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Global phase diagram of $\nu=2$ quantum Hall bilayers in tilted magnetic fields

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We consider a bilayer quantum Hall system at total filling fraction $\nu=2$ in tilted magnetic field allowing for charge imbalance as well as tunneling between the two layers. Using an "unrestricted Hartree Fock," previously discussed by Burkov and MacDonald [Phys. Rev. B **66**, 115323 (2002)], we examine the zero-temperature global phase diagrams that would be accessed experimentally by changing the in-plane field and the bias voltage between the layers while keeping the tunneling between the two layers fixed. In accordance with previous work, we find symmetric and ferromagnetic phases as well as a first-order transition between two canted phases with spontaneously broken U(1) symmetry. We find that these two canted phases are topologically connected in the phase diagram and, reminiscent of a first-order liquid-gas transition, the first-order transition line between these two phases ends in a quantum critical point. We develop a physical picture of these two phases and describe in detail the physics of the transition.

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I. INTRODUCTION

Over the last fifteen years, one of the most exciting frontiers in two-dimensional electron physics has been the study of quantum Hall bilayers.[1] Stacking two quantum Hall systems a small distance away from each other serves two main purposes: First of all, it is a way of creating a multicomponent system in which the layer index of the electron plays the role of an isospin. Secondly, it is a step in adding another dimension to the traditionally two-dimensional quantum Hall medium.

A striking phenomena that illustrates the richness of the physics of bilayer quantum Hall systems is the commensurate-incommensurate transition observed in systems with filling fraction $\nu=1$ subjected to tilted magnetic fields.[1–6] For typical parameters in these systems, the spin degrees of freedom are frozen out and the only important discrete degree of freedom is the layer index (isospin). Thus only two processes are important: interlayer tunneling and Coulomb interactions. When such a bilayer system is subjected to tilted magnetic fields, the electrons tunneling between the layers enclose flux quanta. In the presence of a sufficiently strong in-plane magnetic field it may be favorable for the system to forgo tunneling in order to escape destructive interference induced by the in-plane field. This creates an opportunity for a phase transition to occur between a phase in which tunneling plays the central role and a phase in which tunneling is effectively zero.

Recently, it has been shown[7] that a somewhat different phase transition can occur in bilayer systems with a total filling fraction $\nu=2$. The $\nu=2$ bilayer system is enriched not only by the presence of two electrons per flux quantum, but most importantly by the role of real spin. In contrast to $\nu=1$ bilayers where the real spin degrees of freedom are frozen out, in $\nu=2$ bilayers, spin degrees of freedom are *entangled* with the isospin degrees of freedom as a result of Fermi statistics (Pauli exclusion principle, in other words). Thus, the $\nu=2$ bilayers exhibit a richer phase diagram even in perpendicular field, and, as we will see later, an even more intriguing phase diagram in the tilted field.

In 1997, Zheng, Radtke, and Das Sarma (ZRD)[8,9] predicted that the bilayer quantum Hall systems at total filling fraction $\nu=2$ can exhibit a novel spontaneously symmetry-broken phase. ZRD performed a time-dependent Hartree-Fock study of spin-density excitations of the $\nu=2$ bilayers and found that, under experimentally attainable conditions (at finite Zeeman energy and interlayer tunneling), the spin-density mode softens, signaling a phase transition to a symmetry-broken quantum Hall state. The state was found to have a finite magnetization, similar to the simple ferromagnetic state (F) that occurs in the limit of large Zeeman energy; it also exhibited interlayer phase coherence, similar to the spin-singlet state (S) stabilized by strong interlayer tunneling. In addition, the state was found to possess antiferromagnetic correlations. ZRD dubbed the novel state "canted," since schematically the state can be viewed as one in which the spins in the opposite layers are canted away from the magnetic field in opposite directions. The canted ground state breaks the U(1) symmetry associated with rotations around the direction of the magnetic field; this spontaneously broken symmetry is behind the formation of the soft (Goldstone) spin-density mode found by ZRD.

The canted state is a pure many-body state, stabilized by Coulomb interactions. In the absence of the interactions, there would be a first-order phase transition between the ferromagnetic phase and the spin-singlet state. Coulomb interactions effectively mix the ferromagnetic state and the spin-singlet state around the would-be first-order phase transition (when the energy splitting between these states is small), giving rise to the canted phase. Because Coulomb interactions somewhat favor the ferromagnetic state, a finite amount of tunneling is crucial for the stability of the canted phase.

