost and the known cases of the Mumford–Tate conjecture. We also discuss some strengthenings of the theorem of Bost.

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CHAPTER 1

Introduction

Many problems in arithmetic geometry concern the existence of certain algebraic cycles or subvarieties and one strategy to prove such existence is to construct analytic objects first and then to develop suitable criteria which use arithmetic properties to show the algebraicity. These criteria are originated from the classical Borel–Dwork criterion, which asserts that a nice formal power series with rational coefficients is the power series expansion of a rational function if the product of its convergence radii at all places is larger than 1. Here by a nice power series, we mean that the set of primes dividing some of the denominators of the coefficients is finite. Dwork used this criterion to prove that the zeta function of a smooth projective variety over a finite field is rational, which was part of the Weil conjectures.

Informally speaking, generalizations of the Borel–Dwork criterion concern the algebraicity of analytic subvarieties of smooth algebraic varieties defined over number fields. There are many instances in arithmetic geometry where the algebraicity of certain analytic subvarieties is desired, as illustrated in the following examples.

The first example is the Grothendieck–Katz p-curvature conjecture, which concerns vector bundles with flat connections. This conjecture is a local-global principle of the algebraicity of the solutions of an arithmetic linear homogenous differential equation. The p-curvature is an invariant of the differential equation modulo p and its vanishing is equivalent to the existence of a full set of mod p rational solutions. Under the assumption of the vanishing of p-curvatures for all but finitely primes, one needs to show the algebraicity of the formal solutions of the differential equation.

The second example is a conjecture of Ogus, which is a crystalline analogue of the Mumford–Tate conjecture. Ogus defined absolute Tate cycles using the structure of de Rham and crystalline cohomologies and conjectured that these cycles coincide with Hodge cycles. A variant of Ogus' conjecture for abelian varieties over number fields would follow from the conjectural algebraicity of certain formal subschemes of the moduli space of principally polarized abelian varieties.

We use generalizations of the Borel–Dwork criterion (see chapter 2) to prove:

(1) Variants of the Grothedieck–Katz p-curvature conjecture under the assumption of vanishing p-curvature at all primes (see chapter 3);

(2) The conjecture of Ogus for some classes of abelian varieties over number fields under the assumption that all absolute Tate cycles lie in Betti cohomology with real coefficients (see chapter 4).

In chapter 5, we discuss a conjecture arising naturally from our study of the conjecture of Ogus.

The Grothendieck–Katz p-curvature conjecture

Let X be a smooth variety over a number field K and (M, ∇) a vector bundle with a flat connection over X. The Grothendieck–Katz p-curvature conjecture predicts that (M, ∇) has finite monodromy if and only if, for all but finitely many primes $\mathfrak{p}$, (M, ∇) modulo $\mathfrak{p}$ has vanishing p-curvature. It is known that it suffices to prove the conjecture when $X = \mathbb{P}^1_K - \{0, 1, \infty\}$. We prove a variant of the conjecture for $X = \mathbb{P}^1_K - \{0, 1, \infty\}$ where the condition for all but finitely many $\mathfrak{p}$ is replaced by a condition *for all* $\mathfrak{p}$. A slightly informal formulation of our result is the following:

Theorem 1 (Theorem 3.2.1)**.** *Let (M, ∇) be a vector bundle with a connection over $X = \mathbb{P}^1_K - \{0, 1, \infty\}$. If the p-curvature of (M, ∇) vanishes for all $\mathfrak{p}$, then (M, ∇) is trivial, that is, $M^{\nabla=0}$ generates M as an $\mathcal{O}_X$-module.*

Let us explain the meaning of the condition of vanishing p-curvature at all primes $\mathfrak{p}$: at primes where the p-curvature is either not defined or non-vanishing, we impose a condition on the p-adic radii of convergence of the horizontal sections of (M, ∇). When (M, ∇) has an integral model at a prime $\mathfrak{p}$ so that one can make sense of its reduction mod $\mathfrak{p}$, this convergence condition is implied by the vanishing of the p-curvature.

One can extend the notion of vanishing p-curvature for all $\mathfrak{p}$ to vector bundles with connections over smooth algebraic curves equipped with either a semistable model over $\mathcal{O}_K$ or a flat model over $\mathcal{O}_K$ with a smooth $\mathcal{O}_K$-point. However, the property of all p-curvatures vanishing is not preserved under push-forward along finite maps from the curve in question to $\mathbb{P}^1 - \{0, 1, \infty\}$. Therefore, one cannot deduce from Theorem 1 that vanishing p-curvature for all $\mathfrak{p}$ implies trivial monodromy in the case of arbitrary algebraic curves. Nevertheless, when X is an elliptic curve with j-invariant 1728 minus its identity point, we prove:

Theorem 2 (Theorem 6.1.1)**.** *Let $X \subset \mathbb{A}^2_{\mathbb{Z}}$ be the affine curve defined by $y^2 = x(x-1)(x+1)$ and let (M, ∇) be a vector bundle with a connection over X_K. If the p-curvature of (M, ∇) vanishes for all $\mathfrak{p}$, then (M, ∇) has finite monodromy. That is, there exists a finite étale morphism $f : Y \to X$ such that $f^*(M, \nabla)$ is trivial.*

Unlike in Theorem 1, passing to a finite étale cover is necessary. In the setting of Theorem 2, there is an example of an (M, ∇) with monodromy group equal to $\mathbb{Z}/2\mathbb{Z}$.

The main tools used to prove Theorem 1 and Theorem 2 are the algebraicity results of André [**And04a**, Thm. 5.4.3] and Bost–Chambert-Loir [**BCL09**, Thm. 6.1, Thm. 7.8]. André and Bost used these techniques to prove the p-curvature conjecture when one knows a priori that the monodromy group of (M, ∇) is solvable. Our estimates of archimedean radii use the properties of theta functions, the Chowla–Selberg formula, and works of Hempel [**Hem79**] and Eremenko [**Ere11**].

The conjecture of Ogus

The Mumford–Tate conjecture asserts that, via the Betti–étale comparison isomorphism, the $\mathbb{Q}_\ell$-linear combinations of Hodge cycles coincide with the ℓ-adic Tate cycles. As a crystalline analogue, Ogus defined the notion of absolute Tate cycles for any smooth projective variety X over a number field K and predicted that for any embedding $\sigma : K \to \mathbb{C}$, via the de Rham–Betti comparison isomorphism

$$c_{\mathrm{BdR}} : H^i_{\mathrm{B}}(X_\sigma(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \cong H^i_{\mathrm{dR}}(X/K) \otimes_{K,\sigma} \mathbb{C},$$

absolute Tate cycles coincide with absolute Hodge cycles ([**Ogu82**, Hope 4.11.3]). For any finite extension L of K, an element in the tensor algebra of

$$\bigoplus_{i=0}^{2\dim X} H^i_{\mathrm{dR}}(X/K) \otimes L$$

is called an *absolute Tate cycle* if it is fixed by all but finitely many crystalline Frobenii φ_v. When v is unramified, φ_v can be viewed as acting on $H^i_{\mathrm{dR}}(X/K) \otimes K_v$ via the canonical isomorphism between de Rham and crystalline cohomologies.

Ogus proved that all Hodge cycles are absolute Tate for abelian varieties and verified the agreement of absolute Hodge cycles and absolute Tate cycles when X is a product of abelian varieties with complex multiplication, Fermat hypersurfaces, and projective spaces ([**Ogu82**, Thm. 4.16]).

It is natural to take the archimedean places into account: complex conjugation on the Betti cohomology can be viewed as the analogue of the Frobenii acting on the crystalline cohomology. We define the *de Rham–Tate cycles* to be those absolute Tate cycles which, for any embedding $\sigma : K \to \mathbb{C}$, lie in the tensor algebra of

$$c_{\mathrm{BdR}}(\bigoplus_{i=0}^{2\dim X} H^i_{\mathrm{B}}(X_\sigma(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R}).$$

Our first result is the following:

Theorem 3 (Theorem 8.2.4)**.** *If A is a polarized abelian variety over $\mathbb{Q}$ and its ℓ-adic algebraic monodromy group G_ℓ is connected, then the Mumford–Tate conjecture for A implies that the de Rham–Tate cycles coincide with the Hodge cycles.*

The Mumford–Tate conjecture for abelian varieties is known in many cases. When the abelian variety A over K satisfies $\mathrm{End}_{\bar{K}}(A) = \mathbb{Z}$, Pink proved that the conjecture holds when $2\dim A$ is not in the set ([**Pin98**])

$$S_{\mathrm{Pink}} = \{a^{2b+1}, \binom{4b+2}{2b+1} | a, b \in \mathbb{N}\backslash\{0\}\}.$$

To show this, he constructed a $\mathbb{Q}$-model of G°_ℓ which is independent of ℓ and "looks like" the Mumford–Tate group G_{MT} in the following sense. The group G_{MT} (resp. the $\mathbb{Q}$-model of G°_ℓ) with its tautological faithful absolutely irreducible representation $H^1_{\mathrm{B}}(A, \mathbb{Q})$ (resp. $H^1_{\text{ét}}(A_{\bar{K}}, \mathbb{Q}_\ell)$) is an (absolutely) irreducible *strong Mumford–Tate pair* over $\mathbb{Q}$: the group is reductive and generated over $\mathbb{Q}$ by the image of a cocharacter of weights $(0, 1)$. Based on the work of Serre, Pink gave a classification of irreducible Mumford–Tate pairs; see [**Pin98**, Prop. 4.4, 4.5, and Table 4.6]. This classification unconditionally shows that G_ℓ is of a very restricted form.

In the crystalline setting, we define the de Rham–Tate group G_{dR} of a polarized abelian variety A over K to be the algebraic subgroup of $\mathrm{GL}(H^1_{\mathrm{dR}}(A/K))$ stabilizing all of the de Rham–Tate cycles. This group is reductive by our assumption that de Rham–Tate cycles are fixed by complex conjugation. We show that Pink's classification also applies to G_{dR} in the following situation:

Theorem 4 (Theorem 8.2.6)**.** *Let A be a polarized abelian variety over $\mathbb{Q}$ and assume that its ℓ-adic algebraic monodromy group is connected. If $\mathrm{End}_{\bar{\mathbb{Q}}}(A) = \mathbb{Z}$, then the neutral connected component of G_{dR} with its tautological representation is an irreducible strong Mumford–Tate pair over $\mathbb{Q}$.*

A key input to the proofs of both theorems is:

Proposition 5. *Let M be a set of rational primes of natural density one and let A be a polarized abelian variety over K. If $s \in \mathrm{End}(H^1_{\mathrm{dR}}(A/K) \otimes L)$ satisfies that $\varphi_v(s) = s$ for all v lying over some $p \in M$, then s comes from an algebraic cycle over L.*

Bost proved such algebraicity of s assuming $\varphi_v(s) = s$ for all but finitely many v ([**Bos06**, Thm. 6.4]). Both results may be viewed as analogues of Faltings' isogeny theorem. Based on Bost's work, on [**Gas10**], and on [**Her12**], we prove a strengthening (Corollary 11.1.2, Remark 11.1.3) only assume the density of M to be strictly larger than $1 - \frac{1}{2(\dim A+1)}$ for general A or $3/4$ for A absolutely simple.

Before we present a result valid for a general number field K, we explain the main difficulty in going beyond the $K = \mathbb{Q}$ case in Theorem 3 and Theorem 4. For simplicity, we focus on the case when $\mathrm{End}_{\bar{K}}(A) = \mathbb{Z}$. Pink's classification applies to connected reductive groups with an absolutely irreducible representation. Though we can deduce the irreducibility of $H^1_{\mathrm{dR}}(A/K)$ as a G_{dR}-representation from Bost's theorem, G_{dR} is a priori not known to be connected. In the ℓ-adic setting, Serre, using the Chebotarev density theorem, showed that G_ℓ will be connected after passing to a finite extension ([**Ser13**]). There seems to be no easily available analogous argument for G_{dR}. However, when $K = \mathbb{Q}$, the absolute Frobenii coincide with the relative ones. Thus the connectedness of G_ℓ implies that G_{dR} is almost connected: $\varphi_p \in G^\circ_{\mathrm{dR}}(\mathbb{Q}_p)$ for all p in a set of natural density 1. In other words, although one cannot prove directly that elements fixed by G°_{dR} are de Rham–Tate cycles, such elements are fixed by φ_p for all p in a density one set.

Beyond the $K = \mathbb{Q}$ case, we have proved the following result.

Theorem 6. *Let A be an abelian variety over some number field such that it is isogenous to $\prod_{i=1}^n A_i^{n_i}$, where A_i is absolutely simple and A_i is not isogenous to A_j over any number field for $i \neq j$. Assume that each A_i is one of the following cases:*

(1) A_i is an elliptic curve or has complex multiplication.

(2) The dimension of A_i is a prime number and $\mathrm{End}_{\bar{K}}(A_i)$ is not $\mathbb{Z}$.

(3) The polarized abelian variety A_i of dimension g with $\mathrm{End}_{\bar{K}}(A_i) = \mathbb{Z}$ is defined over a finite Galois extension K over $\mathbb{Q}$ such that $[K : \mathbb{Q}]$ is prime to $g!$ and $2g \notin S_{\mathrm{Pink}}$.

and that if there is an A_i of case (2) with $\mathrm{End}_{\bar{K}}(A_i)\otimes\mathbb{Q}$ being an imaginary quadratic field, then all the other A_j are not of type IV. Then the de Rham–Tate cycles of A coincide with its Hodge cycles.

Case (1) was known before our work: Ogus proved the case of abelian varieties with complex multiplication and the case of elliptic curves is a direct consequence of the Serre–Tate theory. For the rest, the main task is to show that the centralizer of G_{dR}° in $\mathrm{End}(H^1_{\mathrm{dR}}(A/K))$ coincides with that of G_{dR}. In case (2), since the Mumford–Tate group is not too large, we use Bost's theorem to show that otherwise G_{dR}° must be a torus. Then we deduce that A must have complex multiplication by a theorem of Noot ([**Noo96**, Thm. 2.8]) on formal deformation spaces at a point of ordinary reduction and hence we reduce this case to case (1). To exploit Proposition 5 to tackle case (3), we need to understand φ_v for all v lying over $p\in M$, where M is a set of rational primes of natural density 1. While Serre's theorem on the ranks of Frobenius tori only provides information about completely split primes, we prove a refinement when $G_\ell=\mathrm{GSp}_{2g}$ that takes into account the other primes. This refinement asserts that the Frobenius tori are of maximal rank for all v lying over $p\in M$. The rest of the argument is similar to that of case (2). In order to prove the result for the product of abelian varieties in these three cases, we record a proof of the Mumford–Tate conjecture for abelian varieties studied in the theorem following the idea of [**Lom15**].

A relative version of Bost's theorem

In the description of ℓ-adic Tate cycles over some number field L, one uses relative Frobenii instead of the absolute ones. It is natural to use relative Frobenii acting on the crystalline cohomology to define an analogous notion of absolute Tate cycles (see Definition 10.1.1). In analogy with the Mumford–Tate conjecture and the conjecture of Ogus, one may expect that such cycles are L-linear combinations of the absolute Hodge cycles. In particular, we expect the following counterpart of Bost's theorem (see Proposition 5) for an abelian variety A over a number field K.

Conjecture 7. *Let L be a finite extension of K and for any finite place v with residue characteristic p, write $m_v=[L_v:\mathbb{Q}_p]$. If $s\in\mathrm{End}(H^1_{\mathrm{dR}}(A_L/L))$ is fixed by all but finitely many relative Frobenii $\varphi_v^{m_v}$, then s is an L-linear combination of algebraic cycles.*

The validity of this conjecture implies that the agreement of de Rham–Tate cycles and Hodge cycles is a consequence of the Mumford–Tate conjecture, generalizing Theorem 3. The full validity of this conjecture seems difficult. Nevertheless, we prove

Theorem 8 (section 10.2)**.** *Conjecture 7 is valid when A is an elliptic curve, has complex multiplication, or is an abelian surface with quaternion multiplication.*

Notation and convention. Let K be a number field and $\mathcal{O}_K$ its ring of integers. For a place v of K, either archimedean or finite, let K_v be the completion of K with respect to v. When v is finite, we denote by $\mathfrak{p}$, $\mathcal{O}_v$, and k_v the corresponding prime ideal, the ring of integers, and residue field of K_v. We also denote by p_v the characteristic of k_v and when there is no confusion, we will also write p for p_v. When we say all places or any place v of K, this v can be both archimedean and finite. If there is no specific indication, L denotes a finite extension of K.

For any vector space or vector bundle V, let $V^\vee$ be its dual and we denote $V^{\otimes m} \otimes (V^\vee)^{\otimes n}$ by $V^{m,n}$. For a vector space V, we use $\mathrm{GL}(V), \mathrm{GSp}(V), \ldots$ to denote the algebraic groups rather than the rational points of these algebraic groups. For any scheme X or vector bundle/space V over $\mathrm{Spec}(R)$, we denote by X'_R or V'_R the base change to $\mathrm{Spec}\, R'$ for any R-algebra R'. For any archimedean place σ of K and any variety X over K, we use X_σ to denote the base change of X to $\mathbb{C}$ via a corresponding embedding $\sigma : K \to \mathbb{C}$.

A reductive algebraic group here could be nonconnected.

Given an algebraic group G, we use G° to denote its neutral connected component and use $Z(G)$ to denote its center. We use $Z^\circ(G)$ to denote the connected component of $Z(G)$.

For any field F, we use $\bar{F}$ to denote a chosen algebraic closure of F. For any finite dimensional vector space V over F and any subset S of V, we use $\mathrm{Span}_F(S)$ to denote the smallest sub F-vector space of V containing S.

For an Hermitian vector bundle E over an $\mathcal{O}_K$-scheme X, we may use E to denote both the vector bundle over X and that over X_K. If necessary, we may use $\mathcal{E}$ and E to distinguish the one over X and the one over X_K.

CHAPTER 2

Algebraicity criteria

In section 1, we state results on formal power series by André and Bost–Chambert-Loir which will be used in chapter 3. In section 2, we discuss results used in chapter 4 and chapter 5 on formal subschemes in a given quasi-projective scheme over K. The key method to prove these results is the slope method due to Bost, which will be briefly reviewed in section 2.1.

1. Formal power series

We denote by $K[[x]]$ the ring of formal power series in variable x with coefficients in K. We say y is algebraic (resp. rational) if y is the Taylor series of some algebraic (resp. rational) function.

1.1. The algebraicity criterion of André. For simplicity, we only discuss the formal power series in one variable. André proved his theorem for the multi-variable situation.

1.1.1. Let $y \in K[[x]]$, and let v be a place of K. Let $|\cdot|_v$ be the v-adic norm normalized so that $|p|_v = p^{-\frac{[K_v:\mathbb{Q}_p]}{[K:\mathbb{Q}]}}$ if v is finite, and $|x|_v = |x|_\infty^{-\frac{[K_v:\mathbb{R}]}{[K:\mathbb{Q}]}}$ for $x \in K$, if v is archimedean, where $|x|_\infty$ denotes the Euclidean norm on K_v. When there is no confusion, we will also write $|\cdot|$ for $|\cdot|_\infty$. For a positive real number R, we denote by $D_v(0,R)$ the rigid analytic z-disc of v-adic radius R. That is $D_v(0,R)$ is defined by the inequality $|z|_v < R$.

We first state the definition of v-adic uniformization and the associated radius R_v defined in André's paper ([**And04a**, Def. 5.4.1]).

Definition 1.1.2.

(1) For $R \in \mathbb{R}^+$, a *v-adic uniformization* of y by $D_v(0,R)$ is a pair of meromorphic v-adic functions $g(z)$, $h(z)$ on $D_v(0,R)$ such that $h(0) = 0$, $h'(0) = 1$ and $y(h(z))$ is the germ at 0 of the meromorphic function $g(z)$.

(2) Let R_v be the supremum of the set of positive real R for which a v-adic uniformization of y by $D_v(0,R)$ exists. We call R_v the v-adic *radius (of uniformizability)*.

1.1.3. In order to state the algebraicity criterion, we need to introduce two constants $\tau(y)$, $\rho(y)$, which play similar roles as the condition in the Borel–Dwork criterion that all of the coefficients of y are in $\mathcal{O}_K[\frac{1}{N}]$ for some $N \in \mathbb{Z}$. Let $y = \sum_{n=0}^{\infty} a_n x^n$. We define

$$\tau(y) = \inf_l \limsup_n \sum_{v,\, p \geq l} \frac{1}{n} \sup_{j \leq n} \log^+ |a_j|_v, \quad \rho(y) = \sum_v \limsup_n \frac{1}{n} \sup_{j \leq n} \log^+ |a_j|_v,$$

where $\log^+$ is the positive part of log, that is $\log^+(a) = \log(a)$ if $a > 1$ and is zero otherwise. The following is a slight reformulation of André's criterion.

Theorem 1.1.4. *(*[**And04a**, Thm. 5.4.3]*) Let $y \in K[[x]]$ such that $\tau(y) = 0$ and $\rho(y) < \infty$. Let R_v be the v-adic radius of y. If $\prod_v R_v > 1$, then y is algebraic over $K(x)$.*

In general the v-adic radius R_v may be infinity or zero. We refer the reader to [**And04a**] for a precise definition of the infinite product in such situations. In our applications of this theorem, R_v will always be non-zero.

Remark 1.1.5. Suppose that y is a (component of a) formal solution of a vector bundle with an integrable connection (M, ∇). By [**And04a**, Cor. 5.4.5], if the p-curvatures of (M, ∇) vanish for all but finitely many places, then $\tau(y) = 0$ and $\rho(y) < \infty$.

1.2. The rationality criterion of Bost and Chambert-Loir. We now review the definition of adélic tube adapted to a given point, the definition of capacity norms for the special case we need, and the rationality criterion in [**BCL09**].

Definition 1.2.1. ([**BCL09**, Def. 5.16]) Let Y be a smooth projective curve over K, and let (x_0) be the divisor corresponding to a given point $x_0 \in Y(L)$ for some number field $L \supset K$. For each finite place w of L, let Ω_w be a rigid analytic open subset of Y_{L_w} containing x_0. For each archimedean place w, we choose one embedding $\sigma : L \to \mathbb{C}$ corresponding to w and we let Ω_w be an analytic open set of $Y_\sigma(\mathbb{C})$ containing x_0. The collection (Ω_w) is *an adélic tube* adapted to (x_0) if the following conditions are satisfied:

(1) for an archimedean place, the complement of Ω_w is non-polar (e.g. a finite collection of closed domains and line segments); if w is real, we further assume that Ω_w is stable under complex conjugation.

(2) for a finite place, the complement of Ω_w is a nonempty affinoid subset;

(3) for almost all finite places, Ω_w is the tube of the specialization of x_0 in the special fiber of Y. That is, Ω_w, is the open unit disc with center at x_0.

We call (Ω_w) a *weak* adélic tube if we drop the condition that Ω_w is stable under complex conjugation when w is real.

1.2.2. Now let Y be $\mathbb{P}^1_{\mathcal{O}_K}$ and X be $\mathbb{P}^1_{\mathcal{O}_K} - \{0, 1, \infty\}$. The weak adélic tube that we will use in chapter 3 can be described as follows:

(1) For an archimedean place, Ω_w will be an open simply connected domain inside $X_w(\mathbb{C})$.

(2) For a finite place, Ω_w will be chosen to be an open disc of form $D(x_0, \rho_w)$.

(3) For almost all finite places, $\rho_w = 1$.

1.2.3. For Ω_w as above, Bost and Chambert-Loir have defined the local capacity norms $||\cdot||_w^{\mathrm{cap}}$ (see [**BCL09**, Chp. 5]). These are norms on the tangent bundle $T_{x_0}X$ over $\mathrm{Spec}(\mathcal{O}_L)$. The Arakelov degree of the line bundle $T_{x_0}X$ (with respect to these norms)

$$\widehat{\deg}(T_{x_0}X, ||\cdot||^{\mathrm{cap}}) = \sum_w -\log(||s||_w^{\mathrm{cap}}), \text{ where } t \text{ is a section of } T_{x_0}X$$

plays the same role as $\log(\prod R_w)$ in Theorem 1.1.4. Note that this degree is independent of the choice of t by the product formula. We will use the section $\frac{d}{dx}$, in which case one has the following simple description of local capacity norms:

(1) For an archimedean place, let $\phi\colon D(0, R) \to \Omega_w$ be a holomorphic isomorphism that maps 0 to x_0, then $||\frac{d}{dx}||_w^{\mathrm{cap}} = |R\phi'(0)|_w^{-1}$ (see [**Bos99**, Example 3.4]).

(2) For a finite place, $||\frac{d}{dx}||_w^{\mathrm{cap}} = \rho_w^{-1}$ (see [**BCL09**, Example 5.12]).

Theorem 1.2.4. *(*[**BCL09**, Theorem 7.8]*) Let (Ω_w)be an adélic tube adapted to (x_0). A formal power series y over X centered at x_0 is rational if y satisfies the following conditions:*

(1) For all w, y extends to an analytic meromorphic function on Ω_w;

(2) The formal power series y is algebraic over the function field $K(X)$.

(3) The Arakelov degree $\widehat{\deg}(T_{x_0}X, ||\cdot||^{\mathrm{cap}})$ is positive.

Remark 1.2.5. Bost and Chambert-Loir ([**BCL09**, Thm. 7.9]) showed that the condition (2) can be deduced from (1) and (3) under certain assumption on y similar to the assumption that both $\tau(y) = 0$ and $\rho(y) < \infty$ in Theorem 1.1.4 by using the slope method.

Corollary 1.2.6. *The theorem still holds if we only assume that* (Ω_w) *is a weak adélic tube.*

Proof. The idea is implicitly contained in the discussion in [**Bos99**, section 4.4]. We only need to prove that y is rational over $X_{L'}$, where L'/L is a finite extension which we may assume does not have any real places. Let w be a place of L and w' a place of L' over w.

For w is archimedean, choose the embedding $\sigma' \colon L' \to \mathbb{C}$ corresponding to w' which extends the chosen embedding $\sigma : L \to \mathbb{C}$ corresponding to w. We have a natural identification $Y_{\sigma'}(\mathbb{C}) = Y_\sigma(\mathbb{C})$, and we take $\Omega_{w'} := \Omega_w$. If w is a finite place, we set $\Omega_{w'} = \Omega_w \otimes_{L_w} L_{w'}$.

Since L' does not have any real places, the weak adélic tube $(\Omega_{w'})$ is an adélic tube. The first two conditions in Theorem 1.2.4 still hold and the Arakelov degree of $T_{x_0}X$ with respect to (Ω'_w) is the same as that of $T_{x_0}X$ with respect to (Ω_w). We can apply Theorem 1.2.4 to y over $X_{L'}$ and conclude that y is rational. $\square$

2. Formal subschemes

In this section, we prove a strengthening (Corollary 2.2.8) of the following theorem due to Bost following closely the arguments in [**Bos01**, **Gas10**, **Her12**].

Theorem 2.0.1 ([**Bos01**, Thm. 2.3])**.** *Let G be a commutative algebraic group over a number field K and let W be a K-sub vector space of* $\operatorname{Lie} G$. *If for all but finitely many finite places v of K, the k_v-Lie algebra $W \otimes k_v$*[1] *is closed under the p-th power map of derivatives, then W is the Lie algebra of some algebraic subgroup of G.*

Although the proof of Corollary 2.2.8 only involves the study of formal subschemes of commutative algebraic groups, we start from the general setting of algebraicity criteria.

2.0.2. Let X be a geometrically irreducible quasi-projective variety of dimension N over some number field K and let P be a K-point of X. We denote by $\widehat{X}_{/P}$ the formal completion of X at P. Let $\widehat{V}$ be a smooth formal subvariety of $\widehat{X}_{/P}$ of dimension d. Throughout this section, we will

[1]This makes sense after choosing a spread out of W and G over $\mathcal{O}_K[\frac{1}{N}]$ and that this assumption holds for all but finitely many v is independent of the choice of the spread out.

assume that for any place v, the base change to K_v: $\widehat{V}_{K_v} \subset X_{K_v}$ is analytic. That is, the power series defining $\widehat{V}_{K_v}$ have positive radii of convergence. We say $\widehat{V}$ is *algebraic* if the smallest Zariski closed subset Y of X containing P such that $\widehat{V} \subset \widehat{Y}_{/P}$ has the same dimension as $\widehat{V}$. Without loss of generality, we assume in this subsection that $\widehat{V}$ is Zariski dense in X.

2.1. The slope method of Bost. In this section, we briefly recall the slope method by Bost ([**Bos01**, Sec. 4]). See also [**Gas10**, Sec. 2].

2.1.1. We fix a choice of $\mathcal{X}$ flat projective scheme over $\mathrm{Spec}(\mathcal{O}_K)$ such that $\overline{X} := \mathcal{X}_K$ is some compactification of X. We also fix a choice of a relatively ample Hermitian line bundle $(\mathcal{L}, \{||\cdot||_\sigma\}_\sigma)$ on $\mathcal{X}$. We denote by L the restriction of $(\mathcal{L}, \{||\cdot||_\sigma\}_\sigma)$ on $\mathcal{X}$ to $\overline{X}$.

For $D \in \mathbb{N}$, let E_D be the finitely generated projective $\mathcal{O}_K$-module $\Gamma(\mathcal{X}, \mathcal{L}^D)$. For $n \in \mathbb{N}$, let V_n be the n-th infinitesimal neighborhood of P in $\widehat{V}$ and let V_{-1} be $\emptyset$. We define a decreasing filtration on E_D as follows: for $i \in \mathbb{N}$, let E_D^i be the sub $\mathcal{O}_K$-module of E_D consisting of elements vanishing on V_{i-1}. We consider

$$\phi_D^i : E_D^i \to \ker(\mathcal{L}^{\otimes D}|_{V_i} \to \mathcal{L}^{\otimes D}|_{V_{i-1}}) \cong S^i(T_P\widehat{V})^\vee \otimes (\mathcal{L}_P)^{\otimes D},$$

where the first map is evaluation on V_i and S^i denotes the i-th symmetric power. We will also use ϕ_D^i to denote its linear extension $E_D^i \otimes K \to S^i(T_P\widehat{V})^\vee \otimes (\mathcal{L}_P)^{\otimes D}$.

2.1.2. To define the height $h(\phi_D^i)$, we need to specify the structure of the source and the target of ϕ_D^i as Hermitian vector bundles (over $\mathcal{O}_K$) . Notice that the choice of $\mathcal{X}$ gives rise to a projective $\mathcal{O}_K$-module $\mathcal{T}^\vee$ in $(T_P\widehat{V})^\vee$. More precisely, since $\mathcal{X}$ is projective, there is a unique extension $\mathcal{P}$ of P over $\mathcal{O}_K$, we take $\mathcal{T}^\vee$ to be the image of $\mathcal{P}^*\Omega_{\mathcal{X}/\mathcal{O}_K}$ in $(T_P\widehat{V})^\vee$. Moreover, $\mathcal{P}^*\mathcal{L}$ is a projective $\mathcal{O}_K$-module in $\mathcal{L}_P$. Then for any finite place v, we have a unique norm $||\cdot||_v$ on $E_D^i \otimes K$ (resp. $S^i(T_P\widehat{V})^\vee \otimes (\mathcal{L}_P)^{\otimes D}$) such that for any element s, $||p^m s||_v \leq p^{-m[K_v:\mathbb{Q}_p]}$ if and only if $s \in E_D^i$ (resp. $s \in S^i\mathcal{T}^\vee \otimes (\mathcal{P}^*\mathcal{L})^{\otimes D}$). For an archimedean place σ, given the Hermitian norm on $\mathcal{L}$, we equip $E_D^i \otimes K$ and $\mathcal{L}_P$ with the supremum norm and the restriction norm. We fix a choice of Hermitian norm on $T_P\widehat{V}$ and then obtain the induced norm on $S^i\mathcal{T}^\vee \otimes (\mathcal{P}^*\mathcal{L})^{\otimes D}$.[2] We define

$$h(\phi_D^i) = \frac{1}{[K:\mathbb{Q}]} \sum_{\text{all places } v} h_v(\phi_D^i), \text{ where } h_v(\phi_D^i) = \sup_{s \in E_D^i, ||s||_v \leq 1} \log ||\phi_D^i(s)||_v.$$

[2]To obtain the norm on $S^i\mathcal{T}^\vee$, we view it as a quotient of $(\mathcal{T}^\vee)^{\otimes i}$.

