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Universality of General β-Ensembles

Paul Bourgade[1] László Erdős[2*] Horng-Tzer Yau[1†]

Department of Mathematics, Harvard University
Cambridge MA 02138, USA
bourgade@math.harvard.edu htyau@math.harvard.edu [1]

Institute of Mathematics, University of Munich,
Theresienstrasse 39, D-80333 Munich, Germany
lerdos@math.lmu.de [2]

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Abstract

We prove the universality of the β-ensembles with convex analytic potentials and for any $\beta > 0$, i.e. we show that the spacing distributions of log-gases at any inverse temperature β coincide with those of the Gaussian β-ensembles.

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1 Introduction

The central concept of the random matrix theory as envisioned by E. Wigner is the general hypothesis that the distributions of eigenvalue spacings of large complicated quantum systems are universal in the sense that they depend only on the symmetry classes of the physical systems but not on other detailed structures. The simplest case for this hypothesis is for ensembles of large but finite dimensional matrices. The general hypothesis in this setting thus asserts that the eigenvalue spacing distributions of random matrices should be independent of the probability distribution of the ensemble, up to scaling. This is generally referred to as the universality of random matrices. In this paper we will focus only on the bulk behavior i.e., on eigenvalue distribution in the interior of the spectrum, although similar questions regarding the edge distribution are also important.

Over the past two decades, spectacular progress (see, e.g., [5, 10, 11, 24, 25, 9, 22] and [2, 8, 9] for a review) on bulk universality was made for classical invariant ensembles, i.e., matrix models with probability measure given by $e^{-N\beta \mathrm{Tr} V(H)/2}/Z$ where N is the size of the matrix H, V is a real valued potential and Z is the normalization. It is well-known that the probability distribution of the ordered eigenvalues of H on the simplex determined by $\lambda_1 \leq \cdots \leq \lambda_N$ is given by

$$\mu^{(N)} \sim e^{-\beta N \mathcal{H}}, \quad \mathcal{H} = \sum_{k=1}^{N} \frac{1}{2} V(\lambda_k) - \frac{1}{N} \sum_{1 \leq i < j \leq N} \log(\lambda_j - \lambda_i), \tag{1.1}$$

where the parameter $\beta = 1, 2, 4$ is determined by the symmetry type of the matrix, corresponding respectively to the classical orthogonal, unitary or symplectic ensemble. With β taking these special values, the correlation functions can be explicitly expressed in terms of polynomials orthogonal to the measure $e^{-\beta V(x)/2}$. Thus the analysis of the correlation functions relies heavily on the asymptotic properties of the corresponding orthogonal polynomials. In the pioneering work of Gaudin, Mehta and Dyson (see [23] for a review), the potential V is the quadratic polynomial $V(x) = x^2$ and the orthogonal polynomials are the Hermite polynomials for which asymptotic properties are well-known. The major input of the recent work is the asymptotic analysis of the orthogonal polynomials w.r.t. the measure $e^{-\beta V(x)/2}$ for general classes of potentials. The formulas for orthogonal and symplectic cases, i.e., $\beta = 1, 4$, are much more difficult to use than the one for the unitary case. While universality for $\beta = 2$ was proved for very general potential, the best results for $\beta = 1, 4$ [9, 21, 26] are still restricted to analytic V with additional conditions.

For non-classical values of β, i.e., $\beta \notin \{1, 2, 4\}$, one can still consider the measure (1.1), but there is no simple expression of the correlation functions in terms of orthogonal polynomials. Furthermore, the measure (1.1) does not arise from mean-field type matrix models like Wigner matrices with independent entries. Nevertheless, μ is a Gibbs measure of particles in $\mathbb{R}$ with a logarithmic interaction, where the parameter β is interpreted as the inverse temperature and a priori can be an arbitrary positive number. These measures are called general β-ensembles. We will often refer to the variables λ_j as particles or points and the system is called log-gas. It was proved [12] that in the Gaussian case, i.e., when V is quadratic, the measure (1.1) describes eigenvalues of tri-diagonal matrices. This observation allowed one to establish detailed properties, including the local spacing distributions of the Gaussian β-ensembles [27].

Gibbs measures in the continuum with long range or singular interactions are notoriously hard to analyze since they are very far from the perturbative regime. For non-classical values of β, and if we are not in the Gaussian case $V(\lambda) = \lambda^2$, no simple explicit formula is known to express the correlation functions in terms of orthogonal polynomials, and one cannot rely on any explicit known matrix model. In this paper we undertake the direct analysis of the Gibbs measure and we prove the universality for invariant models for any $\beta > 0$. In other words, we will prove that the local spacing distributions of (1.1) are independent of the potential V for certain class of V. There are two major ingredients in our new approach.

Step 1. Uniqueness of local Gibbs measures with logarithmic interactions. The main result in this step asserts that if the particles are not too far from their classical locations then the spacing distributions are given by the corresponding Gaussian ones (We will take the uniqueness of the spacing distributions as our definition of the uniqueness of Gibbs state). More precisely, denote by ρ the limiting density of the particles under the measure $\mu^{(N)}$ (1.1) as $N \to \infty$. Let $\gamma_j = \gamma_{j,N}$ denote the location of the j-th point under ρ, i.e., γ_j is defined by

$$N \int_{-\infty}^{\gamma_j} \rho(x)\mathrm{d}x = j, \qquad 1 \le j \le N. \tag{1.2}$$

We will call γ_j the *classical location* of the j-th particle. The basic assumption is the following:

Assumption A. For some $\mathfrak{b} < \frac{1}{38}$ and any $\alpha > 0$, there exists $\varepsilon_0 > 0$ such that

$$\mathbb{P}_{\mu^{(N)}}(|\lambda_k - \gamma_k| \le N^{-1+\mathfrak{b}}) \ge 1 - \exp(-N^{\varepsilon_0}) \tag{1.3}$$

for large enough N and any $k \in [\alpha N, (1-\alpha)N]$.

Under this assumption (under some minor and easily verifiable assumptions near the edges of the limiting measure), we will prove that the spacing distributions of μ are given by the corresponding Gaussian model with $V(x) = x^2$. We will use the Gaussian case as our reference ensemble only for the convenience of definiteness. In fact, no detailed properties of the Gaussian measures are used in the proof and any other reference ensemble would have worked as well. Furthermore, in this step we make no assumption on the convexity of V, which is needed in the next step.

Step 2. Particle location estimate. The second step is to verify Assumption A. For non-classical β, Assumption A is only proved for $\mathfrak{b}$ near one [20, 25, 21] for analytic potential V under certain constraint. This is far from sufficient to complete Step 1. We will prove Assumption A for all $\beta > 0$ under the assumption that V is convex and analytic. Our method uses the following three ideas: (1) The analysis of the loop equation in [20, 21, 26] to control the density. (2) The logarithmic Sobolev inequality guaranteed by the convexity of V. (3) A multiscale analysis of the probability measures of invariant ensembles. We note that the assumption of analyticity on V is needed only for using the loop equation in (1).

The basic idea of our proof is to use the following tool from [14]: For two probability measures μ and ω define the Dirichlet form by

$$D(\mu \mid \omega) := \frac{1}{2N} \int \left| \nabla \sqrt{\frac{\mathrm{d}\mu}{\mathrm{d}\omega}} \right|^2 \mathrm{d}\omega.$$

Then the difference of the local spacing distributions of the two measures is negligible provided that the Dirichlet form per particle is sufficiently small in the large N limit [14]. Notice that if we used the relative entropy of the two measures, then the uniqueness of the Gibbs measures would require the total entropy, which is an extensive quantity, to be small. To apply this Dirichlet form inequality, we first localize the measure by fixing λ_j for j outside, say, the interval $[L+1, L+K]$ for L in the bulk and $K = N^k$ for some $k > 0$. We will call these data of λ_j outside the interval $[L+1, L+K]$ the boundary condition. We then compare this measure to a local Gaussian β-ensemble with a fixed boundary condition by showing that the Dirichlet form per particle of these two measures is small for typical boundary conditions w.r.t. μ.

Our approach shares some philosophy from the recent method on the universality of Wigner matrices [14, 15]. In this approach, the key condition to establish is

Assumption III. There exists an $\mathfrak{a} > 0$ such that we have

$$\mathbb{E}_{\mu_W} \frac{1}{N} \sum_{j=1}^{N} (x_j - \gamma_j)^2 \leq C N^{-1-2\mathfrak{a}} \tag{1.4}$$

with a constant C uniformly in N. Here μ_W is the law given by the Wigner ensemble.

Under this assumption, a strong estimate on the local ergodicity of Dyson Brownian motion (DBM) was established in [14, 15]. DBM [13] establishes a dynamical interpolation between Wigner matrices and the invariant equilibrium measure μ. This estimate then implies the universality of Wigner matrices. Thus the main task in proving the universality of Wigner matrices is reduced to verifying Assumption III.

There are several similarities between the method used for the universality of Wigner matrices [14, 18] and the current proof for β-ensembles: (i) Both rely on crude estimates such as (1.3) and (1.4) on the location of the eigenvalues to establish the local spacing distributions are the same as in the Gaussian cases. (ii) Both use estimates on the Dirichlet form to identify the local spacing distributions. (iii) The main model dependent argument is to prove these crude bounds on the eigenvalues. The precision of these a-priori estimates on the eigenvalues is weaker than the local spacing, but better than previously known results on eigenvalue locations: we have to develop new methods to prove (1.3) and (1.4).

There are, however, substantial differences between the proofs of universalities for Wigner and β-ensembles. First, since the β-ensembles are already in equilibrium, there is no dynamical relaxation mechanism to exploit and the local statistics need to be identified directly without dynamical argument. Second, we obtain the crude estimate (1.3) by a method completely different from the Wigner matrices, as there is no underlying matrix ensemble with independent entries to analyze. The accuracy result we obtain by this new method is actually optimal, i.e. (1.3) will be shown to hold for any $\mathfrak{b} > 0$.

2 Statement of the main result

Consider a probability measure

$$\mu_{\beta,V}^{(N)} = \mu^{(N)}(\mathrm{d}\lambda) = \frac{1}{Z_N} \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^\beta \prod_{k=1}^{N} e^{-N \frac{\beta}{2} V(\lambda_k)} \mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_N, \tag{2.1}$$

where $\lambda = (\lambda_1, \ldots, \lambda_N)$, $\lambda_1 \leq \cdots \leq \lambda_N$. Here the inverse temperature satisfies $\beta > 0$ and the external potential V is any convex real analytic function in $\mathbb{R}$, and such that

$$\varpi = \frac{\beta}{2} \inf_{x \in \mathbb{R}} V''(x) > 0. \tag{2.2}$$

For such a convex potential, as noted in the next section the equilibrium measure, denoted by $\rho(s)\mathrm{d}s$, is supported on a single interval $[A, B]$. In the following, we omit the superscript N and we will write μ for $\mu^{(N)}$. We will use $\mathbb{P}_\mu$ and $\mathbb{E}_\mu$ to denote the probability and the expectation with respect to μ.

The Gaussian case corresponds to $V(\lambda) = \lambda^2$; the expectation with respect to this Gaussian measure will be denoted by $\mathbb{E}_{\text{Gauss}}$, and the equilibrium measure is known to be

$$\rho_{sc}(E) := \frac{1}{2\pi}\sqrt{(4-E^2)_+},$$

the semicircle density. The Gaussian case includes the classical GUE, GOE and GSE ensembles for the special choice of $\beta = 1, 2, 4$, but our result holds for all $\beta > 0$.

Now we state our main theorem which will be proven at the end of Section 4:

Theorem 2.1 *Assume V is any real analytic function with $\inf_{x\in\mathbb{R}} V''(x) > 0$. Let $\beta > 0$. Consider the β-ensemble $\mu = \mu_{\beta,V}$. Let $G : \mathbb{R} \to \mathbb{R}$ be a smooth, compactly supported function. Let $E \in (A, B)$ lie in the interior of the support of ρ, and similarly let $E' \in (-2, 2)$ be inside the support of ρ_{sc}. Define L and L' by*

$$\frac{L}{N} = \int_A^E \rho(x)\mathrm{d}x, \qquad \frac{L'}{N} = \int_{-2}^{E'} \rho_{sc}(x)\mathrm{d}x.$$

Fix a parameter $K = N^k$ where $0 < k \le \frac{1}{2}$ is an arbitrary constant. Let I and I' be two intervals of natural numbers, $I = [L+1, L+K]$, $I' = [L'+1, L'+K]$ with length $K = |I|$. Then

$$\lim_{N\to\infty}\left|\mathbb{E}_\mu \frac{1}{K\rho(E)}\sum_{i\in I} G\Big(\frac{N(\lambda_i-\lambda_{i+1})}{\rho(E)}\Big) - \mathbb{E}_{\text{Gauss}}\frac{1}{K\rho_{sc}(E')}\sum_{i\in I'} G\Big(\frac{N(\lambda_i-\lambda_{i+1})}{\rho_{sc}(E')}\Big)\right| = 0, \tag{2.3}$$

i.e. the appropriately normalized particles gap distribution of the measure $\mu_{\beta,V}$ at the level E in the bulk of the limiting density asymptotically coincides with that for the Gaussian case and it is independent of the value of E in the bulk. In particular the gap distribution is universal.

Remark. The same result (with the same proof) holds for higher order correlation functions of particles gaps. More precisely, fix $n \ge 1$ and an array of positive integers, $\mathbf{m} = (m_1, m_2, \dots, m_n) \in \mathbb{N}_+^n$. Let $G : \mathbb{R}^n \to \mathbb{R}$ be a bounded smooth function with compact support and we define

$$\mathcal{G}_{i,\mathbf{m}}(\boldsymbol{\lambda}) := \frac{1}{\rho(E)^n} G\Big(\frac{N(\lambda_i-\lambda_{i+m_1})}{\rho(E)}, \frac{N(\lambda_{i+m_1}-\lambda_{i+m_2})}{\rho(E)}, \dots, \frac{N(\lambda_{i+m_{n-1}}-\lambda_{i+m_n})}{\rho(E)}\Big). \tag{2.4}$$

Then, under the conditions of Theorem 2.1 and using its notations, we have

$$\lim_{N\to\infty}\left|\mathbb{E}_\mu \frac{1}{K}\sum_{i\in I}\mathcal{G}_{i,\mathbf{m}}(\boldsymbol{\lambda}) - \mathbb{E}_{\text{Gauss}}\frac{1}{K}\sum_{i\in I'}\mathcal{G}'_{i,\mathbf{m}}(\boldsymbol{\lambda})\right| = 0, \tag{2.5}$$

where $\mathcal{G}'_{i,\mathbf{m}}$ is defined exactly as $\mathcal{G}_{i,\mathbf{m}}$ but $\rho(E)$ is replaced with $\rho_{sc}(E')$.

The limit (2.5) can be reformulated as the convergence of the correlation functions. Let $\rho_n^{(N)}$ denote the n-point correlation function of the measure $\mu = \mu_{\beta,V}^{(N)}$ defined by

$$\rho_n^{(N)}(x_1, \dots, x_n) = \int_{\mathbb{R}^{N-n}} \tilde{\mu}(x)\mathrm{d}x_{n+1}\dots\mathrm{d}x_N, \tag{2.6}$$

where $\tilde{\mu}$ is the symmetrized version of μ given in (2.1) but defined on the $\mathbb{R}^N$ instead of the simplex:

$$\tilde{\mu}^{(N)}(\mathrm{d}\lambda) = \frac{1}{N!}\mu(\mathrm{d}\lambda^{(\sigma)}),$$

where $\lambda^{(\sigma)} = (\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(N)})$, with $\lambda_{\sigma(1)} < \dots < \lambda_{\sigma(N)}$.

From (2.5) we have the convergence of the correlation functions, stated as the following corollary. Since the proof is a standard argument and it is essentially identical to the one given in Section 7 of [15], we omit it.

Corollary 2.2 *Under the assumption of Theorem 2.1 and with the same notations, for any smooth test functions O with compact support and for any $0 < k \le \frac{1}{2}$, we have, with $s := N^{-1+k}$, that*

$$\lim_{N\to\infty} \int \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_n \, O(\alpha_1, \dots, \alpha_n) \Bigg[\int_{E-s}^{E+s} \frac{\mathrm{d}x}{2s} \frac{1}{\varrho(E)^n} \rho_n^{(N)} \Big(x + \frac{\alpha_1}{N\varrho(E)}, \dots, x + \frac{\alpha_n}{N\varrho(E)} \Big) - \int_{E'-s}^{E'+s} \frac{\mathrm{d}x}{2s} \frac{1}{\varrho_{sc}(E)^n} \rho_{\mathrm{Gauss},n}^{(N)} \Big(x + \frac{\alpha_1}{N\varrho_{sc}(E)}, \dots, x + \frac{\alpha_n}{N\varrho_{sc}(E)} \Big) \Bigg] = 0 \, .$$

The local statistics of the $\lambda_i' s$ in the Gaussian case have been explicitly computed by Gaudin, Mehta and Dyson (see, e.g., [23]) for the classical value $\beta \in \{1, 2, 4\}$. For general $\beta > 0$, there is an explicit description in terms of some stochastic differential equations, the *Brownian carousel* [27].

Theorem 2.1 will be proved in two steps as explained in the introduction. For logical reasons, we will first present Step 2 on particle location estimates in Section 3 and then Step 1 on the uniqueness of Gibbs measure in a finite interval in Sections 4 and 5.

3 Optimal accuracy for particle locations

Along this section, we assume that V satisfies the same conditions as in Theorem 2.1. Let the typical position γ_k be defined by

$$\int_{-\infty}^{\gamma_k} \rho(s) \mathrm{d}s = \frac{k}{N}.$$

Moreover, all constants in this section depend on the potential V, which is fixed. In the following, we will denote $[\![x, y]\!] = \mathbb{N} \cap [x, y]$,

The purpose of this section is to prove that accuracy holds for the measure μ at the optimal scale $1/N$, in the following sense.

Theorem 3.1 *Take any $\alpha > 0$ and $\varepsilon > 0$. There are constants $\delta, c_1, c_2 > 0$ such that for any $N \ge 1$ and $k \in [\![\alpha N, (1-\alpha)N]\!]$,*

$$\mathbb{P}_\mu \left(|\lambda_k - \gamma_k| > N^{-1+\varepsilon} \right) \le c_1 e^{-c_2 N^\delta}.$$

After some initial estimates relying on large deviations results, the proof consists in comparing μ to some *locally constrained measures* for which better concentration estimates can be proved for the differences between particles. This measure is related to the pseudo-equilibrium measure in [14], but has distinctly different properties. Iterations of these comparisons will give optimal accuracy.

3.1 Initial estimates

The purpose of this paragraph it to prove the following crude estimate. It will be the initial step in the induction of Subsection 3.3.

Proposition 3.2 *For any $\alpha, \varepsilon > 0$ there are constants $c_1, c_2, \delta > 0$ such that for any N and $k \in [\![\alpha N, (1-\alpha)N]\!]$*

$$\mathbb{P}_\mu\left(|\lambda_k - \gamma_k| > N^{-\frac{1}{2}+\varepsilon}\right) \leq c_1 e^{-c_2 N^\delta}. \tag{3.1}$$

This result is a direct consequence of the following equation (3.12) and Corollary 3.5, whose proofs are the purpose of this section. We first state well-known facts about the equilibrium measure.

For convex analytic potential V satisfying the asymptotic growth condition (2.2) (or even with weaker hypotheses on V, see e.g. [6, 1]), the equilibrium measure $\rho(s)\mathrm{d}s$ associated with $(\mu^{(N)})_{N\geq 0}$ can be defined as the unique minimizer (in the set of probability measures on $\mathbb{R}$ endowed with the weak topology) of the functional

$$I(\nu) = \int V(t)\mathrm{d}\nu(t) - \iint \log|t-s|\mathrm{d}\nu(s)\mathrm{d}\nu(t)$$

if $\int V(t)\mathrm{d}\nu(t) < \infty$, and $I(\nu) = \infty$ otherwise. Moreover, ρ has the following properties:

(a) The support of ρ is a single interval $[A, B]$.

(b) This equilibrium measure satisfies

$$\frac{1}{2}V'(t) = \int \frac{\rho(s)\mathrm{d}s}{t-s}. \tag{3.2}$$

for any $t \in (A, B)$.

(c) For any $t \in [A, B]$,

$$\rho(t)\,\mathrm{d}t = \frac{1}{\pi} r(t)\sqrt{(t-A)(B-t)}\mathbb{1}_{[A,B]}\,\mathrm{d}t, \tag{3.3}$$

where r can be extended into an analytic function in $\mathbb{C}$ satisfying

$$r(z) = \frac{1}{2\pi}\int_A^B \frac{V'(z)-V'(t)}{z-t}\frac{\mathrm{d}t}{\sqrt{(t-A)(B-t)}}. \tag{3.4}$$

In particular, for convex V, r has no zero in $\mathbb{R}$.

It is known that the particles locations cannot be far from its classical location [4, 26]: for any $\varepsilon > 0$ there are positive constants C, c, such that, for all $N \geq 1$,

$$\mathbb{P}_\mu\left(\exists k \in [\![1, N]\!] \mid |\lambda_k - \gamma_k| \geq \varepsilon\right) \leq C e^{-cN^c}. \tag{3.5}$$

In order to have density strictly in a compact support, for given $R > 0$, define the following variant of $\mu^{(N)}$ conditioned to have all particles in $[-R, R]$:

$$\mu^{(N,R)}(\mathrm{d}\lambda) = \frac{1}{Z_{N,R}}\prod_{1\leq i<j\leq N}|\lambda_i - \lambda_j|^\beta \prod_{k=1}^N e^{-N\frac{\beta}{2}V(\lambda_k)}\mathbb{1}_{|\lambda_k|<R}\mathrm{d}\lambda_1\ldots\mathrm{d}\lambda_N. \tag{3.6}$$

Let $\rho_k^{(N,R)}$ denote the marginals of the measure $\mu^{(N,R)}$, i.e. the same definition as (2.6), but with $\mu^{(N)}$ replaced by $\mu^{(N,R)}$.

Then Lemma 1 in [6] states that under condition (2.2) there exist some $R > 0$ and $c > 0$, depending only on V, such that for any $|x_1|, \dots, |x_k| \le R$

$$\left|\rho_k^{(N,R)}(x_1, \dots, x_k) - \rho_k^{(N)}(x_1, \dots, x_k)\right| \le \rho_k^{(N,R)}(x_1, \dots, x_k) e^{-cN}, \tag{3.7}$$

and for $|x_1|, \dots, |x_j| \ge R$, $|x_{j+1}|, \dots, |x_k| \le R$,

$$\rho_k^{(N)}(x_1, \dots, x_k) \le e^{-cN\sum_{i=1}^{j} \log|x_k|}. \tag{3.8}$$

The last type of estimates we need are concentration and accuracy of the particles location at scale $N^{-1/2}$, in the bulk. Concentration is a simple consequence of the Bakry-Émery convexity criterion for the logarithmic Sobolev inequality ([3], see also [2]): define $\mathcal{H}$ by $\mu(\mathrm{d}\lambda) = \frac{1}{Z_N} e^{-N\mathcal{H}(\lambda)}\mathrm{d}\lambda$, and assume

$$\nabla^2 \mathcal{H} \ge \sigma \, \mathrm{Id}_N \tag{3.9}$$

in the sense of partial order for positive definite operators. Then μ satisfies a logarithmic Sobolev inequality with constant $2/(\sigma N)$: for any probability density f we have

$$\mathbb{E}_\mu f \log f \le \frac{2}{\sigma N} \mathbb{E}_\mu |\nabla \sqrt{f}|^2. \tag{3.10}$$

It is well-known that the logarithmic Sobolev inequality implies the spectral gap and, together with Herbst's lemma, it also implies that for any $k \in [\![1, N]\!]$ and $x > 0$

$$\mathbb{P}_\mu \left(|\lambda_k - \mathbb{E}_\mu(\lambda_k)| > x\right) \le 2e^{-\sigma N x^2/2}.$$

In our case where μ is defined by (2.1), for any $v \in \mathbb{R}^N$

$$v^*(\nabla^2 H)v = \frac{\beta}{N} \sum_{i<j} \frac{(v_i - v_j)^2}{(\lambda_i - \lambda_j)^2} + \frac{\beta}{2} \sum_i V''(\lambda_i) v_i^2 \ge \varpi |v|^2, \tag{3.11}$$

where ϖ is defined in (2.2). Thus there is a constant $\tilde{c} > 0$ such that for any k

$$\mathbb{P}_\mu \left(|\lambda_k - \mathbb{E}_\mu(\lambda_k)| > x\right) \le 2e^{-\tilde{c} N x^2}, \tag{3.12}$$

i.e. concentration at scale $1/\sqrt{N}$ holds. We now prove that accuracy at the same scale holds inside the bulk.