Brey, Demler, and Das Sarma (BDD),[10] however, showed that the canted phase can be extended to the region of finite Zeeman energy and infinitely small tunneling by creating a charge imbalance between the layers. The charge imbalance can be induced by an external bias voltage applied perpendicularly to the system. In the charge-unbalanced regime,

another many-body phase with a spontaneously broken U(1) symmetry, but with vanishing antiferromagnetic order parameter, was discovered at zero tunneling. This phase—the *I* phase—is continuously (without a phase transition) connected to the antiferromangetic canted phase. MacDonald, Rajaraman, and Jungwirth[11] (MRJ) pointed out in their thorough Hartree-Fock study of the $\nu=2$ bilayer phase diagram that the *I* phase is akin to the spontaneous interlayer phase coherent phase that occurs in the $\nu=1$ bilayers in the absence of tunneling.[2] MRJ therefore suggested that the spontaneous interlayer phase coherence of the *I* phase may lead to interesting effects in tilted fields, closely related to the commensurate-incommensurate transition of the $\nu=1$ bilayers.[3–6]

In a recent paper, Burkov and MacDonald[7] (BM) explore this possibility. Indeed they find that a tilted field applied to a charge-unbalanced $\nu=2$ bilayer system induces a quantum phase transition within the canted phase. Their "unrestricted Hartree-Fock" analysis shows that the phase transition is first order and is between two commensurate phases: At low tilt angles the phase is a simple canted commensurate phase, in which the layer degree of freedom (isospin) is commensurate with the in-plane component of the magnetic field. At higher tilt angles a phase transition occurs to a phase, in which not only the isospin, but also the *spin* becomes commensurate with the in-plane field (Fig. 4). Thus, BM conclude, the phase transition is not a commensurate-incommensurate transition.

In this paper, their results are extended and given a physical explanation. Using the "unrestricted Hartree-Fock" approximation, we find that the first-order phase transition terminates with a quantum critical transition embedded within the canted phase (Fig. 3). The simple commensurate and the spin-isospin commensurate canted phases are continuously connected to each other—very similar to the familiar example of water and its vapor, the two phases possess the same symmetry properties [the spontaneously broken U(1) in our case]. To illustrate the first-order transition and the quantum critical transition that terminates it, we present a series of phase diagrams (Fig. 3). Each phase diagram is obtained for fixed perpendicular-field Zeeman energy and tunneling strength, and the axes on the phase diagrams are the in-plane field and the bias voltage. We find this choice of axes particularly suitable, since current experimental techniques allow us to vary the bias voltage and the in-plane field *in situ* over a wide range of values. All the phases on a given phase diagram can therefore be accessed on a single sample; this should facilitate the detection of the novel phase transitions and the new phases. In an upcoming publication,[12] we obtain the collective modes in various parts of the phase diagram, thus providing signatures of different phases and phase transitions for possible light-scattering experiments.

In addition to our new Hartree-Fock results, we present a detailed physical discussion of the surprising behavior of the $\nu=2$ bilayers in the tilted magnetic field. We show that the exotic spin-isospin commensurate canted phase is closely related to the *I* phase, as illustrated by a comparison of order parameters in the two phases (Fig. 5). In fact, the spin-isospin commensurate phase rapidly converges to the *I* phase as the tilt angle is increased. We use the symmetry properties of the *I* state to give a simple explanation to the spin commensuration at high in-plane fields. The similarities between the novel first-order transition of the $\nu=2$ bilayers and the commensurate-incommensurate transition of the $\nu=1$ bilayers help us understand many aspects of our and BM's numerical findings, such as the confinement of the phase transition to the canted phase and the absence of the new transition in charge-balanced $\nu=2$ bilayers.

This paper is organized as follows. In Sec. II, the bilayer Hamiltonian under the influence of tilted magnetic fields is presented. In Sec. III, the Hartree-Fock procedure, used by Das Sarma, Sachdev, and Zheng[9] (DSZ) to obtain the phase diagram of the charge-balanced $\nu=2$ bilayer system in perpendicular field, is extended to the present case of a charge-unbalanced $\nu=2$ bilayer in tilted field. A similar procedure was outlined by BM,[7] who dubbed it the "unrestricted" Hartree-Fock approximation. In Sec. III, a detailed presentation of this, unrestricted, Hartree-Fock procedure is given.