2.1.3. Let E be an Hermitian vector bundle over $\mathrm{Spec}(\mathcal{O}_K)$. The *Arakelov degree* $\widehat{\deg}(E)$ is defined to be the Arakelov degree[3] of the determinant line bundle $\det(E)$. The *slope* $\mu(E)$ is defined to be $\widehat{\deg}(E) \cdot (\mathrm{rk}(E))^{-1}$ and the *maximal slope* $\mu_{\max}(E)$ is defined to be $\max_F \mu(F)$ where F runs through all sub bundles of E.

We recall some basic properties of the Arakelov degree and the maximal slope.

Proposition 2.1.4 (Slope inequality [**Bos01**, Prop. 4.6, Eqn. (4.18)])**.** *Since $\widehat{V}$ is Zariski dense in X, we have*

$$\widehat{\deg}(E_D) \leq \sum_{i=0}^{\infty} \mathrm{rk}(E_D^i/E_D^{i+1})(\mu_{\max}(S^i\mathcal{T}^\vee \otimes (\mathcal{P}^*\mathcal{L})^{\otimes D}) + h(\phi_D^i)).$$

Here as $E_D^i = 0$ for i large enough, the right hand side is a finite sum.

Proposition 2.1.5. *There exists a positive constant C such that*

(1) (Arithmetic Hilbert–Samuel formula [**Bos01**, Prop. 4.4, Lem. 4.7]*)* $\widehat{\deg}(E_D) \geq -CD^{N+1}$,

(2) ([**Bos01**, Lem. 4.8]*)* $\mu_{\max}(S^i\mathcal{T}^\vee \otimes (\mathcal{P}^*\mathcal{L})^{\otimes D}) \leq C(i + D)$.

2.1.6. Bost reduced the proof of Theorem 2.0.1 to the algebraicity of a certain formal subscheme $\widehat{V}$ of G (see the proof of Corollary 2.2.8 for details) and used the tools in Arakelov geometry to show the algebraicity. We now sketch his proof of the algebraicity result. A modification of this idea will be used in the proof of Theorem 2.2.5. See also [**Gas10**, Thm. 2.2] and its proof.

By Proposition 2.1.5, we have a good control of every term in the slope inequality except $h(\phi_D^i)$. In order to understand $h(\phi_D^i)$, one expresses it as a sum of local terms $h_v(\phi_D^i)$ and uses the arithmetic property of $\widehat{V}$ at each place to obtain an upper bound for $h_v(\phi_D^i)$. For every finite place v, Bost defined a notion of *size* R_v of $\widehat{V}_{K_v}$. This notion plays a similar role to the convergence radius of formal power series. Bost proved that

$$h_v(\phi_D^i) \leq -i \log R_v.$$

For every archimedean place σ, the analytic submanifold V_σ^{an} of $\widehat{V}$ admits a uniformization by $\mathbb{C}^d$. Bost used Schwarz's lemma to show that

$$\limsup_{i/D \to \infty} \frac{1}{i} h_\sigma(\phi_D^i) = -\infty.$$

[3]We use the normalized one independent of the choice of number field K. See section 5.1.

Under the assumption of Theorem 2.0.1, $\sum_v \log R_v$ is finite, and hence we have

$$\limsup_{i/D\to\infty} \frac{1}{i} h(\phi_D^i) = -\infty.$$

Then, under the assumption that $N > d$, one deduces a contradiction to the slope inequality (see [**Bos01**, pp. 204] for details). In the proof of Theorem 2.2.5, $\sum_v \log R_v$ may be infinite and one instead studies the asymptotic behavior of $\dfrac{\sum_v h_v(\phi_D^i)}{i\log i}$ and $\dfrac{h_\sigma(\phi_D^i)}{i\log i}$. This is done in a general setting by Gasbarri using higher dimensional Nevanlinna theory.

2.2. A refinement of a theorem of Gasbarri in a special case. For simplicity, we only work with the classical higher dimensional Nevanlinna theory developed by Griffiths and King [**GK73**]. See also [**Bos01**, Sec. 4.3] and [**Gas10**, Sec. 5.24]. We refer the reader to [**Gas10**, Sec. 5] for the more general setting. The important common features of the formal subschemes $\widehat{V}$ studied in the proofs of Bost's theorem and its strengthening are:

(1) For every complex place, the analytic sub manifold defined by $\widehat{V}$ admits a uniformization map from $\mathbb{C}^{\dim \widehat{V}}$;
(2) $\widehat{V}$ is a formal leaf of some involutive subbundle of the tangent bundle of the commutative group G.

We will only focus on such particular type of formal subschemes.

2.2.1. To bound $h_\sigma(\phi_D^i)$, we fix a complex embedding $\sigma : K \to \mathbb{C}$ for each archimedean place. We assume that there exists an analytic map $\gamma_\sigma : \mathbb{C}^d \to X_\sigma(\mathbb{C})$ which sends 0 to P_σ and maps the germ of $\mathbb{C}^d$ at 0 biholomorphically onto the germ V_σ^{an} of $\widehat{V}$.

Let $z = (z_1, \cdots, z_d)$ be the coordinate of $\mathbb{C}^d$ and the Hermitian norm $||z||$ on $\mathbb{C}^d$ is given by $(|z_1|^2 + \cdots + |z_d|^2)^{1/2}$. Let ω be the Kahler form on $\mathbb{C}^d - \{0\}$ defined by $dd^c \log ||z||^2$. Then ω is the pull-back of the Fubini–Study metric on $\mathbb{P}^{d-1}(\mathbb{C})$ via $\pi : \mathbb{C}^d - \{0\} \to \mathbb{P}^{d-1}(\mathbb{C})$.

Let η is the first Chern form of the fixed Hermitian ample line bundle $\mathcal{L}|_{X_\sigma}$. More precisely, η can be defined locally as follows: choose a generator s of $\mathcal{L}|_{X_\sigma}$ on a small enough open set $U \subset X_\sigma$, $\eta|_U$ is defined to be $-dd^c \log ||s||_\sigma^2$. Notice that this (1,1)-form is independent of the choice of a local generator as $dd^c \log |f|^2 = 0$ for a nowhere vanishing holomorphic function f. We always assume that η is positive, which is possible by a suitable choice of the Hermitian metric.

Definition 2.2.2. We define the *characteristic function* $T_{\gamma_\sigma}(r)$ as follows:

$$T_{\gamma_\sigma}(r) = \int_0^r \frac{dt}{t} \int_{B(t)} \gamma_\sigma^* \eta \wedge \omega^{d-1},$$

where $B(t)$ is the ball around 0 of radius t in $\mathbb{C}^d$.

Definition 2.2.3. We define the *order* ρ_σ of γ_σ to be

$$\limsup_{r \to \infty} \frac{\log T_{\gamma_\sigma}(r)}{\log r}.$$

It is a standard fact that ρ_σ is independent of the choice of an Hermitian ample line bundle on $\overline{X}_\sigma$[4]. When ρ_σ is finite, that γ_σ is of order ρ_σ implies that for any $\epsilon > 0$, we have $T_{\gamma_\sigma}(r) < r^{\rho_\sigma + \epsilon}$ for r large enough. We denote by ρ the maximum of ρ_σ over all archimedean places σ.

2.2.4. Let $\mathcal{F}$ be an involutive subbundle of the tangent bundle TX of X. From now on, we assume that $\widehat{V}$ is the formal leaf of $\mathcal{F}$ passing through P. We may spread out $\mathcal{F}$ and X and assume that they are defined over $\mathcal{O}_K[1/n]$ for some integer n. Let M_{good} be the set of finite places v of K such that $\mathrm{char}(k_v) \nmid n$ and that $\mathcal{F} \otimes k_v$ is stable under p-th power map of derivatives. Let α be the *A-density*[5] of bad places defined by (see [**Her12**, Def. 3.5]):

$$\limsup_{x \to \infty} \left(\sum_{v | p_v \leq x, v \notin M_{good}} \frac{[L_v : \mathbb{Q}_{p_v}] \log p_v}{p_v - 1} \right) \left([L : \mathbb{Q}] \sum_{p \leq x} \frac{\log p}{p-1} \right)^{-1}.$$

Theorem 2.2.5. *Assume that $\widehat{V}$ is a formal leaf and is Zariski dense in X, then*

$$1 \leq \frac{N}{N-d} \rho \alpha.$$

This is a refinement of a special case of [**Gas10**, Thm. 5.21]. To get the better bound here using some ideas from [**Her12**], we need the following auxiliary lemmas.

[4]See [**Gas10**, Thm. 4.13(c) and Prop. 5.9]. Roughly speaking, one first shows that ρ_σ is independent of the choice of an Hermitian metric on a fixed ample line bundle and then shows that ρ_σ is independent of the choice of an ample line bundle. The first part follows from the fact that the difference between two different metrics is bounded. For the second part, let T_i be the characteristic function of L_i, $(i = 1, 2)$ with a suitable choice of metrics that will be specified later. There exists a positive integer D such that $L_1^D \otimes L_2^{-1}$ is ample on $\overline{X}_\sigma$. We choose the metric on L_i such that the first Chern form of the induced metric on $L_1^D \otimes L_2^{-1}$ is positive. Then T_{γ_σ} with respect to $L_1^D \otimes L_2^{-1}$ is non-negative. Hence $DT_1 \geq T_2$ and ρ_σ defined by L_1 is no less than that defined by L_2. The same argument shows the converse is also true and hence ρ_σ is independent of the choice of ample Hermitian line bundles.

[5]Here A stands for arithmetic and this notion is related to the natural density by [**Her12**, Lem. 3.7].

Lemma 2.2.6. *For any $\epsilon > 0$ and any complex embedding σ, there exists a constant C_1 independent of i, D such that*

$$h_\sigma(\phi_D^i) \leq C_1(i+D) - \frac{i}{\rho_\sigma + \epsilon} \log \frac{i}{D}.$$

In particular,

$$\frac{1}{[K:\mathbb{Q}]} \sum_\sigma h_\sigma(\phi_D^i) \leq C_1(i+D) - \frac{i}{\rho + \epsilon} \log \frac{i}{D}.$$

Proof. This is [**Gas10**, Thm. 5.19 and Prop. 5.26]. We sketch a more direct[6] proof for the special case here using the same idea originally due to Bost. See also [**Her12**, Lem. 6.8].

By [**Bos01**, Cor. 4. 16], there exists a constant B_1 only depend on d such that

$$h_\sigma(\phi_D^i) \leq -i \log r + DT_{\gamma_\sigma}(r) + B_1 i.$$

By the definition of ρ_σ, there exists a constant $M > 0$ such that for all $r > M$, we have $T_{\gamma_\sigma}(r) < r^{\rho_\sigma+\epsilon}$. On the other hand, as in the proof of [**Gas10**, Thm. 4.15], $-i \log r + Dr^{\rho_\sigma+\epsilon}$, as a function of r, reaches its minimum in $r_0 = (\frac{i}{(\rho_\sigma+\epsilon)D})^{1/(\rho_\sigma+\epsilon)}$. Therefore, once i/D is large enough so that $r_0 > M$, we have

$$h_\sigma(\phi_D^i) \leq -i \log r_0 + Dr_0^{\rho_\sigma+\epsilon} + B_1 i \leq -\frac{i}{\rho_\sigma + \epsilon} \log \frac{i}{D} + B_2 i,$$

for some constant B_2. In the case when i/D is not large enough, we notice that there exists a constant B_3 such that (see for example [**Bos01**, Prop. 4.12])

$$h_\sigma(\phi_D^i) \leq B_3(i+D).$$

Since $\frac{i}{\rho_\sigma+\epsilon} \log \frac{i}{D} \leq B_4 i$, we have

$$h_\sigma(\phi_D^i) \leq (B_3 + B_4)(i+D) - \frac{i}{\rho_\sigma + \epsilon} \log \frac{i}{D}.$$

We can take C_1 to be $\max\{B_2, B_3 + B_4\}$. □

Lemma 2.2.7. *For any $\epsilon > 0$, there exists a constant C_2 such that*

$$\frac{1}{[K:\mathbb{Q}]} \sum_{\text{all places}} h_v(\phi_D^i) \leq (\alpha + \epsilon) i \log i + C_2(i+D).$$

[6]We use the definition of the order as in [**Bos01**] rather than as in [**Gas10**]. Gasbarri gave a proof showing that two definitions are the same, but in this paper, we only need to work with the definition in [**Bos01**].

Proof. This is [**Her12**, Prop. 3.6]. ∎

Proof of Theorem 2.2.5. We follow [**Her12**, Sec. 6.6]. By Proposition 2.1.4 and Proposition 2.1.5, we have

$$-C_3D^{N+1} \leq \sum_{i=0}^{\infty} \mathrm{rk}(E_D^i/E_D^{i+1})(C_4(i+D)+h(\phi_D^i)).$$

By Lemma 2.2.6 and Lemma 2.2.7, we have

$$-C_3D^{N+1} \leq \sum_{i=0}^{\infty} \mathrm{rk}(E_D^i/E_D^{i+1})(C_5(i+D)+(\alpha+\epsilon-\frac{1}{\rho+\epsilon})i\log i+\frac{i}{\rho+\epsilon}\log D).$$

Let $S_D(\delta)$ be

$$\sum_{i\leq D^\delta} \mathrm{rk}(E_D^i/E_D^{i+1})(-C_5(i+D)+(-\alpha-\epsilon+\frac{1}{\rho+\epsilon})i\log i-\frac{i}{\rho+\epsilon}\log D)$$

and $S'_D(\delta)$ be

$$\sum_{i>D^\delta} rk(E_D^i/E_D^{i+1})(-C_5(i+D)+(-\alpha-\epsilon+\frac{1}{\rho+\epsilon})i\log i-\frac{i}{\rho+\epsilon}\log D).$$

By [**Bos01**, Lem. 4.7 (1)], $\mathrm{rk}(E_D^0/E_D^{i+1})<(i+1)^d$. Hence (see [**Her12**, Lem. 6.14]) if $\delta\geq 1$, then

$$|S_D(\delta)| \leq C_6D^\delta\log D\sum_{i\leq D^\delta}\mathrm{rk}(E_D^i/E_D^{i+1}) \leq C_7D^{(d+1)\delta}\log D.$$

On the other hand, if $\frac{1}{1-(\rho+\epsilon)(\alpha+\epsilon)}<\delta<N$, [**Her12**, Lem. 6.15] shows that for D large enough,

$$S'_D(\delta)\geq C_8D^{N+\delta}\log D.$$

If there exists a δ such that $1\leq\frac{1}{1-(\rho+\epsilon)(\alpha+\epsilon)}<\delta<N/d$, then

$$S'_D(\delta)+S_D(\delta)\geq C_9D^{N+\delta}\log D$$

for D large enough, which contradicts the fact that

$$S'_D(\delta)+S_D(\delta)\leq C_3D^{N+1}.$$

In other words,

$$N/d\leq\frac{1}{1-(\rho+\epsilon)(\alpha+\epsilon)}.$$

As ϵ is arbitrary, we obtain the desired result by rearranging the inequality. $\square$

Corollary 2.2.8. *Given a commutative algebraic group G over K and an K-sub vector space W of* $\mathrm{Lie}\, G$. *Assume that there exists a set M of finite places of L such that:*

(1) for any $v \in M$ over rational prime p, W modulo v is closed under p-th power map,

(2) $$\liminf_{x\to\infty} \left(\sum_{v, p_v \leq x, v \in M} \frac{[L_v : \mathbb{Q}_{p_v}] \log p_v}{p_v - 1} \right) \left([L : \mathbb{Q}] \sum_{p \leq x} \frac{\log p}{p-1} \right)^{-1} = 1.$$

Then W is the Lie algebra of some algebraic subgroup of G.

Proof. The idea is due to Bost. We apply Theorem 2.2.5 to the formal leaf $\widehat{V}$ passing through identity of the involutive subbundle of the tangent bundle of G generated by W via translation. Since the Zariski closure of $\widehat{V}$ is an algebraic subgroup of G, we may replace G by this subgroup and assume that $\widehat{V}$ is Zariski dense in G. We take the uniformization map to be the exponential map $W(\mathbb{C}) \to \mathrm{Lie}\, G(\mathbb{C}) \to G(\mathbb{C})$. It is a standard fact that the order ρ of this uniformization map is finite.[7] On the other hand, the assumptions on W are equivalent to that the A-density of bad primes α is 0. There would be a contradiction with Theorem 2.2.5 if $\widehat{V}$ is not algebraic. $\square$

[7] In [**BW07**, p. 112], they summarized some results of Faltings and Wütholz that may enable us to show ρ is finite by standard complex analytic arguments.

CHAPTER 3

Grothendieck–Katz p-curvature conjecture

In this chapter, we discuss our variant of the p-curvature conjecture (Theorem 1) for a vector bundle with connection (M, ∇) on $\mathbb{P}^1_K - \{0, 1, \infty\}$, where K is a number field: if the p-curvature vanishes for all finite places, then all formal horizontal sections of (M, ∇) are rational. In section 3, we formulate our main result and in particular the condition which substitutes for the vanishing of the p-curvature when it does not make sense to reduce (M, ∇) mod $\mathfrak{p}$.

The proof is given in section 4: we first apply Theorem 1.1.4 to the formal horizontal sections of (M, ∇) centered at a specific point x_0 to show its algebraicity. Then we are allowed to apply Corollary 1.2.6 and deduce that these formal sections are rational. For the first step, the interpretation of $\mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$ as the moduli space of elliptic curves with level 2 structure enables us to define a uniformization of $\mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$ by the unit disc and this uniformization gives a lower bound for the v-adic radii of uniformizability at archimedean places. The chosen point x_0 corresponds to the elliptic curve with smallest stable Faltings' height and we use the Chowla-Selberg formula to deduce the lower bound. The link between our lower bound of archimedean radii and the stable Faltings' height is given in section 5. For the second step, we choose the archimedean component of the adelic tube to be the image in $\mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$ of a standard fundamental domain for $\Gamma(2)$ under the uniformization mentioned above and give a lower bound for its local capacity.

Katz has shown in [**Kat82**, Thm. 10.2] that if the p-curvature conjecture holds, then for any vector bundle with a flat connection (M, ∇) on a smooth variety X over K, the Lie algebra $\mathfrak{g}_{\mathrm{gal}}$ of the differential Galois group G_{gal} of (M, ∇) is in some sense generated by the p-curvatures. Namely, the p-curvature conjecture implies that $\mathfrak{g}_{\mathrm{gal}}$ is the smallest algebraic Lie subalgebra of $\mathfrak{gl}_n(K(X))$ such that for all but finitely many $\mathfrak{p}$ the reduction of $\mathfrak{g}_{\mathrm{gal}}$ mod $\mathfrak{p}$ contains the p-curvature, where $K(X)$ is the function field of X.

We use Theorem 1 to prove a result (Theorem 3.2.5) analogous to Katz's theorem when $X = \mathbb{P}^1_K - \{0, 1, \infty\}$ in section 3. Of course, this result involves a condition at every $\mathfrak{p}$, but as a compensation we describe G_{gal} and not only $\mathfrak{g}_{\text{gal}}$. When (M,∇) is the relative de Rham cohomology with the Gauss–Manin connection, this extra local condition is often vacuous. In section 4, we discuss the example of the Legendre family (Remark 4.2.2) and show that a variant of our result implies that $\mathfrak{g}_{\text{gal}}$ is generated by the p-curvatures, which recovers a result of Katz.

In section 6, we discuss some variants on the p-curvature conjecture of vector bundles with connection over X when X is certain affine elliptic curve with j-invariant 1728 (Theorem 2) or $\mathbb{A}^1 - \{\pm 1, \pm i\}$. As in section 3, we define the notion of p-curvature vanishing at bad primes using local convergence condition. Using the property of theta functions and Weierstrass-$\wp$ functions, we deduce from a result of Eremenko [**Ere11**] a lower bound of the archimedean radii, which enables us to prove our results by Theorem 1.1.4. We give an example of an (M,∇) over the affine elliptic curve such that its p-curvatures vanish for all $\mathfrak{p}$ but its G_{gal} is $\mathbb{Z}/2\mathbb{Z}$. We also give an example to show that even when (M,∇) has good reduction everywhere over $\mathbb{A}^1 - \{\pm 1, \pm i\}$ and all its p-curvatures vanish, it can still have local monodromies of order two around the singularities $\pm 1, \pm i, \infty$.

3. Statement of the main results

Let X be $\mathbb{P}^1_{\mathcal{O}_K} - \{0,1,\infty\}$ and M a vector bundle with a connection $\nabla\colon M \to \Omega^1_{X_K} \otimes M$ over X_K. For Σ a finite set of finite rational primes, we set $\mathcal{O}_{K,\Sigma} = \mathcal{O}_K[1/p]_{p\in\Sigma} \subset K$.

3.1. The p-curvature and p-adic differential Galois groups.

3.1.1. For Σ, as above, sufficiently large, (M,∇) extends to a vector bundle with connection (again denoted (M,∇)) over $X_{\mathcal{O}_{K,\Sigma}}$. In particular, if $p \notin \Sigma$ we can consider the pull back of (M,∇) to $X \otimes \mathbb{Z}/p\mathbb{Z}$. If D is a derivation on $X \otimes \mathbb{Z}/p\mathbb{Z}$, so is D^p. Let $\nabla(D)$ be the map $(D \otimes \mathrm{id}) \circ \nabla$. Then on $X \otimes \mathbb{Z}/p\mathbb{Z}$, the *p-curvature* is given by (see [**Kat82**, Sec. VII] for details) [1]

$$\psi_p(D) := \nabla(D^p) - \nabla(D)^p \in \mathrm{End}_{\mathcal{O}_{X\otimes\mathbb{Z}/p\mathbb{Z}}}(M \otimes \mathbb{Z}/p\mathbb{Z}).$$

In particular, $\psi_p\left(\frac{d}{dx}\right) = -\left(\nabla\left(\frac{d}{dx}\right)\right)^p$. Since $\psi_p(D)$ is p-linear in D, for $X = \mathbb{P}^1_{\mathcal{O}_K} - \{0,1,\infty\}$, the equation $\psi_p \equiv 0$ is equivalent to $-\left(\nabla\left(\frac{d}{dx}\right)\right)^p \equiv 0$.

In general, the ψ_p depends on the choice of extension of (M,∇) over $X_{\mathcal{O}_{K,\Sigma}}$. However, any two such extensions are isomorphic over $X_{\mathcal{O}_{K,\Sigma'}}$ for some sufficiently large Σ'.

[1] We could have defined the p-curvatures by considering derivations on X_{k_v} for v a place of K. For primes which are unramified in K, the two definitions are essentially equivalent, and the present definition will allow us to formulate the inequalities which arise below in a more uniform manner.

3.1.2. Let L be a finite extension of K and w a place of L over v. We view L as a subfield of $\mathbb{C}_p$ via w. Fix an $x_0 \in X(L_w)$. Given a positive real number r, we denote by $D(x_0, r)$ the open rigid analytic disc of radius r, with center x_0. Thus

$$D(x_0, r) = \{x \in X(\mathbb{C}_p) \text{ such that } |x - x_0|_p < r\},$$

where $|\cdot|_p$ is normalized so that $|p|_p = p^{-1}$.

It is naturally endowed with the connection such that for any local sections m, l of M and $M^\vee$ respectively,

$$d\langle l, m\rangle = \langle \nabla_{M^\vee}(l), m\rangle + \langle l, \nabla_M(m)\rangle.$$

Definition 3.1.3. If (V, ∇) is a vector bundle with connection over some scheme or rigid space, we denote by $\langle V, \nabla\rangle^{\otimes}$, or simply $\langle V\rangle^{\otimes}$, if there is no risk of confusion regarding the connection ∇, the category of ∇-stable sub quotients of all the tensor products $V^{m,n}$ for $m, n \geq 0$. If the scheme or rigid space over which V is a vector bundle is connected, then this is a Tannakian category.

Definition 3.1.4. Let F_w be the field of fractions of the ring of all rigid analytic functions on $D(x_0, r)$ and $\eta_w \colon \mathrm{Spec}(F_w) \to X$ the natural map. Consider the fiber functor

$$\eta_w \colon \langle M|_{D(x_0,r)}\rangle^{\otimes} \to \mathrm{Vec}_{F_w}; \quad V \mapsto V_{\eta_w}.$$

The *p-adic differential Galois group* $G_w(x_0, r)$ is defined to be the automorphism group $\underline{\mathrm{Aut}}^{\otimes}\eta_w$ of η_w.

For $v|p$ a finite place of K, we will say that (M, ∇) has *good reduction* at v if (M, ∇) extends to a vector bundle with connection on $X_{\mathcal{O}_v}$. The following lemma gives the basic relation between the p-curvature and the p-adic differential Galois group.

Lemma 3.1.5. *Let $x_0 \in X(\mathcal{O}_{L_w})$ and suppose that (M, ∇) has good reduction at v. If the p-curvature vanishes, then the local differential Galois group $G_w(x_0, p^{-\frac{1}{p(p-1)}})$ is trivial.*

Proof. To show that $G_w(x_0, p^{-\frac{1}{p(p-1)}})$ is trivial, we have to show that the restriction of M to $D(x_0, p^{-\frac{1}{p(p-1)}})$ admits a full set of solutions. It is well known that this is the case when $\psi_p \equiv 0$, but for the convenience of the reader we sketch the argument. See [**Bos01**, section 3.4.2, prop. 3.9] for related arguments.

Assume there is an extension of (M, ∇) to a vector bundle with connection $(\mathcal{M}, \nabla)$ over $X_{\mathcal{O}_v}$. If m_0 is any section of $\mathcal{M}$, then a formal section in the kernel of ∇ is given by

$$m = \sum_{i=0}^{\infty} \nabla\left(\frac{d}{dx}\right)^i (m_0)\frac{(x-x_0)^i}{i!}(-1)^i.$$

Since $\psi_p \equiv 0$ (recall that this means the p-curvature vanishes on $X_{\mathcal{O}_v} \otimes \mathbb{Z}/p\mathbb{Z}$), we have $\nabla(\frac{d}{dx})^p(\mathcal{M}) \subset p\mathcal{M}$. Hence $\nabla(\frac{d}{dx})^i(m_0) \subset p^{\left[\frac{i}{p}\right]}\mathcal{M}$, and one sees easily that the series defining m converges on $D(x_0, p^{-\frac{1}{p(p-1)}})$. □

Remark 3.1.6.

(1) Unlike the notion of p-curvature, the definition of $G_w(x_0, r)$ does not require (M, ∇) to have good reduction. It depends only on the $\mathcal{O}_v$-model of X (which we of course always take to be $\mathbb{P}^1_{\mathcal{O}_v} - \{0, 1, \infty\}$), which is used to define $D(x_0, r)$, but not on how (M, ∇) is extended.

(2) If (M, ∇) has good reduction with respect to $X_{\mathcal{O}_v}$ and it admits a Frobenius structure with respect to some Frobenius lifting on $X_{\mathcal{O}_v}$, then $G_w(x_0, 1)$ is trivial whenever $x_0 \in X(\mathcal{O}_v)$. See for example [**Ked10**, 17.2.2, 17.2.3].

From now on we set $x_0 = \frac{1+\sqrt{3}i}{2}$, which corresponds to the elliptic curve with smallest stable Faltings height. In section 5, we will give a theoretical explanation of why this choice gives the best possible estimates. We set $G_w = G_w\big(\frac{1+\sqrt{3}i}{2}, p^{-\frac{1}{p(p-1)}}\big)$, and we take L to be a number field containing $K(\sqrt{3}i)$.

By Lemma 3.1.5, the local differential Galois group G_w is trivial when the vector bundle with connection (M, ∇) has good reduction over v, and $\psi_p \equiv 0$. This motivates the following definition:

Definition 3.1.7. We say that *the p-curvatures of* (M, ∇) *vanish for all* p if

(1) $\psi_p \equiv 0$ for all but finitely many p,

(2) $G_w = \{1\}$ for all primes w of L.

By what we have just seen, for all but finitely many p, the condition (1) makes sense, and implies (2). Thus (2) is only an extra condition at finitely many primes. As above, the definition does not depend on the extension of (M, ∇) to $X_{\mathcal{O}_{K,\Sigma}}$ or the choice of primes Σ.

3.2. The main theorem and a Tannakian consequence.

Theorem 3.2.1. *Let* (M, ∇) *be a vector bundle with a connection over* $X_K = \mathbb{P}^1_K - \{0,\, 1,\, \infty\}$*, and suppose that the p-curvatures of* (M, ∇) *vanish for all p. Then* (M, ∇) *admits a full set of rational solutions.*

The proof of this theorem is the subject of section 4.

Remark 3.2.2. By varying the conditions on the radii of convergence in (2), one can prove variants of Theorem 3.2.1, whose conclusion is that (M, ∇) has finite monodromy. See Remark 4.2.2 for details.

André has pointed out that, if one replaces (2) in Definition 3.1.7 by the condition that the so called *generic radii* of all formal horizontal sections of (M, ∇) are at least $p^{-\frac{1}{p(p-1)}}$, then the analogue of Theorem 3.2.1 admits an easier proof. Indeed if $w|p$, and the w-adic *generic radius* is at least $p^{-\frac{1}{p(p-1)}}$, then by [**BS82**, Sec. IV], p cannot divide the (finite by (1) and Katz's theorem [**Kat70**, Thm. 13.0]) order of the local monodromies. If this condition holds for all w, then the local monodromies around $0, 1, \infty$ are all trivial and hence the global monodromy is trivial.

Once one uses (1) to show that the local monodromies are finite, this argument is 'prime by prime'. We do not know if Theorem 3.2.1 admits a similar proof, which avoids global arguments, although this seems to us unlikely. In any case, our method allows us to deal with some cases when X is an affine elliptic curve or the projective line minus more than three points. See Theorem 6.1.1 and Proposition 6.4.1. The conclusion of both results is that (M, ∇) has finite monodromy and we will give examples with nontrivial monodromy. It seems unlikely that these results can be proved with a 'prime by prime' argument.

Applying Lemma 3.1.5, we have the following corollary:

Corollary 3.2.3. *If* (M, ∇) *is defined over* $X_{\mathbb{Z}}$ *and the p-curvature vanishes for all primes, then* (M, ∇) *admits a full set of rational solutions.*

3.2.4. As in [**Kat82**], we can use our main theorem to give a description of the differential Galois group of any vector bundle with a connection (M, ∇) over X_K.

Let $K(X)$ be the function field of X_K. Let ω be the fibre functor on $\langle M\rangle^{\otimes}$ given by restriction to the generic point of X_K. Write $G_{\mathrm{gal}} = \mathrm{Aut}^{\otimes}\,\omega \subset \mathrm{GL}(M_{K(X)})$ for the corresponding differential Galois group (see [**Kat82**, Ch. IV] and [**And04a**, 1.3, 1.4]).

Let G be the smallest closed subgroup of $\mathrm{GL}(M_{K(X)})$ such that:

(1) For almost all p, the reduction of $\mathrm{Lie}\,G$ mod p contains ψ_p.

(2) $G \otimes F_w$ contains G_w for all w, where, as above, F_w is the field of fractions of the ring of rigid analytic functions on $D\big(x_0, p^{-\frac{1}{p(p-1)}}\big)$.

Let $\mathfrak{g}$ be the smallest Lie subalgebra of $\mathrm{GL}(M_{K(X)})$ such that for almost all p, the reduction of $\mathfrak{g}$ mod p contains ψ_p. As proved in [**Kat82**, Prop. 9.3], $\mathfrak{g}$ is contained in Lie G_{gal}. Moreover, G_w is contained in $G_{\mathrm{gal}} \otimes F_w$ by definition. Hence G is a subgroup of G_{gal}. We will see from the proof of the following theorem that (in the presence of the condition (1)), to define G we only need to impose the condition (2) at finitely many primes.