The proof of the following lemma is based on an argument in [20] for the polynomial case. In the form presented here, it follows very closely the proof in [26] for the analytic case except that we use the logarithmic Sobolev inequality to have a more precise estimate. We now introduce some notations needed in the proof.

- m_N is the Stieljes transform of $\rho_1^{(N)}(s)(\mathrm{d}s)$, evaluated at some z with $\mathrm{Im}(z) > 0$, and m its limit:

$$m_N(z) = \mathbb{E}_\mu \left(\frac{1}{N} \sum_{k=1}^{N} \frac{1}{z - \lambda_i} \right) = \int_{\mathbb{R}} \frac{1}{z - t} \rho_1^{(N)}(t) \mathrm{d}t, \;\; m(z) = \int_{\mathbb{R}} \frac{1}{z - t} \rho(t) \mathrm{d}t.$$

  It is well-known that uniformly in any $\{\mathrm{Im}(z) > \varepsilon\}$, $\varepsilon > 0$, $|m_N - m| \to 0$ (see e.g. [2]). Along the proof of the next Lemma 3.3 we will see that this convergence holds at speed $1/N$.

- $s(z) = -2r(z)\sqrt{(A-z)(B-z)}$, where the square root is defined such that

$$f(z) = \sqrt{(A-z)(B-z)} \sim z \quad \text{as} \quad z \to \infty;$$

- $b_N(z)$ is an analytic function defined by

$$b_N(z) = \int_{\mathbb{R}} \frac{V'(z) - V'(t)}{z-t} (\rho_1^{(N)} - \rho)(t)\,\mathrm{d}t;$$

- finally, $c_N(z) = \frac{1}{N^2} k_N(z) + \frac{1}{N}\left(\frac{2}{\beta} - 1\right) m_N'(z)$, where

$$k_N(z) = \mathrm{var}_\mu \left(\sum_{k=1}^N \frac{1}{z - \lambda_i} \right).$$

  Here the var of a complex random variable denotes $\mathrm{var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$, i.e. without absolute value unlike for the usual variance. Note that $|\mathrm{var}(X)| \le \mathbb{E}(|X - \mathbb{E}(X)|^2)$.

The equation used by Johansson (which can be obtained by a change of variables in (2.1) [20] or by integration by parts [26]), is a variation of the loop equation (see, e.g., [19]) used in physics literatures and it takes the form

$$(m_N - m)^2 + s(m_N - m) + b_N = c_N. \tag{3.13}$$

Equation (3.13) expresses the difference $m_N - m$ in terms of $(m_N - m)^2$, b_N and c_N. In the regime $|m_N - m|$ is small, we can neglect the quadratic term. The term b_N is the same order as $|m_N - m|$ and is difficult to treat. As observed in [1, 26], for analytic V, this term vanishes when we perform a contour integration. So we have roughly the relation

$$(m_N - m) \sim \frac{1}{N^2} \mathrm{var}_\mu \left(\sum_{k=1}^N \frac{1}{z - \lambda_k} \right), \tag{3.14}$$

where we dropped the less important error involving $m_N'(z)/N$ due to the extra $1/N$ factor. In the convex setting, the variance can be estimated by the logarithmic Sobolev inequality and we immediately obtain an estimate on $m_N - m$. We then follow the method in [16] to use the Helffer-Sjöstrand functional calculus to have an estimate on the particle locations. Although it is tempting to use this new accuracy information on the particle locations to estimate the variance again in (3.14), this naive bootstrap is difficult to implement. The main reason is, roughly speaking, that the particle location estimate obtained from knowing only the size of $m_N - m$ is not strong enough in the bootstrap. The key idea in this section is the observation that *accuracy information on particle locations can be used to improve the local convexity of the measure μ in the direction involving the differences of particle locations*, see Lemma 3.8. Now we are able to complete the bootstrap argument and obtain a more accurate estimate on $m_N - m$. Since this argument can be repeated, we can estimate the locations of particles up to the optimal scale in the bulk.

Lemma 3.3 *Let $\delta > 0$. For $z = E + \mathrm{i}\eta$ with $A + \delta < E < B - \delta$ assume that*

$$\frac{1}{N^2} k_N(z) \to 0 \tag{3.15}$$

as $N \to \infty$ uniformly in $\eta \geq N^{-1+a}$ for some $0 < a < 1$. Then there is a constant $c > 0$ such that for any $\eta \geq N^{-1+a}$, $A + \delta < E < B - \delta$,

$$|m_N(z) - m(z)| \leq c\left(\frac{1}{N\eta} + \frac{1}{N^2}k_N(z)\right). \tag{3.16}$$

Proof. First, for technical contour integration reasons, it will be easier to consider the measure (3.6) instead of $\mu^{(N)}$ here. More precisely, define

$$\begin{aligned}
m_N^{(R)}(z) &= \mathbb{E}_{\mu^{(N,R)}}\left(\frac{1}{N}\sum_{k=1}^{N}\frac{1}{z-\lambda_i}\right) = \int_{\mathbb{R}}\frac{1}{z-t}\rho_1^{(N,R)}(t)\mathrm{d}t,\\
k_N^{(R)}(z) &= \mathrm{var}_{\mu^{(N,R)}}\left(\sum_{k=1}^{N}\frac{1}{z-\lambda_i}\right),\\
c_N^{(R)}(z) &= \frac{1}{N^2}k_N^{(R)}(z) + \frac{1}{N}\left(\frac{2}{\beta}-1\right)m_N^{(R)\prime}(z).
\end{aligned}$$

Then it is a direct consequence of (3.7) and (3.8) that there are constants $c > 0$ and $R > 0$ such that uniformly on $\eta \geq N^{-10}$ (or any power of N),

$$|m_N^{(R)} - m_N| = \mathrm{O}\left(e^{-cN}\right), \quad |k_N^{(R)} - k_N| = \mathrm{O}(e^{-cN}). \tag{3.17}$$

Consider the rectangle with vertices $2R + \mathrm{i}N^{-10}, -2R + \mathrm{i}N^{-10}, -2R - \mathrm{i}N^{-10}, 2R - \mathrm{i}N^{-10}$, call $\mathcal{L}$ the corresponding clockwise closed contour and $\mathcal{L}'$ the one consisting only in the horizontal pieces, with the same orientation. From (3.13), we obviously have, for $z \notin \mathcal{L}'$,

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{L}'}\frac{(m_N(\xi)-m(\xi))^2 + s(\xi)(m_N(\xi)-m(\xi)) + b_N(\xi) - c_N(\xi)}{r(\xi)(z-\xi)}\mathrm{d}\xi = 0.$$

Note that the above expression makes sense for large enough N, because then r has no zero on $\mathcal{L}$. Using (3.17), this implies, for $\eta \geq N^{-1}$,

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{L}'}\frac{(m_N^{(R)}(\xi)-m(\xi))^2 + s(\xi)(m_N^{(R)}(\xi)-m(\xi)) + b_N(\xi) - c_N^{(R)}(\xi)}{r(\xi)(z-\xi)}\mathrm{d}\xi = \mathrm{O}(e^{-cN}).$$

Now, as $\rho_1^{(N,R)}$ and ρ are supported on $[-R, R]$, $m_N^{(R)} - m$ and $c_N^{(R)}$ are uniformly $\mathrm{O}(1)$ in the vertical segments of $\mathcal{L}$. Consequently, from the above equation

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{L}}\frac{(m_N^{(R)}(\xi)-m(\xi))^2 + s(\xi)(m_N^{(R)}(\xi)-m(\xi)) + b_N(\xi) - c_N^{(R)}(\xi)}{r(\xi)(z-\xi)}\mathrm{d}\xi = \mathrm{O}(N^{-10}).$$

As b_N and r are analytic inside $\mathcal{L}$, and z is outside we get

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{L}}\frac{(m_N^{(R)}(\xi)-m(\xi))^2 + s(\xi)(m_N^{(R)}(\xi)-m(\xi)) - c_N^{(R)}(\xi)}{r(\xi)(z-\xi)}\mathrm{d}\xi = \mathrm{O}(N^{-10}).$$

Remember we define a $f(z)=\sqrt{(A-z)(B-z)}$ uniquely by $f(z)\sim z$ as $z\to\infty$. Moreover, $|m_N^{(R)}-m|(z)=\mathrm{O}(z^{-2})$ as $|z|\to\infty$ because ρ and $\rho_1^{(N,R)}$ are compactly supported:

$$|m_N^{(R)}(z)-m(z)|=\left|\int_{-R}^{R}\frac{\rho(t)-\rho^{(N,R)}(t)}{z-t}\mathrm{d}t\right|$$
$$=\left|\int_{R}^{R}(\rho(t)-\rho^{(N,R)}(t))\left(\frac{1}{z}+\mathrm{O}\left(\frac{1}{z^2}\right)\right)\mathrm{d}t\right|=\mathrm{O}\left(z^{-2}\right).$$

Consequently, the function $s(m_N^{(R)}-m)/r=-2f(m_N^{(R)}-m)$ is $\mathrm{O}(z^{-1})$ as $|z|\to\infty$. Moreover, it is analytic outside $\mathcal{L}$, so the Cauchy integral formula yields

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{L}}\frac{s(\xi)(m_N^{(R)}(\xi)-m(\xi))}{r(\xi)(z-\xi)}\mathrm{d}\xi=-2f(z)(m_N^{(R)}-m)(z),$$

proving

$$-2f(z)(m_N^{(R)}(z)-m(z))=-\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{L}}\frac{(m_N^{(R)}(\xi)-m(\xi))^2-c_N^{(R)}(\xi)}{r(\xi)(z-\xi)}\mathrm{d}\xi+\mathrm{O}(N^{-10}). \tag{3.18}$$

Consider now the following rectangular contours, defined by their vertices:

$$\begin{aligned}&\mathcal{L}_1: R+\varepsilon+\mathrm{i}\varepsilon, -R-\varepsilon+\mathrm{i}\varepsilon, -R-\varepsilon-\mathrm{i}\varepsilon, R+\varepsilon-\mathrm{i}\varepsilon,\\&\mathcal{L}_2: R+2\varepsilon+2\mathrm{i}\varepsilon, -R-2\varepsilon+2\mathrm{i}\varepsilon, -R-2\varepsilon-2\mathrm{i}\varepsilon, R+2\varepsilon-2\mathrm{i}\varepsilon,\end{aligned} \tag{3.19}$$

where $\varepsilon>0$ is fixed, small enough such that all zeros of r are strictly outside $\mathcal{L}_2$. For z inside $\mathcal{L}_2$ and $\mathrm{Im}(z)\geq N^{-1}$, by the Cauchy formula, equation (3.18) implies that

$$\begin{aligned}&-2f(z)(m_N^{(R)}(z)-m(z))\\&=-(m_N^{(R)}(z)-m(z))^2+c_N^{(R)}(z)-\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{L}_2}\frac{(m_N^{(R)}(\xi)-m(\xi))^2-c_N^{(R)}(\xi)}{r(\xi)(z-\xi)}\mathrm{d}\xi+\mathrm{O}(N^{-10}).\end{aligned} \tag{3.20}$$

In the above expression, if now z is on $\mathcal{L}_1$, $|z-\xi|>\varepsilon$, and on $\mathcal{L}_2$ $|r|$ is separated away from zero by a positive universal constant. Moreover, $c_N^{(R)}(\xi)$ can be bounded in the following way: for any constants $\alpha_1,\dots,\alpha_N\in[-R-\varepsilon,R+\varepsilon]$,

$$\begin{aligned}\frac{1}{N^2}\left|\mathrm{var}_{\mu^{(N,R)}}\left(\sum_{k=1}^N\frac{1}{\xi-\lambda_k}\right)\right|&\leq\frac{1}{N^2}\mathbb{E}_{\mu^{(N,R)}}\left(\left|\sum_{k=1}^N\frac{1}{\xi-\lambda_k}-\frac{1}{\xi-\alpha_k}\right|^2\right)\\&\leq\frac{1}{\varepsilon^4N}\sum_{k=1}^N\mathbb{E}_{\mu^{(N,R)}}\left(|\lambda_k-\alpha_k|^2\right),\end{aligned}$$

because for any k, we have $|\xi-\lambda_k|>\varepsilon$, $|\xi-\alpha_k|>\varepsilon$. Now, choose $\alpha_k=\mathbb{E}_\mu(\lambda_k)$. By (3.8), for large enough N any α_k, $1\leq k\leq N$, is in $[-R-\varepsilon,R+\varepsilon]$ indeed. Moreover, by (3.7),

$$\left|\mathbb{E}_{\mu^{(N,R)}}\left(|\lambda_k-\alpha_k|^2\right)-\mathbb{E}_\mu\left(|\lambda_k-\alpha_k|^2\right)\right|\leq e^{-cN}\mathbb{E}_{\mu^{(N,R)}}\left(|\lambda_k-\alpha_k|^2\right),$$

and, by the spectral gap inequality for μ, $\mathbb{E}_\mu\left(|\lambda_k-\alpha_k|^2\right)=\mathrm{O}(N^{-1})$. This together proves that $k_N^{(R)}(\xi)$ is $\mathrm{O}(N^{-1})$, uniformly on the contour $\mathcal{L}_2$. Moreover, $\frac{1}{N}{m_N^{(R)}}'=\mathrm{O}(N^{-1})$, so finally $c_N^{(R)}(\xi)$ is uniformly $\mathrm{O}(N^{-1})$ on $\mathcal{L}_2$ and (3.20) implies

$$-2f(z)(m_N^{(R)}(z)-m(z))=-(m_N^{(R)}(z)-m(z))^2(z)+\mathrm{O}\left(\sup_{\mathcal{R}_2}|m_N^{(R)}-m|^2\right)+\mathrm{O}(N^{-1}).$$

Moreover, from the maximum principle for analytic functions, $\sup_{\mathcal{L}_2}|m_N^{(R)}-m|\leq\sup_{\mathcal{L}_1}|m_N^{(R)}-m|$, so the previous equation implies

$$\sup_{\mathcal{L}_1}|m_N^{(R)}-m|=\mathrm{O}\left(\sup_{\mathcal{L}_1}|m_N^{(R)}-m|^2+\frac{1}{N}\right).$$

We know that $\rho_1^{(N)}(s)\mathrm{d}s$ converges weakly to $\rho(s)\mathrm{d}s$ (see [2]), so by (3.7) and (3.8) $\rho_1^{(N,R)}(s)\mathrm{d}s$ converges weakly to $\rho(s)\mathrm{d}s$. On $\mathcal{L}_1$, z is at distance at least ε from the support of both $\rho_1^{(N,R)}(s)\mathrm{d}s$ and $\rho(s)\mathrm{d}s$ so, on $\mathcal{L}_1$, $m_N^{(R)}-m$ converges uniformly to 0. This together with the above equation implies that

$$\sup_{\mathcal{L}_1}|m_N^{(R)}-m|=\mathrm{O}\left(\frac{1}{N}\right).$$

By the maximum principle the same estimate holds outside $\mathcal{L}_1$, in particular on $\mathcal{L}_2$, so equation (3.20) implies that for z inside $\mathcal{L}_1$

$$-2f(z)(m_N^{(R)}(z)-m(z))=-(m_N^{(R)}(z)-m(z))^2+c_N^{(R)}(z)+\mathrm{O}\left(\frac{1}{N}\right). \tag{3.21}$$

Moreover,

$$\begin{aligned}&\frac{1}{N}|{m_N^{(R)}}'(z)|=\frac{1}{N^2}\left|\mathbb{E}_{\mu^{(N,R)}}\sum_j\frac{1}{(z-\lambda_j)^2}\right|\\ &\leq\frac{1}{N\eta}\mathrm{Im}\,m_N^{(R)}(z)\leq\frac{1}{N\eta}|m_N^{(R)}(z)-m(z)|+\frac{1}{N\eta}|\mathrm{Im}\ m(z)|\leq\frac{1}{N\eta}|m_N^{(R)}(z)-m(z)|+\frac{c}{N\eta}\end{aligned} \tag{3.22}$$

for some constant c. We used the well-known fact that Im m is uniformly bounded on the upper half plane[1]. On the set $A+\delta<E<B-\delta$ and $|\eta|<\varepsilon$, we have $\inf|f|>0$. Therefore (3.21) takes the form

$$\left(1+\mathrm{O}\left(\frac{1}{N\eta}\right)\right)(m_N^{(R)}(z)-m(z))=\mathrm{O}\left(|m_N^{(R)}(z)-m(z)|^2+\frac{1}{N^2}k_N^{(R)}(z)+\frac{1}{N\eta}\right). \tag{3.23}$$

From the hypothesis (3.15), if $N^{-1+a}\leq\eta\leq\varepsilon$ and $A+\delta<E<B-\delta$, then

$$|m_N^{(R)}-m|\leq c|m_N^{(R)}-m|^2+\varepsilon_N, \tag{3.24}$$

for some $c>0$ and $\varepsilon_N\to0$ as $N\to\infty$. For large N, (3.24) implies that $|m_N^{(R)}-m|\leq2\varepsilon_N$ or $|m_N^{(R)}-m|\geq1/c-2\varepsilon_N$. This together with $|m_N^{(R)}-m|(E+\varepsilon\mathrm{i})\to0$ and the continuity of $|m_N^{(R)}-m|$

[1] This follows for example from properties of the Cauchy operator, see p 183 in [8].

in the upper half plane, this implies that $|m_N^{(R)} - m| \le 2\varepsilon_N$ and therefore $|m_N^{(R)} - m| \to 0$ uniformly on $N^{-1+a} \le \eta \le \varepsilon$, $A + \delta < E < B - \delta$. Consequently, using (3.23), this proves that there is a constant $c > 0$ such that for any $\eta \ge N^{-1+a}$, $A + \delta < E < B - \delta$,

$$|m_N^{(R)}(z) - m(z)| \le c\left(\frac{1}{N\eta} + \frac{1}{N^2} k_N^{(R)}(z)\right).$$

The same conclusion remains when substituting $m_N^{(R)}$ (resp. $k_N^{(R)}$) by m_N (resp. k_N) thanks to (3.7) and (3.8).

□

To prove accuracy results for μ, the above Lemma 3.3 will be combined with the following one.

Lemma 3.4 *Let $\delta < (B-A)/2$ and $E \in [A+\delta, B-\delta]$ and $0 < \eta < \delta/2$. Define a function $f = f_{E,\eta}$: $\mathbb{R} \to \mathbb{R}$ such that $f(x) = 1$ for $x \in (-\infty, E - \eta]$, $f(x)$ vanishes for $x \in [E + \eta, \infty)$, moreover $|f'(x)| \le c\eta^{-1}$ and $|f''(x)| \le c\eta^{-2}$, for some constant c. Let $\widetilde{\rho}$ be an arbitrary signed measure and let $S(z) = \int (z-x)^{-1}\widetilde{\rho}(x)\mathrm{d}x$ be its Stieltjes transform. Assume that, for any $x \in [A+\delta/2, B-\delta/2]$,*

$$|S(x+iy)| \le \frac{U}{Ny} \text{ for } \eta < y < 1, \text{ and } |Im S(x+iy)| \le \frac{U}{Ny} \text{ for } 0 < y < \eta. \tag{3.25}$$

Assume moreover that $\int_{\mathbb{R}} \widetilde{\rho}(\lambda)\mathrm{d}\lambda = 0$ and that there is a real constant $\mathcal{T}$ such that

$$\int_{[-\mathcal{T},\mathcal{T}]^c} |\lambda\widetilde{\rho}(\lambda)|\mathrm{d}\lambda \le \frac{U}{N}. \tag{3.26}$$

Then for some constant $C > 0$, independent of N and $E \in [A + \delta, B - \delta]$, we have

$$\left|\int f_E(\lambda)\widetilde{\rho}(\lambda)\mathrm{d}\lambda\right| \le \frac{CU|\log\eta|}{N}.$$

Proof. Our starting point, relying on the Helffer-Sjöstrand functional calculus, is formula (B.13) in [16]:

$$\left|\int_{-\infty}^{\infty} f_E(\lambda)\widetilde{\rho}(\lambda)\mathrm{d}\lambda\right| \le C\left|\iint y f_E''(x)\chi(y)\mathrm{Im}S(x+\mathrm{i}y)\mathrm{d}x\mathrm{d}y\right| \tag{3.27}$$

$$+ C\iint (|f_E(x)| + |y||f_E'(x)|)\,|\chi'(y)|\,|S(x+\mathrm{i}y)|\,\mathrm{d}x\mathrm{d}y, \tag{3.28}$$

for some universal $C > 0$, and where χ is a smooth cutoff function with support in $[-1, 1]$, with $\chi(y) = 1$ for $|y| \le 1/2$ and with bounded derivatives.

Using (3.25) and (3.26), the support of χ' being included in $1/2 \le |y| \le 1$, and the fact that f_E' is $\mathrm{O}(\eta^{-1})$ on an interval of size $\mathrm{O}(\eta)$, the term (3.28) is easily bounded by $\mathrm{O}\left(\frac{U}{N}\right)$. Concerning the right hand side of (3.27), following [16] we split it depending on $0 < y < \eta$ and $\eta < y < 1$. Note that by symmetry we only need to consider positive y. The integral on the first integration regime is easily bounded by

$$\left|\iint_{0<y<\eta} y f_E''(x)\chi(y)\mathrm{Im}S(x+\mathrm{i}y)\mathrm{d}x\mathrm{d}y\right| = \mathrm{O}\left(\iint_{|x-E|<\eta, 0<y<\eta} y\eta^{-2}\frac{U}{Ny}\mathrm{d}x\mathrm{d}y\right) = \mathrm{O}\left(\frac{U}{N}\right).$$

For the second integral, as f_E'' and χ are real, we can substitute $\mathrm{Im} m$ by m and use the analyticity of m when integrating by parts (first in x, then in y):

$$\begin{aligned}\left|\iint_{\eta<y} y f_E''(x)\chi(y)\mathrm{Im}S(x+\mathrm{i}y)\mathrm{d}x\mathrm{d}y\right| &\leq \left|\iint_{\eta<y} y f_E''(x)\chi(y)S(x+\mathrm{i}y)\mathrm{d}x\mathrm{d}y\right| \\ &= \left|\iint_{\eta<y} y f_E'(x)\chi(y)S'(x+\mathrm{i}y)\mathrm{d}x\mathrm{d}y\right| \\ &\leq \left|\iint_{\eta<y} \partial_y(y\chi(y)) f_E'(x)S(x+\mathrm{i}y)\mathrm{d}x\mathrm{d}y\right| \\ &\quad + \left|\int \eta f_E'(x)\chi(\eta)S(x+\mathrm{i}\eta)\mathrm{d}x\right|.\end{aligned}$$

This last integral is easily bounded by $\mathrm{O}(U/N)$ using (3.25). Concerning the previous one, as $f_E' = \mathrm{O}(\eta^{-1})$, $|x-E|<\eta$ for non vanishing f_E', $\partial_y(y\chi(y)) = \mathrm{O}(1)$ and $S(x+\mathrm{i}y) = \mathrm{O}(U/(Ny))$, this is bounded by

$$\mathrm{O}\left(\frac{U}{N}\int_\eta^1 \frac{\mathrm{d}y}{y}\right) = \mathrm{O}\left(\frac{U|\log\eta|}{N}\right),$$

concluding the proof. □

As a corollary of Lemmas 3.3 and 3.4, we get the accuracy at scale $1/\sqrt{N}$ for the λ_k's in the bulk.

Corollary 3.5 *For any $\alpha>0$ and $\varepsilon>0$ we have*

$$|\gamma_k^{(N)} - \gamma_k| = \mathrm{O}\left(N^{-1/2+\varepsilon}\right)$$

uniformly in $k \in [\![\alpha N, (1-\alpha)N]\!]$ where $\gamma_k^{(N)}$ and γ_k are defined by

$$\int_{-\infty}^{\gamma_k^{(N)}} \rho_1^{(N)}(s)\mathrm{d}s = \frac{k}{N}, \quad \textit{and} \quad \int_{-\infty}^{\gamma_k} \rho(s)\mathrm{d}s = \frac{k}{N}.$$

Proof. We will apply Lemma 3.4 to $\widetilde{\rho} = \rho - \rho_1^{(N)}$ with $\eta = N^{-1/2+\varepsilon}$, and check the conditions on $S = m - m_N$. We denote $z = x + \mathrm{i}y$.

By the spectral gap inequality for the measure μ, we get

$$\frac{1}{N^2}\left|\mathrm{var}_\mu\left(\sum_{k=1}^N \frac{1}{z-\lambda_k}\right)\right| \leq \frac{c}{N^3}\mathbb{E}_\mu\left(\left|\nabla\left(\sum_{k=1}^N \frac{1}{z-\lambda_k}\right)\right|^2\right) \leq \frac{c}{N^2y^4}. \tag{3.29}$$

Together with Lemma 3.3, this implies that uniformly in $N^{-1/2+\varepsilon} \leq y \leq 1$ and $x \in [A+\delta/2, B-\delta/2]$, we have

$$|m_N(z) - m(z)| = \mathrm{O}\left(\frac{1}{N} + \frac{1}{N^2y^4}\right) = \mathrm{O}\left(\frac{\sqrt{N}}{Ny}\right).$$

For $0<y<N^{-1/2+\varepsilon}$, m is uniformly bounded and

$$y \mapsto y\,\mathrm{Im} m_N(x+\mathrm{i}y) = \int \frac{y^2}{(x-t)^2+y^2}\rho^{(N)}(t)\mathrm{d}t$$

is an increasing function, so denoting $y_0 = N^{-1/2+\varepsilon}$ we have

$$y\,|\mathrm{Im}(m_N(x+\mathrm{i}y)-m(x+\mathrm{i}y))| \leq y_0\,\mathrm{Im} m_N(x+\mathrm{i}y_0)+\mathrm{O}(y) \leq y_0|\mathrm{Im}(m_N(x+\mathrm{i}y_0)-m(x+\mathrm{i}y_0))|+\mathrm{O}(y_0).$$

Therefore, for any $0 < y < N^{-1/2+\varepsilon}$,

$$|\mathrm{Im}(m_N(x+\mathrm{i}y)-m(x+\mathrm{i}y))| = \mathrm{O}\left(\frac{N^{1/2+\varepsilon}}{Ny}\right).$$

Finally, the condition (3.26) with $U = \mathrm{O}(N^{1/2+\varepsilon})$ and with the choice of any $\mathcal{T} \geq \max(|A|,|B|)+\delta$ follows from the large deviation estimate (3.5).

Using the conclusion of Lemma 3.4 for functions f_E and $f_{E+\eta}$ defined in the same lemma, and subtracting both results, we get that uniformly in $E \in [A+\delta, B-\delta]$,

$$\left|\int_{-\infty}^{E}(\rho^{(N)}(t)-\rho(t))\mathrm{d}t\right| = \mathrm{O}(N^{-1/2+\varepsilon}), \tag{3.30}$$

so if $\gamma_k^{(N)} \in [A+\delta, B-\delta]$, then $|\gamma_k^{(N)} - \gamma_k| = \mathrm{O}\left(N^{-1/2+\varepsilon}\right)$. This estimate holds uniformly in $k \in [\![\alpha N, (1-\alpha)N]\!]$: as a consequence of (3.30) and the smooth form (3.3) of ρ, for any $k \in [\![\alpha N, (1-\alpha)N]\!]$ and sufficiently large N we have $\gamma_k^{(N)} \in [A+\delta, B-\delta]$, for $\delta > 0$ small enough, concluding the proof. □

Lemma 3.6 *For any $\varepsilon > 0$ there exists c_1, c_2, ε' positive constants such that for any $N^{3/5+\varepsilon} \leq j \leq N - N^{3/5+\varepsilon}$, we have*

$$\mathbb{P}_\mu\left(|\lambda_j - \gamma_j| \geq N^{-4/15+\varepsilon}\right) \leq c_1 e^{-c_2 N^{\varepsilon'}}.$$

Proof. We will assume that $j < N/2$ in the following, i.e. we will estimate the accuracy near the edge A, the proof close to the other edge B being analogous. We follow the notations used in Corollary 3.5 and Lemma 3.4. For $E \in [A-\delta, A+\delta] \cup [B-\delta, B+\delta]$, we have $\inf |f|(z) \geq \sqrt{\eta}$. Therefore we can divide $-2f(z)$ on both side of (3.21) to have

$$\left(1+\mathrm{O}\left(\frac{1}{N\eta^{3/2}}\right)\right)(m_N^{(R)}(z)-m(z)) = \mathrm{O}\left(\frac{|m_N^{(R)}(z)-m(z)|^2}{\sqrt{\eta}} + \frac{1}{N^2\sqrt{\eta}}k_N^{(R)}(z) + \frac{1}{N\eta^{3/2}}\right). \tag{3.31}$$

By (3.29), (3.7) and (3.8) we can bound the variance term by

$$\frac{1}{N^2}k_N^{(R)}(z) \leq \frac{c}{N^2\eta^4}. \tag{3.32}$$

for $\eta \geq N^{-10}$ for example. Following the same continuity argument in the proof of Lemma 3.3, we obtain that for any $\varepsilon > 0$

$$|\mathrm{Im}(m_N - m)(x+\mathrm{i}\eta)| = \mathrm{O}\left(\frac{N^\varepsilon}{N^2\eta^{9/2}}\right),$$

provided that

$$\frac{1}{\sqrt{\eta}}\left[\frac{1}{N^2\sqrt{\eta}}k_N^{(R)}(z) + \frac{1}{N\eta^{3/2}}\right] \leq \frac{c}{N^2\eta^5} \ll 1. \tag{3.33}$$

We can now follow the argument in the proof of Corollary 3.5 so that (3.25) holds with $U = N^{3/5}$. Since the condition (3.26) is easy to verify, we thus have

$$\left| \int f_E(t) \left[\rho^{(N)}(t) - \rho(t) \right] \mathrm{d}t \right| \leq \frac{C|\log \eta|}{N^{2/5}}, \quad \eta = N^{-2/5},$$

where f_E is defined in Lemma 3.4. This proves that

$$\int_{-\infty}^{E-\eta} \rho^{(N)}(t) \mathrm{d}t \leq \int_{-\infty}^{E+\eta} \rho(t) \mathrm{d}t + \frac{C|\log \eta|}{N^{2/5}}.$$

In particular,

$$\frac{j}{N} - \frac{C|\log \eta|}{N^{2/5}} = \int_{-\infty}^{\gamma_j^{(N)}} \rho^{(N)}(t) \mathrm{d}t - \frac{C|\log \eta|}{N^{2/5}} \leq \int_{-\infty}^{\gamma_j^{(N)}+2\eta} \rho(t) \mathrm{d}t,$$

and we have, by definition of γ_i, that

$$\gamma_{j-N^{3/5+\varepsilon}} \leq \gamma_j^{(N)} + 2\eta.$$

Similarly, the reverse inequality holds and we have

$$\gamma_{j-N^{3/5+\varepsilon}} - 2\eta \leq \gamma_j^{(N)} \leq \gamma_{j+N^{3/5+\varepsilon}} + 2\eta.$$

Since $\int_A^E \rho(t) \mathrm{d}t \sim (E-A)^{3/2}$, for $j \geq N^{3/5+\varepsilon}$ we have

$$|\gamma_{j-N^{3/5+\varepsilon}} - \gamma_j| \leq C \left(\frac{j}{N} \right)^{-1/3} N^{-2/5+\varepsilon/2} \leq N^{-4/15+\varepsilon}.$$

Moreover, by (3.12), λ_j is concentrated around $\mathbb{E}_\mu(\lambda_j)$ at scale $N^{-1/2}$, so $|\mathbb{E}_\mu(\lambda_j) - \gamma_j^{(N)}| = \mathrm{O}(N^{-1/2})$, concluding the proof of the lemma. □

3.2 The locally constrained measures

In this section some arbitrary $\varepsilon, \alpha > 0$ are fixed. Let θ be a continuous nonnegative function with $\theta = 0$ on $[-1,1]$ and $\theta'' \geq 1$ for $|x| > 1$. We can take for example $\theta(x) = (x-1)^2 \mathbb{1}_{x>1} + (x+1)^2 \mathbb{1}_{x<-1}$ in the following.

Definition 3.7 *For a given $k \in [\![\alpha N, (1-\alpha)N]\!]$ and any integer $1 \leq M \leq \alpha N$, we denote $I^{(k,M)} = [\![k-M, k+M]\!]$ and $i_M = |I^{(k,M)}| = 2M+1$. Moreover, let*

$$\phi^{(k,M)} = \beta \sum_{i<j, i,j \in I^{(k,M)}} \theta \left(\frac{N^{1-\varepsilon}(\lambda_i - \lambda_j)}{i_M} \right).$$

We define the probability measure

$$\mathrm{d}\omega^{(k,M)} := \frac{1}{Z} e^{-\phi^{(k,M)}} \mathrm{d}\mu, \tag{3.34}$$

where $Z = Z_{\omega^{(k,M)}}$. The measure $\omega^{(k,M)}$ will be referred to as locally constrained transform of μ, around k, with width M. The dependence of the measure on ε will be suppressed in the notation.

We will also frequently use the following notation for block averages in any sequence $x_1, x_2, \dots$

$$x_k^{[M]} := \frac{1}{i_M} \sum_{i \in I^{(k,M)}} x_i. \tag{3.35}$$

The reason for introducing these locally constrained measures is that they improve the convexity in $I^{(k,M)}$ up to a common shift, as explained in the following lemma.

Lemma 3.8 *Consider the previously defined probability measure*

$$\omega^{(k,M)} = \frac{1}{Z} e^{-\phi^{(k,M)}} \mathrm{d}\mu = \frac{1}{\tilde{Z}} e^{-N(\mathcal{H}_1 + \mathcal{H}_2)} \mathrm{d}\lambda,$$

where we denote

$$\mathcal{H}_1 := \frac{1}{N}\phi^{(k,M)} - \frac{\beta}{N} \sum_{i<j, i,j \in I^{(k,M)}} \log|\lambda_i - \lambda_j|,$$

$$\mathcal{H}_2 := -\frac{\beta}{N} \sum_{(i,j) \in J^{(k,M)}} \log|\lambda_i - \lambda_j| + \frac{\beta}{2} \sum_{i=1}^{N} V(\lambda_i)$$

where $J^{(k,M)}$ is the set of pairs of points $i < j$ in $[\![1, N]\!]$ such that i or j is not in $I^{(k,M)}$, and $\mathcal{H}_1 = \mathcal{H}_1(\lambda_{k-M}, \dots, \lambda_{k+M})$. Then $\nabla^2 \mathcal{H}_2 > 0$ and denoting $v = (v_i)_{i \in I^{(k,M)}}$, we also have

$$v^*(\nabla^2 \mathcal{H}_1) v \geq \frac{\beta}{2} \frac{N^{1-2\varepsilon}}{i_M} \sum_{i,j \in I^{(k,M)}} (v_i - v_j)^2. \tag{3.36}$$

Proof. Since V is convex, to prove the convexity of $\mathcal{H}_2$, it suffices to prove it for the Coulomb interaction terms; this relies on the calculation, for any $u \in \mathbb{R}^N$,

$$u^*(\nabla^2 \mathcal{H}_2(\lambda)) u = \frac{\beta}{N} \sum_{J^{(k,M)}} \frac{(u_i - u_j)^2}{(\lambda_i - \lambda_j)^2} \geq 0.$$

Moreover, for any $v \in \mathbb{R}^{i_M}$, a similar calculation yields

$$v^*(\nabla^2 \mathcal{H}_1) v \geq \frac{\beta}{N} \sum_{i<j, i,j \in I^{(k,M)}} \frac{(v_i - v_j)^2}{(\lambda_i - \lambda_j)^2} + \beta \frac{N^{1-2\varepsilon}}{i_M^2} \sum_{i<j, i,j \in I^{(k,M)}} (v_i - v_j)^2 \theta'' \left(\frac{N^{1-\varepsilon}(\lambda_i - \lambda_j)}{i_M} \right). \tag{3.37}$$

From our definition of θ,

$$\frac{1}{(\lambda_i - \lambda_j)^2} + \frac{N^{2-2\varepsilon}}{i_M^2} \theta'' \left(\frac{N^{1-\varepsilon}(\lambda_i - \lambda_j)}{i_M} \right) \geq \frac{N^{2-2\varepsilon}}{i_M^2},$$

which implies

$$v^*(\nabla^2 \mathcal{H}_1) v \geq \beta \frac{N^{1-2\varepsilon}}{i_M^2} \sum_{i<j, i,j \in I^{(k,M)}} (v_i - v_j)^2. \tag{3.38}$$

which completes the proof of (3.36), noting that the above factor $1/2$ comes from the strict ordering of i and j indexes in (3.38). $\square$

The above convexity bound, associated with the following local criterion for the logarithmic Sobolev inequality, will yield a strong concentration for $\sum_{i\in I^{(k,M)}} v_i\lambda_i$ under $\omega^{(k,M)}$, if $\sum_i v_i = 0$. This lemma is a consequence of the Brascamp-Lieb inequality [7]. Notice that the original inequality applied only to measures on $\mathbb{R}^N$, but a mollifying argument in Lemma 4.4 of [17] has extended it to the measures on the simplex $\{\lambda_1 < \lambda_2 < \ldots < \lambda_N\}$ considered in this paper.

Lemma 3.9 *Decompose the coordinates $\lambda = (\lambda_1, \ldots, \lambda_N)$ of a point in $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^{N-m}$ as $\lambda = (x, y)$, where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^{N-m}$. Let $\omega = \frac{1}{Z}e^{-N\mathcal{H}}$ be a probability measure on $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^{N-m}$ such that $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$, with $\mathcal{H}_1 = \mathcal{H}_1(x)$ depending only on the x variables and $\mathcal{H}_2 = \mathcal{H}_2(x, y)$ depending on all coordinates. Assume that, for any $\lambda \in \mathbb{R}^N$, $\nabla^2\mathcal{H}_2(\lambda) > 0$. Assume moreover that $\mathcal{H}_1(x)$ is independent of $x_1 + \ldots + x_m$, i.e., $\sum_{j=1}^m \partial_{x_j}\mathcal{H}_1 = 0$, and that for any $x, v \in \mathbb{R}^m$,*

$$v^*(\nabla^2\mathcal{H}_1(x))v \geq \frac{\xi}{m}\sum_{i,j=1}^m |v_i - v_j|^2 \tag{3.39}$$

with some positive $\xi > 0$. Then for any function of the form $f(\lambda) = F(\sum_{i=1}^m v_ix_i)$, where $\sum_i v_i = 0$ and $F : \mathbb{R} \to \mathbb{R}$ is any smooth function, we have

$$\int f^2 \log f^2 \mathrm{d}\omega - \left(\int f^2 \mathrm{d}\omega\right)\log\left(\int f^2\mathrm{d}\omega\right) \leq \frac{1}{\xi N}\int |\nabla f|^2 \mathrm{d}\omega. \tag{3.40}$$

Proof. In the space $\mathbb{R}^m$ we introduce new coordinates $(z, w) = M^*(x_1, \ldots, x_m)$ with $z = (z_1, \ldots, z_{m-1}) \in \mathbb{R}^{m-1}$,

$$w := m^{-1/2}\sum_i x_i, \tag{3.41}$$

and M is an orthogonal matrix. Since $\mathcal{H}_1(x)$ is independent of $x_1 + \ldots + x_m$, we can define $\widehat{\mathcal{H}}_1(z) := \mathcal{H}_1(x)$. Similarly, the function $f(\lambda) = F(\sum_{i=1}^m v_ix_i)$ with $\sum_i v_i = 0$ depends only on the z coordinates, i.e. it can be written as $g(z) = f(\lambda)$. Hence we can rewrite

$$\int_{\mathbb{R}^N} f^2 \log f^2\mathrm{d}\omega = \int_{\mathbb{R}^{m-1}} g^2\log g^2 \mathrm{d}\nu, \qquad \int_{\mathbb{R}^N} f^2\mathrm{d}\omega = \int_{\mathbb{R}^{m-1}} g^2\mathrm{d}\nu,$$

where $\mathrm{d}\nu = \nu(z)\mathrm{d}z$ with

$$\nu(z) := \frac{1}{\tilde{Z}}e^{-N\tilde{\mathcal{H}}(z)} = \frac{1}{Z}\int_{\mathbb{R}\times\mathbb{R}^{N-m}} e^{-N\mathcal{H}(x,y)}\mathrm{d}w\mathrm{d}y.$$

Introduce the variable $q = (w, y) \in \mathbb{R} \times \mathbb{R}^{N-m}$ and denote by $\mathcal{H}_{qq}, \mathcal{H}_{zq}, \mathcal{H}_{zz}$ the matrices of second partial derivatives. As $\mathcal{H}_2$ is convex, the Brascamp-Lieb inequality yields

$$\tilde{\mathcal{H}}_{zz} \geq \frac{1}{Z}\int_{\mathbb{R}\times\mathbb{R}^{N-m}} e^{-N\mathcal{H}(x,y)}\Big[\mathcal{H}_{zz} - \mathcal{H}_{zq}[\mathcal{H}_{qq}]^{-1}\mathcal{H}_{zq}\Big]\mathrm{d}w\mathrm{d}y.$$

Since $\mathcal{H}_1$ is independent of q, we have, by assumption of the positivity of the Hessian of $\mathcal{H}_2$, that for any q fixed,

$$(\mathcal{H}_2)_{zz} - \mathcal{H}_{zq}[\mathcal{H}_{qq}]^{-1}\mathcal{H}_{zq} = (\mathcal{H}_2)_{zz} - (\mathcal{H}_2)_{zq}[(\mathcal{H}_2)_{qq}]^{-1}(\mathcal{H}_2)_{zq} \geq 0. \tag{3.42}$$

Thus we have, for any $u \in \mathbb{R}^{m-1}$, that

$$u^*\tilde{\mathcal{H}}_{zz}u \geq u^*(\hat{\mathcal{H}}_1)_{zz}u = u^*\widetilde{M}^*(\mathcal{H}_1)_{xx}\widetilde{M}u \geq \frac{\xi}{m}\sum_{i,j}[(\widetilde{M}u)_i - (\widetilde{M}u)_j]^2, \tag{3.43}$$

where $\widetilde{M}$ denotes the first $m-1$ columns of M. Since the last column of M is parallel with $(1,1,\dots,1)\in\mathbb{R}^m$ and M is an orthogonal matrix, we have $\sum_i(\widetilde{M}u)_i=0$ and

$$\frac{\xi}{m}\sum_{i,j=1}^{m}[(\widetilde{M}u)_i-(\widetilde{M}u)_j]^2=2\xi\sum_{i=1}^{m}[(\widetilde{M}u)_i]^2=2\xi\sum_{i=1}^{m-1}u_i^2. \tag{3.44}$$

Hence the measure $\nu\sim\exp(-N\widetilde{\mathcal{H}})$ is log-concave with a lower bound $2N\xi$ on the Hessian of $N\widetilde{\mathcal{H}}$, and we can apply the Bakry-Emery argument to prove the logarithmic Sobolev inequality for ν. Without loss of generality we can assume that $\int f^2\mathrm{d}\omega=\int g^2\mathrm{d}\nu=1$. Therefore, we have

$$\int_{\mathbb{R}^N} f^2\log f^2\mathrm{d}\omega=\int_{\mathbb{R}^{m-1}} g^2\log g^2\mathrm{d}\nu\leq\frac{1}{N\xi}\int_{\mathbb{R}^{m-1}}|\nabla_z g|^2\mathrm{d}\nu=\frac{1}{N\xi}\int_{\mathbb{R}^N}|\nabla_x f|^2\mathrm{d}\omega, \tag{3.45}$$

where we have used the orthogonality of M to show that $|\nabla_z g|^2=|\nabla_x f|^2$. This proves the estimate (3.40). □

It is now immediate, from Lemma 3.8, Lemma 3.9 and Herbst's lemma, that the following concentration holds.

Corollary 3.10 *For any function $f(\{\lambda_i, i\in I^{(k,M)}\})=\sum_{I^{(k,M)}}v_i\lambda_i$ with $\sum_i v_i=0$ we have*

$$\mathbb{P}_{\omega^{(k,M)}}(|f-\mathbb{E}_{\omega^{(k,M)}}(f)|>x)\leq 2\exp\left(-\frac{\beta}{4}\frac{N^{2-2\varepsilon}}{i_M|v|^2}x^2\right).$$

Choosing $v_j=-v_{j+1}=1$ and all other v_i's being zero, this corollary shows that the particle differences $\lambda_j-\lambda_{j+1}$ concentrate around their mean with respect to the $\omega^{(k,M)}$ measure. By choosing ε small and M almost order one, we obtain concentration almost up to the optimal scale $1/N$. If we can justify that the measures $\omega^{(k,M)}$ and μ are very close (in a sense to be defined), we will have concentration of differences at the optimal scale for μ. We will then separately show, by using the loop equation, that accuracy will hold at the same scale as well. This is the purpose of the next subsection, through an inductive argument.

3.3 The induction

The purpose of this paragraph is to prove the following proposition: if accuracy holds at scale N^{-1+a}, it holds also at scale $N^{-1+\frac{3}{4}a}$.

Proposition 3.11 *Assume that for some $a\in(0,1)$ the following property holds: for any $\alpha,\varepsilon>0$, there are constants $\delta,c_1,c_2>0$ such that for any $N\geq1$ and $k\in[\![\alpha N,(1-\alpha)N]\!]$,*

$$\mathbb{P}_\mu\left(|\lambda_k-\gamma_k|>N^{-1+a+\varepsilon}\right)\leq c_1e^{-c_2N^\delta}. \tag{3.46}$$

Then the same property holds also replacing a by $3a/4$: for any $\alpha,\varepsilon>0$, there are constants $\delta,c_1,c_2>0$ such that for any $N\geq1$ and $k\in[\![\alpha N,(1-\alpha)N]\!]$, we have

$$\mathbb{P}_\mu\left(|\lambda_k-\gamma_k|>N^{-1+\frac{3}{4}a+\varepsilon}\right)\leq c_1e^{-c_2N^\delta}.$$

Proof of Theorem 3.1. This is an immediate consequence of the initial estimate, Proposition 3.2, and iterations of Proposition 3.11. □

Two steps are required in the proof of the above Proposition 3.11. First we will prove that concentration holds at the smaller scale $N^{-1+\frac{a}{2}}$.

Proposition 3.12 *Assume that (3.46) holds. Then for any $\alpha > 0$ and $\varepsilon > 0$, there are constants $c_1, c_2, \delta > 0$ such that for any $N \geq 1$ and $k \in [\![\alpha N, (1-\alpha)N]\!]$,*

$$\mathbb{P}_\mu\left(|\lambda_k - \mathbb{E}_\mu(\lambda_k)| > \frac{N^{\frac{a}{2}+\varepsilon}}{N}\right) \leq c_1 e^{-c_2 N^\delta}.$$

The above step builds on the locally constrained measures of the previous subsection. Then, knowing this better concentration, the accuracy can be improved to the scale $N^{-1+\frac{3a}{4}}$.