Theorem 3.2.5. *Let (M, ∇) be a vector bundle with a connection defined over $X_K = \mathbb{P}^1_K - \{0, 1, \infty\}$. Then $G = G_{\mathrm{gal}}$.*

Proof. We follow the idea of the proof of Theorem 10.2 in [**Kat82**]. See also [**And04a**, Prop. 3.2.2].

By a theorem of Chevalley, there exists W in $\langle M\rangle^{\otimes}$ and a line $L' \subset W_{K(X)}$ such that G is the intersection of G_{gal} with the stabilizer of L'. Let W' be the smallest ∇-stable submodule of $W_{K(X)}$ containing L'. Then W' has a $K(X)$-basis of the form $\{l, \nabla l, \cdots, \nabla^{r-1} l\}$ where $l \in L'$, $r = \mathrm{rk} W'$, and we have written $\nabla^i l$ for $\nabla(\frac{d}{dx})^i(l)$. Replacing W by $W' \cap W$, we may assume that $W_{K(X)} = W'$. Then $L = L' \cap W$ is a line bundle in W.

As above, let $\mathfrak{g}$ be the smallest algebraic Lie subalgebra of $\mathrm{GL}(M_{K(X)})$ such that for almost all p the reduction of $\mathfrak{g}$ mod p contains ψ_p. Let Σ be a finite set of primes of $\mathbb{Q}$ such that (M, ∇) extends to a vector bundle $\mathcal{M}$ with connection $\nabla\colon \mathcal{M} \to \mathcal{M} \otimes \Omega_{X_{\mathcal{O}_{K,\Sigma}}}$ over $X_{\mathcal{O}_{K,\Sigma}}$, and $\mathfrak{g}$ mod p contains ψ_p for $p \notin \Sigma$. We also assume that Σ contains all primes $p \le r$.

Let $U \subset X_{\mathcal{O}_{K,\Sigma}}$ be a non-empty open subset such that $l \in L|_U$, L and W extend to vector bundles with connection $\mathcal{L}$ and $\mathcal{W}$ respectively, in $\langle \mathcal{M}|_U\rangle^{\otimes}$, and $\{l, \nabla l, \cdots, \nabla^{r-1} l\}$ forms a basis of $\mathcal{W}$. Let $\mathcal{N} := \mathrm{Sym}^r\,\mathcal{W} \otimes (\det \mathcal{W}^{\vee})$ with the induced connection. The argument in [**Kat82**] implies that for $p \notin \Sigma$, the p-curvature of $(\mathcal{N}, \nabla)$ vanishes. Let $N := \mathcal{N}_{X_K \cap U}$. We will use the condition

(2) in the definition of G to show that G_w acts trivially on N_{η_w}. We already know this for $p \notin \Sigma$, by Lemma 3.1.5. Thus we will only need to use (2) for $p \in \Sigma$. Assuming this for a moment, we can apply Theorem 3.2.1 to (N, ∇) and conclude that it has trivial global monodromy. Hence G_{gal} acts as a scalar on W. In particular, G_{gal} stabilizes L so, by the definition of L, $G_{\mathrm{gal}} = G$,

Use D to denote $D(x_0, p^{-\frac{1}{p(p-1)}})$. Recall that the category $\langle M|_{D(x_0,r)}\rangle^{\otimes} \otimes F_w$ is obtained from $\langle M|_{D(x_0,r)}\rangle^{\otimes}$ by taking the same collection of objects and tensoring the morphisms by F_w. By the definition of L, the group G_w acts as a character χ on L_{η_w}. The morphism $L_{\eta_w} \to W_{\eta_w}$ is a map between G_w-representations. By the equivalence of categories between $\langle M|_{D(x_0,r)}\rangle^{\otimes} \otimes F_w$ and the category of linear representations of G_w over F_w, this morphism is a finite F_w-linear combination of maps $L|_D \to W_D$ in $\langle M|_{D(x_0,r)}\rangle^{\otimes}$. In other words, there are a finite number of ∇-stable line bundles $W_i \subset W_D$, with G_w acting on W_{i,η_w} as χ such that $L|_D \subset \sum W_i$. In particular, $l|_D = \sum a_i \cdot w_i$, where $a_i \in F_w$ and $w_i \in W_i$. Since $\sum W_i$ is ∇-stable, $\nabla^n l \in \sum W_i$ and G_w acts as χ on $\nabla^n l|_D$. As W_{η_w} is generated by $\{l,\ \nabla l, \cdots, \nabla^{r-1} l\}|_D$, the group G_w acts as χ on W_{η_w}. Hence G_w acts trivially on N_{η_w}. $\square$

Using the same idea as in the last paragraph of the proof above, we have the following lemma which is of independent interest.

Lemma 3.2.6. *Let $H_w \subset G_{\mathrm{gal}}$ be the smallest closed subgroup such that $G_w \subset H_w \otimes_{K(X)} F_w$. Then H_w is normal in G_{gal}.*

Proof. We need the following fact (see [**And92**, Lem. 1]): Assume that G is a algebraic group over some field E. Let $H \subset G$ be a closed subgroup and V an E-linear faithful algebraic representation of G. Then H is a normal subgroup of G if for every tensor space $V^{m,n}$, and for every character χ of H over E, G stabilizes $(V^{m,n})^{\chi}$, the subspace of $V^{m,n}$ where H acts as χ. If G is connected, then these two conditions are equivalent.

We apply this result to $H_w \subset G_{\mathrm{gal}}$ and $V = M_{K(X)}$. Let $L \subset V^{m,n}$ be a line, and $W \subset V^{m,n}$ the smallest ∇-stable subspace containing L. It suffices to show that, if H_w acts via χ on L, then H_w acts via χ on W. This shows that $(V^{m,n})^{\chi}$ is ∇-stable, and hence that G_{gal} stabilizes $(V^{m,n})^{\chi}$.

As in the proof of the theorem above, G_w acts on W via χ. Hence H_w is contained in the subgroup of G_{gal} which acts on W via χ. $\square$

4. The proof: an application of theorems due to André and Bost–Chambert-Loir

As the coordinate ring of X_K a principal ideal domain, M is free. Hence we may view ∇ as a system of first-order homogeneous differential equations. Thus $M \cong \mathcal{O}_{X_K}^m$ and

$$\nabla(\frac{d}{dx})\boldsymbol{y} = \frac{d\boldsymbol{y}}{dx} - A(x)\boldsymbol{y},$$

where $\boldsymbol{y}$ is a section of M, x is the coordinate of X, and $A(x)$ is an $m \times m$ matrix with entries in $\mathcal{O}_{X_K} = K[x^{\pm}, (x-1)^{\pm}]$.

As above, we set $x_0 = \frac{1}{2}(1+\sqrt{3}i)$. If $\boldsymbol{y}_0 \in L^m$, there exists $\boldsymbol{y} \in L[[x-x_0]]^m$ such that $\boldsymbol{y}(x_0) = \boldsymbol{y}_0$ and $\nabla(\boldsymbol{y}) = 0$. Our goal is to show that if the p-curvatures of (M, ∇) vanishes for all p, then $\boldsymbol{y}$ is rational.

We will first apply Theorem 1.1.4 to show that $\boldsymbol{y}$ is algebraic and then apply Corollary 1.2.6 to conclude.

4.1. Estimate of the radii at archimedean places.

Lemma 4.1.1. *Suppose that $\phi\colon D(0,1) \to \mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$ is a holomorphic map such that $\phi(0) = x_0$. Then for any archimedean place w of the number field L where the connection and the initial conditions x_0, $\boldsymbol{y}_0$ are defined, $R_w \geq |\phi'(0)|_w$.*

Proof. Let z be the complex coordinate on $D(0,1)$. Consider the formal power series $\phi^*\boldsymbol{y}$. The vector valued power series $\boldsymbol{g} = \phi^*\boldsymbol{y}$ is a formal solution of the differential equations $\frac{d\boldsymbol{g}}{dz} = (\phi'(z))^{-1}A(\phi(z))\boldsymbol{g}$ which is associated to the vector bundle with connection $(\phi^*M, \phi^*\nabla)$. Since $D(0,1)$ is simply connected, $\boldsymbol{g}$ arises from a vector valued holomorphic function on $D(0,1)$ which we again denote by $\boldsymbol{g}$.

Let $t = \phi'(0)z$, and set $R = |\phi'(0)|_\infty$. Then we may identify $D(0,1)$ with the t-disc $D(0,R) = D_w(0, |\phi'(0)|_w)$ and the map ϕ with a map

$$\tilde{\phi} : D(0,R) \to \mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$$

which satisfies $\tilde{\phi}'(0) = 1$. By the definition of R_w, we have $R_w \geq |\phi'(0)|_w$. $\square$

4.1.2. Given x_0, the upper bound (in terms of x_0) of $|\phi'(0)|$ for all such ϕ in the above lemma has been studied by Landau and other people. Based on the work of Landau and Schottky, Hempel gave an explicit upper bound (see [**Hem79**, Thm. 4]) that can be reached when $x_0 = \frac{-1+\sqrt{3}i}{2}$. For the completeness of our paper, we give some details on the computation of $|\phi'(0)|$.

4.1.3. We recall the definition of θ-functions and their classical relation with the uniformization of $\mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$. Following the notation of [**Igu62**] and [**Igu64**], let

$$\theta_{00}(t) = \sum_{n\in\mathbb{Z}} \exp(\pi i n^2 t),\ \theta_{01}(t) = \sum_{n\in\mathbb{Z}} \exp(\pi i(n^2 t + n)), \theta_{10}(t) = \sum_{n\in\mathbb{Z}} \exp(\pi i(n + \frac{1}{2})^2 t)$$

These series converge pointwise to holomorphic functions on $\mathcal{H}$, which we denote by the same symbols.

Lemma 4.1.4. *(*[**Igu64**, p. 243]*) These holomorphic functions $\theta_{00}^4, \theta_{01}^4, \theta_{10}^4$ are modular forms of weight 2 and level $\Gamma(2)$. Moreover, there is an isomorphism from the ring of modular forms of level $\Gamma(2)$ to $\mathbb{C}[X, Y, Z]/(X - Y - Z)$ given by sending $\theta_{00}^4, \theta_{01}^4$ and θ_{10} to X, Y and Z respectively.*

4.1.5. Let $\lambda = \frac{\theta_{00}^4(t)}{\theta_{01}^4(t)} : \mathcal{H} \to \mathbb{P}^1(\mathbb{C})$ and $t_0 = \frac{1}{2}(-1 + \sqrt{3}i)$. Then $\lambda : \mathcal{H} \to \mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}$ is a covering map with $\Gamma(2)$ as the deck transformation group ([**Cha85**], VII, §7). In particular, the projective curve defined by $v^2 = u(u-1)(u-\lambda(t))$ is an elliptic curve. Moreover, it is isomorphic to the elliptic curve $\mathbb{C}/(\mathbb{Z} + t\mathbb{Z})$ (see *loc. cit.*).

We need the following basic facts mentioned in [**Igu62**, p. 180] and [**Igu64**, p. 244] in this section and section 5:

Lemma 4.1.6.

(1) Let η be the Dedekind eta function defined by $\eta = q^{1/24} \prod(1-q^n)$, where $q = e^{2\pi i t}$. We have $2^8\eta^{24} = (\theta_{00}\theta_{01}\theta_{10})^8$. In particular, the holomorphic functions $\theta_{00}, \theta_{01}, \theta_{10}$ are everywhere nonzero on the upper half plane.

(2) The derivative $\lambda'(t_0) = \pi i (\frac{\theta_{00}(t_0)\theta_{10}(t_0)}{\theta_{01}(t_0)})^4$.

(3) The holomorphic function $\frac{1}{2}(\theta_{00}^8 + \theta_{01}^8 + \theta_{10}^8)$ is the weight 4 Eisenstein form of level $\mathrm{SL}_2(\mathbb{Z})$ with constant term 1 in its Fourier expansion; the holomorphic function $\frac{1}{2}(\theta_{00}^4 + \theta_{01}^4)(\theta_{00}^4 + \theta_{10}^4)(\theta_{01}^4 - \theta_{10}^4)$ is the weight 6 Eisenstein form of level $\mathrm{SL}_2(\mathbb{Z})$ with constant term 1 in its Fourier expansion.

Lemma 4.1.7. *The map λ sends t_0 to x_0.*

PROOF. Since the automorphism group of the lattice $\mathbb{Z}+t_0\mathbb{Z}$, hence that of the elliptic curve $\mathbb{C}/(\mathbb{Z}+t_0\mathbb{Z})$ is of order 6, the automorphism group of the elliptic curve $v^2 = u(u-1)(u-\lambda(t_0))$ must also be of order 6. In particular, λ must send t_0 to either $\frac{1}{2}(1+\sqrt{3}i)$ or $\frac{1}{2}(1-\sqrt{3}i)$ (the roots of $0 = j(t_0) = 2^8\frac{(\lambda(t_0)^2-\lambda(t_0)+1)^3}{\lambda(t_0)^2(\lambda(t_0)-1)^2}$). Moreover, from the definition of θ, we can easily see that $\lambda(t_0)$ has positive imaginary part. $\square$

Proposition 4.1.8. *Let y be a component of the formal solution of the differential equations. Then* $R_w^{\frac{[L:\mathbb{Q}]}{[L_w:\mathbb{R}]}} \geq \frac{3\Gamma(1/3)^6}{2^{8/3}\pi^3} = 5.632\cdots$.

PROOF. Consider the map $\lambda\circ\alpha\colon D(0,1)\to X_{\mathbb{C}}$, where $\alpha\colon D(0,1)\to\mathcal{H}$ is a holomorphic isomorphism such that $\alpha(0)=t_0$, that is, $\alpha\colon z\mapsto -\frac{1}{2}+\frac{\sqrt{3}i}{2}\frac{z+1}{1-z}$. We would like to apply Lemma 4.1.1 to the map $\lambda\circ\alpha$, which maps $0\in D(0,1)$ to x_0 since $\lambda(t_0)=\lambda(\frac{1}{2}(-1+\sqrt{3}i)) = x_0$ by Lemma 4.1.7.

Note that $|x_0| = |1-x_0| = 1$, so we have $|\theta_{00}(t_0)| = |\theta_{01}(t_0)| = |\theta_{10}(t_0)|$. By Lemma 4.1.6, we have

$$|\lambda'(t_0)| = |\pi i(\frac{\theta_{00}(t_0)\theta_{10}(t_0)}{\theta_{01}(t_0)})^4| = \pi|\theta_{00}(t_0)|^4 = \pi|2^8\eta^{24}(t_0)|^{1/6}.$$

We now apply the Chowla–Selberg formula (see **[SC67]**) to $\mathbb{Q}(\sqrt{3}i)$:

$$|\eta(t_0)|^4\Im(t_0) = \frac{1}{4\pi\sqrt{3}}\left(\frac{\Gamma(1/3)}{\Gamma(2/3)}\right)^3.$$

Then we have

$$|\lambda'(t_0)| = \pi|2^8\eta^{24}(t_0)|^{1/6} = \frac{\pi 2^{4/3}}{4\pi\sqrt{3}\Im(t_0)}\left(\frac{\Gamma(1/3)}{\Gamma(2/3)}\right)^3.$$

We get

$$|(\lambda\circ\alpha)'(0)| = |\lambda'(t_0)|\cdot|\alpha'(0)| = \frac{\pi 2^{4/3}}{4\pi\sqrt{3}\Im(t_0)}\left(\frac{\Gamma(1/3)}{\Gamma(2/3)}\right)^3\cdot 2\Im(t_0) = \frac{3\Gamma(1/3)^6}{2^{8/3}\pi^3}$$

by the fact $\Gamma(1/3)\Gamma(2/3) = \frac{2\pi}{\sqrt{3}}$. $\square$

4.2. Algebraicity of the formal solutions.

Proposition 4.2.1. *Let (M,∇) be a vector bundle with a connection over $\mathbb{P}^1_K - \{0,1,\infty\}$, and assume that the p-curvatures of (M,∇) vanish for all p. Then (M,∇) is locally trivial with respect to the étale topology of $\mathbb{P}^1_K-\{0,1,\infty\}$.*

Proof. Consider $\boldsymbol{y} \in L[[(x-x_0)]]$. By Proposition 4.1.8, we have

$$\prod_{w|\infty} R_w \geq 5.632\cdots.$$

If $w|p$ is a finite place of L, then since G_w is trivial, (M,∇) has a full set of solutions over $D(x_0, |p|^{\frac{1}{p(p-1)}})$. In particular, $\boldsymbol{y}$ is analytic on $D(x_0, |p|^{\frac{1}{p(p-1)}})$. Hence

$$\prod_{w|p} R_w \geq \prod_{w|p} |p|_w^{-\frac{1}{p(p-1)}} = p^{-\frac{1}{p(p-1)}}.$$

and

$$\log(\prod_{w} R_w) \geq \log 5.6325\cdots - \sum_p \frac{\log p}{p(p-1)} > 0.967\cdots.$$

Applying Theorem 1.1.4, we have that $\boldsymbol{y}$ is algebraic. Hence (M,∇) is étale locally trivial. $\square$

Remark 4.2.2. It is possible to define G_w using different radii such that the proof of the above proposition continues to hold. Here are two examples:

(1) Set $G'_w := G_w(x_0, \frac{1}{4})$ for all primes $w|2$ and $G'_w = G_w(x_0,1)$ for other w. We can define G' in the same way as G in section 3.2.4 but replacing G_w by G'_w. In this situation, we have $\log(\prod_w R_w) \geq \log 5.6325\cdots - \log 4 > 0.342\cdots$. Applying the same argument as in Theorem 3.2.5, we have $\operatorname{Lie} G' = \operatorname{Lie} G_{\mathrm{gal}}$.

In particular, if (M,∇) is a vector bundle with connection on X_K such that $\psi_p \equiv 0$ for almost all p, and $G'_w = \{1\}$ for all w, then (M,∇) has finite monodromy. This result cannot be proved 'prime by prime' because the condition at $w|2$ is too weak to imply that 2 does not divide the order of the local monodromies.

The equality $\operatorname{Lie} G' = \operatorname{Lie} G_{\mathrm{gal}}$ fails in general, if one drops condition (1) in section 3.2.4, and defines G' using just the analogue of condition (2) (that is with G_w replaced by G'_w). (The condition (1) is used to guarantee the assumption that $\tau(y) = 0, \rho(y) < \infty$ in Theorem 1.1.4.)

To see this, we consider the Gauss–Manin connection on H^1_{dR} of the Legendre family of elliptic curves. Since the Legendre family has good reduction at primes $w \nmid 2$, H^1_{dR} admits a Frobenius structure at such primes, so that $G_w = \{1\}$ (see Remark 3.1.6). For $w|2$ we have $G_w(x_0, \frac{1}{4}) = \{1\}$ by a direct computation: as in section 5.2 below, we see that the matrix giving the connection lies in $\frac{1}{2}\operatorname{End}(M_{\mathcal{O}_K}) \otimes \Omega^1_{X_{\mathcal{O}_K}}$ and a formal horizontal section of a general differential equation of this form will have convergence radius $\frac{1}{4}$. Hence, the smallest group containing all p-adic differential Galois groups is trivial while $\operatorname{Lie} G_{\mathrm{gal}} = \mathfrak{sl}_2$. In particular, G' (defined with the condition (1)) is the smallest group containing almost all ψ_p and we recover a special case of [**Kat82**, thm. 11.2].

(2) We now consider a variant of our result when X equals to $\mathbb{P}^1$ minus more than three points. Let D be the union of $\{0\}$ and all 8-th roots of unity and let $X = \mathbb{A}^1 - D$. Let u_0 be one of the preimages of x_0 of the covering map $f : X \to \mathbb{P}^1 - \{0, 1, \infty\}, u \mapsto x = -\frac{1}{4}(u^4 + u^{-4} - 2)$. We may assume that the number field L contains u_0.

We consider the following weaker version of p-curvature conjecture:

Proposition 4.2.3. *Let (N, ∇) be a vector bundle with connection over X. Assume that the p-curvatures vanish for almost all $\mathfrak{p}$ and that for any finite place v, all the formal horizontal sections of (N, ∇) converges over the largest disc around u_0 in X_{L_w}. Then (N, ∇) must be étale locally trivial.*

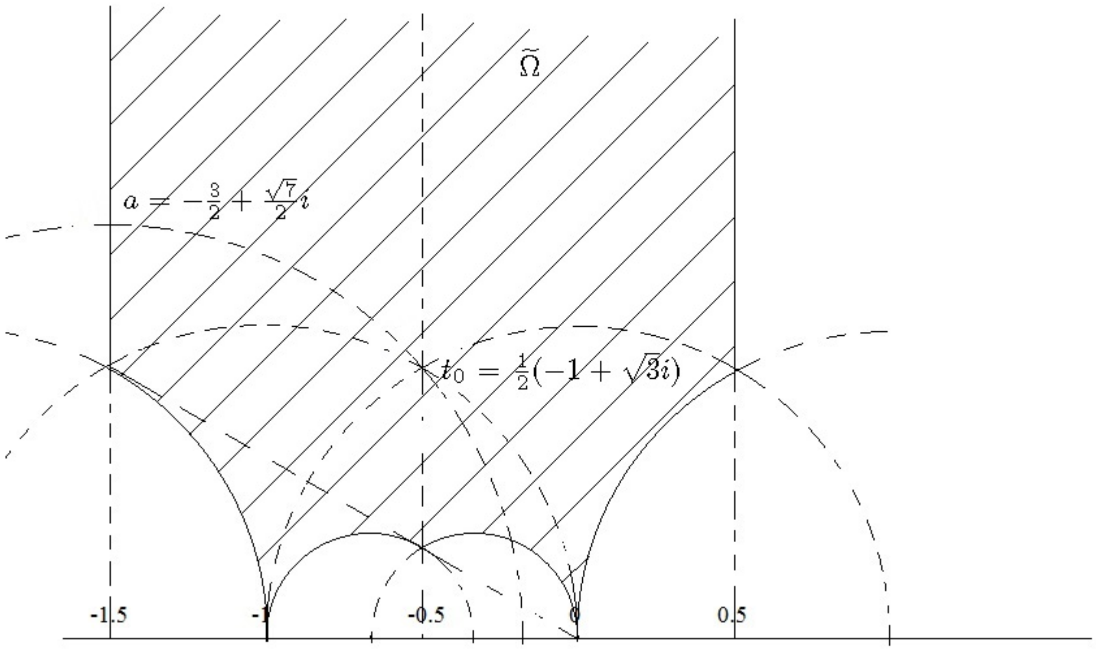
By direct calculation, the w-adic distance from u_0 to D is $|2|_w^{\frac{1}{4}}$ when w is finite. Then our assumption means that all the formal horizontal sections of (N, ∇) centered at u_0 converge over $D(u_0, |2|_w^{\frac{1}{4}})$.

Proof of the proposition. By applying Theorem 1.1.4 to the formal horizontal sections around u_0, one only need to show that $\prod_{w|\infty} R_w \geq 2^{1/4}$. Since the uniformization $\lambda \circ \alpha : D(0, 1) \to \mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}$ factors through $f : \mathbb{A}^1(\mathbb{C}) - D \to \mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}$, then for the formal horizontal sections of (N, ∇), we have $R_w \geq |5.632 \cdots|_w / |f'(u_0)|_w$ by the chain rule and Lemma 4.1.1. A direct computation shows that $\prod_{w|\infty} |f'(u_0)|_w = 4$ and then $\prod_{w|\infty} R_w \geq 5.6325.../4 > 2^{1/4}$. $\square$

If one replaces the assumption in Proposition 4.2.3 by that the generic radii of all formal horizontal sections of (N, ∇) are at least $|2|_w^{1/4}$ for all w finite, the results in [**BS82**] does not apply directly due to the fact that the points in D are too close to each other in L_w when $w|2$. However, one may modify the argument there, especially a modified version of eqn. (3) in *loc. cit.*, to see that the condition on generic radii would imply trivial monodromy of (N, ∇).

4.3. Proof of Theorem 3.2.1. Let y be the algebraic formal function which is one component of the formal horizontal section $\boldsymbol{y}$ of (M, ∇) over X_K.

Lemma 4.3.1. *The formal power series of y centered at x_0 has convergence radius equal to 1 for almost all finite places.*



Proof. Since the covering induced by y is finite étale over X_L, by Proposition 4.2.1, it is étale over $X_{\mathcal{O}_w}$ at x_0 for almost all places. For such places, we have $\rho_w = 1$ by lifting criterion for étale maps. $\square$

4.3.2. We now define an adélic tube (Ω_w) adapted to x_0. For an archimedean place w, we choose the embedding $\sigma : L \to \mathbb{C}$ corresponding to w such that $\sigma(x_0) = (1 + \sqrt{3}i)/2$. Let $\widetilde{\Omega}$ be the open region in the upper half plane cut out by the following six edges (see the attached figure): $\Re t = -\frac{3}{2}$, $|t + 2| = 1$, $|t + \frac{2}{3}| = \frac{1}{3}$, $|t + \frac{1}{3}| = \frac{1}{3}$, $|t - 1| = 1$, and $\Re t = \frac{1}{2}$. This is a fundamental domain of the arithmetic group $\Gamma(2) \subset \mathrm{SL}_2(\mathbb{Z})$. We define Ω_w to be $\lambda(\widetilde{\Omega})$. For w finite, we choose Ω_w to be $D(x_0, 1)$ if y is étale over $X_{\mathcal{O}_w}$ at x_0; otherwise, we choose Ω_w to be $D(x_0, p^{-\frac{1}{p(p-1)}})$.

The collection (Ω_w) is a weak adélic tube and y extends to an analytic (in particular meromorphic) function on each Ω_w by Lemma 4.3.1, Lemma 4.1.1, and Lemma 3.1.5.

Lemma 4.3.3. *The Arakelov degree of $T_{x_0}X$ with respect to the adélic tube (Ω_w) in 4.3.2 is positive.*

Proof. We want to give a lower bound of $(||\frac{d}{dx}||_w^{\mathrm{cap}})^{-1}$, the capacity of Ω_w. Let $a = -\frac{3}{2} + \frac{\sqrt{7}}{2}i$. On the line $\Re(t) = -\frac{3}{2}$, the point a is the point closest to $t_0 = \frac{1}{2}(-1 + \sqrt{3}i)$ with respect to the Poincaré metric. The stabilizer of t_0 in $\mathrm{SL}_2(\mathbb{Z})$ has order 3, and permutes the geodesics $\Re t = -\frac{3}{2}$, $|t + \frac{2}{3}| = \frac{1}{3}$, $|t - 1| = 1$, and this action preserves the Poincaré metric. Using this, together with the fact that the distance to t_0 is invariant under $z \mapsto -1-\bar{z}$, one sees that the distance from any point on the boundary of $\widetilde{\Omega}$ to t_0 is at least that from a to t_0. Since $\alpha\colon D(0,1) \to \mathcal{H}$ (defined in the proof of Prop. 4.1.8) preserves the Poincaré metrics, $\alpha^{-1}(\widetilde{\Omega})$ contains a disc with respect to the Poincaré radius equal to the distance from t_0 to a.

In $D(0,1)$, a disc with respect to Poincaré metric is also a disc in the Euclidean sense. Hence $\alpha^{-1}(\widetilde{\Omega})$ contains a disc of Euclidean radius

$$|\alpha^{-1}(a)| = |(a-t_0)/(a-\bar{t}_0)| = 0.45685\cdots .$$

Since λ maps the fundamental domain $\widetilde{\Omega}$ isomorphically onto Ω_w, by 1.2.3, the local capacity $(||\frac{d}{dx}||_w^{\mathrm{cap}})^{-1}$ is at least $|(a-t_0)/(a-\bar{t}_0)| \cdot |\lambda'(\frac{1}{2}(-1+\sqrt{3}i))|$.

By 1.2.3, we have $-\log(||\frac{d}{dx}||_w^{\mathrm{cap}}) \geq -\frac{\log p}{p(p-1)}$ when $w|p$. By Proposition 4.1.8, we have $|\lambda'(\frac{1}{2}(-1+\sqrt{3}i))| = 5.632\cdots$. Since

$$\sum_w -\log(||\frac{d}{dx}||_w^{\mathrm{cap}}) > \log(5.6325\cdots \times 0.45685\cdots) - \sum_p \frac{\log p}{p(p-1)} > 0.184\cdots ,$$

we have that $\widehat{\deg}(T_{x_0}X, ||\cdot||^{\mathrm{cap}})$ is postive. $\square$

Proof of Theorem 3.2.1. Applying Proposition 4.2.1, we have a full set of algebraic solutions $\boldsymbol{y}$. Choosing the weak adélic tube as in 4.3.2 and applying Corollary 1.2.6 (the assumptions are verified by 4.3.2 and Lemma 4.3.3), we have that these algebraic solutions are actually rational.

This shows that (M,∇) has a full set of rational solutions over X_L. Since formation of $\ker(\nabla)$ commutes with the finite extension of scalars $\otimes_K L$, this implies that (M,∇) has a full set of rational solutions over X_K. $\square$

5. Interpretation using the Faltings height

In this section, we view $X_{\mathbb{Z}[\frac{1}{2}]}$ as the moduli space of elliptic curves with level 2 structure. Let $\lambda_0 \in X(\bar{\mathbb{Q}})$ and E the corresponding elliptic curve. Using the Kodaira–Spencer map, we will relate the Faltings height of E with our lower bound for the product of radii of uniformizability (see section 4) at archimedean places of the formal solutions in $\widehat{\mathcal{O}}_{X_K,\lambda_0}$. We will focus mainly on the case when $\lambda_0 \in X(\bar{\mathbb{Z}})$ and sketch how to generalize to $\lambda_0 \in X(\bar{\mathbb{Q}})$ at the end of this section. In this section, unlike the previous sections, we will use λ as the coordinate of X.

5.1. Hermitian line bundles and their Arakelov degrees.

5.1.1. Recall that an *Hermitian line bundle* $(L, ||\cdot||_\sigma)$ over $\mathrm{Spec}(\mathcal{O}_K)$ is a line bundle L over $\mathrm{Spec}(\mathcal{O}_K)$, together with an Hermitian metric $||\cdot||_\sigma$ on $L\otimes_\sigma\mathbb{C}$ for each archimedean place $\sigma\colon K\to\mathbb{C}$.

Given an Hermitian line bundle $(L, ||\cdot||_\sigma)$, its (normalized) *Arakelov degree* is defined as:

$$\widehat{\deg}(L) := \frac{1}{[K:\mathbb{Q}]}\left(\log(\#(L/s\mathcal{O}_K)) - \sum_{\sigma:K\to\mathbb{C}} \log||s||_\sigma\right),$$

where s is any section.

For a finite place v over p, the integral structure of L defines a norm $||\cdot||_v$ on L_{K_v}. More precisely, if s_v is a generator of $L_{\mathcal{O}_{K_v}}$ and n is an integer, we define $||p^n s_v||_v = p^{-n[K_v:\mathbb{Q}_p]}$. We obtain a norm on $\mathcal{O}_v$ by viewing it as the trivial line bundle. We will use $||\cdot||_v$ for the norms on different line bundle as no confusion would arise. We may rewrite the Arakelov degree using the p-adic norms:

$$\widehat{\deg}(L) = \frac{1}{[K:\mathbb{Q}]}\left(-\sum_v \log||s||_v\right),$$

where v runs over all places of K. It is an immediate corollary of the product formula that the right hand side does not depend on the choice of s.

5.1.2. Let E be an elliptic curve over K, and denote by $e\colon \mathrm{Spec}\,K\to E$ and $f\colon E\to\mathrm{Spec}\,K$ the identity and structure map respectively. For each $\sigma\colon K\to\mathbb{C}$, we endow $e^*\Omega^1_{E/K} = f_*\Omega_{E/K}$ with the Hermitian norm given by

$$||\alpha||_\sigma = \left(\frac{1}{2\pi}\int_{E_\sigma(\mathbb{C})}|\alpha\wedge\bar{\alpha}|\right)^{\frac{\epsilon_\sigma}{2}},$$

where ϵ_σ is 1 for real embeddings and 2 otherwise.