Proposition 3.13 *Assume that (3.46) holds. Then for any $\alpha > 0$ and $\varepsilon > 0$, there is a constant $c > 0$ such that for any $N \geq 1$ and $k \in [\![\alpha N, (1-\alpha)N]\!]$,*

$$\left|\gamma_k^{(N)} - \gamma_k\right| \leq c\frac{N^{\frac{3a}{4}+\varepsilon}}{N}.$$

Proposition 3.11 is an immediate consequence of the last two propositions. The proofs of these two propositions are postponed to the end of this section, after the following necessary series of lemmas.

Lemma 3.14 *Take any $\varepsilon > 0$ and $\alpha > 0$. There are constants $c_1, c_2 > 0$ such that for any $N \geq 1$, any integers $1 \leq M_1 \leq M \leq \alpha N$, any $k \in [\![\alpha N, (1-\alpha)N]\!]$, and $\omega^{(k,M)}$ from Definition 3.7 associated with k, M, ε,*

$$\mathbb{P}_{\omega^{(k,M)}}\left(\left|\lambda_k^{[M_1]} - \lambda_k^{[M]} - \mathbb{E}_{\omega^{(k,M)}}\left(\lambda_k^{[M_1]} - \lambda_k^{[M]}\right)\right| > \frac{x}{N^{1-\varepsilon}}\sqrt{\frac{M}{M_1}}\right) \leq c_1 e^{-c_2 x^2}.$$

Proof. Note that $\lambda_k^{[M_1]} - \lambda_k^{[M]}$ is of type $\sum_{I^{(k,M)}} v_i \lambda_i$ with $\sum v_i = 0$ and

$$|v|^2 = \sum_1^{M_1}\left(\frac{1}{M_1} - \frac{1}{M}\right)^2 + \sum_{M_1+1}^{M}\frac{1}{M^2} \leq \sum_1^{M_1}\left(\frac{2}{M_1^2} + \frac{2}{M^2}\right) + \sum_{M_1+1}^{M}\frac{2}{M^2} \leq \frac{4}{M_1}.$$

This together with Corollary 3.10 concludes the proof. □

Lemma 3.15 *Assume that for μ accuracy holds at scale N^{-1+a}, i.e. (3.46). Take arbitrary $\alpha, \varepsilon > 0$. There exist constants $c_1, c_2, \delta > 0$ such that for any $N \geq 1$, for any integer M satisfying $N^a \leq M \leq \alpha N/2$, for any $k \in [\![\alpha N, (1-\alpha)N]\!]$ and for any $j \in [\![1, N]\!]$, we have*

$$|\mathbb{E}_\mu(\lambda_j) - \mathbb{E}_{\omega^{(k,M)}}(\lambda_j)| \leq c_1 e^{-c_2 N^\delta},$$

where the measure $\omega^{(k,M)}$ is defined in (3.34) from Definition 3.7 with parameters k, M, ε.

Proof. First, the total variation norm is bounded by the square root of the entropy, and by (3.8) the particles are bounded with very high probability, so we have

$$|\mathbb{E}_\mu(\lambda_j) - \mathbb{E}_{\omega^{(k,M)}}(\lambda_j)| \leq \tilde{C}\sqrt{S(\mu \mid \omega^{(k,M)})} + \mathrm{O}(e^{-\tilde{c}N})$$

for some $\tilde{c}, \tilde{C} > 0$ independent of k, j. For the measures we are interested in, using the logarithmic Sobolev inequality for μ, we have for some $c, C > 0$

$$S(\mu \mid \omega^{(k,M)}) \leq CN^c \, \mathbb{E}_\mu \left(\theta' \left(\frac{(\lambda_{k+M} - \lambda_{k-M}) N^{1-\varepsilon}}{i_M} \right)^2 \right).$$

Now, as $\theta''(x) = 0$ if $|x| < 1$ and $\theta'(x)^2 < 4x^2$, for some new and universal constants $c, C > 0$

$$\begin{aligned} S(\mu \mid \omega^{(k,M)}) &\leq CN^c \, \mathbb{E}_\mu \left((\lambda_{k+M} - \lambda_{k-M})^2 \mathbf{1}_{|\lambda_{k+M} - \lambda_{k-M}| > \frac{i_M}{N^{1-\varepsilon}}} \right) \\ &\leq CN^c \left[\mathbb{E}_\mu \left((\lambda_{k+M} - \lambda_{k-M})^4 \right) \right]^{1/2} \left[\mathbb{P}_\mu \left(|\lambda_{k+M} - \lambda_{k-M}| > \frac{i_M}{N^{1-\varepsilon}} \right) \right]^{1/2}. \end{aligned} \tag{3.47}$$

This moment of order 4 is polynomially bounded, for example just by concentration of order $N^{-1/2}$ for all λ_j's under μ. Concerning the above probability, as $|\gamma_{k+M} - \gamma_{k-M}| = \mathrm{O}(M/N)$, for sufficiently large N if $|\lambda_{k+M} - \lambda_{k-M}| > \frac{i_M}{N^{1-\varepsilon}}$ then either $|\lambda_{k+M} - \gamma_{k+M}| > M/N^{1-\varepsilon}$ or $|\lambda_{k-M} - \gamma_{k-M}| > M/N^{1-\varepsilon}$. But accuracy holds at scale $N^{-1+a} < M/N$, so both previous events have exponentially small probabilities, uniformly in k. Indeed, one has $k - M, k + M \in [\![\alpha N/2, (1-\alpha/2)N]\!]$ and by (3.46)there are constants $\delta, c_1, c_2 > 0$ such that for any $N \geq 1$ and $k \in [\![\alpha N/2, (1-\alpha/2)N]\!]$,

$$\mathbb{P}_\mu \left(|\lambda_k - \gamma_k| > M/N^{1-\varepsilon} \right) \leq c_1 e^{-c_2 N^\delta}.$$

This concludes the proof. □

Lemma 3.16 *Assume that for μ accuracy and concentration hold at scale N^{-1+a}. Take arbitrary $\alpha, \tilde{\varepsilon} > 0$. There are constants $c_1, c_2, \delta > 0$ such that for any $N \geq 1$, any integers $N^a \leq M \leq \alpha N$, $1 \leq M_1 \leq M$, and $k \in [\![2\alpha N, (1-2\alpha)N]\!]$,*

$$\mathbb{P}_\mu \left(\left| \lambda_k^{[M_1]} - \lambda_k^{[M]} - \mathbb{E}_\mu \left(\lambda_k^{[M_1]} - \lambda_k^{[M]} \right) \right| > \frac{N^{\tilde{\varepsilon}}}{N} \sqrt{\frac{M}{M_1}} \right) \leq c_1 e^{-c_2 N^\delta}.$$

Proof. Consider the measure $\omega^{(k,M)}$ associated with the choice $\varepsilon = \tilde{\varepsilon}/2$. First note that, by Lemma 3.15,

$$\left| \mathbb{E}_\mu \left(\lambda_k^{[M_1]} - \lambda_k^{[M]} \right) - \mathbb{E}_{\omega^{(k,M)}} \left(\lambda_k^{[M_1]} - \lambda_k^{[M]} \right) \right| < c e^{-CN^{\delta_1}},$$

for some coefficients c, C, δ_1, uniformly in N, k, M, M_1. As a consequence of this exponentially small difference of expectations, the probability bound to prove is equivalent to the existence of $c_1, c_2, \delta > 0$ such that

$$\mathbb{P}_\mu (A) \leq c_1 e^{-c_2 N^\delta}, \ A = \left\{ \left| \lambda_k^{[M_1]} - \lambda_k^{[M]} - \mathbb{E}_{\omega^{(k,M)}} \left(\lambda_k^{[M_1]} - \lambda_k^{[M]} \right) \right| > \frac{N^{\tilde{\varepsilon}}}{N} \sqrt{\frac{M}{M_1}} \right\},$$

with the same uniformity requirements. By Lemma 3.14, we know that there are such constants with

$$\mathbb{P}_{\omega^{(k,M)}} (A) \leq c_1 e^{-c_2 N^\delta},$$

so the proof will be complete if we can prove that $|\mathbb{P}_{\omega^{(k,M)}} (A) - \mathbb{P}_\mu (A)|$ is uniformly exponentially small. By the total variation/entropy inequality we have:

$$|\mathbb{P}_{\omega^{(k,M)}} (A) - \mathbb{P}_\mu (A)| \leq \int |\mathrm{d}\omega^{(k,M)} - \mathrm{d}\mu| \leq \sqrt{2S(\mu \mid \omega^{(k,M)})}. \tag{3.48}$$

This entropy was shown to be exponentially small in the proof of Lemma 3.15, see equation (3.47).
□

Lemma 3.17 *Assume that for μ accuracy and concentration hold at scale N^{-1+a}. For any $\tilde{\varepsilon} > 0$ and $\alpha > 0$, there are constants $c_1, c_2, \delta > 0$ such that for any $N \geq 1$ and $k \in [\![2\alpha N, (1-2\alpha)N]\!]$,*

$$\mathbb{P}_\mu\left(\left|\lambda_k - \lambda_k^{[\alpha N]} - \mathbb{E}_\mu(\lambda_k - \lambda_k^{[\alpha N]})\right| > \frac{N^{\frac{a}{2}+\tilde{\varepsilon}}}{N}\right) \leq c_1 e^{-c_2 N^\delta}.$$

Note that in this lemma and its proof, for non-integer M we still write $\lambda_k^{[M]}$ for $\lambda_k^{[\lfloor M \rfloor]}$, where $\lfloor M \rfloor$ means the lower integer part of M.

Proof. Note first that

$$\left|\lambda_k - \lambda_k^{[\alpha N]} - \mathbb{E}_\mu(\lambda_k - \lambda_k^{[\alpha N]})\right| \leq \left|\lambda_k - \lambda_k^{[N^a]} - \mathbb{E}_\mu(\lambda_k - \lambda_k^{[N^a]})\right| + \left|\lambda_k^{[N^a]} - \lambda_k^{[\alpha N]} - \mathbb{E}_\mu(\lambda_k^{[N^a]} - \lambda_k^{[\alpha N]})\right|.$$

By the choice $M_1 = 1$, $M = N^a$ in Lemma 3.16, the probability that the first term is greater than $\frac{N^{\frac{a}{2}+\tilde{\varepsilon}}}{N}$ is exponentially small, uniformly in k, as desired. Concerning the second term, given some $r > 0$ and $q \in \mathbb{N}$ defined by $1 - r \leq a + qr < 1$, it is bounded by

$$\sum_{\ell=0}^{q-1}\left|\lambda_k^{[N^{a+(\ell+1)r}]} - \lambda_k^{[N^{a+\ell r}]} - \mathbb{E}_\mu\left(\lambda_k^{[N^{a+(\ell+1)r}]} - \lambda_k^{[N^{a+\ell r}]}\right)\right| + \left|\lambda_k^{[a+qr]} - \lambda_k^{[\alpha N]} - \mathbb{E}_\mu\left(\lambda_k^{[N^{a+qr}]} - \lambda_k^{[\alpha N]}\right)\right|.$$

By Lemma 3.16, for any $\varepsilon > 0$, each one of these $q+1$ terms has an exponentially small probability of being greater than $\frac{N^{\varepsilon+\frac{r}{2}}}{N}$. Consequently, choosing any ε and r (and therefore q) such that $\varepsilon + \frac{r}{2} < a/2$ concludes the proof. □

Proof of Proposition 3.12. We just need to write

$$|\lambda_k - \mathbb{E}_\mu(\lambda_k)| \leq \left|\lambda_k - \lambda_k^{[\alpha N]} - \mathbb{E}_\mu(\lambda_k - \lambda_k^{[\alpha N]})\right| + \left|\lambda_k^{[\alpha N]} - \mathbb{E}_\mu(\lambda_k^{[\alpha N]})\right|.$$

By Lemma 3.17, the first term has exponentially small probability to be greater than $\frac{N^{\frac{a}{2}+\varepsilon}}{N}$. By the logarithmic Sobolev inequality for μ with constant O(1) (see (2.2)), the second term has an even better concentration, at scale $1/N$:

$$\mathbb{P}\left(|\lambda_k^{[\alpha N]} - \mathbb{E}_\mu(\lambda_k^{[\alpha N]})| > \frac{x}{N}\right) \leq C\, e^{-cx^2}.$$

This concludes the proof of the proposition. □

Proof of Proposition 3.13. Thanks to Lemmas 3.3, 3.4, and reproducing the proof of Corollary 3.5, we know it is sufficient to prove that for any $\delta > 0$

$$\frac{1}{N^2}\left|\mathrm{var}\left(\sum_k \frac{1}{z-\lambda_k}\right)\right|$$

goes uniformly to 0 where $z = E + \mathrm{i}\eta$, $E \in [A+\delta, B-\delta]$ and $\eta \geq N^{-1+\frac{3a}{4}+\varepsilon}$.

Let i_0 be the index in $[0, N]$ such that the typical position $\gamma_{i_0}^N$ is the closest to E. Define the indexes of particles close to E, far from E and in the edge as

$$\begin{aligned}
\mathrm{Int} &= \{i : |i - i_0| < N^{a+\varepsilon}\},\\
\mathrm{Ext} &= \{i : |i - i_0| \geq N^{a+\varepsilon}, i \in [\![\alpha N, (1-\alpha)N]\!]\},\\
\mathrm{Edg} &= \{i : i \notin [\![\alpha N, (1-\alpha)N]\!]\},
\end{aligned}$$

where α is small enough such that

$$\gamma_{\alpha N} < A + \frac{\delta}{2} < A + \delta < E < B - \delta < B - \frac{\delta}{2} < \gamma_{(1-\alpha)N}. \tag{3.49}$$

We choose $\alpha_k = \mathbb{E}_\mu(\lambda_k)$ in the following equations. Then

$$\begin{aligned}
\frac{1}{N^2}\left|\mathrm{var}_\mu\left(\sum_k \frac{1}{z-\lambda_k}\right)\right| \leq{}& \frac{C}{N^2}\mathbb{E}_\mu\left(\left|\sum_{k\in\mathrm{Edg}} \frac{1}{z-\lambda_k} - \frac{1}{z-\alpha_k}\right|^2\right)\\
&+ \frac{C}{N^2}\mathbb{E}_\mu\left(\left|\sum_{k\in\mathrm{Ext}} \frac{1}{z-\lambda_k} - \frac{1}{z-\alpha_k}\right|^2\right) + \frac{C}{N^2}\mathbb{E}_\mu\left(\left|\sum_{k\in\mathrm{Int}} \frac{1}{z-\lambda_k} - \frac{1}{z-\alpha_k}\right|^2\right).
\end{aligned} \tag{3.50}$$

The edge term is bounded by

$$\frac{C}{N}\sum_{\mathrm{Edg}}\mathbb{E}_\mu\left(\left|\frac{1}{z-\lambda_k} - \frac{1}{z-\alpha_k}\right|^2\right) \leq \frac{C'}{N\eta^2}\sum_{\mathrm{Edg}}\mathbb{P}\left(|E-\lambda_k| < \frac{\delta}{3}\right) + \frac{C'}{N\delta^2}\sum_{\mathrm{Edg}}\mathbb{E}_\mu\left(|\lambda_k - \alpha_k|^2\right).$$

From the condition (3.49) and the large deviation estimate (3.5), the above probability is exponentially small. Moreover, the above L^2 moments are $\mathrm{O}(1/N)$ by the spectral gap inequality for μ, see e.g. equation (3.12). Hence the edge term goes to 0 uniformly.

Using the accuracy at scale N^{-1+a} and the concentration at scale $N^{-1+a/2}$ (Proposition 3.12), the second term in (3.50) is bounded, up to constants, for some $c_1, c_2, \delta > 0$ by

$$\begin{aligned}
&\frac{1}{N^2}\mathbb{E}\left(\left|\sum_{k\geq N^{a+\varepsilon}} \frac{\lambda_{i_0+k} - \alpha_{i_0+k}}{(\frac{k}{N})^2}\right|^2\right) + c_1 e^{-c_2 N^\delta}\\
&\qquad \leq N^2\mathbb{E}\left(\sum_{k\geq N^{a+\varepsilon}} \frac{|\lambda_{i_0+k} - \alpha_{i_0+k}|^2}{k^2}\right)\sum_{k\geq N^{a+\varepsilon}}\frac{1}{k^2} + c_1 e^{-c_2 N^\delta} \leq N^2 N^{-2+a}(N^{-a})^2 = N^{-a}.
\end{aligned}$$

In particular, it converges uniformly to 0. For the third term, for some $c > 0$ it is less than

$$\begin{aligned}
\frac{1}{N^2\eta^4}\mathbb{E}\left(\left(\sum_{Int}|\lambda_k - \alpha_k|\right)^2\right) &\leq \frac{c}{N^2\eta^4}N^{a+\varepsilon}\mathbb{E}\left(\sum_{Int}|\lambda_k - \alpha_k|^2\right)\\
&\leq c\frac{(N^{a+\varepsilon})^2}{N^2\eta^4}N^{-2+a} = c\frac{N^{3a+2\varepsilon}}{N^4\eta^4}.
\end{aligned}$$

This goes to 0 if $\eta \gg N^{-1+\frac{3a}{4}+\frac{\varepsilon}{2}}$, concluding the proof. □

4 Local equilibrium measure

4.1 Construction of the local measure

Let $0 < \kappa < 1/2$. Choose $q \in [\kappa, 1-\kappa]$ and set $L = [Nq]$ (integer part). Fix an integer K with $K \le (N-L)/2$, in fact we will always assume that K depends on N as $K = N^k$ with $k < 1$. We will study the local spacing statistics of K consecutive particles

$$\{\lambda_j \ : \ j \in I\}, \qquad I = I_L := [\![L+1, L+K]\!].$$

These particles are typically located near E_q determined by the relation

$$\int_{-\infty}^{E_q} \rho(t)\mathrm{d}t = q.$$

Note that $|\gamma_L - E_q| \le C/N$.

We will distinguish the inside and outside particles by renaming them as

$$(\lambda_1, \lambda_2, \dots, \lambda_N) := (y_1, \dots y_L, x_{L+1}, \dots x_{L+K}, y_{L+K+1}, \dots y_N) \in \Xi^{(N)}, \tag{4.1}$$

but note that they keep their original indices. The notation $\Xi^{(N)}$ refers to the simplex $\{\mathbf{z} \ : \ z_1 < z_2 < \dots < z_N\}$ in $\mathbb{R}^N$. In short we will write

$$\mathbf{x} = (x_{L+1}, \dots x_{L+K}), \qquad \text{and} \qquad \mathbf{y} = (y_1, \dots y_L, y_{L+K+1}, \dots y_N),$$

all in increasing order, i.e. $\mathbf{x} \in \Xi^{(K)}$ and $\mathbf{y} \in \Xi^{(N-K)}$. We will refer to the y's as the external points and to the x's as internal points.

We will fix the external points (often called as boundary conditions) and study the conditional measures on the internal points. We consider the parameters L and K fixed and we will not indicate them in the notation. We first define the *local equilibrium measure* on $\mathbf{x}$ with boundary condition $\mathbf{y}$ by

$$\mu_{\mathbf{y}}(\mathrm{d}\mathbf{x}) = u_{\mathbf{y}}(\mathbf{x})\mathrm{d}\mathbf{x}, \qquad u_{\mathbf{y}}(\mathbf{x}) := u(\mathbf{y}, \mathbf{x}) \left[\int u(\mathbf{y}, \mathbf{x})\mathrm{d}\mathbf{x} \right]^{-1}, \tag{4.2}$$

where u is the density of μ_V. Note that for any fixed $\mathbf{y} \in \Xi^{(N-K)}$, x_j lies in the interval $[y_L, y_{L+K+1}]$.

Given the classical locations, $\gamma = \{\gamma_1, \gamma_2, \dots, \gamma_N\}$ with respect to the μ-measure, we define the *relaxation measure* $\mu_N^{\tau,\gamma} = \mu^\tau$ by

$$\mathrm{d}\mu^\tau := \frac{Z}{Z_{\mu^\tau}} e^{-NQ^\tau}\mathrm{d}\mu, \quad Q^\tau(\mathbf{x}) = \sum_{j \in I} Q_j^\tau(x_j), \qquad Q_j^\tau(x) = \frac{1}{2\tau}(x - \gamma_j)^2, \tag{4.3}$$

where Z_μ is chosen such that μ is a probability measure. Here $0 < \tau < 1$ is a parameter which may even depend on $\mathbf{y}$, i.e., $\tau = \tau(\mathbf{y})$ is allowed. Note that an artificial quadratic confinement has been added to the equilibrium measure. We define *the local relaxation measure* $\mu_{\mathbf{y}}^\tau$ to be conditional measure of μ^τ.

Define the Dyson Brownian motion reversible with respect to $\mu_{\mathbf{y}}^\tau$, by the Dirichlet form

$$D_{\mu_{\mathbf{y}}^\tau}(f) = \sum_{i \in I} \frac{1}{2N} \int (\partial_i f)^2 \mathrm{d}\mu_{\mathbf{y}}^\tau, \tag{4.4}$$

where $\partial_i = \partial_{x_i}$. The Hamiltonian $\mathcal{H}_{\mathbf{y}}^\tau$ of the measure $\mu_{\mathbf{y}}^\tau(\mathrm{d}\mathbf{x}) \sim \exp(-N\mathcal{H}_{\mathbf{y}}^\tau)$ is given by

$$\mathcal{H}_{\mathbf{y}}^\tau(\mathbf{x}) = \sum_{i\in I} \frac{\beta}{2} V_{\mathbf{y}}(x_i) - \frac{\beta}{N} \sum_{\substack{i,j\in I\\ i<j}} \log|x_j - x_i| + \sum_{i\in I} Q_i^\tau(x_i), \tag{4.5}$$

$$V_{\mathbf{y}}(x) = V(x) - \frac{1}{N} \sum_{j\notin I} \log|x - y_j|. \tag{4.6}$$

We now define the set of *good boundary configurations* with a parameter $\varepsilon_0 > 0$ and a parameter $\delta = \delta(N) > 0$ that in the applications may depend on N:

$$\begin{aligned}\mathcal{G}_{\delta,\varepsilon_0} = \mathcal{G} := \Big\{ & \mathbf{y} \in \Xi^{(N-K)} \ : \ |y_k - \gamma_k| \le \delta, \ \forall\, k \in [\![N\kappa/2, L]\!] \cup [\![L+K+1, N(1-\kappa/2)]\!], \\ & \text{and } \ |y_k - \gamma_k| \le 1, \ \forall\, k \in [\![1, N]\!], \\ & \text{and } \ \mathbb{E}_{\mu_{\mathbf{y}}}(x_j - \gamma_j)^2 \le \delta^2 \ \text{ for all } \ j \in [\![L+1, L+K]\!] \\ & \text{and } \ y_L - y_{L-1} \ge \exp(-N^{\varepsilon_0}), \quad y_{L+K+2} - y_{L+K+1} \ge \exp(-N^{\varepsilon_0}) \Big\}.\end{aligned} \tag{4.7}$$

First we show that the good configurations have overwhelmingly large probability

Lemma 4.1 *For any $\varepsilon_0 > 0$ and for any choice $\delta = N^{-d}$ with $d \in (1-k, 1)$, there is an $\varepsilon' > 0$ depending on d such that*

$$\mathbb{P}_\mu(\mathcal{G}^c) \le Ce^{-cN^{\varepsilon'}} + Ce^{-cN^{\varepsilon_0}}. \tag{4.8}$$

Proof. We have proved in Theorem 3.1 that for any choice $\delta = N^{-d}$ with $d \in (0,1)$ the probability that the first condition in (4.7) is violated is bounded by $C\exp(-cN^{\varepsilon'})$ with some $\varepsilon' > 0$ depending on d. Similarly, the second condition is violated with an analogous very small probability by (3.5). To check the probability to violate the third requirement in the definition of $\mathcal{G}$, we use that

$$\begin{aligned}\mathbb{P}_\mu\Big\{\mathbb{E}_{\mu_{\mathbf{y}}}(x_j - \gamma_j)^2 \ge \delta^2\Big\} &\le \mathbb{P}_\mu\Big\{\mathbb{P}_{\mu_{\mathbf{y}}}\{|x_j - \gamma_j| \ge \delta/2\} \ge 3\delta^2/4\Big\} + C\exp(-cN^{\varepsilon'}) \\ &\le C\delta^{-2}\mathbb{E}_\mu \mathbb{P}_{\mu_{\mathbf{y}}}\big\{|x_j - \gamma_j| \ge \delta/2\big\} + C\exp(-cN^{\varepsilon'}) \\ &\le C\delta^{-2}\mathbb{P}_\mu\big\{|x_j - \gamma_j| \ge \delta/2\big\} \le c_1 e^{-c_2 N^{\varepsilon'}},\end{aligned} \tag{4.9}$$

since for $\mathbf{y}$ satisfying the first two conditions of (4.7) we have

$$\mathbb{E}_{\mu_{\mathbf{y}}}(x_j - \gamma_j)^2 \le \delta^2/4 + \mathbb{P}_{\mu_{\mathbf{y}}}\{|x_j - \gamma_j| \ge \delta/2\}$$

as $x_j - \gamma_j \le y_{L+K+1} - \gamma_j \le \delta + \gamma_{L+K+1} - \gamma_1 \le 1$ and also a similar lower bound holds.

Finally, we show that

$$\mathbb{P}_\mu\big(y_L - y_{L-1} \le \exp(-N^{\varepsilon_0})\big) \le Ce^{-cN^{\varepsilon_0}},$$

and a similar bound holds for the other condition in the fourth line of (4.7). For simplicity of the presentation and to avoid introducing new notations, we will actually prove

$$\mathbb{P}_\mu\big(y_{L+1} - y_L \le \exp(-N^{\varepsilon_0})\big) \le Ce^{-cN^{\varepsilon_0}}$$

from which the previous inequality follows just by shifting the indices. With the events

$$A := \{y_{L+1} - y_L \le \exp(-N^{\varepsilon_0})\}, \qquad \Omega := \{y_{L+K+1} - y_L \le 2a\},$$

we write

$$\mathbb{P}_\mu(A) = \mathbb{E}_\mu\big[\mathbf{1}(\Omega)\mathbb{P}_{\mu_{\mathbf{y}}}(A)\big] + \mathbb{P}_\mu(\Omega^c). \tag{4.10}$$

Choosing $a = C_0K/N$ with a sufficiently large fixed constant C_0 Theorem 3.1 and $\delta \ll K/N$ guarantee that $\mathbb{P}_\mu(\Omega^c)$ is subexponentially small.

We will prove that

$$\mathbb{P}_{\mu_{\mathbf{y}}}(x_{L+1} - y_L \le N^{-2}r) \le C_V r \tag{4.11}$$

for any $r \in (0,1)$. The constant depends on V, more precisely

$$C_V = C + C\sup\big\{|V'(x)| \;:\; x \in [y_L, y_{L+K+1}]\big\}. \tag{4.12}$$

From (4.11) the necessary subexponential estimate on the first term in (4.10) follows by choosing $r = N^{-2}\exp(-N^{\varepsilon_0})$.

To prove (4.11), on the set Ω we can shift the measure such that $y_L = -y_{L+K+1}$ and denote $a := -y_L$. Then we have

$$\begin{aligned}\int\cdots\int_{-a+a\varphi}^{a-a\varphi} \mathrm{d}\mathbf{x} \prod_{\substack{i,j\in I\\ i<j}} (x_i - x_j)^\beta e^{-N\frac{\beta}{2}\sum_j V_{\mathbf{y}}(x_j)} \\ = (1-\varphi)^{K+\beta K(K-1)/2}\int\cdots\int_{-a}^{a}\mathrm{d}\mathbf{w}\prod_{i<j}(w_i - w_j)^\beta e^{-N\frac{\beta}{2}\sum_j V_{\mathbf{y}}((1-\varphi)w_j)},\end{aligned}$$

where we set $w_j := (1-\varphi)^{-1}x_{L+j}$, $\mathrm{d}\mathbf{x} = \mathrm{d}x_{L+1}\ldots\mathrm{d}x_{L+K}$ and $\mathrm{d}\mathbf{w} = \mathrm{d}w_1\ldots\mathrm{d}w_K$. By definition,

$$\begin{aligned}e^{-N\frac{\beta}{2}V_{\mathbf{y}}((1-\varphi)w_j)} &= e^{-N\frac{\beta}{2}V((1-\varphi)w_j)}\prod_{i\le L}((1-\varphi)w_j - y_i)^\beta \prod_{i\ge L+K+1}(y_i - (1-\varphi)w_j)^\beta \\ &\ge e^{-N\frac{\beta}{2}V(w_j) - C_V\varphi N}(1-\varphi)^N\prod_{i\le L}(w_j - y_i)^\beta\prod_{i\ge L+K+1}(y_i - w_j)^\beta.\end{aligned}$$

Note that we only used that V is a C^1-function with bounded derivative in performing a Taylor expansion and using that $w_j \le a$ is finite. Hence

$$\frac{1}{Z}\int\cdots\int_{-a+a\varphi}^{a-a\varphi}\mathrm{d}\mathbf{x}\prod_{\substack{i,j\in I\\ i<j}}(x_i - x_j)^\beta e^{-N\frac{\beta}{2}\sum_j V_{\mathbf{y}}(x_j)} \ge (1-\varphi)^{NK+CK^2}e^{-C_V NK\varphi}$$

with

$$Z := \int_{-a}^{a}\mathrm{d}\mathbf{w}\prod_{\substack{i,j\in I\\ i<j}}(w_i - w_j)^\beta e^{-N\frac{\beta}{2}\sum_j V_{\mathbf{y}}(w_j)}.$$

Therefore the $\mu_{\mathbf{y}}$-probability of $y_{L+1} - y_L = x_{L+1} - y_L \ge a(1-\varphi)$ can be estimated by

$$\mathbb{P}_{\mu_{\mathbf{y}}}(x_{L+1} \ge -a + \varphi a) \ge (1-\varphi)^{NK+CK^2}e^{-C_V NK\varphi} \ge 1 - (C_V + C)NK\varphi$$

by using $K \le N$. Choosing $\varphi = N^{-2}r/a$ and recalling that $a \sim K/N$, we arrive at (4.11). □

Proposition 4.2 *Let $\varphi > 0$ be fixed. For any smooth, compactly supported function $G : \mathbb{R} \to \mathbb{R}$ we have*

$$\lim_{N\to\infty} \left| \mathbb{E}_\mu \left[\mathbb{E}_{\mu_{\mathbf{y}}} - \mathbb{E}_{\mu_{\mathbf{y}}^\tau} \right] \frac{1}{K} \sum_{i\in I} G\Big(N(x_i - x_{i+1}) \Big) \right| = 0, \tag{4.13}$$

provided

$$\frac{1}{2}\hat\tau \le \tau(\mathbf{y}) \le 2\hat\tau \qquad \text{for any } \mathbf{y} \in \mathcal{G} \tag{4.14}$$

holds for the function $\tau = \tau(\mathbf{y})$ with some constant $\hat\tau = \hat\tau_N$ such that

$$\frac{N\delta^2}{\hat\tau} \le N^{-\varphi}. \tag{4.15}$$

We remark that, with a slight abuse of notation, the last term, $i = L + K$ in the sum involving the non-existing $x_{i+1} = x_{L+K+1}$ is defined to be zero. We also point out that the notation $\mathbb{E}_\mu \mathbb{E}_{\mu_{\mathbf{y}}}$ means that the law of $\mathbf{y}$ is given by μ in the first expectation and we are using the measure $\mu_{\mathbf{y}}$ in the second one. Of course, we have $\mathbb{E}_\mu = \mathbb{E}_\mu \mathbb{E}_{\mu_{\mathbf{y}}}$.

Proof. For any configuration $\mathbf{y}$, any τ (may depend on $\mathbf{y}$) and for any smooth function G with compact support, we have

$$\left| \left[\mathbb{E}_{\mu_{\mathbf{y}}} - \mathbb{E}_{\mu_{\mathbf{y}}^\tau} \right] \frac{1}{K} \sum_{i\in I} G\Big(N(x_i - x_{i+1}) \Big) \right| \le C \Big(\frac{\tau N^{\varphi/2}}{K} D\big(\mu_{\mathbf{y}} | \mu_{\mathbf{y}}^\tau\big) \Big)^{1/2} + C e^{-cN^{\varepsilon/2}} \sqrt{S(\mu_{\mathbf{y}} | \mu_{\mathbf{y}}^\tau)}, \tag{4.16}$$

Here we also introduced the notations

$$D(\mu \mid \omega) := \frac{1}{2N} \int \Big| \nabla \log \Big(\frac{\mathrm{d}\mu}{\mathrm{d}\omega} \Big) \Big|^2 \mathrm{d}\mu = \frac{1}{2N} \int \Big| \nabla \sqrt{\frac{\mathrm{d}\mu}{\mathrm{d}\omega}} \Big|^2 \mathrm{d}\omega \tag{4.17}$$

and

$$S(\mu \mid \omega) := \int \log \Big(\frac{\mathrm{d}\mu}{\mathrm{d}\omega} \Big) \mathrm{d}\mu$$

for any probability measures μ, ω. The estimate (4.16) follows from our the local relaxation to equilibrium argument that in this form first appeared in Theorem 3.4 of [15]. We will neglect the exponentially small entropy term since it can be estimated by the Dirichlet form, i.e. by the first term as long as $\tau \ge N^{-C}$.

We thus obtain

$$\left| \mathbb{E}_\mu \left[\mathbb{E}_{\mu_{\mathbf{y}}} - \mathbb{E}_{\mu_{\mathbf{y}}^\tau} \right] \frac{1}{K} \sum_{i\in I} G\Big(N(x_i - x_{i+1}) \Big) \right| \le C \Big(\frac{N^{\varphi/2}}{K} \mathbb{E}_\mu \big[\mathbf{1}_{\mathcal{G}}\, \tau(\mathbf{y}) D\big(\mu_{\mathbf{y}} | \mu_{\mathbf{y}}^\tau\big) \big] \Big)^{1/2} + C e^{-cN^{\varepsilon'}}. \tag{4.18}$$

To obtain the estimate (4.18) we separated good and bad configurations; we used (4.16) for $\mathbf{y} \in \mathcal{G}$. On the complement $\mathcal{G}^c$ we just used the trivial estimate on G, and this yields the subexponentially small second term.

Assuming (4.14), we have

$$\frac{1}{K} \mathbb{E}_\mu \big[\mathbf{1}_{\mathcal{G}}\, \tau(\mathbf{y}) D\big(\mu_{\mathbf{y}} | \mu_{\mathbf{y}}^\tau\big) \big] \le \frac{N}{K} \mathbb{E}^\mu \Big[\mathbf{1}_{\mathcal{G}}\, \frac{1}{\tau(\mathbf{y})} \sum_{j\in I} (x_j - \gamma_j)^2 \Big] \le \frac{N\delta^2}{\hat\tau} \le N^{-\varphi} \tag{4.19}$$

by (4.15). Inserting this estimate into (4.18) we completed the proof of the proposition. □

4.2 Matching the boundary conditions

Suppose we have measures σ and μ with potentials W and V given by (2.1) with densities $\rho = \rho_V$ and ρ_W, respectively. For our purpose $W(x) = x^2$, i.e, σ is the Gaussian β-ensemble and $\rho_W(t) = \frac{1}{2\pi}\sqrt{[4-t^2]_+}$ is the Wigner semicircle law. Let the sequence γ_j be the classical location for μ and the sequence θ_j be the classical locations for σ.

We will match the boundary conditions for the local measure on $J_{\mathbf{y}} := [y_L, y_{L+K+1}]$ around E_q with those of the σ measure. For definiteness we choose the interval $J' = [\theta_{L'}, \theta_{L'+K+1}]$ with $L' = \frac{1}{2}(N-K-1)$ as our reference interval. Note that J' is symmetric to the origin. The local density $\rho_V(E_q)$ at the point E_q we look at may be different from the density $\rho_W(0)$ at the origin. Thus the typical length of $J_{\mathbf{y}}$, which is $\gamma_{L+K+1} - \gamma_L \sim [\rho_V(E_q)]^{-1}N^{-1}$, may not be close to the length of J' which is very close to $[\rho_W(0)]^{-1}N^{-1} = \pi N^{-1}$, so we will have to rescale the σ measure by a factor

$$s_q \approx \frac{\rho_V(E_q)}{\rho_W(0)}.$$

In fact, we need to match not only the interval of classical locations γ with J', but the exact interval $I_{\mathbf{y}}$. This requires a $\mathbf{y}$-dependendent scaling factor $s = s(\mathbf{y})$.

From now on we assume that $\mathbf{y}$ is a good boundary condition with a parameter δ that satisfies

$$\frac{\delta N}{K} \to 0. \tag{4.20}$$

We can shift the coordinates so that

$$-y_L = y_{L+K+1}. \tag{4.21}$$

Since our observable is translationally invariant, we will not track the translation and we assume that (4.21) holds. We define

$$s(\mathbf{y}) := \frac{\theta_{L'}}{y_L} = \frac{\theta_{L'+K+1}}{y_{L+K+1}}, \qquad s_q := \frac{\theta_{L'}}{\gamma_L}. \tag{4.22}$$

We have

$$|s(\mathbf{y}) - s_q| = \Big|\frac{\theta_{L'}}{y_L} - \frac{\theta_{L'}}{\gamma_L}\Big| \le C\frac{\delta N}{K} \to 0 \tag{4.23}$$

since

$$\theta_{L'} \approx -[\varrho_W(0)]^{-1}\frac{K}{2N}, \qquad \gamma_L \approx -[\varrho_V(E_q)]^{-1}\frac{K}{2N}, \qquad y_L \approx -[\varrho_V(E_q)]^{-1}\frac{K}{2N}, \tag{4.24}$$

by using $\mathbf{y} \in \mathcal{G}$ and (4.20). Similar formulas hold for $\theta_{L'+K+1}$, γ_{L+K+1} and y_{L+K+1} at the upper edge of the interval. Here the $A \approx B$ is understood in the sense that the approximation error at most of order $(K/N)^2$, recalling that $K = o(N)$.

For simplicity of the presentation, we can first shift the original μ-ensemble such that $E_q = 0$. Second, we can perform an initial rescaling of the Gaussian β-ensemble so that $s_q = 1$.

Lemma 4.3 *Assuming $E_q = 0$, $s_q = 1$, we have*

$$|\gamma_{L+j} - \theta_{L'+j}| \le C\frac{j^2}{N^2} + C\delta, \qquad |j| \le \frac{1}{100}N\kappa. \tag{4.25}$$

Proof. The classical locations γ_{L+j} and $\theta_{L'+j}$ are given by the equation

$$\int_{\gamma_L}^{\gamma_{L+j}} \rho_V(x) = \frac{j}{N}, \qquad \int_{\theta_{L'}}^{\theta_{L'+j}} \rho_W(x) = \frac{j}{N}. \tag{4.26}$$

We will use the approximations

$$\rho_V(x) = \rho_V(0) + O(x), \qquad \rho_W(x) = \rho_W(0) + O(x) \tag{4.27}$$

for small x (to stay away from the spectral edge). Since $|y_j - \gamma_j| \le \delta$, we have

$$\frac{K+1}{N} = \int_{\gamma_L}^{\gamma_{L+K+1}} \rho_V(x)\mathrm{d}x = \int_{y_L}^{y_{L+K+1}} \rho_V(x)\mathrm{d}x + O(\delta) = \rho_V(0)(y_{L+K+1} - y_L) + O\left(\frac{K^2}{N^2}\right) + O(\delta). \tag{4.28}$$

Similarly,

$$\frac{K+1}{N} = \int_{\theta_{L'}}^{\theta_{L'+K+1}} \rho_W(x)\mathrm{d}x = \rho_W(0)(\theta_{L'+K+1} - \theta_{L'}) + O\left(\frac{K^2}{N^2}\right). \tag{4.29}$$

Since $-y_L = y_{L+K+1} = -\theta_{L'}/s = \theta_{L'+K+1}/s$ which is comparable with K/N by (4.24), and since $|s-1| \le C\frac{\delta N}{K}$ from (4.23), we have

$$|\rho_W(0) - \rho_V(0)| \le \frac{CK}{N} + \frac{C\delta N}{K}. \tag{4.30}$$

From (4.26), (4.27) and (4.24) we get

$$\varrho_V(0)(\gamma_{L+j} - \gamma_L) + O\Big(\frac{j^2}{N^2}\Big) = \frac{j}{N} = \varrho_W(0)(\theta_{L'+j} - \theta_{L'}) + O\Big(\frac{j^2}{N^2}\Big),$$

which combining with (4.30) and $\rho_W(0) \ge c$ gives

$$\gamma_{L+j} - \gamma_L = \theta_{L'+j} - \theta_{L'} + O\Big(\frac{j^2}{N^2}\Big) + O(\delta).$$

Since $\gamma_L = \theta_{L'}$, this completes the proof of the lemma. □

4.3 Rescaling of the reference problem

Throughout this section we fix a good boundary configuration. $\mathbf{y} \in \mathcal{G}$ and a number $\tau(\mathbf{y})$ depending on this configuration and satisfying (4.14). We will approximate the local relaxation measure $\mu_{\mathbf{y}}^{\tau}$ on $[y_L, y_{L+K+1}]$ by a fixed reference measure.

Given the collection of classical locations θ_j corresponding to the Gaussian potential $W(x) = x^2$ we define a *reference local relaxation measure* $\sigma_\theta^{\hat\tau}$ via the Hamiltonian

$$\mathcal{H}_\theta^{\hat\tau}(\mathbf{x}) = \sum_{i\in I'} \Big[\frac{\beta}{2}x_i^2 - \frac{\beta}{N}\sum_{j\notin I'} \log|x_i - \theta_j|\Big] - \frac{\beta}{N}\sum_{\substack{i,j\in I'\\ i<j}} \log|x_j - x_i| + \frac{1}{2\hat\tau}\sum_{i\in I'}(x_i - \theta_i)^2, \tag{4.31}$$

on the set $[\theta_{L'}, \theta_{L'+K+1}]$ where $I' := [\![L'+1, L'+K]\!]$. Note that if σ is the equilibrium measure given by (2.1) corresponding to W and $\sigma^{\hat\tau}$ denotes the corresponding relaxation measure

$$\mathrm{d}\sigma^{\hat\tau} := \frac{Z}{Z_{\sigma^{\hat\tau}}} e^{-NQ^{\hat\tau}}\mathrm{d}\sigma, \quad Q^{\hat\tau}(\mathbf{x}) = \sum_{j\in I'} Q_j^{\hat\tau}(x_j), \qquad Q_j^{\hat\tau}(x) = \frac{1}{2\hat\tau}(x-\theta_j)^2, \tag{4.32}$$

defined analogously to (4.3), then $\sigma_\theta^{\hat\tau}$ is the conditional measure of $\sigma^{\hat\tau}$ under the condition that the outside points are exactly at their classical locations, i.e. $\lambda_j = \theta_j$, $j \notin I'$.

We make three simplifications in the presentation. First, as already in Section 4.2, we assume that both the configuration space $[y_L, y_{L+K+1}]$ for the original measure $\mu_{\mathbf{y}}^{\hat\tau}$ and the configuration space $[\theta_{L'}, \theta_{L'+K+1}]$ of the reference measure $\sigma_\theta^{\hat\tau}$ are symmetric around the origin; this can be achieved by an irrelevant shift. Second, we assumed $s_q = 1$, which can be achieved by an irrelevant rescaling of W. Finally, we will set $L' = L$. This last assumption expresses an irrelevant shift in the labelling of one of the ensembles. Strictly speaking, shifting would mean that the original set of particles indices $[\![1, N]\!]$ gets shifted as well. However, in our argument this shift does not play any active role; the only information we use about the set of indices is that L is macroscopically separated from its boundary and that its cardinality is N.

We now rescale the measure $\sigma_\theta^{\hat\tau}$ from $[\theta_L, \theta_{L+K+1}] = [\theta_L, -\theta_L]$ to $[y_L, y_{L+K+1}] = [y_L, -y_L]$ by the factor $s = s(\mathbf{y})$ defined in (4.22) (note that $y_L, \theta_L < 0$). With the rescaled boundary conditions $\theta_j \to \theta_j' := \theta_j/s$, we define the *reference local relaxation measure*, or *reference measure* in short, to be

$$\sigma_\theta^{\hat\tau,s} := \frac{1}{Z^{\hat\tau,\theta,s}} e^{-N\mathcal{H}_\theta^{\hat\tau,s}(\mathbf{x})} \mathrm{d}\mathbf{x}, \tag{4.33}$$

a measure on the set $[y_L, y_{L+K+1}]$ with Hamiltonian

$$\mathcal{H}_\theta^{\hat\tau,s}(\mathbf{x}) = \sum_{i\in I} \Big[\frac{\beta s^2 x_i^2}{2} - \frac{\beta}{N}\sum_{j\notin I} \log|x_i - \theta_j/s|\Big] - \frac{\beta}{N}\sum_{\substack{i,j\in I\\ i<j}} \log|x_j - x_i| + \frac{s^2}{2\hat\tau}\sum_{i\in I}(x_i - \theta_i/s)^2. \tag{4.34}$$

The rescaled potential associated with this Hamiltonian is $W_s(x) = s^2 x^2$.

For any smooth function G with compact support, we have

$$\mathbb{E}_{\sigma_\theta^{\hat\tau,s}} \frac{1}{K}\sum_{i\in I} G\Big(N(x_i - x_{i+1})\Big) = \frac{1}{Z^{\hat\tau,\theta,s}} \int_{\theta_L/s}^{-\theta_L/s} \mathrm{d}\mathbf{x}\, e^{-N\mathcal{H}_\theta^{\hat\tau,s}(\mathbf{x})}\, \frac{1}{K}\sum_{i\in I} G\Big(N(x_i - x_{i+1})\Big), \tag{4.35}$$

where $\int_a^b \mathrm{d}\mathbf{x}$ stands for the K-dimensional integral $\int_{[a,b]^K} \mathrm{d}x_{L+1}\ldots\mathrm{d}x_{L+K}$ and $Z^{\hat\tau,\theta,s}$ is the normalization factor. Let $x_j = w_j/s$, then the right side becomes

$$\begin{aligned}\frac{1}{Z^{\hat\tau,\theta}} \int_{\theta_L}^{-\theta_L} \mathrm{d}\mathbf{w}\, e^{-N\mathcal{H}_\theta^{\hat\tau}(\mathbf{w})} \frac{1}{K}\sum_{i\in I} G\Big(\frac{N(w_i - w_{i+1})}{s}\Big) &= \mathbb{E}_{\sigma_\theta^{\hat\tau}} \frac{1}{K}\sum_{i\in I} G\Big(\frac{N(x_i - x_{i+1})}{s}\Big) \\ &= \mathbb{E}_{\sigma_\theta^{\hat\tau}} \frac{1}{K}\sum_{i\in I} G\Big(N(x_i - x_{i+1})\Big) + o(1),\end{aligned} \tag{4.36}$$

where we renamed the w-variables to x-variables in the first step and in the second step we have used that

$$\Big|G\Big(N(x_i - x_{i+1})/s\Big) - G\Big(N(x_i - x_{i+1})\Big)\Big| \le C|1-s|\|G'\|_\infty$$

by Taylor expansion and from the fact that G is compactly supported. Clearly, the difference vanishes as long as $s \to 1$. Thus we are free to scale the measure with factor converging to 1. The condition $s \to 1$ will be guaranteed by (4.23).

Our main result is the following theorem.