This can be used to define the Faltings height of E, which we only recall the precise definition when E has good reduction over $\mathcal{O}_K$. Denote by $f:\mathcal{E}\to\mathrm{Spec}\,\mathcal{O}_K$ the elliptic curve over $\mathcal{O}_K$ with generic fibre E, and again write e for the identity section of $\mathcal{E}$. The norms $||\alpha||_\sigma$ make $e^*\Omega^1_{\mathcal{E}/\mathrm{Spec}(\mathcal{O}_K)} = f_*\Omega^1_{\mathcal{E}/\mathrm{Spec}(\mathcal{O}_K)}$ into a Hermitian line bundle, and we define the *(stable) Faltings height* by

$$h_F(E_\lambda) = \widehat{\deg}(f_*\Omega^1_{\mathcal{E}/\mathrm{Spec}(\mathcal{O}_K)}).$$

Notice that $h_F(E_\lambda)$ does not depend on the choice of K. Here we use Deligne's definition for convenience [**Del85**, 1.2]. This differs from the original definition of Faltings (see [**Fal86**]) by a constant $\log(\pi)$.

In general, the elliptic curve E will have semi-stable reduction everywhere after some field extension. We assume this is the case and E has a Neron model $f : \mathcal{E} \to \operatorname{Spec} \mathcal{O}_K$ which endows $f_*\Omega^1_{\mathcal{E}/\operatorname{Spec}(\mathcal{O}_K)}$ a canonical integral structure. With the same Hermitian norm defined as above, we have a similar definition of Faltings height in the general case. See [**Fal86**] for details. As in the good reduction case, this definition does not depend on the choice of K.

5.1.3. We will assume that both λ_0 and $\lambda_0 - 1$ are units at each finite place. Given such a λ_0, consider the elliptic curve E_{λ_0} over $\mathbb{Q}(\lambda_0)$ defined by the equation $y^2 = x(x-1)(x-\lambda_0)$. Then E_{λ_0} has good reduction at primes not dividing 2, and potentially good reduction everywhere, since its j-invariant is an algebraic integer. Let K be a number field such that $(E_{\lambda_0})_K$ has good reduction everywhere. We denote by $\mathcal{E}_{\lambda_0}$ the elliptic curve over $\mathcal{O}_K$ with generic fiber E_{λ_0}.

5.1.4. To express our computation of radii in terms of Arakelov degrees, we endow the $\mathcal{O}_K$-line bundle $T_{\lambda_0}(X_{\mathcal{O}_K})$, the tangent bundle of $X_{\mathcal{O}_K}$ at λ_0, with the structure of an Hermitian line bundle as follows. For each archimedean place $\sigma\colon K \to \mathbb{C}$, we have the universal covering $\lambda : \mathcal{H} \to \sigma X$, introduced in 4.1.5. The $\mathrm{SL}_2(\mathbb{R})$-invariant metric $\frac{dt}{2\Im(t)}$ on the tangent bundle of $\mathcal{H}$ induces the desired metric on the tangent bundle via push-forward. As in the proof of Proposition 4.1.8, our lower bound on the radius of the formal solution is $|2\Im(t_0)\lambda'(t_0)|^{\epsilon_\sigma} = ||\frac{d}{d\lambda}||_\sigma^{-1}$, where t_0 is a point on $\mathcal{H}$ mapping to λ_0. It is easy to see the left hand side does not depend on the choice of t_0. Under the assumptions in 5.1.3, the tangent vector $\frac{d}{d\lambda}$ is an $\mathcal{O}_K$-basis vector for the tangent bundle $T_{\lambda_0}(X_{\mathcal{O}_K})$, and we have

$$\widehat{\deg}(T_{\lambda_0}X) = \frac{1}{[K:\mathbb{Q}]}\Big(-\sum_{\sigma:K\to\mathbb{C}} \log||\frac{d}{d\lambda}||_\sigma\Big) \leq \frac{1}{[K:\mathbb{Q}]}\log\Big(\prod_\sigma R_\sigma\Big),$$

where the R_σ are the radius of uniformization discussed in section 4.1.

5.2. The Kodaira–Spencer map. Consider the Legendre family of elliptic curves $E \subset \mathbb{P}^2_{\mathbb{Z}[\frac{1}{2}]} \times X_{\mathbb{Z}[\frac{1}{2}]}$ over $X_{\mathbb{Z}[\frac{1}{2}]}$ given by $y^2 = x(x-1)(x-\lambda)$. We have the Kodaira–Spencer map ([**FC90**, Ch. III,9],[**Kat72**, 1.1]):

$$(5.2.1) \qquad KS\colon (f_*\Omega^1_{E/X_{\mathbb{Z}[\frac{1}{2}]}})^{\otimes 2} \to \Omega^1_{X_{\mathbb{Z}[\frac{1}{2}]}},\ \alpha\otimes\beta \mapsto \langle \alpha, \nabla\beta\rangle,$$

where ∇ is the Gauss–Manin connection and $\langle\cdot,\cdot\rangle$ is the pairing induced by the natural polarization.

5.2.1. Following Kedlaya's notes ([**Ked**, Sec. 1,3]), we choose $\{\frac{dx}{2y}, \frac{xdx}{2y}\}$ to be an integral basis of $H^1_{dR}(E/X)|_{\lambda_0}$ and compute the Gauss–Manin connection:

$$\nabla\frac{dx}{2y} = \frac{1}{2(1-\lambda)}\frac{dx}{2y}\otimes d\lambda + \frac{1}{2\lambda(\lambda-1)}\frac{xdx}{2y}\otimes d\lambda.$$

The Kodaira–Spencer map then sends $(\frac{dx}{2y})^{\otimes 2}$ to $\frac{1}{2\lambda(\lambda-1)}d\lambda$.

This computation shows:

Lemma 5.2.2. *Given v a finite place not lying over 2, the Kodaira–Spencer map (5.2.1) preserves the $\mathcal{O}_v$-generators of $(f_*\Omega^1_{E/X_{\mathbb{Z}[\frac{1}{2}]}})^{\otimes 2}|_{\lambda_0}$ and $\Omega^1_{X_{\mathbb{Z}[\frac{1}{2}]}}|_{\lambda_0}$ when λ_0 and λ_0-1 are both v-units.*

5.2.3. For the archimedean places σ, we consider $f_*\Omega^1_{\sigma E/\operatorname{Spec}\mathbb{C}}$ with the metrics $||\alpha||_\sigma$ defined in section 5.1, and we endow $\Omega^1_{X_{\mathbb{Z}}}|_{\lambda_0}$ the Hermitian line bundle structure as the dual of the tangent bundle.

To see that the Kodaira–Spencer map preserves the Hermitian norms on both sides, one may argue as follows. Notice that the metrics on $(f_*\Omega^1_{\sigma E/\operatorname{Spec}\mathbb{C}})^{\otimes 2}$ and $\Omega^1_{X_{\mathbb{Z}}}$ are $\mathrm{SL}_2(\mathbb{R})$-invariant (see for example [**ZP09**, Remark 3 in Sec. 2.3]). Hence they are the same up to a constant and we only need to compare them at the cusps. To do this, one studies both sides for the Tate curve. See for example [**MB90**, 2.2] for a related argument and Lemma 4.1.6 (2) for relation between θ-functions and Ω^1_X.

Here we give another argument:

Lemma 5.2.4. *The Kodaira–Spencer map preserves the Hermitian metrics:*

$$||(\frac{dx}{2y})^{\otimes 2}||_\sigma = ||\frac{d\lambda}{2\lambda_0(\lambda_0-1)}||_\sigma.$$

Proof. Let dz be an invariant holomorphic differential of $\mathbb{C}/(\mathbb{Z} \oplus t_0\mathbb{Z})$, where $\lambda(t_0) = \lambda_0$. By the theory of the Weierstrass-$\wp$ function, we have a map from the complex torus to the elliptic curve

$$u^2 = 4v^3 - g_2(t_0)v - g_3(t_0)$$

such that dz maps to $\frac{dv}{u}$. Here g_2 is the weight 4 modular form of level $\mathrm{SL}_2(\mathbb{Z})$ with $\frac{4\pi^4}{3}$ as the constant term in its Fourier series and g_3 is the weight 6 modular form with $\frac{8\pi^6}{27}$ as the constant term. Using Lemma 4.1.6 (3), we see that the right hand side has three roots: $\frac{\pi^2}{3}(\theta_{00}^4(t_0) + \theta_{01}^4(t_0)), -\frac{\pi^2}{3}(\theta_{00}^4(t_0) + \theta_{10}^4(t_0)), \frac{\pi^2}{3}(\theta_{10}^4(t_0) - \theta_{01}^4(t_0))$. Hence this curve is isomorphic to $y^2 = x(x-1)(x-\lambda_0)$ via the map

$$x = \frac{v - \frac{1}{3}\pi^2(\theta_{00}^4(t_0) + \theta_{01}^4(t_0))}{-\pi^2\theta_{01}^4(t_0)}, \ y = \frac{u}{2(-\pi^2\theta_{01}^4(t_0))^{3/2}}, \tag{5.2.2}$$

and we have

$$\frac{dx}{2y} = \pi i \theta_{01}^2(t_0)\frac{dv}{u} = \pi i \theta_{01}^2(t_0)dz.$$

Hence

$$||(\frac{dx}{2y})^{\otimes 2}||_\sigma = |\pi^2\theta_{01}^4(t_0) \cdot (\frac{1}{2\pi}\int_{E(\mathbb{C})} |dz \wedge d\bar{z}|)|^{\epsilon_\sigma} = |\pi\theta_{01}^4(t_0)\Im(t_0)|^{\epsilon_v}.$$

On the other hand, using Lemma 4.1.6 (2), we have

$$||\frac{d\lambda}{2\lambda_0(\lambda_0-1)}||_\sigma^{1/\epsilon_\sigma} = \left|\frac{2\Im(t_0)|\lambda'(t_0)|}{2\lambda_0(\lambda_0-1)}\right| = \left|\frac{\Im(t_0)\pi\theta_{00}^4(t_0)\theta_{10}^4(t_0)}{\theta_{01}^4(t_0)\lambda_0(\lambda_0-1)}\right| = |\pi\theta_{01}^4(t_0)\Im(t_0)|.$$

$\square$

Proposition 5.2.5. *If λ_0 and $\lambda_0 - 1$ are both units at every finite places, we have $\widehat{\deg}(T_{\lambda_0}X) = -2h_F(E_{\lambda_0}) + \frac{\log 2}{3}$.*

Proof. By lemma 5.2.2 and lemma 5.2.4, we have

$$
\begin{aligned}
-\widehat{\deg}(T_{\lambda_0}X) =& \widehat{\deg}(\Omega^1_{X_{O_K}}|_{\lambda_0}) \\
=& \frac{1}{[K:\mathbb{Q}]}\left(-\sum_v \log ||\frac{d\lambda}{2\lambda(\lambda-1)}||_v\right) \\
=& \frac{1}{[K:\mathbb{Q}]}\left(-\sum_{v|\infty} \log ||\frac{d\lambda}{2\lambda(\lambda-1)}||_v - \sum_{v\nmid\infty} \log ||\frac{d\lambda}{2\lambda(\lambda-1)}||_v\right) \\
=& \frac{1}{[K:\mathbb{Q}]}\left(-\sum_{v|\infty} \log ||(\frac{dx}{2y})^{\otimes 2}||_v - \sum_{v\nmid 2\infty} \log ||(\frac{dx}{2y})^{\otimes 2}||_v - \sum_{v|2} \log ||1/2||_v\right) \\
=& 2h_F(E_{\lambda_0}) + \frac{1}{[K:\mathbb{Q}]}\sum_{v|2} \log ||(\frac{dx}{2y})^{\otimes 2}||_v - \log 2.
\end{aligned}
\tag{5.2.3}
$$

Now we study $||(\frac{dx}{2y})^{\otimes 2}||_v$ given $v|2$. The sum $\frac{1}{[K:\mathbb{Q}]}\sum_{v|2} \log ||(\frac{dx}{2y})^{\otimes 2}||_v$ does not change after extending K, hence we may assume that $\mathcal{E}_{\lambda_0}$ over $\mathcal{O}_v$ has the Deuring normal form $u^2+auw+u=w^3$ (see **[Sil09]** Appendix A Prop. 1.3 and the proof of Prop. 1.4 shows in the good reduction case, a is a v-integer). An invariant differential generating $f_*\Omega^1_{\mathcal{E}_{\lambda_0}/\operatorname{Spec}\mathcal{O}_K[\frac{1}{3}]}$ is $\frac{dw}{2u+aw+1}$.

Because both $\frac{dw}{2u+aw+1}$ and $\frac{dx}{2y}$ are invariant differentials, we have

$$
||\frac{dx}{2y}||_v = ||\Delta_1/\Delta_2||_v^{\frac{1}{12}}||\frac{dw}{2u+aw+1}||,
$$

where Δ_1 and Δ_2 are the discriminant of the Deuring normal form and that of the Legendre form respectively. Since E has good reduction, $||\Delta_1||_v = 1$ (see the proof of *loc. cit.*). Hence $||\frac{dx}{2y}||_v = ||\frac{dw}{2u+aw+b}||_v \cdot ||1/16||_v^{1/12} = ||2||_v^{-1/3}$.

Hence $\widehat{\deg}(T_{\lambda_0}X) = -2h_F(E_{\lambda_0}) - \frac{2}{3}\log 2 + \log 2 = -2h_F(E_{\lambda_0}) + \frac{\log 2}{3}$. □

5.2.6. **[Del85**, 1.5] mentioned that the point $\frac{1+\sqrt{3}i}{2}$ corresponds to the elliptic curve with smallest height. Hence, our choice $\frac{1+\sqrt{3}i}{2}$ gives the largest $\widehat{\deg}(T_{\lambda_0}X)$ among those λ_0 such that λ_0 and λ_0-1 are units at every prime.

5.3. The general case. When $\lambda_0 \in X(\bar{\mathbb{Q}})$, a similar argument as in section 5.2 shows that

$$
\begin{aligned}
\frac{1}{[K:\mathbb{Q}]}\left(-\sum_{\sigma:K\to\mathbb{C}} \log ||\frac{d}{d\lambda}||_\sigma\right) \leq& -2h_F(E_{\lambda_0}) + \frac{\log 2}{3} \\
&+ \frac{1}{[K:\mathbb{Q}]}\left(\sum_{v \text{ finite}} \log^+ ||\lambda_0||_v + \log(|\operatorname{Nm}\lambda_0(\lambda_0-1)|)\right)
\end{aligned}
\tag{5.3.1}
$$

and equality holds if and only if $\lambda_0 \in X(\bar{\mathbb{Z}}_2)$. As discussed in 5.1.4, the left hand side is the sum of the logarithms of our estimates of the radii of uniformizability at archimedean places.

We also need to modify the estimate of the radii at finite places in Lemma 3.1.5. A possible estimate for R_v is $p^{-\frac{1}{p(p-1)}} \cdot \min\{||\lambda_0||_v, ||\lambda_0 - 1||_v, 1\}$. One explanation of the factor $\min\{||\lambda_0||_v, ||\lambda_0 - 1||_v, 1\}$ is that we cannot rule out the possibility that one has local monodromy at $0, 1, \infty$ merely from the information of p-curvature at v.

Compared to the case when $\lambda_0 \in X(\bar{\mathbb{Z}})$, our estimate for the sum of the logarithms of the archimedean radii increases by at most

$$\frac{1}{[K:\mathbb{Q}]}\Big(\sum_{v \text{ finite}} \log^+ ||\lambda_0||_v + \log(|\operatorname{Nm} \lambda_0(\lambda_0 - 1)|)\Big),$$

while the estimate for the sum of logarithms of the radii at finite places becomes smaller by

$$\sum_v \max\{\log^+ ||\lambda_0^{-1}||_v, \log^+ ||(\lambda_0 - 1)^{-1}||_v\}.$$

An explicit computation shows that the later is larger than the former. Hence the estimate for the product of the radii does not become larger than the case when $\lambda_0 \in X(\bar{\mathbb{Z}})$.

6. The affine elliptic curve case and examples

6.1. Statement of the main result of the affine elliptic curve case. Let $X \subset \mathbb{A}^2_{\mathbb{Z}}$ be the affine curve over $\mathbb{Z}$ defined by the equation $y^2 = x(x-1)(x+1)$. The generic fiber $X_{\mathbb{Q}}$ is an elliptic curve (with j-invariant 1728) minus its identity point. Given a vector bundle with connection over X_K, we will define the notion of vanishing p-curvature for *all* finite places along the same lines as in section 3.1. The main result of this section is:

Theorem 6.1.1. *Let (M, ∇) be a vector bundle with connection over X_K. Suppose that the p-curvatures of (M, ∇) vanish for all p. Then (M, ∇) is étale locally trivial.*

Remark 6.1.2. This theorem cannot be deduced from applying Theorem 3.2.1 to the push-forward of (M, ∇) via some finite étale map from an open subvariety of the affine elliptic curve to $\mathbb{P}^1_K - \{0, 1, \infty\}$. Unlike the $\mathbb{P}^1_K - \{0, 1, \infty\}$ case, the conclusion here allows the existence of (M, ∇) with finite nontrivial monodromy. See section 6.3.

Now we explain the meaning of vanishing p-curvature for all p.

6.1.3. We fix $x_0 = (0,0) \in X(\mathbb{Z})$ and denote by $(x_0)_K$ and $(x_0)_{k_v}$ the images of x_0 in $X(K)$ and $X(k_v)$. Let $y : X \to \mathbb{A}^1_{\mathbb{Z}}$ be the projection to the y-coordinate. It is easy to check that this map is étale along x_0 and hence induces isomorphisms between the tangent spaces $T_{x_0}X \cong T_0\mathbb{A}^1_{\mathbb{Z}}$ and between the formal schemes $\widehat{X_{K/(x_0)_K}} \cong \widehat{\mathbb{A}^1_{K/0}}$. In particular, we have an analytic section s_v of the projection y from $D(0,1) \subset \mathbb{A}^1(K_v)$ to $X(K_v)$ such that $s_v(0) = x_0$ for any finite place v by the lifting criterion for étale maps. By definition, the image $s_v(D(0,1))$ is the open rigid analytic disc in $X(K_v)$ which is the preimage of $(x_0)_{k_v}$ under the reduction map $X(K_v) \to X(k_v)$.

By choosing a trivialization of M in some neighborhood of $(x_0)_K$, we can view a formal horizontal section m of (M, ∇) around $(x_0)_K$ as a formal function in $\widehat{\mathcal{O}}^r_{X_K,(x_0)_K} \cong \widehat{\mathcal{O}}^r_{\mathbb{A}^1_K,0}$, where r is the rank of M. We denote $f \in \widehat{\mathcal{O}}^r_{\mathbb{A}^1_K,0}$ to be the image and the goal of the next subsection is to prove that the formal power series f is algebraic.

Let U be $X - \{(0,1),(0,-1)\}$. It is a smooth scheme over $\mathbb{Z}$. Our chosen point x_0 is a $\mathbb{Z}$-point of U and $s_v(D(0,1)) \subset U(K_v)$. For v a finite place of K with residue characteristic p, we say that (M, ∇) has *good reduction* at v if (M, ∇) extends to a vector bundle with connection on $U_{\mathcal{O}_v}$. Similar to Lemma 3.1.5, we have:

Lemma 6.1.4. *Suppose that (M, ∇) has good reduction at v. If the p-curvature ψ_p vanishes*[2]*, then the formal power series f is the germ of some meromorphic function on the disc $D(0, p^{-\frac{1}{p(p-1)}}) \subset \mathbb{A}^1$.*

Proof. Let $(\mathcal{M}, \nabla)$ be an extension of (M, ∇) over $X_{\mathcal{O}_v}$. Since y is étale, the derivation $\frac{\partial}{\partial y}$ is regular over some Zariski open neighborhood $\bar{V}$ of $x_0 \in X \otimes \mathbb{Z}/p\mathbb{Z}$. Let $V \subset X(K_v)$ be the preimage of $\bar{V}$ under reduction map. Since the p-curvature vanishes, we have $\nabla(\frac{\partial}{\partial y})^p(\mathcal{M}|_V) \subset p\mathcal{M}|_V$. Notice that $s_v(D(0,1)) \subset V$. Then the proof of Lemma 3.1.5 shows the existence of horizontal sections of M on $s_v(D(0, p^{-\frac{1}{p(p-1)}}))$. Via a local trivialization of M and the isomorphism of formal neighborhoods of x_0 and 0, we see that f is meromorphic over $D(0, p^{-\frac{1}{p(p-1)}})$. $\square$

This lemma motivates the following definition:

Definition 6.1.5. We say that *the p-curvatures of (M, ∇) vanish for all p* if

(1) the p-curvature ψ_p vanishes for all but finitely many p,

[2]This means $\psi_p \equiv 0$ on $X_{\mathcal{O}_v} \otimes \mathbb{Z}/p\mathbb{Z}$ as in section 3.1.1.

(2) all formal horizontal sections around x_0, when viewed as formal functions in $\widehat{\mathcal{O}}^r_{\mathbb{A}^1_K,0}$, are the germs of some meromorphic functions on $D(0, p^{-\frac{1}{p(p-1)}})$ for all finite places v.

Remark 6.1.6. The second condition does not depend on the choice of local trivialization of M. Moreover, for each v, this condition remains the same if we replace the projection y by any map $g : W_{\mathcal{O}_v} \to \mathbb{A}^1_{\mathcal{O}_v}$ such that $W_{\mathcal{O}_v}$ is a Zariski open neighborhood of $(x_0)_{\mathcal{O}_v}$ in $X_{\mathcal{O}_v}$ and that g is étale.

6.2. Estimate at archimedean places and algebraicity. Let $\sigma : K \to \mathbb{C}$ be an archimedean place. Let $\phi : D(0,1) \to X(\mathbb{C})$ be a uniformization map such that $\phi(0) = x_0$. We have the following lemma whose proof is the same as that of Lemma 4.1.1:

Lemma 6.2.1. *The σ-adic radius R_σ (see Definition 1.1.2) of the formal functions f in 6.1.3 is at least $|(y \circ \phi)'(0)|_\sigma$.*

Let $t_0 = \frac{1+i}{2}$. A direct manipulation of the definition shows $\lambda(t_0) = -1$, where λ is defined in 4.1.5. Let $F : D(0,1) \to \mathbb{C} - (\mathbb{Z} + t_0\mathbb{Z})$ be a uniformization map such that $F(0) = \frac{1}{2}$.

Lemma 6.2.2 (Eremenko)**.** *The derivative* $|F'(0)| = 2^{-3/2}\pi^{-3/2}\Gamma(1/4)^2 = 0.8346...$

Proof. From [**Ere11**, Sec. 2], we have $F'(0) = \frac{2^{5/2}}{B(1/4,1/4)}|(\lambda^{-1})'(i)|^3$, where B is the Beta function. By Lemma 4.1.6, the Chowla-Selberg formula ([**SC67**])

$$|\eta(i)| = 2^{-1}\pi^{-3/4}\Gamma(1/4), \tag{6.2.1}$$

and the fact that $\theta_{00}^4(i) = 2\theta_{01}^4(i) = 2\theta_{10}^4(i)$, we have

$$|(\lambda^{-1})'(i)| = |\pi i (\frac{\theta_{01}(i)\theta_{10}(i)}{\theta_{00}(i)})^4| = \pi|\eta(i)|^4 = \frac{\Gamma(1/4)^4}{2^4\pi^2}.$$

We obtain the desired formula by noticing that $B(1/4, 1/4) = \pi^{-1/2}\Gamma(1/4)^2$. □

Lemma 6.2.3. *Let α be the constant $2(-\pi^2\theta_{01}^4(t_0))^{3/2}$ and $\wp$ be the Weierstrass-$\wp$ function. We have $y \circ \phi = \alpha^{-1}\wp' \circ F$, up to some rotation on $D(0,1)$.*

Proof. The map $g := (\wp, \wp')$ maps $\mathbb{C} - (\mathbb{Z} + t_0\mathbb{Z})$ to the affine curve $u^2 = 4v^3 - g_2(t_0)v - g_3(t_0)$. Let s be the isomorphism from this affine curve to $X(\mathbb{C})$ given by (5.2.2). Since both $s \circ g(1/2)$ and

[3] The choice of λ there is different. We have $\lambda(i) = 2$ here.

x_0 are the unique point fixed by the four automorphisms of $X(\mathbb{C})$, we have $s \circ g(1/2) = x_0$. Hence $s \circ g \circ F(0) = x_0 = \phi(0)$ and then the uniformizations $s \circ g \circ F$ and ϕ are the same up to some rotation. Then we have $y \circ \phi = y \circ s \circ g \circ F = \alpha^{-1} \wp' \circ F$ by (5.2.2). $\square$

Proposition 6.2.4. *The σ-adic radius* $R_\sigma^{\frac{[K:\mathbb{Q}]}{[K_\sigma:\mathbb{R}]}} \geq 2^{-5/2}\pi^{-2}\Gamma(1/4)^4 = 3.0949\cdots$.

Proof. Differentiate both sides of $(\wp'(z))^2 = 4(\wp(z))^3 - g_2(t_0)\wp(z) - g_3(t_0)$, we have

$$\wp''(1/2) = 6\wp(1/2)^2 - g_2(t_0)/2 = -g_2(t_0)/2,$$

where the second equality follows from that

$$\wp(1/2) = \pi^2(\theta_{00}^4(t_0) + \theta_{01}^4(t_0))/3 = \pi^2\theta_{01}^4(t_0)(\lambda(t_0)+1)/3 = 0.$$

By Lemma 4.1.6 and the fact that $\theta_{00}^4(t_0) = -\theta_{01}^4(t_0) = \theta_{10}^4(t_0)/2$, we have

$$|g_2(t_0)| = \frac{4\pi^4}{3} \cdot \frac{1}{2}|\theta_{00}^8(t_0) + \theta_{01}^8(t_0) + \theta_{10}^8(t_0)| = 4\pi^4|\theta_{01}^8(t_0)|.$$

Then by Lemma 6.2.3, the absolute value of the derivative of $y \circ \phi$ at 0 is

$$\begin{aligned}
|\alpha^{-1}\wp''(1/2) \cdot F'(0)| &= 2^{-1}\pi^{-3}|\theta_{01}(t_0)|^{-6} \cdot 2\pi^4|\theta_{01}(t_0)|^8 \cdot |F'(0)| \\
&= \pi|\theta_{01}(t_0)|^2 \cdot 2^{-3/2}\pi^{-3/2}\Gamma(1/4)^2 \text{ (by Lemma 6.2.2)} \\
&= 2\pi \cdot 2^{-2}\pi^{-3/2}\Gamma(1/4)^2 \cdot 2^{-3/2}\pi^{-3/2}\Gamma(1/4)^2 \\
&= 2^{-5/2}\pi^{-2}\Gamma(1/4)^4 = 3.0949\cdots,
\end{aligned} \tag{6.2.2}$$

where the third equality follows from

$$|\theta_{01}(t_0)| = 2^{-1/12}|\theta_{00}(t_0)\theta_{01}(t_0)\theta_{10}(t_0)|^{1/24} = 2^{1/4}|\eta(t_0)| = 2^{1/2}|\eta(i)|,$$

and (6.2.1). $\square$

Proof of Theorem 6.1.1. By Proposition 6.2.4, we have $\prod_{v|\infty} R_v \geq 3.0949\cdots$. By Definition 6.1.5, we have $\log(\prod_{v\nmid\infty} R_v) \geq -\sum_p \frac{\log p}{p(p-1)} = -0.761196\cdots$. Hence

$$\log(\prod_v R_v) \geq \log 3.0949\cdots - 0.761196\cdots = 0.3685\cdots > 0.$$

We conclude by applying Theorem 1.1.4. $\square$

6.3. An example with vanishing p-curvature for all $\mathfrak{p}$ and nontrivial G_{gal}. Let K be $\mathbb{Q}(\sqrt{-1})$, $X \subset \mathbb{A}^2_{\mathbb{Z}}$ be the affine curve defined by $y^2 = x(x-1)(x+1)$, E be the elliptic curve defined as the compactification of X_K, and $f : E \to E$ be a degree two self isogeny of E. We will also use f to denote the restriction of f to $X_K \backslash \{P\}$, where P is the non-identity element in the kernel of f.

Let (M, ∇) be $f_*(\mathcal{O}_{X_K \backslash \{P\}}, d)$. By definition, G_{gal} is $\mathbb{Z}/2\mathbb{Z}$.

Proposition 6.3.1. *The p-curvature of (M, ∇) vanishes for all finite places.*

Proof. Notice that f extends to a degree two étale cover from E to E over $\mathbb{Z}[\frac{i}{2}]$. Then for finite $v \nmid 2$, the p-curvature of (M, ∇) coincides with that of $f^*(M, \nabla)$ by the fact that p-curvatures remain the same under étale pull back[4]. Hence the p-curvature of (M, ∇) vanishes as $f^*(M, \nabla)$ is trivial.

For $v|2$, we write (M, ∇) out explicitly. Without loss of generality, we may assume that the isogeny f from the curve $y^2 = x(x-1)(x+1)$ to the curve $s^2 = t(t-1)(t+1)$ is given by

$$t = -\frac{i}{2}(x - \frac{1}{x}), \quad s = \frac{(1+i)y}{4x}(x + \frac{1}{x}).$$

Locally around $(t, s) = (0, 0)$, the sections $1, x$ is an $\mathcal{O}_{X_K}$ basis of $f_*\mathcal{O}_{X_K}$ and this basis gives rise to a natural Zariski local extension of (M, ∇) over $X_{\mathcal{O}_{\mathfrak{p}}}$. Direct calculation shows that

$$\nabla(1) = 0, \nabla(x) = \frac{2s}{(t^2-1)(3t^2-1)}ds + \frac{2st(1+2i)}{(t^2-1)(3t^2-1)}xds.$$

Therefore, $\nabla(f_1 + f_2 x) \equiv df_1 + xdf_2 \pmod 2$ and the p-curvature of (M, ∇) vanishes. □

Remark 6.3.2. In the above proof, we show that (M, ∇) has all p-curvatures vanishing in the strict sense: there is an extension of (M, ∇) over $X_{\mathcal{O}_K}$ such that its p-curvatures are all vanishing. However, given the argument for $v \nmid 2$, in order to to apply Theorem 6.1.1, we do not need to construct an extension of (M, ∇) but only need to check that x, locally as a formal power series of s, converges on $D(0, 2^{-1/2})$ for $v|2$. This is not hard to see from the facts: x, as a power series of t, converges when $|t|_v < |2|_v$; and t, as a power series of s, converges when $|s|_v < |2|_v^{1/2}$ and the image of $|s|_v < |2|_v^{1/2}$ is contained in $|t|_v < |2|_v$.

[4]Because $p \neq 2$ is unramified in K and (M, ∇) has good reduction at $\mathfrak{p}$, the notion of p-curvature here is classical.

6.4. A variant of the main theorems. We now prove a variant of the main theorems when $X = \mathbb{A}^1_{\mathbb{Q}} - \{\pm 1, \pm i\}$. Similar to Theorem 6.1.1, the conclusion is that (M, ∇) has finite monodromy and we give an example with nontrivial finite monodromy.

In order to define the local convergence conditions for bad primes, we take $x_0 = 0$.

Proposition 6.4.1. *Let (M, ∇) be a vector bundle with connection over X with p-curvature vanishes for all finite places. We further assume that the formal horizontal sections around x_0 converge over $D(x_0, 1)$ for all finite places $v|15$. Then (M, ∇) is étale locally trivial.*

Proof. By Lemma 6.2.2, we have $R_\infty \geq 2 \cdot 0.8346 \cdots$. By the assumptions on finite places, we have $\log(\prod_{v \nmid \infty} R_v) \geq -\sum_{p \neq 3,5} \frac{\log p}{p(p-1)} = -0.4976 \cdots$. We conclude by applying Theorem 1.1.4. $\square$

Example 6.4.2. Let s be the algebraic function $(1 - x^4)^{1/2}$. It is the solution of the differential equation $\frac{ds}{dx} = \frac{-2x^3}{1-x^4}$. Consider the connection on $\mathcal{O}_X$ given by $\nabla(f) = df + \frac{2x^3}{1-x^4} dx$. It has p-curvature vanishing for all p: $\nabla(f) \equiv df \pmod 2$ and $\nabla(f) \equiv df + (p+1)\frac{2x^3}{1-x^4} dx \pmod p$ with solution $s \equiv (1 - x^4)^{(p+1)/2} \pmod p$ when $p \neq 2$. In conclusion, $(\mathcal{O}_X, \nabla)$ satisfies the assumptions in the above proposition while it has nontrivial monodromy of order two.