Theorem 4.4 *Let $0 < \varphi \le \frac{1}{38}$. Fix $K = N^k$, $\delta = N^{-d}$, $\hat\tau = N^{-t}$ with $d = 1-\varphi$, $t = 2d-1-\varphi = 1-3\varphi$ and $k = \frac{39}{2}\varphi$, in particular such that* (4.15), (4.20) *are satisfied. Then*

$$\left| \mathbb{E}_\mu \mathbb{E}_{\mu_{\mathbf{y}}^{\hat\tau}} \frac{1}{K}\sum_{i\in I} G\Big(N(x_i - x_{i+1})\Big) - \mathbb{E}_{\sigma_\theta^{\hat\tau}} \frac{1}{K}\sum_{i\in I} G\Big(N(x_i - x_{i+1})\Big) \right| \to 0 \tag{4.37}$$

as $N\to\infty$ for any smooth and compactly supported test function G. Here the law of $\mathbf{y}$ is given by μ in the expectation.

Proof. From the rescaling estimates, (4.35)-(4.36), it suffices to prove that

$$\mathbb{E}_\mu \big[\mathbb{E}_{\mu_{\mathbf{y}}^{\hat\tau}} - \mathbb{E}_{\sigma_\theta^{\hat\tau, s(\mathbf{y})}} \big] \frac{1}{K}\sum_{i\in I} G\Big(N(x_i - x_{i+1})\Big) \to 0 \tag{4.38}$$

as $N\to\infty$. Notice that after the rescaling both measures $\mu_{\mathbf{y}}^{\hat\tau}$ and $\sigma_\theta^{\hat\tau, s(\mathbf{y})}$ live on the same interval $[y_L, y_{L+K+1}]$. In Proposition 4.2 we already showed that

$$\mathbb{E}_\mu \big[\mathbb{E}_{\mu_{\mathbf{y}}^{\hat\tau}} - \mathbb{E}_{\mu_{\mathbf{y}}^{\hat\tau/s(\mathbf{y})^2}} \big] \frac{1}{K}\sum_{i\in I} G\Big(N(x_i - x_{i+1})\Big) \to 0, \tag{4.39}$$

since (4.23) with $s_q = 1$ and $\delta N/K \to 0$ guarantee that $\tau(\mathbf{y}) := \hat\tau / s(\mathbf{y})^2$ satisfies (4.14). Thus the limit (4.38) will follow from the following Proposition that we will prove in Sections 5:

Proposition 4.5 *Under the assumptions of Theorem 4.4, we have*

$$\mathbb{E}_\mu \big[\mathbb{E}_{\mu_{\mathbf{y}}^{\hat\tau/s(\mathbf{y})^2}} - \mathbb{E}_{\sigma_\theta^{\hat\tau, s(\mathbf{y})}} \big] \frac{1}{K}\sum_{i\in I} G\Big(N(x_i - x_{i+1})\Big) \to 0. \tag{4.40}$$

This completes the proof of Theorem 4.4. □

Proof of Theorem 2.1. Finally, combining Theorem 4.4 with Proposition 4.2 and noticing that (4.15) is satisfied since $t = 2d - 1 - \varphi$, we have

$$\left| \mathbb{E}_\mu \frac{1}{K}\sum_{i\in I} G\Big(N(x_i - x_{i+1})\Big) - \mathbb{E}_{\sigma_\theta^{\hat\tau}} \frac{1}{K}\sum_{i\in I} G\Big(N(x_i - x_{i+1})\Big) \right| \to 0 \tag{4.41}$$

as $N\to\infty$. This holds for $K = N^k$ with any $0 < k \le \frac{1}{2}$ by selecting a suitable φ in Theorem 4.4. However, the measure $\sigma_\theta^{\hat\tau}$ is independent of V, the only information we used was that the local density matches. So we obtain that any two measures $\mu_{\beta,V}$ and $\mu_{\beta,W}$ have the same local gap statistics assuming that the local densities of the two ensembles coincide. □

5 Comparison with the reference problem

In this section we prove Proposition 4.5. On the set $\mathbf{y}\in\mathcal{G}^c$ with subexponentially small probabality (4.8) a trivial estimate on G suffices. For the sequel we therefore assume that $\mathbf{y}\in\mathcal{G}$ and we set $\tau(\mathbf{y}) := \hat\tau/s(\mathbf{y})^2$ which clearly satisfies (4.14). In the first step we will soften the boundary condition $\mathbf{y}$ for the measure local relaxation measure $\mu_{\mathbf{y}}^{\tau}$.

5.1 Regularizing the boundary conditions

We know that the boundary condition $\mathbf{y} \in \mathcal{G}$ is regularly spaced on the scale $\delta = N^{-d}$, but it does not exclude that $N\delta = N^{1-d} \gg 1$ points of the colletion $\mathbf{y}$ pile up near the edges of the interval $[y_L, y_{L+K+1}]$. This would substantially influence the local relaxation measure $\mu_{\mathbf{y}}^{\tau}$ near the corresponding edge inside $[y_L, y_{L+K+1}]$. We therefore first replace the boundary conditions near the edges by the regularly spaced ones given by $\theta' = \theta/s(\mathbf{y})$. This change will be controlled only in the entropy sense. The local relaxation measure with regularized boundary conditions will then be compared with the reference measure in the stronger Dirichlet form sense.

Set a parameter

$$B = N^b \quad \text{with} \quad 1 + \varphi - d \le b < k, \tag{5.1}$$

in particular $\delta N \ll B \ll K$. Given a boundary condition $\mathbf{y} \in \mathcal{G}$, we define a new boundary condition $\mathbf{y}^B = \{y_i^B \ : \ i \notin I\}$ as

$$y_i^B := \begin{cases} \max\{\theta_i', y_{L-4B}\} & \text{for} \quad L - 4B \le i \le L \\ y_i & \text{for} \quad i < L - 4B, \quad \text{or} \quad i > L + K + 4B \\ \min\{\theta_i', y_{L+K+4B}\} & \text{for} \quad L + K + 1 \le i \le L + K + 4B, \end{cases} \tag{5.2}$$

i.e., we replace at most $4B$ boundary conditions y_i with the rescaled classical ones $\theta_i' = \theta_i/s(\mathbf{y})$ near the edges of the interval $[y_L, y_{L+K+1}] = [\theta_L', \theta_{L+K+1}']$. Note that the configuration space is unchanged. We have

$$y_{L-4B} \le \gamma_{L-4B} + \delta \le \theta_{L-4B}' + CB^2 N^{-2} + C\delta \le \theta_{L-2B}',$$

where we used that $\mathbf{y} \in \mathcal{G}$ in the first step and (4.25) in the second. In the last inequality we used that $\theta_{L-2B}' - \theta_{L-4B}' \ge cBN^{-1}$ (by regular spacing) and the definition of B from (5.1). Thus we obtain

$$y_i^B = \theta_i', \quad L - 2B \le i \le L, \tag{5.3}$$

and similarly at the upper edge. In other words, we do replace at least $2B$ boundary condition points near the edges with the classical ones. Although it may happen that a few y_i^B pile up, but this occurs away from the edges. The key property of the family y_i^B is the following bound

$$\#\{i \ : \ y_i^B \in J\} \le CN|J| \tag{5.4}$$

for any interval J such that $|J| \ge cN^{-1}$ and $c|J| \le \text{dist}(J, [y_L, y_{L+K+1}]) \le |J|/c$ with some small constant c.

Consider the *regularized local relaxation measure*, which is defined as the probability measure

$$\mu_{\mathbf{y}}^{B,\tau}(\mathrm{d}\mathbf{x}) = Z^{-1} e^{-N\mathcal{H}_{\mathbf{y}}^{B,\tau}} \mathrm{d}\mathbf{x} \tag{5.5}$$

of K ordered points $\mathbf{x} = (x_{L+1}, \ldots, x_{L+K})$ in $[y_L, y_{L+K+1}]$, with Hamiltonian

$$\mathcal{H}_{\mathbf{y}}^{B,\tau}(\mathbf{x}) := \sum_{i \in I} \frac{\beta}{2} V_{\mathbf{y}^B}^i(x_i) - \frac{\beta}{N} \sum_{\substack{i,j \in I \\ i<j}} \log|x_j - x_i| + \sum_{i \in I} Q_i^{\tau}(x_i), \tag{5.6}$$

with a quadratic confinement $Q_i^{\tau}(x) = (2\tau(\mathbf{y}))^{-1}(x - \theta_i')^2$ as in (4.34) and $\tau(\mathbf{y}) = \hat{\tau}/s(\mathbf{y})^2$. The potential V^i is given by

$$V_{\mathbf{y}^B}^i(x) = V(x) - \frac{2}{N} \sum_{j \le L} \log|x - y_j^B| - \frac{2}{N} \sum_{j \ge L+K+1} \log|x - y_j| \qquad \text{for} \quad L + 1 \le i \le L + 4B$$

$$V^i_{\mathbf{y}^B}(x) = V(x) - \frac{2}{N}\sum_{j\le L}\log|x-y_j| - \frac{2}{N}\sum_{j\ge L+K+1}\log|x-y_j| \qquad \text{for} \quad L+4B+1\le i\le L+K-4B$$

and

$$V^i_{\mathbf{y}^B}(x) = V(x) - \frac{2}{N}\sum_{j\le L}\log|x-y_j| - \frac{2}{N}\sum_{j\ge L+K+1}\log|x-y^B_j| \qquad \text{for} \quad L+K-4B+1< i\le L+K.$$

In other words, we replace the boundary condition $\mathbf{y}$ with $\mathbf{y}^B$ for the points x_i with $L+1\le i\le L+4B$ at the lower edge and similarly for the other edge. The boundary conditions for the middle points x_i with $L+4B+1\le i\le L+K-4B$ remain unchanged. Recalling (4.5), we have in particular

$$\mathcal{H}^{B,\tau}_{\mathbf{y}}(\mathbf{x}) - \mathcal{H}^{\tau}_{\mathbf{y}}(\mathbf{x}) = \frac{2}{N}\sum_{L-4B\le j<L}\;\sum_{L<i\le L+4B}\big[-\log|x_i-y^B_j| + \log|x_i-y_j|\big] + \big(\text{Upper edge}\big), \quad (5.7)$$

where *(Upper edge)* refers to an analogous term collecting interactions near the upper edge.

Lemma 5.1 *Let $\mathbf{y}\in\mathcal{G}$. The relative entropies of the measures $\mu^\tau_{\mathbf{y}}$ and $\mu^{B,\tau}_{\mathbf{y}}$ satisfy*

$$S(\mu^\tau_{\mathbf{y}}|\mu^{B,\tau}_{\mathbf{y}}) + S(\mu^{B,\tau}_{\mathbf{y}}|\mu^\tau_{\mathbf{y}}) \le CB^2\log N. \quad (5.8)$$

Proof. We start with the following lemma that estimates the relative entropy of any two measures:

Lemma 5.2 *Suppose $\mu_i(\mathrm{d}x) = Z_i^{-1}e^{-H_i}\mathrm{d}x$, $i=1,2$ are probability measures with Hamiltonians H_i on a common measure space. Then*

$$S(\mu_1|\mu_2) \le \mathbb{E}_{\mu_1}[H_2-H_1] + \mathbb{E}_{\mu_2}[H_1-H_2]. \quad (5.9)$$

We also have the inequality

$$\mathbb{E}_{\mu_2}[H_2-H_1] \le \log Z_1 - \log Z_2 \le \mathbb{E}_{\mu_1}[H_2-H_1]. \quad (5.10)$$

Proof. By Jensen inequality, we have

$$\begin{aligned} 0\le S(\mu_1|\mu_2) = \int \mathrm{d}\mu_1\log\left(\frac{\mathrm{d}\mu_1}{\mathrm{d}\mu_2}\right) &= \int\mathrm{d}\mu_1[H_2-H_1] + \log\left(\frac{Z_2}{Z_1}\right)\\ &\le \mathbb{E}_{\mu_1}[H_2-H_1] - \log\left[\int e^{-H_1}\frac{\mathrm{d}x}{\int e^{-H_2}\mathrm{d}x}\right]\\ &\le \mathbb{E}_{\mu_1}[H_2-H_1] + \mathbb{E}_{\mu_2}[H_1-H_2]. \end{aligned}$$

This completes the proof of Lemma 5.2. □

Hence we have $S(\mu^\tau_{\mathbf{y}}|\mu^{B,\tau}_{\mathbf{y}})\le\beta\Omega_1$, where

$$\Omega_1 := \Big(\mathbb{E}_{\mu^\tau_{\mathbf{y}}} - \mathbb{E}_{\mu^{B,\tau}_{\mathbf{y}}}\Big)\sum_{L-4B\le j<L}\;\sum_{L<i\le L+4B}\big[-\log(x_i-y^B_j) + \log(x_i-y_j)\big] + \big(\text{Upper edge}\big). \quad (5.11)$$

Using that $x_i - y_j \ll 1$, we clearly have

$$\Omega_1 \le - \sum_{L-4B\le j<L} \sum_{L<i\le L+4B} \Big[\mathbb{E}_{\mu_{\mathbf{y}}^\tau} \log(x_i - y_j^B) + \mathbb{E}^{\mu_{\mathbf{y}}^{B,\tau}} \log(x_i - y_j)\Big] + \big(\text{Upper edge}\big)$$
$$\le CB^2 \log N - B^2 \mathbb{E}_{\mu_{\mathbf{y}}^{B,\tau}} \log(x_{L+1} - y_L) + \big(\text{Upper edge}\big). \qquad (5.12)$$

In the first term we used the trivial estimate $x_i - y_j^B \ge \theta'_L - \theta'_{L-1} \ge cN^{-1}$ for any $j < L$. The second term will be estimated by Lemma 5.3 below and this completes the estimate for $S(\mu_{\mathbf{y}}^\tau|\mu_{\mathbf{y}}^{B,\tau})$. The other relative entropy, $S(\mu_{\mathbf{y}}^{B,\tau}|\mu_{\mathbf{y}}^\tau)$ can be treated similarly and this proves Lemma 5.1. □

Lemma 5.3 *Suppose $\tau \ge N^{-1}$, then for any $p \ge 1$ we have*

$$\mathbb{E}_{\mu_{\mathbf{y}}^\tau} |\log(x_{L+1} - y_L)|^p \le C_p \log N \qquad (5.13)$$

and the same estimate holds w.r.t the measure $\mu_{\mathbf{y}}^{B,\tau}$.

Proof. We will need that

$$\mathbb{P}_{\mu_{\mathbf{y}}^\tau}(x_{L+1} - y_L \le N^{-3} r) \le Cr \qquad (5.14)$$

for any $r \in (0,1)$. Then (5.13) follows from integrating in r from 0 to 1 and treating the regime $x_{L+1} - y_L \ge N^{-3}$ trivially by using $x_{L+1} - y_L \le y_{L+K+1} - y_L \le CK/N \le 1$.

The estimate (5.14) can be proven essentially in the same way as (4.11), just the potential $\frac{\beta}{2}V(x_j)$ of the j-th point in that proof is replaced with $\frac{\beta}{2}V(x_j) + Q_j^\tau(x_j)$. The final estimate is somewhat weaker since now the bound on the constant C_V defined in (4.12) deteriorates to $C_V \le C\tau^{-1} \le CN$. This accounts for the change from N^{-2} to N^{-3} in (5.14). The argument for the measure $\mu_{\mathbf{y}}^{B,\tau}$ is analogous and this proves Lemma 5.3. □

5.2 Regularization does not change spacing statistics

Given that the local relaxation measure $\mu_{\mathbf{y}}^\tau$ and its regularized version $\mu_{\mathbf{y}}^{B,\tau}$ are close in relative entropy sense, the next proposition shows that their local spacing statistics coincide.

Proposition 5.4 *Let $\mathbf{y} \in \mathcal{G}$, $\tau = \tau(\mathbf{y}) = \hat\tau/s(\mathbf{y})^2$ and assume that for the parameters $B = N^b$, $K = N^k$ and $\hat\tau = N^{-t}$ it holds that*

$$1 + 2b - t - k < 0. \qquad (5.15)$$

Then

$$\left| \big[\mathbb{E}_{\mu_{\mathbf{y}}^\tau} - \mathbb{E}_{\mu_{\mathbf{y}}^{B,\tau}}\big] \frac{1}{K} \sum_{i\in I} G\Big(N(x_i - x_{i+1})\Big) \right| \to 0 \qquad (5.16)$$

as $N \to \infty$ for any smooth and compactly supported test function G.

Proof. Since Lemma 5.1 and (5.15) guarantee that

$$\frac{NS(\mu_{\mathbf{y}}^{B,\tau} \mid \mu_{\mathbf{y}}^\tau)\tau}{K} \le \frac{CNB^2\tau}{K} \log N \le N^{-\varepsilon'} \qquad (5.17)$$

with some $\varepsilon' > 0$, Proposition 5.4 is a direct consequence of the following comparison lemma which was first stated in a remark after Lemma 3.4 in [14], see also Lemma 4.4 in [15]. □

Lemma 5.5 *Let $G : \mathbb{R} \to \mathbb{R}$ be a bounded smooth function with compact support and let a sequence E_i be fixed. Let I be an interval of indices with $|I| = K$. Consider a measure ω with relaxation time τ and let $q\mathrm{d}\omega$ be another probability measure. Then for any $\varepsilon' > 0$ and for any smooth compactly supported function we have*

$$\Big|\frac{1}{K}\sum_{i\in I}\int G\big(N(x_i - E_i)\big)[q-1]\mathrm{d}\omega\Big| \le C\sqrt{\frac{N^{1+\varepsilon'}S_\omega(q)\tau}{K}} + Ce^{-cN^{\varepsilon'}}\sqrt{S_\omega(q)} \tag{5.18}$$

and

$$\Big|\frac{1}{K}\sum_{i\in I}\int G\big(N(x_i - x_{i+1})\big)[q-1]\mathrm{d}\omega\Big| \le C\sqrt{\frac{N^{1+\varepsilon'}S_\omega(q)\tau}{K}} + Ce^{-cN^{\varepsilon'}}\sqrt{S_\omega(q)}, \tag{5.19}$$

where $S_\omega(q) := S(q\omega \mid \omega)$.

Proof. Let q evolve by the dynamics $\partial_t q_t = \mathcal{L}q_t$, where $\mathcal{L}$ is the generator defined by

$$\int -f\mathcal{L}f\mathrm{d}\omega = D_\omega(f) = \frac{1}{2N}\int |\nabla f|^2\mathrm{d}\omega. \tag{5.20}$$

Let $\tau' = N^\varepsilon \tau$. Since $q_{\tau'}$ is already subexponentially close to ω in entropy sense, $S_\omega(q_{\tau'}) \le C\exp(-cN^\varepsilon)S_\omega(q)$, and the total variation norm can be estimated by the relative entropy, we only have to compare q with $q_{\tau'}$.

By differentiation, we have (the summation over i always runs $i \in I$)

$$\int \frac{1}{K}\sum_i G\Big(N(x_i - E_i)\Big)q_{\tau'}\mathrm{d}\omega - \int \frac{1}{K}\sum_i G\Big(N(x_i - E_i)\Big)q\mathrm{d}\omega \tag{5.21}$$

$$= \int_0^{\tau'}\mathrm{d}s\int \frac{1}{K}\sum_i \partial_i G\Big(N(x_i - E_i))\Big)\partial_i q_s\mathrm{d}\omega. \tag{5.22}$$

Here we used the definition of $\mathcal{L}$ from (5.20) and note that the $1/N$ factor present in (5.20) cancels the factor N from the argument of G. From the Schwarz inequality and $\partial q = 2\sqrt{q}\partial\sqrt{q}$, the last term is bounded by

$$\Big[\frac{N}{K^2}\int_0^{\tau'}\mathrm{d}s\int\sum_i\Big[\partial_i G\big(N(x_i - E_i)\big)\Big]^2 q_s\mathrm{d}\omega\Big]^{1/2}\left[\int_0^{\tau'}\mathrm{d}s\int\frac{1}{N}\sum_i(\partial_i\sqrt{q_s})^2\mathrm{d}\omega\right]^{1/2}$$

$$\le C\sqrt{\frac{NS_\omega(q)\tau'}{K}} \tag{5.23}$$

by integrating $\partial_s S_\omega(q_s) = -4D_\omega(\sqrt{q_s})$. This proves (5.18) and the proof of (5.19) is analogous. □

5.3 Accuracy of block averages

In the next Section 5.5 we will compare the regularized local relaxation measure $\mu_{\mathbf{y}}^{B,\tau}$ with the reference measure $\sigma_\theta^{\hat{\tau},s(\mathbf{y})}$ in Dirichlet form sense. As a preparation for this step, we give an estimate on the location of the block averages $x_j^{[B]}$. Recall their definition

$$x_j^{[B]} := \frac{1}{2B+1}\sum_{|k-j|\le B} x_k$$

for any $j \in [\![L+B+1, L+K-B]\!]$. The following lemma shows concentration on a scale ζ for $x_j^{[B]}$ w.r.t. $\mu_{\mathbf{y}}^{\tau}$ and $\mu_{\mathbf{y}}^{B,\tau}$. The scale ζ is larger than δ but will be smaller than K/N, the length of configuration space interval. Thus that the accuracy of the position of x_j decreases from δ to ζ, but the accuracy of y_k is still δ.

Lemma 5.6 *Set $\zeta = N^{-z}$, $t = 2d - 1 - \varphi$ and fix $\mathbf{y} \in \mathcal{G}$. For any $j \in [\![L+B+1, L+K-B]\!]$ we have*

$$\mathbb{P}_{\mu_{\mathbf{y}}^{\tau}}\Big(\big|x_j^{[B]} - \mathbb{E}_{\mu_{\mathbf{y}}^{\tau}} x_j^{[B]}\big| \geq \zeta\Big) \leq c_1 e^{-c_2 N^{\varepsilon'}} \tag{5.24}$$

and

$$\mathbb{P}_{\mu_{\mathbf{y}}^{B,\tau}}\Big(\big|x_j^{[B]} - \mathbb{E}_{\mu_{\mathbf{y}}^{B,\tau}} x_j^{[B]}\big| \geq \zeta\Big) \leq c_1 e^{-c_2 N^{\varepsilon'}} \tag{5.25}$$

provided

$$z \leq -\varphi + \min\Big(d - \frac{b}{2} - \frac{\varphi}{2},\ d - \frac{k}{2} + \frac{b}{2}\Big) \tag{5.26}$$

for some $\varepsilon' = \varepsilon'(d,\varphi) > 0$ depending only on d and φ. Furthermore, we have

$$\Big|\mathbb{E}_{\mu_{\mathbf{y}}^{B,\tau}} x_j^{[B]} - \gamma_j^{[B]}\Big| \leq 5\zeta, \qquad \Big|\mathbb{E}_{\mu_{\mathbf{y}}^{\tau}} x_j^{[B]} - \gamma_j^{[B]}\Big| \leq 5\zeta, \qquad \Big|\mathbb{E}_{\mu_{\mathbf{y}}} x_j^{[B]} - \gamma_j^{[B]}\Big| \leq 5\zeta. \tag{5.27}$$

Proof. We will need two standard inequalities from probability theory. The first one is

$$\mathbb{P}_{\mu}(A) \cdot \log \frac{1}{\mathbb{P}_{\nu}(A)} \leq \log 2 + S(\mu|\nu) \tag{5.28}$$

for any set A and probability measures μ, ν. This can be obtained from the entropy inequality

$$\int f \mathrm{d}\mu \leq S(\mu|\nu) + \log\Big[\int e^f \mathrm{d}\nu\Big]$$

by choosing $f(x) = b \cdot \mathbf{1}_A(x)$ with $b = -\log \mathbb{P}^{\nu}(A)$. Using Lemma 5.1 we thus obtain

$$\mathbb{P}_{\mu_{\mathbf{y}}^{B,\tau}}(A) \leq \frac{\log 2 + CB^2 \log N}{-\log \mathbb{P}_{\mu_{\mathbf{y}}^{\tau}}(A)}. \tag{5.29}$$

The second inequality is a concentration estimate. Suppose that the probability measure ω satisfies the logarithmic Sobolev inequality (LSI), i.e.

$$S_{\omega}(f) \leq C_{\mathrm{s}} \int |\nabla \sqrt{f}|^2 \mathrm{d}\omega \tag{5.30}$$

holds for any $f \geq 0$ with $\int f \mathrm{d}\omega = 1$. Then for any random variable X with $\mathbb{E}_{\omega} X = 0$ and any number $T > 0$ we have

$$\mathbb{E}_{\omega} e^{TX} \leq \mathbb{E}_{\omega} \exp\Big(\frac{C_{\mathrm{s}} T^2}{2} |\nabla X|^2\Big). \tag{5.31}$$

Since the Hamiltonian $\mathcal{H}_{\mathbf{y}}^{\tau}$ is convex with $\nabla^2 \mathcal{H}_{\mathbf{y}}^{\tau} \geq \tau^{-1}$, by the Bakry-Eméry criterion the measure $\mu_{\mathbf{y}}^{\tau} \sim \exp(-N\mathcal{H}_{\mathbf{y}}^{\tau})$ satisfies (5.30) with Sobolev constant $C_s = 2\tau/N$, i.e.

$$S(\nu \mid \mu_{\mathbf{y}}^{\tau}) \leq 4\tau D(\nu \mid \mu_{\mathbf{y}}^{\tau}) \tag{5.32}$$

for any probability measure ν (recall that the definition of the Dirichlet form (4.17) contains a $1/2N$ prefactor). The same statements hold for the regularized measure $\mu_{\mathbf{y}}^{B,\tau}$.