Remark 6.4.3. If we replace our assumption by similar conditions on generic radii, the above example shows that one could have order two local monodromy around $\pm 1, \pm i$. The reason is [**BS82**, III eqn. (3)] does not hold in this situation and a modification of their argument would show that an order two local monodromy is possible.

CHAPTER 4

The conjecture of Ogus

Let A be a polarized abelian variety of dimension g defined over a number field K. Let L be a finite extension of K. For any field F containing K, we denote by $H^i_{\mathrm{dR}}(A, F)$ the de Rham cohomology group

$$H^i_{\mathrm{dR}}(A_F/F) = \mathbb{H}^i(A_F, \Omega^\bullet_{A_F/F}) = H^i_{\mathrm{dR}}(A/K) \otimes_K F.$$

We consider the filtered vector space $H^1_{\mathrm{dR}}(A, L)^{m,n}$ and the following semi-linear actions.

Let v be a finite place of L and k_v be the residue field. If A_L has good reduction at v and v is unramified in $L/\mathbb{Q}$, we use φ_v to denote the crystalline Frobenius acting on $H^1_{\mathrm{dR}}(A, L_v)^{m,n}$ via the canonical isomorphism to the crystalline cohomology $(H^1_{\mathrm{cris}}(A_{k_v}/W(k_v)) \otimes L_v)^{m,n}$. For any archimedean place σ corresponding to an embedding $\sigma \colon L \to \mathbb{C}$, let φ_σ be the map on $H^1_{\mathrm{dR}}(A, \mathbb{C})^{m,n}$ induced by the complex conjugation on $(H^1_{\mathrm{B}}(A_\sigma(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{C})^{m,n}$ via c_{BdR}. As mentioned in the introduction, these semi-linear actions define special elements, namely de Rham–Tate cycles (Definition 7.1.1), in $H^1_{\mathrm{dR}}(A, L)^{m,n}$.[1]

Theorem 3 and Theorem 4 are proved in section 8 and Theorem 6 is proved in section 9. For Theorem 3 and Theorem 6, the goal is to prove that G_{dR} and G_{MT} are the same. As a first step, we prove in section 7 that G_{dR} is reductive and reformulate Bost's theorem (see Proposition 5) as that the centralizer of G_{dR} in $\mathrm{End}(H^1_{\mathrm{dR}}(A/K))$ coincides with that of G_{MT}. To do this, we follow the construction of motives of absolute Hodge cycles due to Deligne and construct the category of motives generated by A with morphisms being the de Rham–Tate cycles. We prove that this category is a semisimple Tannakian category whose fundamental group is $G_{\mathrm{dR}} \subset \mathrm{GSp}(H^1_{\mathrm{dR}}(A/K))$. Then we use the techniques mentioned in the introduction to further show that the centralizer of G°_{dR} in $\mathrm{End}(H^1_{\mathrm{dR}}(A/K))$ coincides with that of G_{MT}. The Mumford–Tate conjecture (recalled in section 8) is an input: we show that this conjecture implies that the rank of G°_{dR} equals to that of G_{MT}. This allows us to conclude by a lemma of Zarhin. The extra inputs of the proof of Theorem 4

[1] Here we implicitly take into account the Tate twist as $L \otimes \mathbb{Q}(1)$ is a direct summand of some $H^1_{\mathrm{dR}}(A, L)^{m,n}$

are the weakly admissibility of certain filtered φ-modules of geometric origin and the Riemann hypothesis part of the Weil conjectures.

We also discuss a natural variant of de Rham–Tate cycles, the relative de Rham–Tate cycles (Definition 7.4.1). Here the word 'relative' means that to define these cycles, we use relative Frobenii $\varphi_v^{[L_v:\mathbb{Q}_p]}$ instead of φ_v. In next chapter, we will discuss some results on the conjectural analogue (Conjecture 7) of Theorem 7.3.7 for relative de Rham–Tate cycles.

Throughout this chapter, we will use $\mathrm{End}^{\circ}_{?}(A)$ to denote $\mathrm{End}_{?}(A) \otimes \mathbb{Q}$, where ? can be K, L or $\bar{K}$. The subscription is omitted if $\mathrm{End}_K(A) = \mathrm{End}_{\bar{K}}(A)$.

7. De Rham–Tate cycles and a result of Bost

In this section, we define de Rham–Tate cycles (section 7.1), de Rham–Tate groups (section 7.2) and their relative version (section 7.4) and discuss their basic properties. We recall previous results on absolute Tate cycles in section 7.1 and discuss Proposition 5 in section 7.3.

7.1. De Rham–Tate cycles.

Definition 7.1.1. An element $s \in (H^1_{\mathrm{dR}}(A, L))^{m,n}$ is called a *de Rham–Tate cycle* of the abelian variety A (over L) if there exists a finite set Σ of finite places of L such that for all places $v \notin \Sigma$, $\varphi_v(s) = s$.

Remark 7.1.2.

(1) Similar arguments as in [**Ogu82**, Cor. 4.8.1, 4.8.3] show that $s \in (H^1_{\mathrm{dR}}(A, K))^{m,n}$ is de Rham–Tate if and only if its base change in $(H^1_{\mathrm{dR}}(A, L))^{m,n}$ is de Rham–Tate and that the set of de Rham–Tate cycles over L is stable under the natural action of $\mathrm{Gal}(L/K)$ (on the coefficient of de Rham cohomology groups).

(2) Due to [**Ogu82**, Cor. 4.8.2], although one could define de Rham–Tate cycles over arbitrary field L containing K, we only need to consider cycles over number fields since any de Rham–Tate cycle must be defined over $\bar{\mathbb{Q}}$ and hence over some number field.

We have the following important fact, whose proof we sketch for completeness.

Lemma 7.1.3 ([**Ogu82**, Prop. 4.15])**.** *If $s \in H^1_{\mathrm{dR}}(A, L)^{m,n}$ is fixed by infinitely many φ_v (for example, when s is de Rham–Tate), then s lies in $\mathrm{Fil}^0 H^1_{\mathrm{dR}}(A, L)^{m,n}$. Moreover, if such s lies in $\mathrm{Fil}^1 H^1_{\mathrm{dR}}(A, L)^{m,n}$, then $s = 0$.*

Proof. By [**Maz73**, Thm. 7.6] and the extension of the result to $H^1_{\mathrm{dR}}(A,L)^{m,n}$ in the proof by Ogus[2] , we have that for all but finitely many v, the mod $\mathfrak{p}$ filtration $\mathrm{Fil}^j(H^1_{\mathrm{cris}}(A_{k_v}/W(k_v)\otimes k_v)^{m,n})$ is the set $\{\xi \bmod \mathfrak{p} \mid \xi \in (H^1_{\mathrm{cris}}(A_{k_v}/W(k_v)))^{m,n}$ with $\varphi_v(\xi)\in p^j(H^1_{\mathrm{cris}}(A_{k_v}/W(k_v)))^{m,n}\}$. Then for the infinitely many v such that $\varphi_v(s)=s$, we have that the reduction

$$s \bmod \mathfrak{p} \in \mathrm{Fil}^0((H^1_{\mathrm{cris}}(A_{k_v}/W(k_v))\otimes k_v)^{m,n})$$

and if $s\in \mathrm{Fil}^1 H^1_{\mathrm{dR}}(A,L)^{m,n}$, then s is 0 modulo $\mathfrak{p}$. Since the Hodge filtration over L is compatible with the Hodge filtration over k_v, we obtain the desired assertions. $\square$

The main conjecture that we study in this chapter is the following:

Conjecture 7.1.4. *The set of de Rham–Tate cycles of an abelian variety A defined over K coincides with the set of Hodge cycles via the isomorphism between Betti and de Rham cohomologies.*

Remark 7.1.5.

(1) Our conjecture is weaker than the conjectures of Ogus [**Ogu82**, Problem 2.4, Hope 4.11.3]. Therefore, Conjecture 7.1.4 was known when A has complex multiplication ([**Ogu82**, Thm. 4.16]). It was also known when A is an elliptic curve. See [**And04b**, 7.4.3.1] for an explanation using Serre–Tate theory.

(2) This conjecture reduces to the case when A is principally polarizable. To see this, note that after passing to some finite extension of K, the abelian variety A is isogenous to a principally polarizable one. Moreover this conjecture is insensitive to base change and the conjectures for two isogenous abelian varieties are equivalent.

Theorem 7.1.6 ([**Del82a**, Thm. 2.11], [**Ogu82**, Thm. 4.14], [**Bla94**])**.** *For any abelian variety, every Hodge cycle is de Rham–Tate.*

Therefore, to prove Conjecture 7.1.4, one only need to show that all of the de Rham–Tate cycles are Hodge cycles.

[2]The dual of $H^1_{\mathrm{cris}}(A_{k_v}/W(k_v))$ has a natural $W(k_v)$-structure, although φ_v on the dual does not preserve this integral structure. In order to apply Mazur's argument to the dual, Ogus passes to a suitable Tate twist of the dual such that the new φ_v acts integrally.

7.2. The de Rham–Tate group. We fix an isomorphism of K-vector spaces $H^1_{\mathrm{dR}}(A,K)$ and K^{2g}. Then the algebraic group $\mathrm{GL}_{2g,K}$ acts on $H^1_{\mathrm{dR}}(A,K)$ and hence on $H^1_{\mathrm{dR}}(A,L)^{m,n}$.

Definition 7.2.1. We define G_{dR} to be the algebraic subgroup of $\mathrm{GL}_{2g,\bar{K}}$ such that for any $\bar{K}$-algebra R, the set of R-valued points $G_{\mathrm{dR}}(R)$ is the subgroup of $\mathrm{GL}_{2g}(R)$ which fixes all de Rham–Tate cycles. We call G_{dR} the *de Rham–Tate group* of the abelian variety A.

Remark 7.2.2. The de Rham–Tate group G_{dR} is naturally defined over K by Remark 7.1.2 (1). From now on, we use G_{dR} to denote the K-algebraic group.

Lemma 7.2.3. *There exists a smallest number field K^{dR} containing K such that all of the de Rham–Tate cycles are defined over K^{dR}. Let $\{s_\alpha\}$ be a finite set of de Rham–Tate cycles such that the algebraic group G_{dR} is the stabilizer of all these s_α. Then K^{dR} is the smallest number field such that all these s_α are defined. Furthermore, K^{dR} is Galois over K.*

Proof. Let K^{dR} be the smallest number field over which all s_α in the finite set are defined. We need to show that if $t \in (H^1_{\mathrm{dR}}(A,\bar{K}))^{m,n}$ is de Rham–Tate, then t is defined over K^{dR}. Let L be a number field such that t is defined and we may assume L is Galois over K^{dR}. Let W be the smallest sub vector space of $(H^1_{\mathrm{dR}}(A,K^{\mathrm{dR}}))^{m,n}$ such that $t \in W \otimes L$. Let Γ be the Galois group $\mathrm{Gal}(L/K^{\mathrm{dR}})$. Then $W \otimes L$ is spanned by γt for $\gamma \in \Gamma$. By Remark 7.1.2 (1), these γt are de Rham–Tate, and hence $W \otimes L$ is spanned by de Rham–Tate cycles. Then by definition, $G_{\mathrm{dR}}(L)$ acts on $W \otimes L$ trivially and hence so does $G_{\mathrm{dR}}(K^{\mathrm{dR}})$ on W. On the other hand, since $\{s_\alpha\} \cup \{t\}$ is a finite set, for all but finitely many finite places v of L, we have $\varphi_v(s_\alpha) = s_\alpha$ and $\varphi_v(t) = t$. Let p be the residue characteristic of v and let m_v be $[K^{\mathrm{dR}}_v : \mathbb{Q}_p]$. The K^{dR}_v-linear action $\varphi_v^{m_v}$ lies in $G_{\mathrm{dR}}(K^{\mathrm{dR}}_v)$ since it fixes all s_α and hence acts on $W \otimes K^{\mathrm{dR}}_v$ trivially. By definition, $\varphi_v(t) = t$ and hence t is stable by $\varphi_v^{m_v}$. Therefore, t is defined over K^{dR} by the Chebotarev density theorem (applying to the coefficients of t expressed in terms of a K^{dR}-base of W). The last assertion of the lemma comes from Remark 7.1.2. ∎

7.2.4. Before we reformulate Conjecture 7.1.4 in terms of algebraic groups following Deligne, we recall the definition and basic properties of the Mumford–Tate group G_{MT}. See [**Del82a**, Sec. 3] for details. When we discuss G_{MT} and Hodge cycles, we always fix an embedding $\sigma : K \to \mathbb{C}$. We denote $H^1_{\mathrm{B}}(A_\sigma(\mathbb{C}),\mathbb{Q})$ by V_B, which has a natural polarized Hodge structure of type $((1,0),(0,1))$.

Let $\mu : \mathbb{G}_{m,\mathbb{C}} \to \mathrm{GL}(V_{B,\mathbb{C}})$ be the *Hodge cocharacter*, through which $z \in \mathbb{C}^\times$ acts by multiplication with z on $V^{1,0}_{B,\mathbb{C}}$ and trivially on $V^{0,1}_{B,\mathbb{C}}$. The *Mumford–Tate group* G_{MT} of the abelian variety A is the smallest algebraic subgroup defined over $\mathbb{Q}$ of $\mathrm{GL}(V_B)$ such that its base change to $\mathbb{C}$ containing the image of μ. The Mumford–Tate group is the algebraic subgroup of $\mathrm{GL}(V_B)$ which fixes all Hodge cycles.[3] Since all Hodge cycles are absolute Hodge cycles in the abelian variety case ([**Del82a**, Thm. 2.11]), the algebraic group G_{MT} is independent of the choice of σ.

Corollary 7.2.5. *Via the de Rham–Betti comparison, we have $G_{\mathrm{dR},\mathbb{C}} \subset G_{\mathrm{MT},\mathbb{C}}$ and the Hodge cocharacter μ factors through $G_{\mathrm{dR},\mathbb{C}}$.*

PROOF. It follows from Theorem 7.1.6 and Lemma 7.1.3. □

7.2.6. The Mumford–Tate group G_{MT} is reductive ([**Del82a**, Prop. 3.6]) and the fixed part of G_{MT} in $V_B^{m,n}$ is the set of Hodge cycles. Conjecture 7.1.4 is equivalent to the following conjecture, which we will mainly focus on from now on.

Conjecture 7.2.7. *Via the de Rham–Betti comparison, we have $G_{\mathrm{dR},\mathbb{C}} = G_{\mathrm{MT},\mathbb{C}}$.*

PROOF OF EQUIVALENCE. Conjecture 7.1.4 implies this conjecture. Conversely, by the discussion in 7.2.6, the isomorphism of these two groups implies that every $\mathbb{C}$-linear combination of de Rham–Tate cycles maps to a $\mathbb{C}$-linear combination of Hodge cycles via the de Rham–Hodge comparison. Then we conclude by Theorem 7.1.6 and Prop. 4.9 in [**Ogu82**], which shows that all de Rham–Tate cycles are $\mathbb{C}$-linearly independent. □

Remark 7.2.8. This conjecture implies that G_{dR} is connected and reductive. We will show that G_{dR} is reductive using the same idea of the proof of [**Del82a**, Prop. 3.6]. However, there seems no direct way to show that G_{dR} is connected without proving the above conjecture first.

Lemma 7.2.9. *The de Rham–Tate group G_{dR} is reductive.*

PROOF. Fix an embedding $\sigma : K \to \mathbb{C}$. By Corollary 7.2.5, we view $G_{\mathrm{dR},\mathbb{C}}$ as a subgroup of $G_{\mathrm{MT},\mathbb{C}} \subset \mathrm{GL}(V_{B,\mathbb{C}})$. Since all de Rham–Tate cycles are fixed by φ_σ, the subgroup $G_{\mathrm{dR},\mathbb{C}}$ is stable under the action of φ_σ. Therefore, both μ and its complex conjugate $\bar{\mu}(= \varphi_\sigma \circ \mu)$ factor through

[3]Here we consider Hodge cycles as elements in $V_B^{m,m-2i}(i) \subset V_B^{m',n'}$ for some choice of m', n' as Tate twists is a direct summand of the tensor algebra of V_B.

$G_{\mathrm{dR},\mathbb{C}}$ by Corollary 7.2.5 and so does $h = \mu \cdot \bar{\mu}$. Let ψ be the polarization on V_B and $G^1_{\mathrm{dR},\mathbb{C}}$ be the subgroup of $G_{\mathrm{dR},\mathbb{C}}$ acting trivially on the Tate twist. Then ψ is invariant under $G^1_{\mathrm{dR},\mathbb{C}}$. Let C be $h(i) \in G^1_{\mathrm{dR},\mathbb{C}}$ and let $\phi(x,y)$ be $\psi(x, Cy)$. Then the positive definite form ϕ on $V_{B,\mathbb{R}}$ is invariant under $\mathrm{ad}C(G^1_{\mathrm{dR},\mathbb{C}})(\mathbb{R})$[4]. Therefore, $G^1_{\mathrm{dR},\mathbb{C}}$ has a compact real form $\mathrm{ad}C(G^1_{\mathrm{dR},\mathbb{C}})(\mathbb{R})$ and is reductive. Then $G_{\mathrm{dR}} = G^1_{\mathrm{dR}} \cdot Z(G_{\mathrm{dR}})$ is reductive. □

7.3. The centralizer of the de Rham–Tate group. The following proposition, whose proof uses the construction of a Tannakian category of de Rham–Tate cycles, provides a description of the centralizer of G_{dR} in $\mathrm{End}(H^1_{\mathrm{dR}}(A,K))$. We will use this proposition to reformulate a result of Bost. Moreover, at the end of this subsection, we use Corollary 2.2.8 to prove a strengthening of the result of Bost that will be used to describe the centralizer of G°_{dR}.

Proposition 7.3.1. *Let s be an element in $(H^1_{\mathrm{dR}}(A,L))^{m,n}$ for some number field L containing K^{dR}. The de Rham–Tate group G_{dR} fixes s if and only if s is a L-linear combination of de Rham–Tate cycles.*

7.3.2. We now construct the category $\mathcal{M}_{\mathrm{dRT},L}$ of motives of de Rham–Tate cycles of the abelian variety A, where L is a field algebraic over K. We follow the idea of the construction of the motive of absolute Hodge cycles in [**Del82b**, Sec. 6]. Let $\langle A\rangle^{\otimes}$ be the set of varieties generated by A under finite product and disjoint union, and let $H_{\mathrm{dR}}(X)$ be the direct sum of $H^i_{\mathrm{dR}}(X,L)$ for all i.

The objects in the category $\mathcal{M}_{\mathrm{dRT},L}$ are of the form

$$M = (X, n, pr), \text{ where } X \in \langle A\rangle^{\otimes},\ n \in \mathbb{Z},\ pr \in \mathrm{End}(H_{\mathrm{dR}}(X,L)) \text{ idempotent de Rham–Tate.}$$

Let $M_i = (X_i, n_i, pr_i)$, $i = 1,2$. The set of morphisms $\mathrm{Hom}(M_1, M_2)$ is defined to be

$$\{f : H_{\mathrm{dR}}(X_1)(n_1) \to H_{\mathrm{dR}}(X_2)(n_2) \text{ de Rham–Tate such that } f \circ pr_1 = pr_2 \circ f\}/\sim,$$

where $\sim$ is defined by modulo $\{f : f \circ pr_1 = 0 = pr_2 \circ f\}$.

[**Ogu82**, Prop. 4.9] shows that that $\mathcal{M}_{\mathrm{dRT},L}$ is $\mathbb{Q}$-linear with $\mathrm{End}(\mathbb{I}) = \mathbb{Q}$, where $\mathbb{I} = (pt, 0, id)$ and that $\mathrm{Hom}(M_1, M_2)$ is a finite dimensional $\mathbb{Q}$-vector space. Moreover, by the above construction, the category $\mathcal{M}_{\mathrm{dRT},L}$ is a pseudo-abelian rigid tensor category (see also [**And04b**, 4.1.3, 4.1.4]). Since the de Rham–Tate cycles lie in the image of the Betti cohomology with real coefficients

[4] One can check by definition that the $\mathbb{C}$-sub group $\mathrm{ad}C(G^1_{\mathrm{dR},\mathbb{C}})$ is an $\mathbb{R}$-subgroup of $G_{\mathrm{MT},\mathbb{C}}$.

under the Betti–de Rham comparison, $pr(H_{\mathrm{dR}}(X, \mathbb{C}))$ has a real Hodge structure. By [**Del82b**, Prop. 6.2] and the fact that absolute Hodge cycles are de Rham–Tate cycles, $pr(H_{\mathrm{dR}}(X, \mathbb{C}))$ is polarized. Hence $\mathrm{End}(M)$ is semi-simple by [**Del82b**, Prop. 4.5, Prop. 6.3]. Therefore, we use [**Jan92**, Lem. 2] to conclude that $\mathcal{M}_{\mathrm{dRT},L}$ is a rigid abelian tensor category. By [**Del90**, Thm. 1.12], this is a Tannakian category with a fiber functor $\omega_L : M \mapsto pr(H_{\mathrm{dR}}(X, L))$ over L. Let G_{dR}^L be the Tannakian fundamental group $\underline{\mathrm{Aut}}^{\otimes}(\omega_L)$. Since $\mathcal{M}_{\mathrm{dRT},L}$ is semi-simple, G_{dR}^L is a reductive algebraic group over L.

7.3.3. We now describe the relation between de Rham–Tate groups of cycles over different fields. One can define de Rham–Tate cycles on zero dimensional varieties as in Definition 7.1.1 and define the motive $\mathcal{M}^0_{\mathrm{dRT},L}$ as above. This category is the category of Artin motives and we denote by $\Gamma(L)$ its Tannakian fundamental group, which is an L-form of the Galois group $\mathrm{Gal}(\bar{L}/L)$. A modification of the proof of [**Del82b**, Prop. 6.23] shows that the following sequence is exact:

$$1 \to G_{\mathrm{dR}}^{\bar{L}} \to G_{\mathrm{dR}}^L \to \overline{\Gamma(L)} \to 1,$$

where $\overline{\Gamma(L)}$ is a quotient of $\Gamma(L)$. More precisely, $\overline{\Gamma(L)}$ is the Tannakian fundamental group of $\mathcal{M}_{\mathrm{dRT},L} \cap \mathcal{M}^0_{\mathrm{dRT},L}$, the full subcategory of $\mathcal{M}_{\mathrm{dRT},L}$ whose objects are Artin motives.

The category that we will mainly focus on is $\mathcal{M}_{\mathrm{dRT},\bar{K}}$, which is equivalent to $\mathcal{M}_{\mathrm{dRT},K^{\mathrm{dR}}}$ by Lemma 7.2.3, and we will denote them by $\mathcal{M}_{\mathrm{dRT}}$.

Proof of Proposition 7.3.1. Let $\{s_\alpha\}$ be the set of de Rham–Tate cycles. We view $G_{\mathrm{dR}}^{K^{\mathrm{dR}}}$ as a subgroup of $\mathrm{GL}(H^1_{\mathrm{dR}}(A, K))$ (a priori, $G_{\mathrm{dR}}^{K^{\mathrm{dR}}}$ is only defined over K^{dR}, but it descends to K by Remark 7.1.2(1)). Since $\mathcal{M}_{\mathrm{dRT}}$ is Tannakian, we have an equivalence of categories

$$\mathcal{M}_{\mathrm{dRT}} \otimes L \cong Rep_L(G_{\mathrm{dR}}^{K^{\mathrm{dR}}}).$$

Hence that s is an L-linear combination of s_α is equivalent to that s is fixed by $G_{\mathrm{dR}}^{K^{\mathrm{dR}}}$ and it remains to prove that $G_{\mathrm{dR}}^{K^{\mathrm{dR}}} = G_{\mathrm{dR}}$. Since G_{dR} is defined to be the stabilizer of all s_α, the above equivalence of categories shows that $G_{\mathrm{dR}}^{K^{\mathrm{dR}}} \subset G_{\mathrm{dR}}$. Since $G_{\mathrm{dR}}^{K^{\mathrm{dR}}}$ is reductive, then by [**Del82a**, Prop. 3.1 (b)], $G_{\mathrm{dR}}^{K^{\mathrm{dR}}}$ is the stabilizer of a line in some direct sum of $(H^1_{\mathrm{dR}}(A, L))^{m,n}$. By the definition of $G_{\mathrm{dR}}^{K^{\mathrm{dR}}}$, this line must be an L-linear combination of some s_α and hence $G_{\mathrm{dR}}^{K^{\mathrm{dR}}} = G_{\mathrm{dR}}$ because G_{dR} stabilizes all linear combinations of s_α. $\square$

Remark 7.3.4.

(1) It follows from the proof that G_{dR} is reductive. This argument is essentially the same as the one we gave before since the key input for both arguments is that de Rham–Tate cycles are fixed by φ_σ.

(2) There is a variant of Proposition 7.3.1 when L is not assumed to contain K^{dR}. More precisely, G_{dR}^L is the largest subgroup of $\mathrm{GL}(H^1_{\mathrm{dR}}(A, L))$ that stabilizes every de Rham–Tate cycle over L and $s \in (H^1_{\mathrm{dR}}(A, L))^{m,n}$ is an L-linear combination of de Rham–Tate cycles over L if and only if s is fixed by the action of G_{dR}^L.

The following definition is motivated by [**Her12**, Def. 3.5].

Definition 7.3.5. An element s of $(H^1_{\mathrm{dR}}(A, L))^{m,n}$ is called a *β-de Rham–Tate cycle* if the A-density of the set of primes such that $\phi_v(s) \neq s$ is at most $1 - \beta$. More explicitly, it means

$$\beta \leq \liminf_{x\to\infty} \left(\sum_{v, p_v \leq x, \varphi_v(s)=s} \frac{[L_v : \mathbb{Q}_{p_v}] \log p_v}{p_v - 1} \right) \left([L : \mathbb{Q}] \sum_{p \leq x} \frac{\log p}{p-1} \right)^{-1},$$

where v (resp. p) runs over finite places of L (resp. $\mathbb{Q}$) and p_v is the residue characteristic of v.

Remark 7.3.6.

(1) Absolute Tate cycles and de Rham–Tate cycles are 1-de Rham–Tate cycles by definition.

(2) Let M be a set of rational primes with natural density β and assume that $\forall p \in M, \forall v | p$, one has $\varphi_v(s) = s$. Then s is a β-de Rham–Tate cycle by [**Her12**, Lem. 3.7].

Theorem 7.3.7. *The set of* 1*-de Rham–Tate cycles in* $\mathrm{End}(H^1_{\mathrm{dR}}(A, L))$ *is the image of* $\mathrm{End}_L(A) \otimes \mathbb{Q}$. *In particular, the centralizer of* G_{dR} *in* $\mathrm{End}(H^1_{\mathrm{dR}}(A, L))$ *is* $\mathrm{End}_L(A) \otimes L$.

Proof. The second assertion follows from the first one by Proposition 7.3.1 and the above remark. The first statement restricted to absolute Tate cycles is a direct consequence of Theorem 2.0.1 and we refer the reader to [**And04b**, 7.4.3] for a proof. See also [**Bos06**, Thm. 6.4]. Notice that their argument is valid for 1-de Rham–Tate cycles if one uses Corollary 2.2.8 instead. □

7.4. Relative de Rham–Tate cycles. Let L be a finite extension over K. Let v be a finite place of L with residue characteristic p and define $m_v = [L_v : \mathbb{Q}_p]$. We have an L_v-linear endomorphism, the relative Frobenius $\varphi_v^{m_v}$, of $H^1_{\mathrm{dR}}(A, L_v)$ and hence of $(H^1_{\mathrm{dR}}(A, L_v))^{m,n}$.

Definition 7.4.1. An element $t \in (H^1_{\mathrm{dR}}(A, L))^{m,n}$ is called a *relative de Rham–Tate cycle* (over L) of A if there exists a finite set Σ of finite places of L such that for every finite place $v \notin \Sigma$ and every archimedean place σ, one has $\varphi_v^{m_v}(t) = t$ and $\varphi_v^{m_v}(\varphi_\sigma(t)) = \varphi_\sigma t$.

Remark 7.4.2. By definition, any L-linear combination of de Rham–Tate cycles over L is relatively de Rham–Tate. Moreover, for any $\gamma \in \mathrm{Gal}(L/K)$, the cycle $\gamma(t)$ is relatively de Rham–Tate if (and only if) t is so.

In analogy with the definition of de Rham–Tate groups, we have:

Definition 7.4.3. We define G^L to be the algebraic subgroup of $\mathrm{GL}_{2g,L}$ such that any L-algebra R, its R-points $G^L(R)$ is the subgroup of $\mathrm{GL}_{2g}(R)$ which fixes all relative de Rham–Tate cycles t_α over L. We call G^L the *relative de Rham–Tate group* of the abelian variety A over L.

Lemma 7.4.4. *Similar to the corresponding statements for the de Rham–Tate group, we have:*

(1) The relative de Rham–Tate group G^L is contained in G^L_{dR}.

(2) Every relative de Rham–Tate cycle lies in $\mathrm{Fil}^0((H^1_{\mathrm{dR}}(A, L))^{m,n})$ *and hence the Hodge cocharacter factors through G^L.*

(3) The group G^L is the smallest reductive algebraic subgroup of $\mathrm{GL}_{2g,L}$ *such that*

- *the set of its L_v-points contains $\varphi_v^{m_v}$ for all but finitely many finite places v, and*
- *it is stable under φ_σ for all archimedean places σ.*

(4) Any element in $(H^1_{\mathrm{dR}}(A, L))^{m,n}$ is relatively de Rham–Tate if and only if it is fixed by the action of G^L.

Proof. Part (1) follows from Remark 7.4.2. Lemma 7.1.3 implies part (2). To show that G^L is reductive, we notice that φ_σ fixes the set of relative de Rham–Tate cycles and hence the embedding $G^L_{\mathbb{C}} \subset G_{\mathrm{MT},\mathbb{C}}$ is induced from an embedding of $\mathbb{R}$-groups. Then, combined with part (2), we see that $\mu(i) \cdot \bar{\mu}(i) \in G^L(\mathbb{C})$. Now as in the proof of Lemma 7.2.9, the adjoint action of $\mu(i) \cdot \bar{\mu}(i)$ defines a real form of G^L which is compact moduo center and hence G^L is reductive. The rest of (3) is direct and it implies (4). □

Unlike the de Rham–Tate groups, one can show directly that when L is large enough, the relative de Rham–Tate group G^L is connected. See Corollary 8.1.7.

8. Frobenius Tori and the Mumford–Tate conjecture

In this section, we recall the theory of Frobenius tori initiated by Serre (see section 8.1). The fact that the Frobenius actions on the crystalline and étale cohomology groups have the same characteristic polynomial ([**KM74**]) enables us to view the Frobenius tori as subgroups of both G_{dR} and the ℓ-adic monodromy group G_ℓ. Hence the Frobenius tori serve as bridges between results for G_ℓ and those for G_{dR}. We prove a refinement (Proposition 8.1.11) of results of Serre and Chi on the rank of Frobenius tori. In section 8.2, we recall the Mumford–Tate conjecture and prove Theorem 3 and Theorem 4. In section 8.3, we recall a result of Noot and use it to show that if G_{dR}° of A is a torus, then A has complex multiplication.

From now on, we use Σ to denote a finite set of finite places of K^{dR} containing all ramified places such that for $v \notin \Sigma$, the abelian variety $A_{K^{\mathrm{dR}}}$ has good reduction at v and the Frobenius φ_v stabilizes all of the de Rham–Tate cycles. For any finite extension L of K in question, we still use Σ to denote the finite set of finite places $f^{-1}g(\Sigma)$, where $f : \operatorname{Spec} L \to \operatorname{Spec} K$ and $g : \operatorname{Spec} K^{\mathrm{dR}} \to \operatorname{Spec} K$. When we discuss the relative de Rham–Tate cycles, we also enlarge Σ so that the relative Frobenius $\varphi_v^{m_v}$ stabilizes all relative de Rham–Tate cycles over L.