For $L+B+1 \le j \le L+K-B$ define the event

$$A = A_j = \big\{ |x_j^{[B]} - \mathbb{E}_{\mu_{\mathbf{y}}^\tau} x_j^{[B]}| \ge \zeta \big\}, \qquad \text{with} \quad \zeta = N^{-z}, \tag{5.33}$$

with a parameter $z \in (0,1)$ chosen later. Using (5.31) for $X = \pm(x_j^B - \mathbb{E}_{\mu_{\mathbf{y}}^\tau} x_j^B)$ and noticing that $|\nabla X|^2 = (2B+1)^{-1}$, we obtain

$$\mathbb{P}_{\mu_{\mathbf{y}}^\tau}(A) \le 2e^{-\frac{1}{2}NB\zeta^2\tau^{-1}}. \tag{5.34}$$

Using now (5.29), we get

$$\mathbb{P}_{\mu_{\mathbf{y}}^{B,\tau}}(A) \le \frac{CB\tau}{N\zeta^2} \to 0 \tag{5.35}$$

assuming

$$b - t + 2z - 1 < 0. \tag{5.36}$$

Using $t = 2d - 1 - \varphi$, we need

$$z < d - \frac{b}{2} - \frac{\varphi}{2}. \tag{5.37}$$

Under this condition we have from (5.34) that

$$\mathbb{P}_{\mu_{\mathbf{y}}^\tau}\Big(|x_j^{[B]} - \mathbb{E}_{\mu_{\mathbf{y}}^\tau} x_j^{[B]}| \ge \zeta \Big) \le 2e^{-B^2}. \tag{5.38}$$

Since the measure $\mu_{\mathbf{y}}^{B,\tau}$ is also concentrated by the LSI, we have

$$\mathbb{P}_{\mu_{\mathbf{y}}^{B,\tau}}\Big(|x_j^{[B]} - \mathbb{E}_{\mu_{\mathbf{y}}^{B,\tau}} x_j^{[B]}| \ge \zeta \Big) \le 2e^{-B^2} \to 0$$

and together with (5.35) we have

$$\Big| \mathbb{E}_{\mu_{\mathbf{y}}^{B,\tau}} x_j^{[B]} - \mathbb{E}_{\mu_{\mathbf{y}}^\tau} x_j^{[B]} \Big| \le 2\zeta. \tag{5.39}$$

Therefore $x_j^{[B]}$ is concentrated on a scale ζ around the same point w.r.t both measures $\mu_{\mathbf{y}}^\tau$ and $\mu_{\mathbf{y}}^{B,\tau}$.

Using (5.32) and that $\mathbf{y} \in \mathcal{G}$ we get

$$S(\mu_{\mathbf{y}}|\mu_{\mathbf{y}}^\tau) \le 4\tau D(\mu_{\mathbf{y}}|\mu_{\mathbf{y}}^\tau) \le \frac{4N}{\tau} \mathbb{E}_{\mu_{\mathbf{y}}} \sum_{j \in I} (x_j - \gamma_j)^2 \le \frac{4N\delta^2 K}{\tau}. \tag{5.40}$$

Hence by (5.28) and (5.34) we obtain

$$\mathbb{P}_{\mu_{\mathbf{y}}}(A) \le \frac{\log 2 + \frac{4N\delta^2 K}{\tau}}{-\log \mathbb{P}_{\mu_{\mathbf{y}}^\tau}(A)} \le \frac{C\delta^2 K}{B\zeta^2} \to 0 \tag{5.41}$$

provided that

$$z < d - \frac{k}{2} + \frac{b}{2}. \tag{5.42}$$

Now by the definition of $\mathbf{y} \in \mathcal{G}$ in (4.7) we have

$$\begin{aligned}\mathbb{P}_{\mu_{\mathbf{y}}}\Big(\big|x_j^{[B]} - \mathbb{E}_{\mu_{\mathbf{y}}} x_j^{[B]}\big| \geq \zeta\Big) &\leq \zeta^{-2}\mathbb{E}_{\mu_{\mathbf{y}}}\big|x_j^{[B]} - \mathbb{E}_{\mu_{\mathbf{y}}} x_j^{[B]}\big|^2 \\ &\leq \frac{1}{(2B+1)\zeta^2}\sum_{|k-j|\leq B}\mathbb{E}_{\mu_{\mathbf{y}}}\big|x_k - \mathbb{E}_{\mu_{\mathbf{y}}} x_k\big|^2 \\ &\leq \frac{1}{(2B+1)\zeta^2}\sum_{|k-j|\leq B}\mathbb{E}_{\mu_{\mathbf{y}}}\big|x_k - \gamma_k\big|^2 \leq \frac{\delta^2}{\zeta^2} \to 0\end{aligned}$$

using (5.37). Combining it with (5.41) we obtain

$$\Big|\mathbb{E}_{\mu_{\mathbf{y}}} x_j^{[B]} - \mathbb{E}_{\mu_{\mathbf{y}}^{\tau}} x_j^{[B]}\Big| \leq 2\zeta. \tag{5.43}$$

Finally, since $\mathbf{y} \in \mathcal{G}$, we have

$$\Big(\mathbb{E}_{\mu_{\mathbf{y}}} x_j^{[B]} - \gamma_j^{[B]}\Big)^2 \leq \mathbb{E}_{\mu_{\mathbf{y}}}\Big(x_j^{[B]} - \gamma_j^{[B]}\Big)^2 \leq \delta^2 \leq \zeta^2,$$

which, combined with (5.39) and (5.43), yields (5.27). This completes the proof of Lemma 5.6. □

5.4 Proof of Proposition 4.5

Now we will compare the regularized local relaxation measure $\mu_{\mathbf{y}}^{B,\tau}$ with the reference measure $\sigma_\theta^{\hat{\tau},s}$ in Dirichlet form sense. Recall their definitions from (5.5) and (4.33), respectively, and recall that $\tau = \tau(\mathbf{y}) := \hat{\tau}/s(\mathbf{y})^2$. Here $s = s(\mathbf{y})$ is a function that is approximately 1 for good external configurations $\mathbf{y} \in \mathcal{G}$ (see (4.23)).

The result is the following comparison of local gap statistics. Combining this result with Proposition 5.4 and checking that the condition (5.15) is satisfied with the choice of parameters given below, we arrive at the proof of Proposition 4.5. □

Proposition 5.7 *Fix $\varphi \leq \frac{1}{38}$. Let $\mathbf{y} \in \mathcal{G}$, $\tau = \tau(\mathbf{y}) = \hat{\tau}/s(\mathbf{y})^2$ and assume that for the parameters $\delta = N^{-d}$, $B = N^b$, $K = N^k$ with $d = 1-\varphi$, $b = 8\varphi$, $k = \frac{39}{2}\varphi$. Then with $t = 2d-1-\varphi = 1-3\varphi$ let $\hat{\tau} = N^{-t}$ with $t := 2d-1-\varphi = 1-3\varphi$. Then*

$$\left|\big[\mathbb{E}_{\mu_{\mathbf{y}}^{B,\tau}} - \mathbb{E}_{\sigma_{\mathbf{y}}^{\hat{\tau},s}}\big]\frac{1}{K}\sum_{i\in I} G\Big(N(x_i - x_{i+1})\Big)\right| \to 0 \tag{5.44}$$

as $N \to \infty$ for any smooth and compactly supported test function G.

Proof. The key technical estimate is the following lemma whose proof will take up most of this section.

Lemma 5.8 *Let $\varphi > 0$. Suppose $B = N^b$, $K = N^k$ with $0 < b < k < 1$, and $\delta = N^{-d}$ with $d \in (0,1)$. Suppose that these parameters satisfy*

$$1 - b < -\varphi + \min\Big(d - \frac{b}{2} - \frac{\varphi}{2},\ d - \frac{k}{2} + \frac{b}{2}\Big), \tag{5.45}$$

i.e. one can choose a number $z > 1-b$ and satisfying (5.26). Let $\mathbf{y} \in \mathcal{G} = \mathcal{G}_{\delta,\varepsilon_0}$ be a good configuration. Assume that $\varepsilon_0 \leq \varepsilon'/10$, where $\varepsilon' = \varepsilon'(d,\varphi)$ is obtained in Lemma 5.6. Assume that *the equilibrium measure ρ_V is C^1 away from the edges. Let $\hat\tau = N^{-t}$ with $t = 2d-1-\varphi$. Then the Dirichlet form of $\mu_{\mathbf{y}}^{B,\tau}$ with respect to the reference measure is bounded by*

$$\frac{\tau}{K} D\big(\mu_{\mathbf{y}}^{B,\tau} \mid \sigma_\theta^{\hat\tau,s}\big) \le C\hat\tau(\log N)\Big[\frac{K^2}{N} + \frac{N\delta^2}{\hat\tau^2} + \frac{K^4}{N^3\hat\tau^2} + \frac{\delta^2 N^3}{BK} + N^{3/5+\varphi}\Big] + c_1 e^{-c_2 N^{\varepsilon'/3}}. \quad (5.46)$$

The prefactor τ/K is for convenience; the local gap statistics of two measures are approximately the same if $\tau D/K \to 0$. More precisely, we have the following general theorem which is a slight modification of Lemma 3.4 [14] (see also Theorem 4.1 in [15]).

Lemma 5.9 *Let $G : \mathbb{R} \to \mathbb{R}$ be a bounded smooth function with compact support. Let I be an interval of indices with $|I| = K$. Consider a measure ω with relaxation time τ and let $q\mathrm{d}\omega$ be another probability measure. Then for any $\varepsilon' > 0$ and for any smooth compactly supported function we have*

$$\Big|\frac{1}{K}\sum_{i\in I}\int G\big(N(x_i - x_{i+1})\big)[q-1]\mathrm{d}\omega\Big| \le C\sqrt{\frac{N^{1+\varepsilon'}D_\omega(q)\tau}{K}} + Ce^{-cN^{\varepsilon'}}\sqrt{S_\omega(q)}, \quad (5.47)$$

where $D_\omega(q) := D(q\omega \mid \omega)$.

Proof. As in the proof of Lemma 5.5, let q evolve by the dynamics $\partial_t q_t = \mathcal{L} q_t$, where $\mathcal{L}$ is the generator defined by (5.20). Let $\tau' = N^\varphi \tau$. Since $q_{\tau'}$ is already subexponentially close to ω, $S_\omega(q_{\tau'}) \le C\exp(-cN^\varphi)S_\omega(q)$, we only have to compare q with $q_{\tau'}$.

By differentiation, we have

$$\int \frac{1}{K}\sum_i G\Big(N(x_i - x_{i+1})\Big)q_{\tau'}\mathrm{d}\omega - \int \frac{1}{K}\sum_i G\Big(N(x_i - x_{i+1})\Big)q\mathrm{d}\omega \quad (5.48)$$

$$= \int_0^{\tau'}\mathrm{d}s \int \frac{1}{K}\sum_i \partial_i G\Big(N(x_i - x_{i+1})\Big)\Big)[\partial_i q_s - \partial_{i+1} q_s]\mathrm{d}\omega. \quad (5.49)$$

Here we used the definition of $\mathcal{L}$ from (5.20) and note that the $1/N$ factor present in (5.20) cancels the factor N from the argument of G. From the Schwarz inequality and $\partial q = 2\sqrt{q}\partial\sqrt{q}$, the last term is bounded by

$$\Big[\frac{N^2}{K^2}\int_0^\tau \mathrm{d}s \int \sum_i \Big[\partial_i G\big(N(x_i - x_{i+1})\big)\Big]^2 (x_i - x_{i+1})^2\, q_s \mathrm{d}\omega\Big]^{1/2}$$

$$\times \Big[\int_0^\tau \mathrm{d}s \int \frac{1}{N^2}\sum_i \frac{1}{(x_i - x_{i+1})^2}[\partial_i\sqrt{q_s} - \partial_{i+1}\sqrt{q_s}]^2 \mathrm{d}\omega\Big]^{1/2}$$

$$\le\; C\sqrt{\frac{D_\omega(\sqrt{q})\tau}{K}}, \quad (5.50)$$

which completes the proof of Lemma 5.9. □

The proof of Proposition 5.7 now follows from Lemma 5.8 and Lemma 5.9 with $\omega = \sigma_\theta^{\hat\tau,s}$ and $q\mathrm{d}\omega = \mu_{\mathbf{y}}^{B,\tau}$. The parameters $b, k, d \in (0,1)$ have to satisfy the following relations from (5.45) and from the requirement that the right side of (5.46) converges to zero:

$$
\begin{aligned}
b &< k \\
1 - b + \varphi &< d - \frac{b}{2} - \frac{\varphi}{2} \\
1 - b + \varphi &< d - \frac{k}{2} + \frac{b}{2} \\
1 - 2d + \varphi + 2k - 1 &< 0 \\
1 - 2d + (2d - 1 - \varphi) &< 0 \\
4k - 3 + (2d - 1 - \varphi) &< 0 \\
1 - 2d + \varphi - 2d + 3 - b - k &< 0 \\
1 - 2d + 2\varphi + \frac{3}{5} &< 0.
\end{aligned}
$$

It is easy to check that all these conditions are satisfied if, e.g.

$$
d = 1 - \varphi, \qquad b = 8\varphi, \qquad k = \frac{39}{2}\varphi, \qquad 0 < \varphi \le \frac{1}{38}.
$$

This choice is not optimal for the above system of inequalities, but we took into account that the parameters will also have to satisfy (5.15) so that we could combine Proposition 5.7 and Proposition 5.4 to arrive at Proposition 4.5.

Finally, the entropy term $S\big(\mu_{\mathbf{y}}^{B,\tau} \mid \sigma_\theta^{\hat{\tau},s}\big)$ in (5.47) can be estimated by the Dirichlet form via the logarithmic Sobolev inequality. This completes the proof of Proposition 5.7. □

5.5 Dirichlet form estimate: proof of Lemma 5.8

By definition,

$$
\frac{\tau}{K} D\big(\mu_{\mathbf{y}}^{B,\tau} \mid \sigma_\theta^{\hat{\tau},s}\big) = \frac{\tau}{2NK} \int \Big|\nabla \log\Big(\frac{\mu_{\mathbf{y}}^{B,\tau}}{\sigma_\theta^{\hat{\tau},s}}\Big)\Big|^2 \mathrm{d}\mu_{\mathbf{y}}^{B,\tau} \le \frac{\tau N}{K} \int \sum_{L+1 \le j \le L+K} Z_j^2 \mathrm{d}\mu_{\mathbf{y}}^{B,\tau},
$$

where Z_j is defined as follows: For $L + 1 < j \le L + 4B$, we set

$$
Z_j := \frac{\beta}{2} V'(x_j) - \frac{\beta}{N} \sum_{\substack{k < L-2B \\ k > L+K}} \frac{1}{x_j - y_k^B} - \frac{\beta}{2} W_s'(x_j) + \frac{\beta}{N} \sum_{\substack{k < L-2B \\ k > L+K}} \frac{1}{x_j - \theta_k'} + \frac{\gamma_j - \theta_j'}{\tau}
$$

(recall that $\theta_j' = \theta_j/s$ and we set $W_s(x) = s^2 x^2$). Note that the summation at the lower edge is only for $k < L - 2B$ instead of $k \le L$ because the interaction terms near the boundary cancel by (5.3). Moreover, notice that the linear terms, coming from the derivative of the quadratic confinements (see (4.3) and (4.34)), cancel each other

$$
\frac{s(\mathbf{y})^2}{\hat{\tau}}(x_j - \theta_j') - \frac{1}{\tau}(x_j - \gamma_j) = \frac{\gamma_j - \theta_j'}{\tau}
$$

by the choice of $\tau(\mathbf{y}) = \hat{\tau}/s(\mathbf{y})^2$.

Similarly, for $L+K-4B<j\le L+K$, we set

$$Z_j := \frac{\beta}{2}V'(x_j) - \frac{\beta}{N}\sum_{\substack{k>L+K+2B\\ k<L}}\frac{1}{x_j-y_k^B} - \frac{\beta}{2}W_s'(x_j) + \frac{\beta}{N}\sum_{\substack{k>L+K+2B\\ k<L}}\frac{1}{x_j-\theta_k'} + \frac{\gamma_j-\theta_j'}{\tau}.$$

Finally, for $L+4B<j\le L+K-4B$, we define

$$Z_j := \frac{\beta}{2}V'(x_j) - \frac{\beta}{N}\sum_{\substack{k<L\\ k>L+K+1}}\frac{1}{x_j-y_k} - \frac{\beta}{2}W_s'(x_j) + \frac{\beta}{N}\sum_{\substack{k<L\\ k>L+K+1}}\frac{1}{x_j-\theta_k'} + \frac{\gamma_j-\theta_j'}{\tau}.$$

Notice that here y_k is not replaced with y_k^B since only interactions for x_j's near the edges have been regularized. Moreover, the interactions with the boundary terms y_k, with $k=L$ and $k=L+K+1$ cancel out since $y_L=\theta_L'$ and $y_{L+K+1}=\theta_{L+K+1}'$ by the matching construction.

Now we estimate the size of Z_j in each case.

Case 1: $L+4B<j\le L+K-4B$. The first step is to decompose Z_j as

$$Z_j = \beta\sum_{a=1}^{5}\Omega_j^a, \tag{5.51}$$

where

$$\begin{aligned}
\Omega_j^1 &:= \left[\frac{1}{2}V'(x_j) - \int \mathrm{d}y\frac{\rho_V(y)}{x_j-y}\right] - \left[\frac{1}{2}W_s'(x_j) - \int \mathrm{d}y\frac{\rho_{W_s}(y)}{x_j-y}\right]\\
\Omega_j^2 &= \Omega_j^{2,low} + \Omega_j^{2,up}\\
&:= -\left(\frac{1}{N}\sum_{k<L}\frac{1}{x_j-y_k} - \int_{-\infty}^{y_L}\frac{\rho_V(y)}{x_j-y}\mathrm{d}y\right) - \left(\frac{1}{N}\sum_{k>L+K+1}\frac{1}{x_j-y_k} - \int_{y_{L+K+1}}^{\infty}\frac{\rho_V(y)}{x_j-y}\mathrm{d}y\right)\\
\Omega_j^3 &= \Omega_j^{3,low} + \Omega_j^{3,up}\\
&:= \left(\frac{1}{N}\sum_{k<L}\frac{1}{x_j-\theta_k'} - \int_{-\infty}^{\theta_L'}\frac{\rho_{W_s}(y)}{x_j-y}\mathrm{d}y\right) + \left(\frac{1}{N}\sum_{k>L+K+1}\frac{1}{x_j-\theta_k'} - \int_{\theta_{L+K+1}'}^{\infty}\frac{\rho_{W_s}(y)}{x_j-y}\mathrm{d}y\right)\\
\Omega_j^4 &:= \int_{y_L}^{y_{L+K+1}}\frac{\rho_V(y)-\rho_{W_s}(y)}{x_j-y}\mathrm{d}y\\
\Omega_j^5 &:= \frac{\gamma_j-\theta_j'}{\beta\tau}.
\end{aligned} \tag{5.52}$$

Here we also used that $[y_L, y_{L+K+1}] = [\theta_L', \theta_{L+K+1}']$ when establishing the limits of integrations. By the equilibrium relation (3.2) between V and ρ_V, we have

$$\Omega_j^1 = 0. \tag{5.53}$$

From (4.25), we have

$$[\Omega_j^5]^2 = C\frac{(\gamma_j-\theta_j')^2}{\tau^2} \le \frac{C}{\tau^2}\left[\delta^2 + \frac{K^4}{N^4}\right]. \tag{5.54}$$

Since $\rho_V \in C^1$ away from the edge, and so is the semicircle density ρ_{W_s}, we have by Taylor expansion

$$
\begin{aligned}
|\Omega_j^4| &= \Big| \int_{y_L}^{y_{L+K+1}} \frac{\rho_V(y) - \rho_{W_s}(y)}{x_j - y} \mathrm{d}y \Big| \qquad (5.55) \\
&\leq \Bigg| \int_{y_L}^{y_{L+K+1}} \frac{\rho_V(x_j) - \rho_{W_s}(x_j) + O(x_j - y)}{x_j - y} \mathrm{d}y \Bigg| \\
&\leq C\big[|\log(x_j - y_L)| + |\log(y_{L+K+1} - x_j)| \big] \left[\frac{K}{N} + \frac{\delta N}{K} \right].
\end{aligned}
$$

Here we used (4.27) and (4.30) and the fact that $\rho_{W_s}(x) - \rho_W(x) = O(|s-1|)$ away from the edge together with (4.23) to estimate

$$
|\rho_V(x) - \rho_{W_s}(x)| \leq C \left[\frac{K}{N} + \frac{\delta N}{K} \right]
$$

for any $x \in [y_L, y_{L+K+1}]$. The logarithmic terms after taking square and expectation w.r.t. will give rise to an irrelevant $\log N$ factor by using Lemma 5.3

$$
\mathbb{E}_{\mu_{\mathbf{y}}^{B,\tau}} \big[|\log(x_j - y_L)| + |\log(y_{L+K+1} - x_j)| \big]^2 \leq C \log N.
$$

We now estimate the main error Ω_j^2 and we will deal with the first term only, coming from the lower edge, the second one can be treated similarly. We write it as

$$
\Omega_j^{2,low} = -\left(\frac{1}{N} \sum_{k<L} \frac{1}{x_j - y_k} - \int_{-\infty}^{y_L} \frac{\rho_V(y)}{x_j - y} \mathrm{d}y \right) = \Omega_j^{2,1} + \Omega_j^{2,2} + \Omega_j^{2,3}
$$

with

$$
\begin{aligned}
\Omega_j^{2,1} &:= -\left(\frac{1}{N} \sum_{k<L} \frac{1}{x_j - \gamma_k} - \int_{-\infty}^{\gamma_L} \frac{\rho_V(y)}{x_j - y} \mathrm{d}y \right) \\
\Omega_j^{2,2} &:= \int_{\gamma_L}^{y_L} \frac{\rho_V(y)}{x_j - y} \mathrm{d}y \\
\Omega_j^{2,3} &:= \frac{1}{N} \sum_{k<L} \Big[\frac{1}{x_j - \gamma_k} - \frac{1}{x_j - y_k} \Big]. \qquad (5.56)
\end{aligned}
$$

With $\zeta = N^{-z}$ with z is given in Lemma 5.8, define the event

$$
\Lambda = \Big\{ |x_i^{[B]} - \gamma_i^{[B]}| \leq 6\zeta, \quad \forall i \in [\![L+B+1, L+K-B]\!] \Big\},
$$

then its complement has very small probability,

$$
\mathbb{P}_{\mu_{\mathbf{y}}^{B,\tau}} \big(\Lambda^c \big) \leq c_1 e^{-c_2 N^{\varepsilon'}}
$$

from (5.24) and (5.27). On the event Λ^c we simply estimate

$$
\Big| \frac{1}{N} \sum_{k<L} \frac{1}{x_j - y_k} - \int_{-\infty}^{y_L} \frac{\rho_V(y)}{x_j - y} \mathrm{d}y \Big| \leq \frac{1}{y_L - y_{L-1}} + C \big| \log(x_j - y_L) \big|,
$$

therefore

$$\mathbb{E}_{\mu_{\mathbf{y}}^{B,\tau}}\mathbf{1}(\Lambda^c)\Big|\frac{1}{N}\sum_{k<L}\frac{1}{x_j-y_k}-\int_{-\infty}^{y_L}\frac{\rho_V(y)}{x_j-y}\mathrm{d}y\Big|^2$$

$$\le C\Big(\frac{1}{(y_L-y_{L-1})^2}+\mathbb{E}_{\mu_{\mathbf{y}}^{B,\tau}}\big|\log(x_j-y_L)\big|^4\Big)^{1/2}\big(\mathbb{P}_{\mu_{\mathbf{y}}^{B,\tau}}\big(\Lambda^c\big)\big)^{1/2}\le c_1e^{-c_2N^{\varepsilon'/3}} \tag{5.57}$$

by using Lemma 5.3 and $|y_L-y_{L-1}|\ge\exp(-N^{\varepsilon_0})$ from $\mathbf{y}\in\mathcal{G}$. Here we used that $\varepsilon_0\le\varepsilon'/10$.

Now we continue the estimate on the set Λ and we consider the three terms in (5.56) separately. For the first term we write

$$\Omega_j^{2,1}=\frac{1}{N}\sum_{k<L}\Big[\frac{1}{x_j-\gamma_k}-\int_{\gamma_k}^{\gamma_{k+1}}\frac{N\rho_V(y)}{x_j-y}\mathrm{d}y\Big]=\frac{1}{N}\sum_{k<L}\int_{\gamma_k}^{\gamma_{k+1}}\Gamma_j^kN\rho_V(y)\mathrm{d}y$$

where we have used that $\int_{\gamma_k}^{\gamma_{k+1}}N\rho_V=1$ and

$$\Gamma_j^k=\frac{\gamma_k-y}{(x_j-y)(x_j-\gamma_k)}.$$

Recall that $L\ge\kappa N\gg\delta$ and $x_j\in[y_L,y_{L+K+1}]=[\gamma_L,\gamma_{L+K+1}]+O(\delta)$. For $k\le\frac{1}{2}\kappa N$ we know that $|\gamma_k-x_j|\ge c$ with some positive constant. Hence we have

$$\frac{1}{N}\sum_{k\le\kappa N/2}\Gamma_j^k\le\frac{C}{N}\sum_{k\le\kappa N/2}\int_{\gamma_k}^{\gamma_{k+1}}|\gamma_k-y|N\rho_V(y)\mathrm{d}y\le\frac{C}{N}\sum_{k\le\kappa N/2}|\gamma_{k+1}-\gamma_k|\le CN^{-1},$$

since $\gamma_{k+1}-\gamma_k\le CN^{-2/3}k^{-1/3}$ near a square root singularity of ρ_V at the edge. For the regime $k\ge\frac{1}{2}\kappa N$ we can use $|\gamma_{k+1}-\gamma_k|\le CN^{-1}$ to get

$$\begin{aligned}\frac{1}{N}\sum_{\kappa N/2\le k<L}\Gamma_j^k&\le\frac{1}{N}\sum_{\kappa N/2\le k<L}\frac{C}{N}\frac{1}{(x_j-\gamma_k)^2}\\&\le\frac{1}{N}\sum_{\kappa N/2\le k<L}\frac{C}{N}\frac{1}{(x_{j-B}^{[B]}-\gamma_k)^2}\\&\le\frac{C}{N}\frac{1}{(x_{j-B}^{[B]}-\gamma_L)}\le\frac{C}{B}.\end{aligned}$$

Here in the second inequality we used that on the set Λ we have

$$x_j\ge x_{j-B}^{[B]}>\gamma_{j-B}^{[B]}-6\zeta\ge\gamma_{j-2B}-6\zeta\ge\gamma_L+cBN^{-1}>\gamma_k+cBN^{-1} \tag{5.58}$$

for $k<L$ using $j\ge L+4B$ and thus $\gamma_{j-2B}-\gamma_L\ge cBN^{-1}\gg6\zeta$, since $z>1-b$. Therefore $x_j-\gamma_k\ge x_{j-B}^{[B]}-\gamma_k>0$. In the third inequality we performed the summation and used that γ_k is regularly spaced. In the last inequality we again used (5.58). In summary, we have shown that

$$|\Omega_j^{2,1}|\le\frac{C}{B}+\frac{C}{N} \tag{5.59}$$

on the set Λ and we have seen that the contribution from Λ^c is subexponentially small (5.57).

Now we consider $\Omega_j^{2,2}$ on Λ. We have

$$|\Omega_j^{2,2}| \le C \int_{y_L}^{\gamma_L} \frac{\mathrm{d}y}{x_j - y} \le \frac{C\delta}{\gamma_{j-2B} - \gamma_L}, \tag{5.60}$$

by using $x_j - \gamma_L \ge \gamma_{j-2B} - \gamma_L - 6\zeta \ge c(\gamma_{j-2B} - \gamma_L)$ from (5.58) and from $\gamma_{j-2B} - \gamma_L \ge cBN^{-1} \gg 6\zeta$, moreover $x_j - y_L \ge x_j - \gamma_L - \delta \ge c(\gamma_{j-2B} - \gamma_L)$ by $|\gamma_L - y_L| \le \delta$ (from $\mathbf{y} \in \mathcal{G}$) and $\delta \ll BN^{-1}$ (from (5.45)). Thus

$$\sum_{L+4B \le j \le L+K-4B} |\Omega_j^{2,2}|^2 \le \frac{C\delta^2 N^2}{B}.$$

For the third term $\Omega_j^{2,3}$ we have

$$\begin{aligned} \mathbb{E}_{\mu_{\mathbf{y}}^{B,\tau}} \mathbf{1}(\Lambda) & \sum_{L+4B<j\le L+K-4B} [\Omega_j^{2,3}]^2 \\ &\le \mathbb{E}_{\mu_{\mathbf{y}}^{B,\tau}} \mathbf{1}(\Lambda) \sum_{L+4B<j\le L+K-4B} \left[\frac{1}{N} \sum_{k<L} \Big(\frac{1}{x_j - y_k} - \frac{1}{x_j - \gamma_k} \Big) \right]^2 \\ &\le \mathbb{E}_{\mu_{\mathbf{y}}^{B,\tau}} \mathbf{1}(\Lambda) \sum_{L+4B<j\le L+K-4B} \left[\frac{1}{N} \sum_{k<L} \frac{(y_k - \gamma_k)}{(x_j - y_k)(x_j - \gamma_k)} \right]^2 . \end{aligned} \tag{5.61}$$

We split the summation over k into two terms: $\kappa N/2 \le k < L$ and $k < \kappa N/2$ and separate by a Schwarz inequality.

First we consider the case $\kappa N/2 \le k < L$. Expanding the square, we need to bound

$$\begin{aligned} \mathbb{E}_{\mu_{\mathbf{y}}^{B,\tau}} \mathbf{1}(\Lambda) \frac{1}{N^2} & \sum_{\kappa N/2 \le k < L} \sum_{\kappa N/2 \le a < L} \sum_{L+4B<j\le L+K} \frac{|y_k - \gamma_k||y_a - \gamma_a|}{(x_j - y_k)(x_j - \gamma_k)(x_j - y_a)(x_j - \gamma_a)} \\ \le 2\mathbb{E}_{\mu_{\mathbf{y}}^{B,\tau}} \mathbf{1}(\Lambda) \frac{1}{N^2} & \sum_{\kappa N/2 \le k < L} |y_k - \gamma_k|^2 \sum_{L+4B\le j\le L+K} \frac{1}{(x_j - y_k)^2} \sum_{\kappa N/2 \le a < L} \frac{1}{(x_j - \gamma_a)^2}, \end{aligned} \tag{5.62}$$

where we used another Schwarz inequality and the factor 2 accounts for a similar term with the role of k and a interchanged.

In the case $\kappa N/2 \le k < L$ we have $|\gamma_k - y_k| \le \delta$. Then (5.62) is bounded by

$$\begin{aligned} \frac{2\delta^2}{N^2} & \sum_{L+4B\le j\le L+K} \mathbb{E}_{\mu_{\mathbf{y}}^{B,\tau}} \mathbf{1}(\Lambda) \sum_{k<L} \frac{1}{(x_j - y_k)^2} \sum_{a<L} \frac{1}{(x_j - \gamma_a)^2} \\ & \le C\delta^2 \sum_{L+4B\le j\le L+K} \mathbb{E}_{\mu_{\mathbf{y}}^{B,\tau}} \mathbf{1}(\Lambda) \frac{1}{(x_j - y_L)^2} \le C\delta^2 \frac{N^2}{B}. \end{aligned} \tag{5.63}$$

Here we used

$$\frac{1}{N} \sum_{a<L} \frac{1}{(x_j - \gamma_a)^2} \le \frac{C}{x_j - \gamma_L} \le \frac{C}{x_j - y_L}$$

relying on the regularity of γ_a and using, from (5.58), that $x_j - \gamma_a \ge x_j - \gamma_L \ge cBN^{-1}$ which is much larger than the spacing of order N^{-1} of the γ-sequence. In the last estimate $x_j - \gamma_L \gg |\gamma_L - y_L|$

was used (since $BN^{-1} \gg \delta$). Similarly we could perform the k summation

$$\sum_{k<L} \frac{1}{(x_j - y_k)^2} \le \frac{C}{x_j - y_L}$$

since $x_j - y_k \ge x_j - \gamma_k - \delta \ge c(x_j - \gamma_k)$.

To perform the j summation in (5.63), we use

$$\frac{1}{(x_j - y_L)^2} \le \frac{1}{(x_{j-B}^{[B]} - y_L)^2},$$

and then we recall that apart from a set of subexponentially small probability, we have

$$|x_{j-B}^{[B]} - \gamma_{j-B}^{[B]}| \le 6\zeta$$

from Lemma 5.6. Since $\zeta \ll BN^{-1}$ and $x_{j-B}^{[B]} - y_L \ge cBN^{-1}$ from (5.58), we see that

$$\sum_{L+4B \le j \le L+K} \frac{1}{(x_j - y_L)^2} \le \sum_{L+4B \le j} \frac{C}{(\gamma_{j-B}^{[B]} - y_L)^2} \le \frac{CN}{\gamma_{L+3B}^{[B]} - y_L} \le \frac{CN^2}{B}.$$

On the exceptional set one can just use the trivial bound $(x_j - y_L)^{-2} \le C(x_j - \gamma_L)^{-2} \le CN^2B^{-2}$ from (5.58).

Consider now the case $k \le \kappa N/2$ in (5.61). We have

$$\begin{aligned} &\mathbb{E}_{\mu_{\mathbf{y}}^{B,\tau}} \mathbf{1}(\Lambda) \frac{1}{N^2} \sum_{k \le \kappa N/2} \sum_{a \le \kappa N/2} \sum_{L+4B \le j \le L+K} \frac{|y_k - \gamma_k||y_a - \gamma_a|}{(x_j - y_k)(x_j - \gamma_k)(x_j - y_a)(x_j - \gamma_a)} \qquad (5.64) \\ &\le 2\mathbb{E}_{\mu_{\mathbf{y}}^{B,\tau}} \mathbf{1}(\Lambda) \frac{1}{N^2} \sum_{k \le \kappa N/2} |y_k - \gamma_k|^2 \sum_{L+4B \le j \le L+K} \frac{1}{(x_j - y_k)^2} \sum_{a \le \kappa N/2} \frac{1}{(x_j - \gamma_a)^2} \\ &\le \frac{CK}{N} \mathbb{E}_{\mu_{\mathbf{y}}^{B,\tau}} \mathbf{1}(\Lambda) \sum_{k \le \kappa N/2} |y_k - \gamma_k|^2 \le CKN^{-2/5+\varphi}, \end{aligned}$$

where we used that all denominators are separated away from zero and Lemma 3.6. Furthermore, in the last inequality, we have used Lemma 3.6 for $k \ge N^{3/5+\varphi}$ and we used $|y_k - \gamma_k| \le O(1)$ for $k \le N^{3/5+\varphi}$ from $\mathbf{y} \in \mathcal{G}$ and (4.7). Similar comment applies to all edge terms in this proof and we will not repeat it.

Summarizing, we have shown that

$$\mathbb{E}_{\mu_{\mathbf{y}}^{B,\tau}} \mathbf{1}(\Lambda) \sum_{L+4B<j \le L+K-4B} [\Omega_j^{2,3}]^2 \le \frac{C\delta^2 N^2}{B} + CKN^{-2/5+\varphi}. \qquad (5.65)$$

Finally, we need to estimate Ω_j^3 in (5.52). It can be treated exactly as $\Omega_j^{2,1}$ and the result is

$$|\Omega_j^3| \le \frac{C}{B} + \frac{C}{N} \qquad (5.66)$$

on the set Λ and the contribution from Λ^c is subexponentially small as in (5.57).

Case 2: $L < j \le L + 4B$. (There is a third case $L + K - 4B \le j \le L + K$ which is identical to Case 2 and will not be treated separately). We decompose Z_j as before and the only modifications are

$$\begin{aligned}\Omega_j^{2,low} &:= -\bigg(\frac{1}{N}\sum_{k\le L-2B}\frac{1}{x_j - y_k^B} - \int_{-\infty}^{y_{L-2B}}\frac{\rho_V(y)}{x_j-y}\mathrm{d}y\bigg)\\ \Omega_j^{3,low} &:= \frac{1}{N}\sum_{k\le L-2B}\frac{1}{x_j-\theta'_k} - \int_{-\infty}^{\theta'_{L-2B}}\frac{\rho_{W_s}(y)}{x_j-y}\mathrm{d}y\\ \Omega_j^4 &:= \int_{y_{L-2B}}^{y_{L+K+1}}\frac{\rho_V(y)-\rho_{W_s}(y)}{x_j-y}\mathrm{d}y + \int_{y_{L-2B}}^{\theta'_{l-2B}}\frac{\rho_{W_s}(y)}{x_j-y}\mathrm{d}y.\end{aligned}$$

We now estimate the main error term $\Omega_j^{2,low}$, and we write it, as before

$$\Omega_j^{2,low} = \Omega_j^{2,1} + \Omega_j^{2,2} + \Omega_j^{2,3}$$

with

$$\begin{aligned}\Omega_j^{2,1} &:= -\bigg(\frac{1}{N}\sum_{k<L-2B}\frac{1}{x_j-\gamma_k} - \int_{-\infty}^{\gamma_{L-2B}}\frac{\rho_V(y)}{x_j-y}\mathrm{d}y\bigg)\\ \Omega_j^{2,2} &:= \int_{\gamma_{L-2B}}^{y_{L-2B}}\frac{\rho_V(y)}{x_j-y}\mathrm{d}y\\ \Omega_j^{2,3} &:= \frac{1}{N}\sum_{k<L-2B}\Big[\frac{1}{x_j-\gamma_k} - \frac{1}{x_j-y_k}\Big].\end{aligned}\tag{5.67}$$

We have

$$\begin{aligned}|\Omega_j^{2,1}| &= \bigg|\frac{1}{N}\sum_{k<L-2B}\int_{\gamma_k}^{\gamma_{k+1}}\frac{y-\gamma_k}{(x_j-y)(x_j-\gamma_k)}N\rho_V(y)\mathrm{d}y\bigg|\\ &\le \frac{C}{N}\sum_{k<L-2B}\frac{\gamma_{k+1}-\gamma_k}{(x_j-\gamma_k)^2}\\ &\le \frac{C}{B} + \frac{C}{N}\sum_{k\le\kappa N/2}|\gamma_{k+1}-\gamma_k| \le \frac{C}{B}+\frac{C}{N}\end{aligned}$$

using that $x_j \ge y_L \ge \gamma_L - \delta \ge \gamma_{L-2B} + cBN^{-1}$. The estimate of $\Omega_j^{2,2}$ is trivial

$$|\Omega_j^{2,2}| \le \frac{|\gamma_{L-2B} - y_{L-2B}|}{cBN^{-1}} \le \frac{CN\delta}{B}.$$

Finally

$$\begin{aligned}
\mathbb{E}_{\mu_{\mathbf{y}}^{B,\tau}} \sum_{L\le j\le L+4B} [\Omega_j^{2,3}]^2 &\le \mathbb{E}_{\mu_{\mathbf{y}}^{B,\tau}} \sum_{L\le j\le L+2B} \left[\frac{1}{N}\sum_{k<L-2B}\Big(\frac{1}{x_j-y_k}-\frac{1}{x_j-\gamma_k}\Big)\right]^2 \\
&\le \frac{C\delta^2}{N^2}\sum_{L\le j\le L+4B}\mathbb{E}_{\mu_{\mathbf{y}}^{B,\tau}}\sum_{\kappa N/2\le k<L-2B}\frac{1}{(x_j-y_k)^2}\sum_{\kappa N/2\le a<L-2B}\frac{1}{(x_j-\gamma_a)^2} \\
&+\frac{C}{N^2}\sum_{k\le\kappa N/2}|\gamma_k-y_k|^2 \\
&\le C\delta^2\frac{N^2}{B}+CKN^{-2/5+\varphi},
\end{aligned}$$

where we again split the summation over k into $\kappa N/2\le k\le L-2B$ and $k\le\kappa N/2$, yielding the two terms, similarly to (5.63) and (5.64).

The estimate $\Omega_j^{3,low}$ is analogous to that of $\Omega_j^{2,1}$. The first term of Ω_j^4 is estimated as before in (5.55). The additional second term in Ω_j^4 is trivial by recalling $|\gamma_{L-2B}-\theta'_{L-2B}|\le CB^2N^{-2}+C\delta$ from (4.25):

$$\left|\int_{y_{L-2B}}^{\theta'_{L-2B}}\frac{\rho_{W_s}(y)}{x_j-y}\mathrm{d}y\right|\le\frac{|y_{L-2B}-\theta'_{L-2B}|}{cBN^{-1}}\le\frac{C\delta+CB^2N^{-2}}{cBN^{-1}}\le\frac{C\delta N}{B}+\frac{CB}{N},$$

since the denominator can be estimated by using $x_j-y_{L-2B}\ge y_L-y_{L-2B}\ge\gamma_L-\gamma_{L-2B}-2\delta\ge cBN^{-1}$ and

$$x_j-\theta'_{L-2B}\ge y_L-\theta'_{L-2B}\ge\gamma_L-\gamma_{L-2B}-2\delta+(\gamma_{L-2B}-\theta'_{L-2B})\ge cBN^{-1}$$

where we used $\delta\ll BN^{-1}$ and $B\ll N$.

Collecting all the error terms into (5.46) and removing some redundant terms, we have thus proved Lemma 5.8. □

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References

[1] Albeverio, S., Pastur, L., Shcherbina, M.: On the $1/n$ expansion for some unitary invariant ensembles of random matrices, *Commun. Math. Phys.* **224**, 271–305 (2001).

[2] Anderson, G., Guionnet, A., Zeitouni, O.: An Introduction to Random Matrices. *Studies in advanced mathematics*, **118**, Cambridge University Press, 2009.

[3] Bakry, D., Émery, M.: Diffusions hypercontractives. In: *Séminaire de probabilités, XIX, 1983/84, vol. 1123 of Lecture Notes in Math.* Springer, Berlin, 1985, pp. 177–206.

[4] Ben Arous, G., Guionnet, A.: Large deviations for Wigner's law and Voiculescu's non-commutative entropy. *Probab. Theory Related Fields* **108**, 4 (1997), 517–542.

[5] Bleher, P., Its, A.: Semiclassical asymptotics of orthogonal polynomials, Riemann-Hilbert problem, and universality in the matrix model. *Ann. of Math.* **150**, 185–266 (1999).

[6] Boutet de Monvel, A., Pastur, L., Shcherbina, M.: On the statistical mechanics approach in the Random Matrix Theory. Integrated Density of States. *J. Stat. Phys.* **79** , 585–611 (1995).

[7] Brascamp, H,. Lieb, E.: On extensions of the Brunn-Minkowski and Prékopa-Leindler Theorems, Including Inequalities for Log Concave Functions, and with an Application to the Diffusion Equation, *Journal of Functional Analysis* **22**, 366–389, 1976

[8] Deift, P.: Orthogonal polynomials and random matrices: a Riemann-Hilbert approach. *Courant Lecture Notes in Mathematics* **3**, American Mathematical Society, Providence, RI, 1999.

[9] Deift, P., Gioev, D.: Random Matrix Theory: Invariant Ensembles and Universality. *Courant Lecture Notes in Mathematics* **18**, American Mathematical Society, Providence, RI, 2009.

[10] Deift, P., Kriecherbauer, T., McLaughlin, K.T-R, Venakides, S., Zhou, X.: Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory. *Comm. Pure Appl. Math.* **52**, 1335–1425 (1999).

[11] Deift, P., Kriecherbauer, T., McLaughlin, K.T-R, Venakides, S., Zhou, X.: Strong asymptotics of orthogonal polynomials with respect to exponential weights. *Comm. Pure Appl. Math.* **52**, 1491–1552 (1999).

[12] Dumitriu, I., Edelman, A.: Matrix Models for Beta Ensembles, *Journal of Mathematical Physics* **43** (11) (2002), 5830–5847

[13] Dyson, F.J.: A Brownian-motion model for the eigenvalues of a random matrix. *J. Math. Phys.* **3**, 1191–1198 (1962).

[14] Erdős, L., Schlein, B., Yau, H.-T.: Universality of Random Matrices and Local Relaxation Flow. To appear in *Inv. Math.* Preprint `arXiv:0907.5605`

[15] Erdős, L., Schlein, B., Yau, H.-T., Yin, J.: The local relaxation flow approach to universality of the local statistics for random matrices. to appear in *Annales Inst. H. Poincaré (B), Probability and Statistics.* Preprint `arXiv:0911.3687`

[16] Erdős, L., Ramírez, J.-A., Schlein, B., Yau, H.-T.: Universality of sine-kernel for Wigner matrices with a small Gaussian perturbation, *Electronic Journal of Probability,* **15** (2010), Paper no. 18, pages 526–603.

[17] Erdős, L., Knowles, A., Yau, H.-T., Yin, J.: Spectral Statistics of Erdős-Rényi Graphs II: Eigenvalue Spacing and the Extreme Eigenvalues. Preprint `arXiv:1103.3869`

[18] Erdős, L., Yau, H.-T., Yin, J., Rigidity of Eigenvalues of Generalized Wigner Matrices. Preprint `arXiv:1007.4652.`

[19] Eynard, B.: Master loop equations, free energy and correlations for the chain of matrices. *J. High Energy Phys.* **11** 2003, 018

[20] Johansson, K.: On fluctuations of eigenvalues of random Hermitian matrices. *Duke Math. J.* **91** (1998), no. 1, 151–204.

[21] Kriecherbauer, T., Shcherbina, M.: Fluctuations of eigenvalues of matrix models and their applications. Preprint `arXiv:1003.6121`

[22] Lubinsky, D.S.: A New Approach to Universality Limits Involving Orthogonal Polynomials, *Annals of Mathematics* **170** (2009), 915–939.

[23] Mehta, M.L.: Random Matrices. Academic Press, New York, 1991.

[24] Pastur, L., Shcherbina, M.: Universality of the local eigenvalue statistics for a class of unitary invariant random matrix ensembles. *J. Stat. Phys.* **86**, 109–147, (1997)

[25] Pastur, L., Shcherbina M.: Bulk universality and related properties of Hermitian matrix models. *J. Stat. Phys.* **130**, no. 2., 205–250 (2008).

[26] Shcherbina, M.: Orthogonal and symplectic matrix models: universality and other properties. Preprint `arXiv:1004.2765`

[27] Valkó, B.; Virág, B.: Continuum limits of random matrices and the Brownian carousel. *Invent. Math.* **177** (2009), no. 3, 463-508.