8.1. Frobenius Tori.

The following definition is due to Serre. See also [**Chi92**, Sec. 3] and [**Pin98**, Sec. 3] for details.

Definition 8.1.1. Let T_v be the Zariski closure of the subgroup of $G^L_{L_v}$ (hence also of G^L_{dR,L_v}) generated by the L_v-linear map $\varphi_v^{m_v} \in G^L(L_v)$. Since $\varphi_v^{m_v}$ is semisimple, the group T_v° is a torus and is called the *Frobenius torus* associated to v.

Remark 8.1.2. The torus T_v° and its rank are independent of the choice of L.

8.1.3. For every prime ℓ, we have the ℓ-adic Galois representation

$$\rho_\ell : \mathrm{Gal}(\bar{K}/K) \to \mathrm{GL}(H^1_{\text{ét}}(A_{\bar{K}}, \mathbb{Q}_\ell))(\mathbb{Q}_\ell),$$

and we denote by $G_\ell(A)$ the algebraic group over $\mathbb{Q}_\ell$ which is the Zariski closure of the image of $\mathrm{Gal}(\bar{K}/K)$ and call $G_\ell(A)$ the *ℓ-adic monodromy group* of A. If it is clear which variety is concerned, we may just use G_ℓ to denote this group. Serre proved that there exists a smallest finite

Galois extension $K^{\text{ét}}$ of K such that for any ℓ, the Zariski closure of the image of $\mathrm{Gal}(\overline{K^{\text{ét}}}/K^{\text{ét}})$ is connected ([**Ser13**, Sec. 5, p. 15]).

Remark 8.1.4. For $v \nmid l$, we also view T_v as an algebraic subgroup (only well-defined up to conjugation) of G_ℓ in the following sense. Since A has good reduction at v, the action of the decomposition group at v is unramified on $H^1_{\text{ét}}(A_{\bar{K}}, \mathbb{Q}_\ell)$. Since v is unramified, we have an embedding $\mathrm{Gal}(\bar{k}_v/k_v) \cong \mathrm{Gal}(L_v^{ur}/L_v) \to \rho_\ell(\mathrm{Gal}(\bar{K}/K))$ after choosing an embedding $\bar{K} \to \bar{L}_v$. Hence we view the Frobenius $Frob_v$ as an element of G_ℓ. Due to Katz and Messing [**KM74**], the characteristic polynomial of $\varphi_v^{m_v}$ acting on $H^1_{\text{cris}}(A_{k_v}/W(k_v))$ is the same[5] as the characteristic polynomial of $Frob_v$ acting on $H^1_{\text{ét}}(A_{\bar{K}}, \mathbb{Q}_\ell)$. Hence T_v is isomorphic to the algebraic group generated by semi-simple element $Frob_v$ in G_ℓ. From now on, when we view T_v as a subgroup of G_ℓ, we identify T_v with the group generated by $Frob_v$.

Here are some important properties of Frobenius tori.

Theorem 8.1.5 (Serre, see also [**Chi92**, Cor. 3.8])**.** *There is a set M_{max} of finite places of $K^{ét}$ of natural density one and disjoint from Σ such that for any $v \in M_{max}$, the algebraic group T_v is connected and it is a maximal torus of G_ℓ.*

Proposition 8.1.6 ([**Chi92**, Prop. 3.6 (b)])**.** *For L large enough (for instance, containing all the n-torsion points for some $n \geq 3$), all but finitely many T_v are connected.*

Corollary 8.1.7. *For L large enough, the relative de Rham–Tate group G^L is connected and $G^L = G^{L'}$ for $L \subset L'$.*

Proof. Let $(G^L)^\circ$ be the connected component of G^L. It is reductive and φ_σ-stable for all archimedean places σ. By Proposition 8.1.6, for all but finitely many v, the group T_v is connected and hence is contained in $(G^L)^\circ_{L_v}$. Therefore, $\varphi_v^{m_v} \in T_v(L_v) \subset (G^L)^\circ(L_v)$ and $(G^L)^\circ = G^L$ by Lemma 7.4.4(3)(4). Let v' be a place of L' over v. By definition, $T_{v'}$ is a subgroup of T_v of finite index. Since T_v is connected, we have $\phi_v^{m_v} \in T_v = T_{v'} \subset G^{L'}$. We conclude by Lemma 7.4.4(3)(4). $\square$

Remark 8.1.8. One reason to introduce relative de Rham–Tate cycles is that G^L behaves like G_ℓ in the sense that both of them become connected if one replace the base field K by a large enough L.

[5] To compare the two polynomials, we notice that both of them have $\mathbb{Z}$-coefficients.

The following lemma is of its own interest.

Lemma 8.1.9. *The number field K^{dR} is contained in $K^{\acute{e}t}$.*

Proof. For the simplicity of notation, we enlarge K^{dR} to contain $K^{\text{ét}}$ and prove that they are equal. Let v be a finite place of $K^{\text{ét}}$ above p such that p splits completely in $K^{\text{ét}}/\mathbb{Q}$ and we identify $K_v^{\text{ét}}$ with $\mathbb{Q}_p$ via v. Let w be a place of K^{dR} above v. Denote by σ the Frobenius in $\mathrm{Gal}(K_w^{\mathrm{dR}}/\mathbb{Q}_p) = \mathrm{Gal}(K_w^{\mathrm{dR}}/K_v^{\text{ét}})$. We consider the algebraic group T_v generated by $\varphi_v \in G_{\mathrm{dR}}^{K^{\text{ét}}}(K_v^{\text{ét}})$. If $v \in M_{max}$ as in Theorem 8.1.5, then T_v is connected and hence $T_v \subset G_{\mathrm{dR},K_v^{\text{ét}}}$. This implies that $\varphi_v \in G_{\mathrm{dR}}(K_v^{\text{ét}})$. For any m, n, let $W' \subset (H^1_{\mathrm{dR}}(A, K^{\mathrm{dR}}))^{m,n}$ be the K^{dR}-linear span of all de Rham–Tate cycles in $(H^1_{\mathrm{dR}}(A, K^{\mathrm{dR}}))^{m,n}$. By Remark 7.1.2, there exists a K-linear subspace W of $(H^1_{\mathrm{dR}}(A, K))^{m,n}$ such that $W' = W \otimes K^{\mathrm{dR}}$. Since G_{dR} acts trivially on W' and W, the Frobenius φ_v acts on $W \otimes_K K_v^{\text{ét}}$ trivially and ϕ_w acts on $W' \otimes_{K^{\mathrm{dR}}} K_w^{\mathrm{dR}}$ as the σ-linearly extension of φ_v. Hence the elements in W' that are stabilized by φ_w are contained in $W \otimes_K K_v^{\text{ét}}$. That is to say that all de Rham–Tate cycles are defined over $K_v^{\text{ét}}$. As m, n are arbitrary, we have $K_w^{\mathrm{dR}} = K_v^{\text{ét}}$. This implies that p splits completely in $K^{\mathrm{dR}}/\mathbb{Q}$ and hence $K^{\mathrm{dR}} = K^{\text{ét}}$ by the Chebotarev density theorem. $\square$

Remark 8.1.10. From Theorem 7.3.7, we see the definition field of a de Rham–Tate cycle induced from an endomorphism of $A_{\bar{K}}$ is the same as the definition field of this endomorphism. Hence K^{dR} contains the definition field of all endomorphisms. Then K^{dR} and $K^{\text{ét}}$ are the same if the definition field of all endomorphisms is $K^{\text{ét}}$. This is the case when one can choose a set of ℓ-adic Tate cycles all induced from endomorphisms of A to cut out G_ℓ.

Now we discuss some refinements of Theorem 8.1.5 and Proposition 8.1.6. In the rest of this subsection, the definition field K of the polarized abelian variety A is always assumed to be Galois over $\mathbb{Q}$. The main result is:

Proposition 8.1.11. *Assume that $G_\ell^\circ(A)$ is $\mathrm{GSp}_{2g,\mathbb{Q}_\ell}$. Then there exists a set M of rational primes with natural density one such that for any $p \in M$ and any finite place v of K lying over p, the algebraic group T_v generated by $\varphi_v^{m_v}$ (where $m_v = [K_v : \mathbb{Q}_p]$) is of maximal rank. In particular, T_v is connected for such v.*[6]

[6]Our proof is a direct generalization of the proof of Theorem 8.1.5 by Serre. Will Sawin pointed out to me that it is possible to prove this proposition by applying Chavdarov's method (**[Cha97]**) to $\mathrm{Res}_{\mathbb{Q}}^K A$.

The idea of the proof is to apply the Chebotarev density theorem to a suitably chosen Zariski closed subset of the ℓ-adic monodromy group of B, the Weil restriction $\mathrm{Res}_{\mathbb{Q}}^{K} A$ of A. As we are in characteristic zero, the scheme B is an abelian variety over $\mathbb{Q}$. We have $B_K = \prod_{\sigma \in \mathrm{Gal}(K/\mathbb{Q})} A^{\sigma}$, where $A^{\sigma} = A \otimes_{K,\sigma} K$. It is a standard fact that $B^{\vee} = \mathrm{Res}_{\mathbb{Q}}^{K} A^{\vee}$ and hence the polarization on A induces a polarization on B over $\mathbb{Q}$. Moreover, the polarization on A induces a polarization on A^{σ}. Extend σ to a map $\sigma : \bar{K} \to \bar{K}$. The map $\sigma : A(\bar{K}) \to A^{\sigma}(\bar{K}), P \mapsto \sigma(P)$ induces a map on Tate modules $\sigma : T_{\ell}(A) \to T_{\ell}(A^{\sigma})$. This map is an isomorphism between $\mathbb{Z}_{\ell}$-modules.

Lemma 8.1.12. *The map $\sigma : T_{\ell}(A) \to T_{\ell}(A^{\sigma})$ induces an isomorphism between the ℓ-adic monodromy groups $G_{\ell}(A)$ and $G_{\ell}(A^{\sigma})$.*

Proof. Via σ, the image of $\mathrm{Gal}(\bar{K}/K)$ in $\mathrm{End}(T_{\ell}(A^{\sigma}))$ is identified as that of $\mathrm{Gal}(\bar{K}/K)$ in $\mathrm{End}(T_{\ell}(A))$. Hence $G_{\ell}(A) \simeq G_{\ell}(A^{\sigma})$ as $T_{\ell}(-)^{\vee} = H^1_{\text{ét}}((-)_{\bar{K}}, \mathbb{Q}_{\ell})$. □

We start with the following special case to illustrate the idea of the proof of Proposition 8.1.11.

Proposition 8.1.13. *Assume that $G_{\ell}(A) = \mathrm{GSp}_{2g,\mathbb{Q}_{\ell}}$ and that A^{σ} is not geometrically isogenous to A^{τ} for any distinct $\sigma, \tau \in \mathrm{Gal}(K/\mathbb{Q})$. Then there exists a set M of rational primes with natural density 1 such that for any $p \in M$ and any v above p, the group T_v is of maximal rank. That is, the rank of T_v equals to the rank of $G_{\ell}(A)$. In particular, T_v is connected for such v.*

Proof. We use the same idea as in the proof of Theorem 8.1.5 by Serre. His idea is to first construct a proper Zariski closed subvariety $Z \subset G_{\ell}(A)$ as follows (see also [**Chi92**, Thm. 3.7]) and then to apply the Chebotarev density theorem:

(1) Z is invariant under conjugation by $G_{\ell}(A)$, and
(2) if $u \in G_{\ell}(A)(\mathbb{Q}_{\ell}) \setminus Z(\mathbb{Q}_{\ell})$ semisimple, then the algebraic subgroup of G_{ℓ} generated by u is of maximal rank.

Since $G_{\ell}(A)$ is connected, $Z(\mathbb{Q}_{\ell})$ is of measure zero in $G_{\ell}(A)(\mathbb{Q}_{\ell})$ with respect to the usual Haar measure. We will define a Zariski closed subset $W \subset G_{\ell}(B)$ which has similar properties as Z.

Let $G_{\ell}^{K}(B)$ be the Zariski closure of $\mathrm{Gal}(\bar{K}/K)$ in $\mathrm{GL}(H^1_{\text{ét}}(B_{\bar{\mathbb{Q}}}, \mathbb{Q}_{\ell}))$. Via the isomorphism $H^1_{\text{ét}}(B_{\bar{\mathbb{Q}}}, \mathbb{Q}_{\ell}) \cong \oplus H^1_{\text{ét}}(A^{\sigma}_{\bar{K}}, \mathbb{Q}_{\ell})$ of $\mathrm{Gal}(\bar{K}/K)$-modules, we view $G_{\ell}^{K}(B)$ as a subgroup of $\prod G_{\ell}(A^{\sigma})$. By the assumption that A^{σ}'s are not geometrically isogenous to each other and [**Lom15**, Thm. 4.1, Rem. 4.3], we have $G_{\ell}^{K}(B) \cong \mathbb{G}_m \cdot \prod SG_{\ell}(A^{\sigma})$, where $SG_{\ell} \subset G_{\ell}$ is the subgroup of elements with determinant 1. Indeed, $\mathrm{Lie}\, SG_\ell(A^\sigma) = \mathfrak{sp}_{2g,\mathbb{Q}_\ell}$ of type C and the representations are all standard representations and then Rem. 4.3 in *loc. cit.* verified that Lombardo's theorem is applicable in our situation. Then By Lemma 8.1.12, we have $G_\ell^K(B) \simeq \mathbb{G}_m \cdot SG_\ell(A)^{[K:\mathbb{Q}]}$. This is the neutral connected component of $G_\ell(B)$.

The map $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to G_\ell(B)(\mathbb{Q}_\ell) \to G_\ell(B)(\mathbb{Q}_\ell)/G_\ell^K(B)(\mathbb{Q}_\ell)$ induces a surjection $\mathrm{Gal}(K/\mathbb{Q}) \twoheadrightarrow G_\ell(B)(\mathbb{Q}_\ell)/G_\ell^K(B)(\mathbb{Q}_\ell)$. Given $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$, we denote by $\sigma G_\ell^K(B)$ the subvariety of $G_\ell(B)$ corresponding to the image of σ in the above map. Let m be the order of $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$. We consider those p unramified in $K/\mathbb{Q}$ whose corresponding Frobenii in $\mathrm{Gal}(K/\mathbb{Q})$ fall into c_σ, the conjugacy class of σ. We have $m_v = m$ for all v above p.

Consider the composite map $m_\tau : \sigma G_\ell^K(B) \to G_\ell^K(B) \to G_\ell(A^\tau) \simeq G_\ell(A)$, where the first map is defined by $g \mapsto g^m$ and the second map is the natural projection. Let $W_{\sigma,\tau}$ be the preimage of Z and W_σ be $\cup_{\tau\in\mathrm{Gal}(K/\mathbb{Q})} W_{\sigma,\tau}$. Since by definition W_σ is a proper Zariski subvariety of the connected variety $\sigma G_\ell^K(B)$, the measure of $W_\sigma(\mathbb{Q}_\ell)$ is zero.

Claim. If the Frobenius $Frob_p$ (well-defined up to conjugacy) is not contained in the conjugacy invariant set $\cup_{\gamma\in c_\sigma} W_\gamma(\mathbb{Q}_\ell)$, then for any $v|p$, the algebraic subgroup $T_v \subset G_\ell(A)$ is of maximal rank.

Proof. The subvariety $\cup_{\gamma\in c_\sigma} W_\gamma$ is invariant under the conjugation of $G_\ell^K(B)$ because Z is invariant under the conjugation of $G_\ell(A)$. This subvariety is moreover conjugation invariant under the action $G_\ell(B)$ since $\tau W_\sigma \tau^{-1} = W_{\tau\sigma\tau^{-1}}$ by definition. By second property of Z and the definition of the map m_τ, we see that the image of $Frob_p^m$ generates a maximal torus in $G_\ell(A^\tau)$. For each $v|p$, the Frobenius $Frob_v$ is the image of $Frob_p^m$ in $G_\ell(A^\tau)$ for some τ and hence T_v is of maximal rank. $\square$

Let W be $\cup_\sigma W_\sigma$. It is invariant under the conjugation of $G_\ell(B)$. As each $W_{\sigma,\tau}(\mathbb{Q}_\ell)$ is of measure zero in $G_\ell(B)(\mathbb{Q}_\ell)$, so is $W(\mathbb{Q}_\ell)$. By the Chebotarev density theorem (see for example [**Ser12**, Sec. 6.2.1]), we conclude that there exists a set M of rational primes with natural density 1 such that $Frob_p \notin W(\mathbb{Q}_\ell)$. Then the proposition follows from the above claim. $\square$

Remark 8.1.14. The assumption that $G_\ell(A) = \mathrm{GSp}_{2g,\mathbb{Q}_\ell}$ can be weakened. The proof still works if one has $G_\ell^K(B) = \mathbb{G}_m \cdot \prod SG_\ell(A^\sigma)$. In other words, the proposition holds true whenever [**Lom15**, Thm. 4.1, Rem. 4.3] is applicable. For example, when A has odd dimension and is not of type IV in Albert's classification.

The following property of GSp_{2g} is used in an essential way of our proof of Proposition 8.1.11. It is well-known, but we give a proof for the sake of completeness.

Lemma 8.1.15. *If G is an algebraic subgroup of $\mathrm{GL}(H^1_{\acute{e}t}(A_{\bar{K}}, \mathbb{Q}_\ell))$ containing $\mathrm{GSp}_{2g,\mathbb{Q}_\ell}$ as a normal subgroup, then $G = \mathrm{GSp}_{2g,\mathbb{Q}_\ell}$. In particular, $G^\circ_\ell(A) = \mathrm{GSp}_{2g,\mathbb{Q}_\ell}$ implies that $G_\ell(A)$ is connected.*

Proof. Let g be a $\overline{\mathbb{Q}_\ell}$-point of G. Then $\mathrm{ad}(g)$ induces an automorphism of $\mathrm{GSp}_{2g,\overline{\mathbb{Q}_\ell}}$ by the assumption that $\mathrm{GSp}_{2g,\mathbb{Q}_\ell}$ is a normal subgroup. As $\mathrm{ad}(g)$ preserves determinant, we view $\mathrm{ad}(g)$ as an automorphism of $\mathrm{Sp}_{2g,\overline{\mathbb{Q}_\ell}}$. Since $\mathrm{Sp}_{2g,\overline{\mathbb{Q}_\ell}}$ is a connected, simply connected linear algebra group whose Dynkin diagram does not have any nontrivial automorphism, any automorphism of $\mathrm{Sp}_{2g,\overline{\mathbb{Q}_\ell}}$ is inner. Hence $\mathrm{ad}(g) = \mathrm{ad}(h)$ for some $\overline{\mathbb{Q}_\ell}$-point h of $\mathrm{Sp}_{2g,\mathbb{Q}_\ell}$. Then g and h differ by an element in the centralizer of $\mathrm{Sp}_{2g,\overline{\mathbb{Q}_\ell}}$ in $\mathrm{GL}_{2g,\overline{\mathbb{Q}_\ell}}$. Since the centralizer is $\mathbb{G}_m$, we conclude that g is in $\mathrm{GSp}_{2g,\mathbb{Q}_\ell}(\overline{\mathbb{Q}_\ell})$. □

Proof of Proposition 8.1.11. Let B be $\mathrm{Res}^K_{\mathbb{Q}} A$. As in the proof of Proposition 8.1.13, it suffices to construct a Zariski closed set $W \subset G_\ell(B)$ such that

(1) $W(\mathbb{Q}_\ell)$ is of measure zero with respect to the Haar measure on $G_\ell(B)(\mathbb{Q}_\ell)$,
(2) W is invariant under conjugation by $G_\ell(B)$, and
(3) if $u \in G_\ell(B)(\mathbb{Q}_\ell) \setminus W(\mathbb{Q}_\ell)$ is semisimple, then the algebraic subgroup of $G_\ell(B)$ generated by u is of maximal rank.

We first show that, to construct such W, it suffices to construct $W_\sigma \subset \sigma G^K_\ell(B)$ for each $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ such that

(1) $W_\sigma(\mathbb{Q}_\ell)$ is of measure zero with respect to the Haar measure on $G_\ell(B)(\mathbb{Q}_\ell)$,
(2) W_σ is invariant under conjugation by $G^K_\ell(B)$, and
(3) if $u \in \sigma G^K_\ell(B)(\mathbb{Q}_\ell) \setminus W(\mathbb{Q}_\ell)$ is semisimple, then the algebraic subgroup of $G_\ell(B)$ generated by u is of maximal rank.

Indeed, given such W_σ, we define W' to be $\cup_\sigma W_\sigma$. This set satisfies (1) and (3) and is invariant under conjugation by $G^K_\ell(B)$. We then define W to be the $G_\ell(B)$-conjugation invariant set generated by W'. Since $[G_\ell(B) : G^K_\ell(B)]$ is finite, W as a set is a union of finite copies of W' and hence satisfies (1) and (3).

To construct W_σ, let $C \subset \mathrm{Gal}(K/\mathbb{Q})$ be the subgroup generated by σ. Consider $\{A^\tau\}_{\tau \in C}$. We have a partition $C = \sqcup_{1 \leq i \leq r} C_i$ with respect to the $\bar{K}$-isogeny classes of A^τ. These C_i have the same cardinality m/r. For any $\alpha \in \mathrm{Gal}(K/\mathbb{Q})$, the partition of $\alpha C = \sqcup \alpha C_i$ gives the partition of $\{A^\tau\}_{\tau \in \alpha C}$ with respect to the $\bar{K}$-isogeny classes.

Consider the map $m_\alpha : \sigma G_\ell^K(B) \to G_\ell^K(B) \to G_\ell(A^\alpha) \simeq G_\ell(A)$ and define $W_{\sigma,\alpha}$ to be the preimage of Z and W_σ to be $\cup_{\alpha \in \mathrm{Gal}(K/\mathbb{Q})} W_{\sigma,\alpha}$ as in the proof of Proposition 8.1.13. The proof of the claim there shows that W_σ satisfies (2) and (3).

Now we focus on (1). By the assumption and Lemma 8.1.15, the group $G_\ell(A)$ is connected and hence $Z(\mathbb{Q}_\ell)$ is of measure zero. Let γ be σ^r. Consider $r : \sigma G_\ell^K(B) \to \gamma G_\ell^K(B)$ defined by $g \mapsto g^r$ and the composite map $(m/r)_\alpha : \gamma G_\ell^K(B) \to G_\ell^K(B) \to G_\ell(A^\alpha) \simeq G_\ell(A)$, where the first map is defined by $g \mapsto g^{m/r}$ and the second map natural projection. Then $m_\alpha = (m/r)_\alpha \circ r$. Let W_r be $(m/r)_\alpha^{-1}(Z)$. Then $W_{\sigma,\alpha} = r^{-1}(W_r)$. Since any two of $\{A^\tau\}_{\tau = \alpha, \alpha\sigma, \cdots, \alpha\sigma^{r-1}}$ are not geometrically isogenous, the same argument as in the proof of Proposition 8.1.13 shows that if $W_r(\mathbb{Q}_\ell)$ is of measure zero, so is $W_{\sigma,\alpha}(\mathbb{Q}_\ell)$. The rest of the proof is to show that $W_r(\mathbb{Q}_\ell)$ is of measure zero.

Notice that $G_\ell^\circ(B_K) = \mathbb{G}_m \cdot \prod_{\sigma \in \mathbb{I}} SG_\ell(A^\sigma) = \mathbb{G}_m \cdot \mathrm{Sp}_{2g}^{|\mathbb{I}|}$, where $\mathbb{I}$ is a set of representatives of all isogeny classes in $\{A^\sigma\}_{\sigma \in \mathrm{Gal}(K/\mathbb{Q})}$.

Since the centralizer of GSp_{2g} in GL_{2g} is $\mathbb{G}_m$ and $G_\ell^\circ(B)$ is a normal subgroup of $G_\ell(B)$, the map $(m/r)_\alpha$ is up to a constant the same as the following map:

$$\gamma G_\ell^K(B) \to \mathrm{Isom}_{\mathbb{Q}_\ell}(H^1_{\text{ét}}(A^{\alpha\gamma}_{\bar{K}}, \mathbb{Q}_\ell), H^1_{\text{ét}}(A^\alpha_{\bar{K}}, \mathbb{Q}_\ell)) \cong \mathrm{GL}(H^1_{\text{ét}}(A^\alpha_{\bar{K}}, \mathbb{Q}_\ell)) \to \mathrm{GL}(H^1_{\text{ét}}(A^\alpha_{\bar{K}}, \mathbb{Q}_\ell)),$$

where the first map is the natural projection, the middle isomorphism is given by a chosen isogeny between A^α and $A^{\alpha\gamma}$, and the last map is $g \mapsto g^{m/r}$.

The fact that $\gamma G_\ell^K(B)$ normalizes $G_\ell^\circ(B_K)$ allows us to apply Lemma 8.1.15 to the image of the above map and see that the above map factors through

$$\mathrm{GSp}(H^1_{\text{ét}}(A^\alpha_{\bar{K}}, \mathbb{Q}_\ell)) \to \mathrm{GSp}(H^1_{\text{ét}}(A^\alpha_{\bar{K}}, \mathbb{Q}_\ell)), g \mapsto g^{m/r}$$

and hence $W_r(\mathbb{Q}_\ell)$, being the preimage of a measure zero set under the above map, is of measure zero. □

8.2. The Mumford–Tate conjecture and the proofs of Theorem 3 and Theorem 4.

Conjecture 8.2.1 (Mumford–Tate)**.** *For any rational prime ℓ, we have $G_\ell^\circ(A) = G_{\mathrm{MT}}(A) \otimes \mathbb{Q}_\ell$ via the comparison isomorphism between the étale and the Betti cohomologies.*

Lemma 8.2.2. *If Conjecture 8.2.1 holds for the abelian variety A, then the reductive groups $G_\ell(A)$, $G^L(A)$, $G_{\mathrm{dR}}(A)$, and $G_{\mathrm{MT}}(A)$ have the same rank.*

Proof. Conjecture 8.2.1 implies that G_ℓ and G_{MT} have the same rank. Then by Theorem 8.1.5, there are infinitely many finite places v such that the Frobenius torus T_v is a maximal torus of G_{MT}. Since T_v is a subtorus of G^L except for finitely many v, we have that G^L and hence G_{dR} have the same rank as G_{MT} by Corollary 7.2.5. ☐

The assertion of the above lemma is equivalent to the Mumford–Tate conjecture by the following lemma due to Zarhin and the Faltings isogeny theorem (see for example [**Vas08**, Sec. 1.1]).

Lemma 8.2.3 ([**Zar92**, Sec. 5, key lemma])**.** *Let V be a vector space over a field of characteristic zero and $H \subset G \subset \mathrm{GL}(V)$ be connected reductive groups. Assume that H and G have the same rank and the same centralizer in* $\mathrm{End}(V)$*. Then $H = G$.*

Using this lemma, we prove a special case of Conjecture 7.1.4.

Theorem 8.2.4. *Assume that the polarized abelian variety A is defined over $\mathbb{Q}$ and that $G_\ell(A)$ is connected. Then the centralizer of G°_{dR} in $\mathrm{End}(H^1_{\mathrm{dR}}(A,\mathbb{Q}))$ is $\mathrm{End}^\circ(A)$ and moreover, Conjecture 8.2.1 implies Conjecture 7.1.4.*

Proof. The assumption is equivalent to that $K^{\text{ét}} = \mathbb{Q}$. Then by Theorem 8.1.5, we see that T_p is connected for a density one set of rational primes p. Therefore, for such p, the Frobenius $\varphi_p \in T_p(\mathbb{Q}_p) \subset G^\circ_{\mathrm{dR}}(\mathbb{Q}_p)$ and any s lying in the centralizer of G°_{dR} is fixed by φ_p. In other words, s is a 1-de Rham–Tate cycle and by Theorem 7.3.7, $s \in \mathrm{End}^\circ(A)$. The second assertion follows directly from Lemma 8.2.2 and Lemma 8.2.3. ☐

As in [**Pin98**], we show that if the conjecture does not hold, then G_{dR} is of a very restricted form when we assume that A is defined over $\mathbb{Q}$ and that $K^{\text{ét}} = \mathbb{Q}$. We need the following definition to state our result.

Definition 8.2.5 ([**Pin98**, Def. 4.1])**.** A *strong Mumford–Tate pair* (of weight $\{0,1\}$) over K is a pair (G,ρ) of a reductive algebraic group over K and a finite dimensional faithful algebraic representation of G over K such that there exists a cocharacter $\mu : \mathbb{G}_{m,\bar{K}} \to G_{\bar{K}}$ satisfying:

(1) the weights of $\rho \circ \mu$ are in $\{0,1\}$, and

(2) $G_{\bar{K}}$ is generated by $G(\bar{K}) \rtimes \mathrm{Gal}(\bar{K}/K)$-conjugates of μ.

We refer the reader to [**Pin98**, Sec. 4, especially Table 4.6, Prop. 4.7] for the list of strong Mumford–Tate pairs.

Theorem 8.2.6. *If the polarized abelian variety A is defined over $\mathbb{Q}$ and $G_\ell(A)$ is connected, then there exists a normal subgroup G of G°_{dR} defined over $\mathbb{Q}$ such that*

(1) (G, ρ) is a strong Mumford–Tate pair over $\mathbb{Q}$, where ρ is the tautological representation $\rho : G \subset G_{\mathrm{dR}} \to \mathrm{GL}(H^1_{\mathrm{dR}}(A, \mathbb{Q}))$, and

(2) The centralizer of G in $\mathrm{End}(H^1_{\mathrm{dR}}(A, \mathbb{Q}))$ is $\mathrm{End}^\circ(A)$.

If we further assume that $\mathrm{End}(A)$ is commutative, then we can take G to be G°_{dR}.

The following lemma constructs the cocharacter μ.

Lemma 8.2.7. *There exists a cocharacter $\mu : \mathbb{G}_{m,\bar{K}} \to G^L_{\bar{K}}$ such that its induced filtration on $H^1_{\mathrm{dR}}(A, \bar{K})$ is the Hodge filtration. Moreover, different choices of such cocharacters are conjugate by an element of $G^L_{\bar{K}}(\bar{K})$ and for any embedding $\sigma : \bar{K} \to \mathbb{C}$, we have that μ, as a cocharacter of $G^L_{\bar{K}} \otimes_\sigma \mathbb{C}$, is conjugate to the Hodge cocharacter μ_σ.*

Proof. By lemma 7.4.4 (2) and [**Kis10**, Lem. 1.1.1], the subgroup P of $G^L_{\bar{K}}$ preserving the Hodge filtration is parabolic and the subgroup U of $G^L_{\bar{K}}$ acting trivially on the graded pieces of $H^1_{\mathrm{dR}}(A, \bar{K})$ is the unipotent radical of P. Moreover, the action of P on the graded pieces is induced by a cocharacter of P/U. Then given a Levi subgroup of P, one can construct a cocharacter of $G^L_{\bar{K}}$ inducing the desired filtration and vice virsa. Therefore, the assertions follow from the existence of a Levi subgroup over $\bar{K}$ and the fact that two Levi subgroups are conjugate. □

Proof of Theorem 8.2.6. Let μ be some cocharacter constructed in Lemma 8.2.7 and G be the smallest normal $\mathbb{Q}$-subgroup of G°_{dR} such that $G(\bar{\mathbb{Q}})$ contains the image of μ. Notice that different choices of μ are conjugate to each other over $\bar{\mathbb{Q}}$ and hence the definition of G is independent of the choice of μ.

The weights of $\rho \circ \mu$ are 0 or 1 since the non-zero graded pieces of the Hodge filtration on $H^1_{\mathrm{dR}}(A, \bar{K})$ are at 0 and 1. Since the subgroup of G generated by $G^\circ_{\mathrm{dR}}(\bar{\mathbb{Q}}) \rtimes \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-conjugates of μ must be defined over $\mathbb{Q}$ and normal in G°_{dR}, this subgroup coincides with G. Since G°_{dR}

is connected and reductive and the image of μ is contained in G, the set of $G^\circ_{\mathrm{dR}}(\bar{\mathbb{Q}})$-conjugates of μ is the same as the set of $G(\bar{\mathbb{Q}})$-conjugates of μ. Hence (G,ρ) is a strong Mumford–Tate pair over $\mathbb{Q}$.

To show (2), by Theorem 7.3.7, it suffices to show that $\varphi_p \in G(\mathbb{Q}_p)$ for p in a set of natural density 1. By Theorem 8.1.5, it suffices to show that for any $p \in M_{max}$, there exists an integer n_p[7] such that $\varphi_p^{n_p} \in G(\mathbb{Q}_p)$. Let $W \subset (H^1_{\mathrm{dR}}(A,\mathbb{Q}))^{m,n}$ be the largest $\mathbb{Q}$-sub vector space with trivial G-action. Since G is normal in G°_{dR}, the group G°_{dR} acts on W. Then for all $p \in M_{max}$, we have $\varphi_p \in G^\circ_{\mathrm{dR}}(\mathbb{Q}_p)$ acts on $W \otimes \mathbb{Q}_p$. Since G is reductive, it can be defined to be the subgroup of $\mathrm{GL}(H^1_{\mathrm{dR}}(A,\mathbb{Q}))$ acting trivially on finitely many such W. Since $\varphi_v \in G^\circ_{\mathrm{dR}}(\mathbb{Q}_p)$ is semi-simple, in order to show that $\varphi_p^{n_p} \in G(\mathbb{Q}_p)$, it suffices to prove that the eigenvalues of φ_p acting on $W \otimes \mathbb{Q}_p$ are all roots of unity.

Since $W \subset (H^1_{\mathrm{dR}}(A,\mathbb{Q}))^{m,n}$, the eigenvalues of φ_p are all algebraic numbers. Since $Frob_p$ acts on $(H^1_{\text{ét}}(A_{\bar{\mathbb{Q}}},\mathbb{Q}_\ell))^{m,n}$ with all eigenvalues being ℓ-adic units for $\ell \neq p$, the eigenvalues of φ_p are also ℓ-adic units. Now we show that these eigenvalues are p-adic units. Let H_p be the Tannakian fundamental group of the abelian tensor category generated by sub weakly admissible filtered φ-modules of $(H^1_{\mathrm{cris}}(A/W(\mathbb{F}_p)) \otimes \mathbb{Q}_p)^{m,n}$. For $p \in M_{max}$, by [**Pin98**, Prop. 3.13], H_p is connected. By Lemma 7.1.3, every de Rham–Tate cycle generates a trivial filtered φ-module. Then by the definition of H_p, $H_p(\mathbb{Q}_p) \subset G_{\mathrm{dR}}(\mathbb{Q}_p)$ and thus $H_p(\mathbb{Q}_p) \subset G^\circ_{\mathrm{dR}}(\mathbb{Q}_p)$. Hence $W \otimes \mathbb{Q}_p$ is an H_p-representation and then by the Tannakian equivalence, the filtered φ-module $W \otimes \mathbb{Q}_p$ is weakly admissible. By defintion, μ acts on $W \otimes \mathbb{Q}_p$ trivially and hence by Lemma 8.2.7, the filtration on $W \otimes \mathbb{Q}_p$ is trivial. Then the Newton cocharacter is also trivial. In other words, the eigenvalues of φ_p are p-adic units. By the Weil conjecture, the archimedean norms of the eigenvalues are $p^{(m-n)/2}$. Then by the product formula, the weight $\frac{m-n}{2}$ must be zero and all the eigenvalues are roots of unity.

We now prove the last assertion. If $G \neq G^\circ_{\mathrm{dR}}$, then we have $G^\circ_{\mathrm{dR}} = GH$ where H is some nontrivial normal connected subgroup of G°_{dR} commuting with G and $H \cap G$ is finite.[8] Then H is contained in the centralizer of G and by (2), we have $H \subset \mathrm{End}^\circ(A)$. By the assumption on $\mathrm{End}^\circ(A)$, we see that H is commutative and hence $H \subset Z^\circ(G^\circ_{\mathrm{dR}})$. We draw a contradiction by

[7]By standard arguments, one can choose an n independent of p.

[8]To see this, notice that as a connected reductive group, G°_{dR} is the product of central torus and its derived subgroup and the intersection of these two subgroups is finite. Then one uses the decomposition results for tori and semi-simple groups.

showing that $Z^\circ(G^\circ_{\mathrm{dR}}) \subset G$. By Theorem 8.2.4, we have

$$Z(G^\circ_{\mathrm{dR}}) = G^\circ_{\mathrm{dR}} \cap \mathrm{End}^\circ(A) \subset G_{\mathrm{MT}} \cap \mathrm{End}^\circ(A) = Z(G_{\mathrm{MT}}), \text{ and hence } Z^\circ(G^\circ_{\mathrm{dR}}) \subset Z^\circ(G_{\mathrm{MT}}).$$

On the other hand, for all $p \in M_{max}$, the torus $T_p \subset G$. Hence we only need to show that $Z^\circ(G_{\mathrm{MT}}) \subset T_p$. Since this statement is equivalent up to conjugation, we only need to show that $Z^\circ(G_{\mathrm{MT}} \otimes \mathbb{Q}_\ell) \subset T_p \subset \mathrm{GL}(H^1_{\text{ét}}(A_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell))$. Since T_p is a maximal torus, we have $T_p \supset Z^\circ(G_\ell)$. We then conclude by [**Vas08**, Thm. 1.2.1] asserting that $Z^\circ(G_{\mathrm{MT}} \otimes \mathbb{Q}_\ell) = Z^\circ(G_\ell)$. □

8.3. A result of Noot and its consequence. It is well-known that the Mumford–Tate group is a torus if and only if A has complex multiplication. In particular, G°_{dR} is a torus when A has complex multiplication. In this subsection, we will show that the converse is also true.

Lemma 8.3.1. *If G^L commutes with μ_σ for some σ, then A has ordinary reduction at a positive density of primes of degree one (that is, splitting completely over $\mathbb{Q}$).*

Proof. After replacing K by a finite extension, we may assume that μ in lemma 8.2.7 is defined over K. Let v be a finite place of K with residue characteristic p and assume that p splits completely in $K/\mathbb{Q}$. Then $K_v \cong \mathbb{Q}_p$. Let ν_v be the Newton (quasi-)cocharacter and fix a maximal torus $T \subset G^L$. By Lemma 7.4.4 (3), we have that ν_v factors through G^L. As in [**Pin98**, Sec. 1], we define S_μ (resp. S_{ν_v}) to be the set of $G^L(\overline{K}_v) \rtimes \mathrm{Gal}(\overline{K}_v/K_v)$-conjugates of μ (resp. ν_v) factoring through $T(\overline{K}_v)$ in $\mathrm{Hom}((\mathbb{G}_m)_{\overline{K}_v}, T_{\overline{K}_v}) \otimes_{\mathbb{Z}} \mathbb{R}$. Since all the $G^L(\overline{K}_v)$-conjugacy of μ coincide with itself and that μ is defined over K_v, we have $S_\mu = \{\mu\}$. By the weak admissibility, we have that S_{ν_v} is contained in the convex polygon generated by S_μ (see [**Pin98**, Thm. 1.3, Thm. 2.3]). Hence $S_{\nu_v} = S_\mu$. Then we conclude that A has ordinary reduction at v by [**Pin98**, Thm. 1.5] if $v \notin \Sigma$. □

Proposition 8.3.2. *If G^L commutes with μ_σ or μ_v for some σ or v, then A has complex multiplication and hence Conjecture 7.1.4 holds for A.*

Corollary 8.3.3. *The assumption of Proposition 8.3.2 is satisfied when $(G_{\mathrm{dR}})^\circ$ commutes with either μ_σ or μ_v. In particular, A has complex multiplication if and only if $(G_{\mathrm{dR}})^\circ$ is a torus.*

8.3.4. To prove Proposition 8.3.2, we need following theorem of Noot. Let $\{t_\alpha\}$ be a finite set of relative de Rham–Tate cycles over L such that G^L is the stabilizer of $\{t_\alpha\}$. Let v be a finite place of L with residue characteristic p such that $L_v \cong \mathbb{Q}_p$. We assume that $v \notin \Sigma$ and that A_L has ordinary good reduction at v. The later assumption holds for infinitely many v under the assumption of Proposition 8.3.2 by Lemma 8.3.1. Since $m_v = 1$ in our situation, we have $\varphi_v(t_\alpha) = t_\alpha$. Moreover, $t_\alpha \in \mathrm{Fil}^0(H^1_{\mathrm{dR}}(A, L))^{m,n}$ by Lemma 7.4.4 (2). Hence t_α is a 'Tate cycle' in the sense of [**Noo96**]. In the formal deformation space of A_{k_v}, Noot defined the formal locus $\mathcal{N}$ where the horizontal extensions of all t_α are still in $\mathrm{Fil}^0(H^1_{\mathrm{dR}}(A, L))^{m,n}$ (see [**Noo96**, Sec. 2] for details).

On the other hand, for any embedding $\sigma : L \to \mathbb{C}$, the relative de Rham–Tate group G^L, viewed as a subgroup of G_{MT}, is defined over $\mathbb{R}$. Hence G^L defines a sub Hermitian symmetric domain of the one defined by G_{MT}. Let $\mathcal{S}$ be the moduli space of polarized abelian varieties of dimension $\dim A$ and let $[A] \in \mathcal{S}$ be the point corresponding to A. Then the formal scheme associated to the germ of the image of this sub Hermitian symmetric domain in $\mathcal{S}_{\mathbb{C}}$ at $[A]_\sigma$ is the formal subscheme of $(\mathcal{S}_{/[A]})_{\mathbb{C}}$ defined as the formal locus where all the formal horizontal extensions of t_α remain in $\mathrm{Fil}^0(H^1_{\mathrm{dR}}(A, L))^{m,n}$. Under the assumption of Proposition 8.3.2, the sub Hermitian symmetric domain defined by G^L is zero dimensional.

Theorem 8.3.5 ([**Noo96**, Thm. 2.8])**.** [9] *The formal locus $\mathcal{N}$ is a translate of a formal torus by a torsion point. Moreover, the dimension of $\mathcal{N}$ equals to the dimension of the Hermitian symmetric space defined by G^L.*

Proof of Proposition 8.3.2. By Theorem 8.3.5 and the discussion in 8.3.4, A_{L_v} is a torsion point in the formal deformation space. By the Serre–Tate theory, a torsion point corresponds to an abelian variety with complex multiplication. Hence A has complex multiplication and the last assertion comes from Remark 7.1.5. □

9. Proof of the main theorem

In this section, we prove Theorem 6. In section 9.1, we study the irreducible sub representations of G°_{dR} in $H^1_{\mathrm{dR}}(A/K) \otimes \bar{K}$. This part is valid for most abelian varieties without assuming the Mumford–Tate conjecture. To do this, we focus on the crystalline Frobenii action. The result of Pink that G_ℓ with its tautological representation is a weak Mumford–Tate pair over $\mathbb{Q}_\ell$ provides information on étale Frobenii and hence information on crystalline Frobenii by a result of Noot (see9.1.3) relating these two. In section 9.2, we use the results in section 9.1, Theorem 7.3.7 and

[9] Ananth Shankar pointed out to me that one may use Kisin's results in [**Kis10**, Sec. 1.5] and the property of canonical lifting to prove this result.

Proposition 8.1.11 to show that under the assumptions of Theorem 6, the centralizer of G°_{dR} in $\mathrm{End}(H^1_{\mathrm{dR}}(A,\bar{K}))$ coincides with that of G_{dR} and then complete the proof of Theorem 6.

9.1. Group theoretical discussions.

9.1.1. Throughout this section, we assume that $(G_{\mathrm{MT}}(A)_{\bar{\mathbb{Q}}})^{\mathrm{der}}$ does not have any simple factor of type SO_{2k} for $k \geq 4$. This holds under the assumptions of Theorem 6. The reason for this assumption is that we will use a result of Noot on the conjugacy class of Frobenius to avoid the usage of the Mumford–Tate conjecture. It is likely that one can remove this assumption for all the results in this subsection with some extra work.

Let $\rho_{\mathrm{dR}} : G_{\mathrm{dR}} \to \mathrm{GL}(H^1_{\mathrm{dR}}(A,K))$ be the tautological algebraic representation given in Definition 7.2.1. We denote by $\rho_{\bar{K}} : G_{\mathrm{dR},\bar{K}} \to \mathrm{GL}(H^1_{\mathrm{dR}}(A,\bar{K}))$ the representation on $\bar{K}$-points. Assume that the $G_{\mathrm{dR}}(\bar{K})$-representation $\rho_{\bar{K}}$ decomposes as $\rho_{\bar{K}} = \bigoplus_{i=1}^n \rho_{\bar{K},i}$ and that each component decomposes as $\rho_{\bar{K},i}|_{G^\circ_{\mathrm{dR},\bar{K}}} = \bigoplus_{j=1}^{n_i} \rho_{\bar{K},i,j}$, where $\rho_{\bar{K},i}$'s (resp. $\rho_{\bar{K},i,j}$'s) are irreducible representations of $G_{\mathrm{dR}}(\bar{K})$ (resp. $G^\circ_{\mathrm{dR}}(\bar{K})$). We denote the vector space of $\rho_{\bar{K},i}$ (resp. $\rho_{\bar{K},i,j}$) by V_i (resp. $V_{i,j}$).

The following lemma reduces comparing the centralizers of G_{dR} and G°_{dR} to studying the irreducibility of V_i as representations of $G^\circ_{\mathrm{dR},\bar{K}}$.

Lemma 9.1.2. *If V_i and V_j are not isomorphic as $G_{\mathrm{dR},\bar{K}}$-representations, then they are not isomorphic as $G^\circ_{\mathrm{dR},\bar{K}}$-representations. In particular, if all V_i are irreducible representations of $G^\circ_{\mathrm{dR},\bar{K}}$, then G°_{dR} and G_{dR} have the same centralizer in* $\mathrm{End}(H^1_{\mathrm{dR}}(A,\bar{K}))$.

9.1.3. Before proving the above lemma, we explain how to use a result of Noot to translate problems on representations of G°_{dR} into problems on representations of G_ℓ. We fix an embedding $\bar{K} \to \bar{\mathbb{Q}}_\ell$. Since the de Rham and étale cohomologies can be viewed as fiber functors of the category of motives with absolute Hodge cycles, we have an isomorphism of representations of $(G_{\mathrm{MT}})_{\bar{\mathbb{Q}}_\ell}$:

$$H^1_{\mathrm{dR}}(A,\bar{K}) \otimes \bar{\mathbb{Q}}_\ell \simeq H^1_{\text{ét}}(A_{\bar{K}},\mathbb{Q}_\ell) \otimes \bar{\mathbb{Q}}_\ell.$$

By Theorem 7.3.7, the left hand side, as a representation of $(G_{\mathrm{MT}})_{\bar{\mathbb{Q}}_\ell}$, decomposes into irreducible ones $\oplus V_i \otimes \bar{\mathbb{Q}}_\ell$ and $V_i \otimes \bar{\mathbb{Q}}_\ell \cong V_j \otimes \bar{\mathbb{Q}}_\ell$ if and only if they are isomorphic as representations of G_{dR}. Via the above isomorphism, we denote by $V_i^{\text{ét}}$ the image of $V_i \otimes \bar{\mathbb{Q}}_\ell$. Then by Faltings isogeny theorem, $V_i^{\text{ét}}$ are irreducible representations of $(G_\ell)_{\bar{\mathbb{Q}}_\ell}$ and any two of them are isomorphic if and only if they are isomorphic as representations of $(G_{\mathrm{MT}})_{\bar{\mathbb{Q}}_\ell}$.

Now we study the action of the Frobenius torus T_v° on both sides. More precisely, we use $(T_v^\circ)_{\bar{K}_v}$ to denote the base change of the crystalline one acting on the left hand side and use $(T_v^\circ)_{\bar{\mathbb{Q}}_\ell}$ to denote the base change of the étale one acting on the right hand side. By [**Noo09**, Thm. 4.2], after raising both Frobenius actions to high enough power, $\varphi_v^{m_v}$ is conjugate to $Frob_v$ by an element of G_{MT}[10]. Then the weights of the action of T_v° on V_i and $V_i^{\text{ét}}$ coincide and that V_i is isomorphic to V_j as representations of $(T_v^\circ)_{\bar{K}_v}$ is equivalent to that $V_i^{\text{ét}}$ is isomorphic to $V_j^{\text{ét}}$ as representations of $(T_v^\circ)_{\bar{\mathbb{Q}}_\ell}$.

Proof of Lemma 9.1.2. Let $v \in M_{\max}$ as in Theorem 8.1.5. Then T_v is connected and as a subgroup of G_ℓ, it is a maximal torus. Since V_i and V_j are not isomorphic as representations of $G_{\mathrm{dR},\bar{K}}$, then by 9.1.3, their counterparts $V_i^{\text{ét}}$ and $V_j^{\text{ét}}$ are not isomorphic as representations of $(G_\ell)_{\mathbb{Q}_\ell}$. Since T_v is a maximal torus, then $V_i^{\text{ét}}$ and $V_j^{\text{ét}}$ are not isomorphic as representations of $(T_v)_{\bar{\mathbb{Q}}_\ell}$. Then by the theorem of Noot (see 9.1.3), V_i and V_j are not isomorphic as representations of $(T_v)_{\bar{K}_v}$. In particular, they are not isomorphic as representations of G_{dR}°. □

9.1.4. By construction, we have a fiber functor $\omega : \mathcal{M}_{\mathrm{dRT}} \to Vec_{K^{\mathrm{dR}}}$. In other words, ω is a fiber functor over $\operatorname{Spec} K^{\mathrm{dR}}$, viewed as a $\mathbb{Q}$-scheme. The functor $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ is representable by a $\operatorname{Spec} K^{\mathrm{dR}}/\operatorname{Spec}\mathbb{Q}$-groupoid $\mathfrak{G}$ and $\mathfrak{G}$ is faithfully flat over $\operatorname{Spec} K^{\mathrm{dR}} \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec} K^{\mathrm{dR}}$ (see [**Mil92**, Thm. A.8] or [**Del90**, Thm. 1.12])[11]. Let $v \notin \Sigma$ (defined in section 8) be a finite place of K^{dR} giving rise to an embedding $K^{\mathrm{dR}} \to K_v^{\mathrm{dR}}$. Let $\mathfrak{G}_v$ be the $\operatorname{Spec} K_v^{\mathrm{dR}}/\operatorname{Spec}\mathbb{Q}_p$-groupoid obtained by base changing $\mathfrak{G}$ to $\operatorname{Spec} K_v^{\mathrm{dR}} \times_{\operatorname{Spec}\mathbb{Q}_p} \operatorname{Spec} K_v^{\mathrm{dR}}$. Since $\varphi_v(s_\alpha) = s_\alpha$ for all de Rham–Tate cycles $\{s_\alpha\}$, the Frobenius semi-linear morphism φ_v lies in $\mathfrak{G}_v(K_v^{\mathrm{dR}})$. Since $\mathfrak{G}$ acts on G_{dR} by conjugation, the action $\mathrm{ad}(\varphi_v)$, the conjugation by φ_v, is an isomorphism between the neutral connected components

[10] Recall that we use $Frob_v$ to denote the relative Frobenius action on the étale cohomology. Noot shows that after raising to a high enough power, there exists an element $g \in G_{\mathrm{MT}}(\bar{K})$ such that g is conjugate to $\varphi_v^{m_v}$ by some element in $G_{\mathrm{MT}}(\bar{K}_v)$ and that g is conjugate to $Frob_v$ by some element in $G_{\mathrm{MT}}(\bar{\mathbb{Q}}_\ell)$.

[11] Here we use the language of groupoids. One may also view $\mathfrak{G}$ as a Galois gerb in the sense of Langlands–Rapoport for the following reason. Since $\mathfrak{G}$ is a torsor of a smooth algebraic group, it is trivial étale locally and hence $\mathfrak{G}(\bar{\mathbb{Q}}) \to (\operatorname{Spec}\bar{\mathbb{Q}} \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec}\bar{\mathbb{Q}})(\bar{\mathbb{Q}})$ is surjective. We then have the exact sequence (see for example [**Mil92**, pp. 67])

$$1 \to G_{\mathrm{dR}}(\bar{\mathbb{Q}}) \to \mathfrak{G}(\bar{\mathbb{Q}}) \to \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to 1.$$

Moreover, G_{dR}, as a group scheme over $\operatorname{Spec} K^{\mathrm{dR}}$, is the kernel of $\mathfrak{G}$. Let L be the finite extension of K^{dR} such that all the fiber functors over K^{dR} are isomorphic over L. Then the extension of $\mathrm{Gal}(\bar{L}/L)$ by $G_{\mathrm{dR}}(\bar{L})$ induced by the above sequence splits. Hence $\mathfrak{G}$ is a Galois gerb in the sense of Langlands–Rapoport.

$\sigma^* G_{dR,K_v^{\mathrm{dR}}}$ and $G_{dR,K_v^{\mathrm{dR}}}$, where $\sigma : K_v^{\mathrm{dR}} \to K_v^{\mathrm{dR}}$ is the Frobenius. In terms of K_v^{dR}-points, $\mathrm{ad}(\varphi_v)$ is a σ-linear automorphism of both $G_{\mathrm{dR}}(K_v^{\mathrm{dR}})$ and $G^\circ_{\mathrm{dR}}(K_v^{\mathrm{dR}})$[12].

Proposition 9.1.5. *Assume that $A_{\bar{K}}$ is simple. Then all $\rho_{\bar{K},i,j}$ are of the same dimension. Moreover, if we further assume the assumption in 9.1.1 and that a maximal subfield of $\mathrm{End}^\circ_{\bar{K}}(A)$ is Galois over $\mathbb{Q}$ or that $\mathrm{End}^\circ_{\bar{K}}(A)$ is a field, then there exists a choice of decomposition $\bigoplus V_i$ such that $\varphi_v(V_{i,j}) = V_{\sigma(i),\tau_{v,i}(j)}$, where σ is a permutation of $\{1,\cdots,n\}$ and for each i, $\tau_{v,i}$ is a permutation of $\{1,\cdots,n_i\}$.*

Proof. We fix a finite extension L of K such that all the $V_{i,j}$ are defined over L. Let $v \notin \Sigma$ be a place of L. As discussed in 9.1.4, $\mathrm{ad}(\varphi_v)$ preserves the set $G^\circ_{\mathrm{dR}}(L_v)$. Therefore, for any nonzero vector $v_{i,j} \in V_{i,j}$, as $\mathbb{Q}_p$-linear spaces,

$$\begin{aligned}\varphi_v(V_{i,j}) &= \varphi_v(\mathrm{Span}_{L_v}(G^\circ_{\mathrm{dR}}(L_v)(v_{i,j}))) = \mathrm{Span}_{L_v}(\varphi_v(G^\circ_{\mathrm{dR}}(L_v)(v_{i,j}))) \\ &= \mathrm{Span}_{L_v}(\mathrm{ad}(\varphi_v)(G^\circ_{\mathrm{dR}}(L_v)(\varphi_v(v_{i,j})))) = \mathrm{Span}_{L_v}(G^\circ_{\mathrm{dR}}(L_v)(\varphi_v(v_{i,j}))).\end{aligned}$$

In other words, as an L_v-vector space, $\varphi_v(V_{i,j})$ is the same as the space of the irreducible $G^\circ_{\mathrm{dR}}(L_v)$-sub representation generated by $\varphi_v(v_{i,j})$. Similarly for V_i, we have that the vector space $\varphi_v(V_i)$ is the same as the vector space of an irreducible $G_{\mathrm{dR}}(L_v)$-sub representation. In particular, $\varphi_v(V_{i,j})$ is contained in $\bigoplus_{\dim V_{k,l}=\dim V_{i,j}} V_{k,l}$. Let V' be $\bigoplus_{\dim V_{k,l}=\dim V_{i,j}} V_{k,l}$ and V'' be $\bigoplus_{\dim V_{k,l}\neq\dim V_{i,j}} V_{k,l}$. Then $\varphi_v(V') = V'$ and $\varphi_v(V'') = V''$. Let pr' be the projection to V'. Then $\varphi_v(pr') = pr'$ for all $v \notin \Sigma$. By Theorem 7.3.7, pr' is an algebraic endomorphism of A. Since A is simple, pr' cannot be a nontrivial idempotent and then $V'' = 0$, which is the first assertion.

The second assertion is an immediate consequence of the following two lemmas. Indeed, by Lemma 9.1.7, we see that the only sub representations in V_s of G°_{dR} are $V_{s,j}$'s. Since $\varphi_v(V_{i,j})$ is a sub representation of $\varphi_v(V_i) = V_s$ for some s by Lemma 9.1.6, then $\varphi_v(V_{i,j})$ is $V_{s,t}$ for some t. □

Lemma 9.1.6. *Under the assumptions in Proposition 9.1.5, there exists a decomposition $H^1_{\mathrm{dR}}(A,L) = \bigoplus V_i$ where V_i are irreducible representations of G_{dR} such that for any i, as vector spaces, $\varphi_v(V_i) = V_j$ for some j.*

[12]Although $\mathrm{ad}(\varphi_v)$ defines a σ-linear automorphism of $G_{\mathrm{dR}}(K_v^{\mathrm{dR}})$, this fact itself does not imply that $G_{dR,K_v^{\mathrm{dR}}}$ has a $\mathbb{Q}_p$-structure since a priori we do not have the cocycle condition. However, $G_{dR,\overline{K_v^{\mathrm{dR}}}}$ has a $\mathbb{Q}_p$-structure because $\mathcal{M}_{dRT} \otimes K_v^{\mathrm{dR}}$ has a fiber functor over $\mathbb{Q}_p$. The $\mathbb{Q}_p$-fiber functor can be chosen to be the étale realization because all the de Rham–Tate cycles lie in $(H^1_{\text{ét}}(A_{\bar{K}_v},\mathbb{Q}_p))^{m,n}$ via the p-adic de Rham-étale comparison.

Proof. When $\mathrm{End}^\circ(A)$ is a field, the decomposition is unique and any two different V_j's are not isomorphic. In other words, V_j's are the only irreducible sub representations of G_{dR}. Since the vector space $\varphi_v(V_i)$ is the vector space of an irreducible sub representation, it must be V_j for some j.

Now we assume that the maximal subfield of $\mathrm{End}^\circ_{\bar{K}}(A)$ is Galois over $\mathbb{Q}$. Let $\{s_\alpha\}$ be a $\mathbb{Q}$-basis of de Rham–Tate cycles in $\mathrm{End}(H^1_{\mathrm{dR}}(A, K^{\mathrm{dR}}))$. By Theorem 7.3.7, these s_α are algebraic cycles and we use s_α^B to denote their images in $\mathrm{End}(H^1_{\mathrm{B}}(A_{\mathbb{C}}, \mathbb{Q}))$. Since A is simple, $\mathrm{End}^\circ_{\bar{K}}(A)$ is a division algebra D of index d over some field F and $\{s_\alpha^B\}$ is a basis of D as a $\mathbb{Q}$-vector space. Let $E \subset D$ be a field of degree d over F. Then E is a maximal subfield of D and $D \otimes_F E \cong M_d(E)$. Therefore,

$$D \otimes_{\mathbb{Q}} E \cong D \otimes_{\mathbb{Q}} F \otimes_F E \cong D \otimes_F E \otimes_{\mathbb{Q}} F \cong M_d(E)^{[F:\mathbb{Q}]}.$$

Let $e_i \in M_d(E)$ be the projection to the i-th coordinate. Let $e_i^j \in D \otimes_{\mathbb{Q}} E$ be the element whose image in $M_d(E)^{[F:\mathbb{Q}]}$ is $(0, .., 0, e_i, 0, ..., 0)$, where e_i is on the j-th component. Since $\sum e_i^j$ is the identity element in D, there must exist at least one e_i^j such that $\sum_{\tau \in \mathrm{Gal}(E/\mathbb{Q})} \sigma(e_i^j)$ is nonzero, where the Galois group acts on the coordinates when the basis of the E-vector space $D \otimes_{\mathbb{Q}} E$ is chosen to be a basis of D as a $\mathbb{Q}$-vector space.

We write $e_i^j = \sum k_\alpha s_\alpha^B$, where $k_\alpha \in E$, and let $pr_\tau = \sum \tau(k_\alpha) s_\alpha \in \mathrm{End}(H^1_{\mathrm{dR}}(A, \bar{K}))$, for all $\tau \in \mathrm{Gal}(E/\mathbb{Q})$. Since e_i^j is an idempotent, so is pr_τ. Let V_τ be the image of pr_τ. We may assume that L contains E and still use σ to denote the image of the Frobenius via the map $\mathrm{Gal}(L_v/\mathbb{Q}_p) \subset \mathrm{Gal}(L/\mathbb{Q}) \to \mathrm{Gal}(E/\mathbb{Q})$. Then by definition, as vector spaces, $\varphi_v(V_\tau) = V_{\sigma\tau}$.

Now it remains to prove that $\sum V_\tau$ is a direct sum and $\bigoplus V_\tau = H^1_{\mathrm{dR}}(A, \bar{K})$ as representations of G_{dR}. First, since pr_τ lies in the centralizer of G_{dR}, every V_τ is a subrepresentation. Second, since the number of irreducible representations in a decomposition of $H^1_{\mathrm{dR}}(A, \bar{K})$ equals $[E : \mathbb{Q}]$, it suffices to prove that $\sum V_\tau = H^1_{\mathrm{dR}}(A, \bar{K})$. Since the image of $\sum pr_\tau$ is contained in $\sum V_\tau$, it suffices to prove that $\sum pr_\tau$ is invertible. By construction, $\sum pr_\tau$ lies in D (via comparison) and it is nonzero by the choice of e_i^j. Therefore, $\sum pr_\tau$ is invertible since D is a division algebra. $\square$

Lemma 9.1.7. *Under the assumption in 9.1.1, the G°_{dR}-representations $V_{i,j}$ and $V_{i,j'}$ are not isomorphic if $j \neq j'$.*

Proof. Let T_w be a Frobenius torus of maximal rank for some finite place $w \notin \Sigma$. We only need to show that the weights of T_w acting on V_i are all different. By 9.1.3, we may consider T_w as a maximal torus of $G_\ell(A)$ acting on irreducible sub representations of $(G_\ell)_{\bar{\mathbb{Q}}_\ell}$ in $H^1_{\text{ét}}(A_{\bar{K}}, \mathbb{Q}_\ell) \otimes \bar{\mathbb{Q}}_\ell$. [**Pin98**, Thm. 5.10] shows that $(G_\ell(A), \rho_\ell)$ is a weak Mumford–Tate pair over $\overline{\mathbb{Q}}_\ell$. To show the weights on V_i are different, it suffices to show that the weights of the maximal torus of each geometrical irreducible component of $(G_\ell(A), \rho_\ell)$ are different. Furthermore, it reduces to the case of an almost simple component of each irreducible component. They are still weak Mumford–Tate pairs by [**Pin98**, 4.1]. One checks the list of simple weak Mumford–Tate pairs in [**Pin98**, Table 4.2] to see that all the weights are different. □

9.2. Proof of Theorem 6.

9.2.1. Since the Mumford–Tate conjecture holds for all the abelian varieties considered in Theorem 6[13], we focus on comparing the centralizers of G_{dR} and G°_{dR} in $\text{End}(H^1_{\text{dR}}(A, \bar{K}))$. Once we prove that the centralizers of both groups are the same, we conclude the proof of Theorem 6 by Theorem 7.3.7, Lemma 8.2.2, and Lemma 8.2.3 as in the proof of Theorem 8.2.4. We separate the cases using Albert's classification.

Type I. Let F be the totally real field $\text{End}^\circ_{\bar{K}}(A)$ of degree e over $\mathbb{Q}$. [**BGK06**] shows that Conjecture 8.2.1 holds when g/e is odd.

Proposition 9.2.2. *If $e = g$, then Conjecture 7.1.4 holds for A.*

Proof. The $\mathbb{Q}$-vector space $H^1_{\text{B}}(A, \mathbb{Q})$ has the structure of a two-dimensional F-vector space. Therefore, as a $(G_{\text{MT}})_{\bar{\mathbb{Q}}}$-representation, $H^1_{\text{B}}(A, \overline{\mathbb{Q}})$ decomposes into g non-isomorphic irreducible sub representations of dimension two. By 9.1.3, the $G_{\text{dR},\bar{K}}$-representation $H^1_{\text{dR}}(A, \bar{K})$ decomposes into g non-isomorphic irreducible sub representations $V_1, \dots, V_g$. By Lemma 9.1.2 and 9.2.1, we only need to show that all V_i are irreducible G°_{dR}-representations. By Proposition 9.1.5, if any G°_{dR}-representation V_i is reducible, then all $V_1, \dots, V_g$ are reducible. In such situation, all V_i decompose into one-dimensional representations and hence G°_{dR} is a torus. Then by Corollary 8.3.3, A has complex multiplication, which contradicts our assumption. □

[13] The Mumford–Tate conjecture for A_i is well-studied and we will cite the results we need for each case later in this subsection. See for example [**CF16**] for a survey of type I and II cases. The reduction of the conjecture for the product of A_i to the simple case is essentially contained in [**Lom15**, Sec. 4] and we record a proof at the end of this subsection.

Remark 9.2.3. The above proof is still valid if all (equivalently, any) V_i are of prime dimension.

Now we focus on the case when $\mathrm{End}_{\bar{K}}(A) = \mathbb{Z}$. We refer the reader to [**Pin98**] for the study of Conjecture 8.2.1 in this case. In particular, Conjecture 8.2.1 holds when $2g$ is not of the form a^{2b+1} or $\binom{4b+2}{2b+1}$, where $a, b \in \mathbb{N}\backslash\{0\}$ and in this situation, $G_\ell(A) = \mathrm{GSp}_{2g,\mathbb{Q}_\ell}$.

Proposition 9.2.4. *Assume that $G_\ell(A) = \mathrm{GSp}_{2g,\mathbb{Q}_\ell}$. If A is defined over a number field K which is Galois over $\mathbb{Q}$ of degree d prime to $g!$, then Conjecture 7.1.4 holds for A.*

Proof. Conjecture 8.2.1 holds when $G_\ell(A) = \mathrm{GSp}_{2g}$. It suffices to show that $H^1_{\mathrm{dR}}(A, \bar{K})$ is an irreducible $G^\circ_{\mathrm{dR},\bar{K}}$-representation. If not, then by Proposition 9.1.5, $H^1_{\mathrm{dR}}(A, \bar{K})$ would decompose into r sub representations of dimension $2g/r$. By Corollary 8.3.3, r cannot be $2g$ and hence $r \leq g$. Let pr^j be the projection to the jth irreducible component. By Proposition 9.1.5, we have $\varphi_v(pr^j) = pr^k$ for some k and the action of φ_v on all pr^j gives rise to an element s_v in S_r, the permutation group on r elements. On the other hand, by Proposition 8.1.11, there exists a set M of rational primes of natural density 1 such that for any $p \in M$ and any $v|p$, we have that $\varphi_v^{m_v} \in G^\circ_{\mathrm{dR}}$. Hence $s_v^{m_v}$ is the identity in S_r for such v. By the assumption, m_v is prime to $r!$ and hence s_v is trivial in S_r. In other words, pr^j is a 1-de Rham–Tate cycle. Then by Theorem 7.3.7, pr^j is algebraic, which contradicts with that A is simple. $\square$

Type II and III. In this case, $\mathrm{End}^\circ_{\bar{K}}(A)$ is a quaternion algebra D over a totally real field F of degree e over $\mathbb{Q}$. [**BGK06**] and [**BGK10**] show that if $g/(2e)$ is odd, then Conjecture 8.2.1 holds.

Proposition 9.2.5. *If $g = 2e$, then Conjecture 7.1.4 holds for A.*

Proof. The $G_{\mathrm{dR},\bar{K}}$-representation $H^1_{\mathrm{dR}}(A, \bar{K})$ decomposes into $V_1 \oplus \cdots \oplus V_g$ where V_i is two dimensional and V_i is not isomorphic to V_j unless $\{i, j\} = \{2k-1, 2k\}$. Then we conclude by Remark 9.2.3. $\square$

Type IV. In this case, $\mathrm{End}^\circ_{\bar{K}}(A)$ is a division algebra D over a CM field F. Let $[D : F] = d^2$ and $[F : \mathbb{Q}] = e$. Then $ed^2|2g$.

Proposition 9.2.6. *If $\frac{2g}{ed}$ is a prime, then the centralizer of G°_{dR} in $\mathrm{End}_{\bar{K}}(H^1_{\mathrm{dR}}(A, \bar{K}))$ is the same as that of G_{dR}.*

Proof. We view $H^1_{\mathrm{B}}(A,\mathbb{Q})$ as an F-vector space and hence view G_{MT} as a subgroup of $\mathrm{GL}_{2g/e}$. Since the centralizer of G_{MT} is D, then $H^1_{\mathrm{B}}(A,\mathbb{Q})\otimes_F \bar{F}$ decomposes into d representations of dimension $2g/(ed)$. Hence $H^1_{\mathrm{B}}(A,\bar{\mathbb{Q}})$ as a G_{MT}-representation would decompose into de representations of dimension $2g/(de)$. Then we conclude by Remark 9.2.3. □

Corollary 9.2.7. *If g is a prime, Conjecture 7.1.4 holds for A of type IV.*

Proof. Notice that when g is a prime, then d must be 1 and e must be 2 or $2g$. The second case is when A has complex multiplication and Conjecture 7.1.4 is known. In the first case, $\frac{2g}{ed}(=g)$ is a prime. Then Conjecture 7.1.4 is a consequence of Proposition 9.2.6 and the Mumford–Tate conjecture ([**Chi91**, Thm. 3.1]) by Lemma 8.2.3. □

Corollary 9.2.8. *If the dimension of A is a prime and $\mathrm{End}_{\bar{K}}(A)$ is not $\mathbb{Z}$, then Conjecture 7.1.4 holds.*

Proof. If $g=2$, then A has CM or is of type I with $e=g$ or is type II with $g=2e$. If g is an odd prime, then A is of type I with $e=g$ or of type IV. We conclude by Proposition 9.2.2, Proposition 9.2.5, and Corollary 9.2.7. □

Proof of Theorem 6. Conjecture 7.1.4 is equivalent for isogenous abelian varieties and then we may assume that $A=\prod A_i^{n_i}$. By 9.2.1 and the following lemma, it suffices to show that $G^{\circ}_{\mathrm{dR}}(A)$ and $G_{\mathrm{dR}}(A)$ have the same centralizer. By Lemma 9.1.2, the agreement of the centralizers of $G^{\circ}_{\mathrm{dR}}(A)$ and $G_{\mathrm{dR}}(A)$ is equivalent to that all irreducible sub representations in $H^1_{\mathrm{dR}}(A,\bar{K})$ of $G_{\mathrm{dR}}(A)_{\bar{K}}$ are irreducible representations of $G^{\circ}_{\mathrm{dR}}(A)_{\bar{K}}$. Let V be an irreducible representation of $G_{\mathrm{dR}}(A)_{\bar{K}}$. Since the projection $A\to A_i$ is a de Rham–Tate cycle, then there exists A_i such that $V\subset H^1_{\mathrm{dR}}(A_i,\bar{K})$ and V is an irreducible representation of $G_{\mathrm{dR}}(A_i)_{\bar{K}}$. Then by Remark 7.1.5 (1), Corollary 9.2.8, and Proposition 9.2.4, the de Rham–Tate group $G_{\mathrm{dR}}(A_i)$ is connected. Then the surjective map $G_{\mathrm{dR}}(A)\to G_{\mathrm{dR}}(A_i)$ remains surjective when restricted to $G^{\circ}_{\mathrm{dR}}(A)$. This implies that V is an irreducible representation of $G^{\circ}_{\mathrm{dR}}(A)_{\bar{K}}$. □

Lemma 9.2.9. *Let A be as in Theorem 6. Then the Mumford–Tate conjecture holds for A.*

Proof. The idea of the proof is the same as that of [**Lom15**, Thm. 4.7]. For the simplicity of statements, we assume that each simple factor of all abelian varieties mentioned in the proof falls into one of the three cases in the assumption of the theorem. Notice that an absolutely simple abelian variety A_i of type IV either have complex multiplication or is of case (2) with $\mathrm{End}^\circ_{\bar{K}}(A_i)$ being an imaginary quadratic field. Therefore, by assumption, A is either $B \times C_1$ or $B \times C_2^k$ where B has no simple factor of type IV, C_1 has complex multiplication, and C_2 is absolutely simple of case (2) type IV. By [**Lom15**, Prop. 2.8] and corresponding statement for the Mumford–Tate group, it suffices to show that the Mumford–Tate conjecture holds for $B \times C_i$.

We first prove that the Mumford–Tate conjecture holds for B. Let $H_\ell(B)$ be the neutral connected component of the subgroup of $G_\ell(B)$ with determinant 1. Since B does not have simple factor of type IV, the group $H_\ell(B)$ is semisimple. By assumption, the Lie algebra of each simple component of $H_\ell(B)_{\bar{\mathbb{Q}}_\ell}$ is of type C and then by [**Lom15**, Thm. 4.1, Rem. 4.3], the group $H_\ell(B) = \prod H_\ell(B_i)$, where $\{B_i\}$ is a set of all non-isogeny simple factors of B. Since the Mumford–Tate conjecture holds for B_i, then the conjecture holds for B by [**Lom15**, Lem. 3.6].

On the other hand, $H_\ell(C_1)$ is a torus and the Lie algebras of simple factors $H_\ell(C_2)_{\bar{\mathbb{Q}}_\ell}$ are of type A_{p-1} for $p \geq 3$ prime[14]. Since Lie algebra of type A_{p-1} is not isomorphic to that of type C, we apply [**Lom15**, Prop. 3.9, Lem. 3.6] to conclude that the Mumford–Tate conjecture holds for $B \times C_i$. $\square$

[14]Notice that there does not exist a non-CM abelian surface of type IV.

CHAPTER 5

A relative version of Bost's theorem

In section 10, we prove Theorem 8. Roughly speaking, the main idea is to prove that the relative de Rham–Tate group G^L (see section 7.4) coincides with the de Rham–Tate group G_{dR} when L is large enough. In section 11, we prove a result (Corollary 11.1.2) towards Conjecture 7. Given a relative de Rham–Tate cycle s in $\mathrm{End}(H^1_{\mathrm{dR}}(A,L))$, Conjecture 7 predicts that if s is a β-de Rham–Tate cycles (Definition 7.3.5) in $\mathrm{End}(H^1_{\mathrm{dR}}(A,L))$ with $\beta > \frac{1}{2}$, then s is algebraic. We prove that even without the assumption of being a relative de Rham–Tate cycle, as long as $\beta > \frac{3}{4}$, a β-de Rham–Tate cycle in $\mathrm{End}(H^1_{\mathrm{dR}}(A,L))$ is algebraic.

10. The known cases

We observe that for L'/L finite extension, if $L \supset K^{\mathrm{dR}}$, then Conjecture 7 for L' implies the conjecture for L. Hence throughout this section, we may assume that L is large enough so that all but finitely many T_v are connected. Proposition 8.1.6 shows that such L exists and Corollary 8.1.7 shows that G^L is connected. For any finite place v of L, we write $m_v = [L_v : \mathbb{Q}_p]$.

10.1. Relative de Rham–Tate cycles revisited. The cycles considered in Conjecture 7 motivate the following definition. It is weaker than the notion of relative de Rham–Tate cycles as we put no restrictions on archimedean places.

Definition 10.1.1. An element $t \in (H^1_{\mathrm{dR}}(A,L))^{m,n}$ is called a *weakly relative de Rham–Tate cycle* (over L) of A if there exists a finite set Σ of finite places of L such that for every finite place $v \notin \Sigma$, one has $\varphi_v^{m_v}(t) = t$. When there is no risk of confusion, we simply call t a weakly relative cycle.

Conjecture 7 asserts that any weakly relative cycles in $\mathrm{End}(H^1_{\mathrm{dR}}(A,L))$ is an L-linear combination of algebraic cycles. This conjecture can be verified when $K^{\text{ét}} = \mathbb{Q}$ as in the proof of Theorem 8.2.4.

Theorem 10.1.2. *If the polarized abelian variety A is defined over $\mathbb{Q}$ and $G_\ell(A)$ is connected, then Conjecture 7 holds.*

Proof. We may assume that $L/\mathbb{Q}$ is Galois. Let $t \in \mathrm{End}(H^1_{\mathrm{dR}}(A, L))$ be a weakly relative cycle. By definition, for any $\gamma \in \mathrm{Gal}(L/\mathbb{Q})$, the cycle $\gamma(t)$ is also weakly relative. Consider the vector space $V_L = \mathrm{Span}_L(\{\gamma(t)\}_{\gamma\in\mathrm{Gal}(L/\mathbb{Q})})$. There is a vector space $V \subset \mathrm{End}(H^1_{\mathrm{dR}}(A, \mathbb{Q}))$ such that $V_L = V \otimes L$. Let $\{s_\alpha\}$ be a $\mathbb{Q}$-basis of V. By definition, for $v \notin \Sigma$, the relative Frobenius $\varphi_v^{m_v}$ fixes V_L and hence s_α. In other words, for all but finitely primes p, the Frobenius torus T_p° fixes s_α. By Theorem 8.1.5, for $p \in M_{max}$ of natural density one, $\varphi_p \in T_p = T_p^\circ$ fixes s_α. Then by Theorem 7.3.7, s_α is algebraic and hence t is a linear combination of algebraic cycles. □

Remark 10.1.3. If one replaces Theorem 8.1.5 by our refinements of Serre's theorem in section 8.1, the proof of the above theorem shows that, under the assumption of Proposition 8.1.11 or Proposition 8.1.13, one can reduce the verification of Conjecture 7 for any L to the following statement: if $t \in \mathrm{End}(H^1_{\mathrm{dR}}(A, K))$ is fixed by $\varphi_v^{m_v}$ for all v with residue characteristic in a subset of rational primes of natural density one, then t is a linear combination of algebraic cycles. Here we assume that $K/\mathbb{Q}$ is Galois and that $G_\ell(A)$ is connected.

For relative de Rham–Tate cycles, we have:

Proposition 10.1.4. *If the polarized abelian variety A is defined over $\mathbb{Q}$ and $G_\ell(A) \cong G_{\mathrm{MT}}(A)\otimes\mathbb{Q}_\ell$, then all relative de Rham–Tate cycles are linear combinations of Hodge cycles.*

Proof. As in the proof of Theorem 8.2.4, one can show that $\varphi_p \in G^L(\mathbb{Q}_p)$ for a density one set of rational primes and hence G^L has the same centralizer as G_{MT} and then the rest of the proof is the same. □

Proposition 10.1.5. *If the polarized abelian variety A is defined over $\mathbb{Q}$ and $G_\ell(A)$ is connected, then Theorem 8.2.6 still holds with G°_{dR} replaced by G^L.*

Proof. The key steps of the proof of Theorem 8.2.6 are as follows. By Lemma 8.2.7, we construct the smallest normal $\mathbb{Q}$-subgroup G of G°_{dR} containing μ. Then one uses weak admissibility and the Riemann Hypothesis part of the Weil conjecture to show that some power of φ_p lies in G for all p. Finally by Theorem 8.1.5 and Theorem 7.3.7, G has the same centralizer as G_{MT} and one uses a result of Vasiu to see that G is G°_{dR} when the endomorphism ring of A is commutative. Every step is still valid with G°_{dR} replaced by G^L. □

10.2. Special cases of Conjecture 7. In this subsection, we prove that for A as in Theorem 8, all the weakly relative cycles are linear combinations of Hodge cycles. This assertion implies the theorem.

Theorem 10.2.1. *Let A be an abelian variety and $L \supset K^{\mathrm{dR}} = K^{\acute{e}t}$. All weakly relative cycles over L of A are L-linear combinations of Hodge cycles if one of the following holds*

(1) A has complex multiplication;

(2) A is either an elliptic curve or abelian surface with quaternion multiplication.

Proof. For (1), we fix a finite set Σ of finite places of L such that for all $v \notin \Sigma$, the Frobenius $\varphi_v^{m_v}$ fixes all weakly relative cycles over L. By Theorem 8.1.5 and the fact that $G_\ell^\circ = G_{\mathrm{MT}}$ is a torus, there exists a finite place $v \notin \Sigma$ with $m_v = 1$ such that $T_v = G_{\mathrm{MT}} \otimes L_v$. By definition, a weakly relative cycle is fixed by φ_v and hence G_{MT}. In other words, this cycle is a linear combination of Hodge cycles.

For (2), we may assume that A does not have complex multiplication. Let G be the subgroup of $\mathrm{GL}(H^1_{\mathrm{dR}}(A, L))$ that fixes all weakly relative cycles and we shall prove that $G = G_{\mathrm{MT}} \otimes L$. Since $G_\ell^\circ = G_{\mathrm{MT}} \otimes \mathbb{Q}_\ell$, then by Theorem 8.1.5, G contains a maximal torus of $G_{\mathrm{MT}} \otimes L$. Since $G_{\mathrm{MT}} \otimes \bar{\mathbb{Q}} = \mathrm{GL}_2$, if $G \neq G_{\mathrm{MT}} \otimes L$, then G is either a maximal torus or contains a Borel subgroup. If G contains a Borel subgroup, then the set of cycles fixed by G coincides the set of cycles fixed by $G_{\mathrm{MT}} \otimes L$. In other words, the set of weakly relative cycles agrees with the set of L-linear combination of Hodge cycles and $G = G_{\mathrm{MT}} \otimes L$. If G is a torus, then $G^L \subset G$ is a torus. We draw a contradiction by Corollary 8.3.3. □

11. A strengthening of Theorem 7.3.7 and its application

We use $A^\vee, E(A)$ to denote the dual abelian variety and the universal vector extension of A. For simplicity, when there is no risk of confusion, we use β-cycles to indicate β-de Rham–Tate cycles. The main result of this section is Corollary 11.1.2 and the idea is to apply Theorem 2.2.5. At the end of this section, we use Corollary 11.1.2 to study cycles of abelian surfaces.

11.1. A strengthening of Theorem 7.3.7. If $s \in \mathrm{End}(H^1_{\mathrm{dR}}(A, \bar{K}))$ is a β-cycles for some positive β, then $\varphi_v(s) = s$ for infinitely many v and thus $s \in \mathrm{Fil}^0(\mathrm{End}(H^1_{\mathrm{dR}}(A, \bar{K})))$ by Lemma 7.1.3.

In other words, $s(\mathrm{Fil}^1(H^1_{\mathrm{dR}}(A,\bar{K})))$ is contained in $\mathrm{Fil}^1(H^1_{\mathrm{dR}}(A,\bar{K}))$. Since

$$H^1_{\mathrm{dR}}(A,\bar{K})/\,\mathrm{Fil}^1(H^1_{\mathrm{dR}}(A,\bar{K})) \cong \mathrm{Lie}\, A^\vee_{\bar{K}},$$

the cycle s then induces an endomorphism $\bar{s}$ of $\mathrm{Lie}\, A^\vee_{\bar{K}}$.

Theorem 11.1.1. *Assume that $A_{\bar{K}}$ is simple. If $s \in \mathrm{End}(H^1_{\mathrm{dR}}(A,\bar{K}))$ is a β-de Rham–Tate cycle for some $\beta > \frac{3}{4}$, then $\bar{s}$ is the image of some element in $\mathrm{End}^\circ_{\bar{K}}(A^\vee)$.*

Before proving the theorem, we use it to prove a strengthening of Theorem 7.3.7.

Corollary 11.1.2. *Assume that $A_{\bar{K}}$ is simple. If $s \in \mathrm{End}(H^1_{\mathrm{dR}}(A,\bar{K}))$ is a β-de Rham–Tate cycle for some $\beta > \frac{3}{4}$, then s is algebraic.*

Proof. By Theorem 11.1.1, it suffices to show that if s is fixed by infinitely many φ_v and $\bar{s}$ is algebraic, then s is algebraic. Since the restriction to $\mathrm{End}^\circ_L(A)$ of the map

$$\mathrm{Fil}^0\,\mathrm{End}(H^1_{\mathrm{dR}}(A,L)) \to \mathrm{End}(\mathrm{Lie}\, A^\vee_L),\ s \to \bar{s}$$

is the natural identification $\mathrm{End}^\circ_L(A) \cong \mathrm{End}^\circ_L(A^\vee)$, we obtain an algebraic cycle $t \in \mathrm{End}^\circ_{\bar{K}}(A)$ such that $\bar{t} = \bar{s}$. Then for infinitely many v, we have $\varphi_v(s-t) = s-t$ and $s-t \in \mathrm{Fil}^1(\mathrm{End}(H^1_{\mathrm{dR}}(A,\bar{K}))$. By Lemma 7.1.3, $s-t=0$ and hence s is algebraic. $\square$

Remark 11.1.3. The only place where we use the assumption of $A_{\bar{K}}$ being simple is to obtain the first assertion of Lemma 11.1.5. This assertion shows that if s is not algebraic, then the Zariski closure of the g-dimensional formal subvariety that we will construct using s is of dimension $2g$. In general, the Zariski closure of a non-algebraic g-dimensional formal subvariety is of dimension at least $g+1$ and then the same argument as below shows that any β-cycle with $\beta > 1 - \frac{1}{2(g+1)}$ is algebraic.

11.1.4. The proof of this theorem will occupy the rest of this subsection. Since the definition of β-cycle is independent of the choice of a definition field and the property of being a β-cycle is preserved under isogeny, we may assume that A is principally polarized and that s is defined over K. Let X be $A^\vee \times A^\vee$ and e be its identity. The main idea is to apply Theorem 2.2.5 to the following formal subvariety $\widehat{V} \subset \widehat{X}_{/e}$. Consider the sub Lie algebra

$$H = \{(a, \bar{s}(a)) \mid a \in \mathrm{Lie}(A^\vee)\} \subset \mathrm{Lie}(X).$$

This sub Lie algebra induces an involutive subbundle $\mathcal{H}$ of the tangent bundle of X via translation. The formal subvariety $\widehat{V}$ is defined to be the formal leaf passing through e. Recall that in 2.2.4 a finite place v of K is called bad if $\mathcal{H} \otimes k_v$ is not stable under p-th power map of derivatives.

Lemma 11.1.5. *If $\bar{s}$ is not algebraic, then the formal subvariety $\widehat{V}$ is Zariski dense in X. The A-density of bad primes is at most $1 - \beta$.*

Proof. The Zariski closure G of $\widehat{V}$ must be an algebraic subgroup of X. The simplicity of A implies that the only algebraic subgroup of X with dimension larger than g must be X. Hence if $\bar{s}$ is not algebraic, we have $\dim G > g$ and hence $G = X$.

By [**Mum08**, p. 138], given $v \notin \Sigma$, the p-th power map on $\mathrm{Lie}\, E(A^\vee) \otimes k_v = H^1_{\mathrm{dR}}(A, k_v)$ is the same as $\varphi_v \otimes k_v$. Therefore, for those v such that $\varphi_v(s) = s$, we have that the Lie subalgebra $\{(a, s(a)) \mid a \in (\mathrm{Lie}\, E(A^\vee))\} \otimes k_v$ of $\mathrm{Lie}(E(A^\vee) \times E(A^\vee)) \otimes k_v$ is closed under the p-th power map. Then $H = \{(a, \bar{s}(a)) \mid a \in \mathrm{Lie}(A^\vee)\} \otimes k_v$ and $\mathcal{H} \otimes k_v$ are closed under the p-th power map. Therefore, the density of bad primes is at most one minus the density of primes satisfying $\varphi_v(s) = s$. $\square$

11.1.6. Let $\sigma : K \to \mathbb{C}$ be an archimedean place of K. We define γ_σ to be the composition

$$\gamma_\sigma : \mathbb{C}^g \xrightarrow{(id, \bar{s})} \mathbb{C}^g \times \mathbb{C}^g \xrightarrow{(\exp, \exp)} X_\sigma,$$

where exp the uniformization of $\mathbb{C}^g = \mathrm{Lie}\, A^\vee_\sigma \to A^\vee_\sigma$. We choose an ample Hermitian line bundle $\mathcal{L}$ on $A^\vee$ such that the pull back of its first Chern form via exp is $iC_0 \sum_{k=1}^g dz_k \wedge d\bar{z}_k$ where $C_0 > 0$ is some constant. More explicitly, we may choose $\mathcal{L}$ to be the theta line bundle with a translate-invariant metric. See for example [**dJ08**, Sec. 2].

To compute the order of γ_σ (Definition 2.2.3), we fix the ample Hermitian line bundle on X to be $pr_1^*\mathcal{L} \otimes pr_2^*\mathcal{L}$. Then

$${\gamma_\sigma}^*\eta = C_0(i \sum_{k=1}^g dz_k \wedge d\bar{z}_k + s^*(i \sum_{k=1}^g dz_k \wedge d\bar{z}_k)).$$

Thus ${\gamma_\sigma}^*\eta$ has all coefficients of $dz_i \wedge d\bar{z}_j$ being constant functions on $\mathbb{C}^g$.

Lemma 11.1.7. *The order ρ_σ of γ_σ is at most 2. In other words, $\rho \leq 2$.*

Proof. Up to a positive constant,

$$\omega = i\frac{||z||^2 \sum_{k=1}^{g} dz_k \wedge d\bar{z}_k - \sum_{k,l=1}^{g} \bar{z}_k z_l \, dz_k \wedge d\bar{z}_l}{||z||^4}.$$

Since all the absolute values of the coefficients of $dz_k \wedge d\bar{z}_l$ in ω are bounded by $2||z||^{-2}$ and those in ${\gamma_\sigma}^*\eta$ are constant functions, the volume form ${\gamma_\sigma}^*\eta \wedge \omega^{g-1}$ has the absolute value of the coefficient of $\wedge_{k=1}^{g}(dz_k \wedge d\bar{z}_k)$ to be bounded by $C_1||z||^{-2(g-1)}$ for some constant C_1. Hence

$$\begin{aligned} T_{\gamma_\sigma}(r) &= \int_0^r \frac{dt}{t} \int_{B(t)} {\gamma_\sigma}^*\eta \wedge \omega^{g-1} \\ &\leq \int_0^r \frac{dt}{t} \int_{B(t)} C_1||z||^{-2(g-1)}(i^g) \wedge_{k=1}^{g} (dz_k \wedge d\bar{z}_k) \\ &= \int_0^r \frac{dt}{t} \int_0^t C_2 R^{-2(g-1)} vol(S(R)) \, dR \\ &= \int_0^r \frac{dt}{t} \int_0^t C_3 R \, dR = C_4 r^2, \end{aligned}$$

where $S(R)$ is the sphere of radius R in $\mathbb{C}^g$. We conclude by the definition of orders. $\square$

Remark 11.1.8. By a more careful argument, one can see that ρ_σ is 2.

Proof of Theorem 11.1.1. If $\bar{s}$ is not algebraic, then we apply Theorem 2.2.5 with $N = 2g$ and $d = g$. We have

$$1 \leq 2\rho\alpha \leq 2 \cdot 2 \cdot (1 - \beta),$$

which contradicts with $\beta > \frac{3}{4}$. $\square$

11.2. Abelian surfaces. We see from the discussion in section 9 that the only case left for Conjecture 7.1.4 for abelian surfaces is when $\mathrm{End}_{\bar{K}}(A) = \mathbb{Z}$ and K is of even degree over $\mathbb{Q}$. We discuss in this section the case when K is a quadratic extension of $\mathbb{Q}$ and remark that one can deduce similar results when $[K : \mathbb{Q}]$ is $2n$ for some odd integer n by incorporating arguments as in the proof of Proposition 9.2.4.

Assume that A does not satisfy Conjecture 7.1.4. Then by Proposition 9.1.5, we have $H^1_{\mathrm{dR}}(A, \bar{K}) = V_1 \oplus V_2$ as a representation of G°_{dR}, where V_1 and V_2 are irreducible representations of dimension 2. We have the following description of the decomposition of the filtration.

Lemma 11.2.1. *Let $F_i^1 = \mathrm{Fil}^1(H^1_{\mathrm{dR}}(A,\bar{K})) \cap V_i$. Then both F_i^1 are 1-dimensional and $F_1^1 \oplus F_2^1 = \mathrm{Fil}^1(H^1_{\mathrm{dR}}(A,\bar{K}))$.*

Proof. The second assertion follows from the fact that the Hodge cocharacter μ factors through G°_{dR}. To show the first assertion, it suffices to show that neither F_1^1 nor F_2^1 is zero. If not, we may assume that $F_2^1 = 0$. Then $F_1^1 = V_1$ and $\mu|_{V_1}$ has all weights being 1. Then $\mu|_{V_2}$ has all weights being 0. Then G°_{dR} commutes with μ and then by Proposition 8.3.2, A has complex multiplication, which contradicts our assumption. $\square$

Let $\underline{\beta}$ be the inferior density of good primes of pr_1

$$\liminf_{x\to\infty} \left(\sum_{v|p_v\leq x, \varphi_v(pr_1)=pr_1} \frac{[L_v:\mathbb{Q}_{p_v}]\log p_v}{p_v-1} \right) \left([L:\mathbb{Q}] \sum_{p\leq x} \frac{\log p}{p-1} \right)^{-1}$$

and $\overline{\beta}$ be the supreme density

$$\limsup_{x\to\infty} \left(\sum_{v|p_v\leq x, \varphi_v(pr_1)=pr_1} \frac{[L_v:\mathbb{Q}_{p_v}]\log p_v}{p_v-1} \right) \left([L:\mathbb{Q}] \sum_{p\leq x} \frac{\log p}{p-1} \right)^{-1}.$$

By Theorem 8.1.5, for a density one set of split primes v of K, we have $\varphi_v \in G^\circ_{\mathrm{dR}}(K_v)$ and then $\varphi_v(pr_i) = pr_i$ for $i = 1,2$. In other words, we have $\frac{1}{2} \leq \underline{\beta} \leq \overline{\beta}$.

Theorem 11.2.2. *If A does not satisfy Conjecture 7.1.4, then $\underline{\beta} \leq \frac{3}{4} \leq \overline{\beta}$. In particular, if the natural density of good primes of pr_1 exists, then the density must be $\frac{3}{4}$.*

Proof. By definition, pr_1 is a $\underline{\beta}$-de Rham–Tate cycle. If $\underline{\beta} > \frac{3}{4}$, then by Corollary 11.1.2, we have pr_1 is algebraic. As $\mathrm{End}_{\bar{K}}(A) = \mathbb{Z}$, this is a contradiction. Therefore, $\underline{\beta} \leq \frac{3}{4}$.

Let $\theta \in K$ be an element such that $\sigma(\theta) = -\theta$, where σ is the nontrivial element in $\mathrm{Gal}(K/\mathbb{Q})$. We consider $\theta pr_1 - \theta pr_2 \in \mathrm{End}(H^1_{\mathrm{dR}}(A,\bar{K}))$. By Proposition 9.1.5, if $\varphi_v(pr_1) \neq pr_1$, then $\varphi_v(pr_1) = pr_2$ and $\varphi_v(pr_2) = pr_1$. By the σ-linearity of φ_v, when v is inert, we have that if $\varphi_v(pr_1) \neq pr_1$, and then $\varphi_v(\theta pr_1 - \theta pr_2) = \theta pr_1 - \theta pr_2$. For v split, we have $\varphi_v(pr_i) = pr_i$ and hence $\varphi_v(\theta pr_1 - \theta pr_2) = \theta pr_1 - \theta pr_2$. Then by definition, $\theta pr_1 - \theta pr_2$ is a $(\frac{3}{2} - \overline{\beta})$-de Rham–Tate cycle. By Corollary 11.1.2, if $\overline{\beta} < \frac{3}{4}$, then $\theta pr_1 - \theta pr_2$ is algebraic, which contradicts $\mathrm{End}_{\bar{K}}(A) = \mathbb{Z}$. $\square$

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