In Sec. IV, the global phase diagram of the $\nu=2$ bilayers in tilted field is presented. The phase diagram of the $\nu=2$ bilayers in perpendicular field has been previously reported and discussed by several groups of authors.[8–11] MRJ in particular, have obtained the full Hartree-Fock phase diagrams of the $\nu=2$ bilayers for various combinations of parameters.[11] MRJ used a very elucidating reduced Hartree-Fock solution, consisting of only three variational parameters, which elegantly captures the physics of the $\nu=2$ bilayers. The only shortcoming of their approximation is that it does not always give the exact Hartree-Fock ground state that is crucial for obtaining the correct collective mode dispersions from the time-dependent Hartree-Fock solution (as will be presented in Ref. 12). In particular, as we will discuss further below, the approximation of MRJ gives an approximate ground state for the canted phase *C*, in which, as will be shown in Sec. V, the most interesting phenomena happen when the magnetic field is tilted. In addition, some aspects of the phase diagram, such as the properties of the zero-tunneling *I* phase have been essentially uninvestigated. In Sec. IV some of these properties are discussed in detail.

In Sec. V the main results of this paper are presented. In the first part of this section, the physics of the commensurate-incommensurate transition of the $\nu=1$ bilayers in tilted fields is reviewed. In the next part, the possibility of a similar transition in $\nu=2$ is discussed. Next, the quantitative Hartree-Fock results are presented and explained. The results are summarized in Figs. 3–5. Figure 4 illustrates the commensuration of spin with the in-plane field at large tilt angles. Figure 3 presents a set of global phase diagrams of the $\nu=2$ bilayers in tilted magnetic fields, which show the emergence and the evolution of the phase transition, induced by the in-plane field. Figure 5 llustrates the close relationship between the novel commensurate-commensurate transition (with the involvement of the spin) and a "naive" commensurate-incommensurate transition (in which the spin is not involved).

Section VI gives a short summary of the results presented in this paper. We also qualitative discussion of phenomena beyond the Hartree-Fock results as well as discussing possible effects of finite temperature and the possibility of observing similar physics in other related physical systems.

II. THE BILAYER HAMILTONIAN

We model the $\nu=2$ bilayer quantum Hall system in tilted magnetic field by a simple Hamiltonian which includes the five most important aspects of the system: Landau level quantization, Zeeman energy, tunneling between the layers, an external bias voltage, and the Coulomb interactions between electrons. Disorder will be completely neglected throughout this work.

We choose to work in the gauge $\vec{A}(\vec{r})=(0,B_\perp x,-B_\parallel x)$, where $B_\perp$ is the component of the magnetic field perpendicular to the plane of the sample and $B_\parallel$ is the in-plane component of the field (the total field is $B_{\text{total}}=\sqrt{B_\perp^2+B_\parallel^2}$). If the layers are assumed to be infinitely thin (an approximation used throughout this work), the lowest-Landau-level single-electron wave functions for each layer in the gauge $\vec{A}(\vec{r})$ are

$$\phi_X(\vec{r})=\frac{1}{\sqrt{lL_y\sqrt{\pi}}}e^{iXy/l^2}e^{-(x-X)^2/2l^2}, \tag{1}$$

where $X=-k_y l^2$ are the guiding centers of these Landau-gauge wave functions; $l=\sqrt{\hbar c/eB_\perp}$ is the magnetic length and L_y is the length of the system in the y direction. Throughout this paper, we assume that the cyclotron energy is much larger than all other energy scales in the system and restrict our arguments to the lowest Landau level.

In addition to the orbital degrees of freedom X, electrons in the bilayer systems possess a spin and can be localized in either layer. The layer index serves as an additional discrete degree of freedom—the isospin—in bilayer systems. As a result, the electron creation operators $c^\dagger_{\mu sX}$ are labeled by three indices: the orbital index X, the spin index $s=\uparrow$ (or $+1$) and $\downarrow$ (or -1), and the layer index μ that can take on values R (or $+1$) and L (or -1). To facilitate the analogy between the spin and the isospin, we define the electron spin and isospin operators $\vec{S}_X$ and $\vec{I}_X$:

$$\vec{S}_X=\frac{1}{2}\sum_{\mu ss'}c^\dagger_{\mu sX}\vec{\sigma}_{ss'}c_{\mu s'X}, \tag{2}$$

$$\vec{I}_X=\frac{1}{2}\sum_{s\mu\nu}c^\dagger_{\mu sX}\vec{\tau}_{\mu\nu}c_{\nu sX}, \tag{3}$$

where $\vec{\sigma}$ and $\vec{\tau}$ are sets of Pauli matrices, and the spin operator is defined here in the usual manner.[13] The total spin and isospin of the system are defined by the operators $\mathcal{O}=\Sigma_X\mathcal{O}_X$, where $\mathcal{O}=I,S$.

The term in the Hamiltonian representing the Zeeman energy is simply

$$H_Z=-\sum_X\Delta_Z S^z_X=-\frac{1}{2}\Delta_Z(N_\uparrow-N_\downarrow), \tag{4}$$

where $\Delta_Z=g\mu_B B_{\text{total}}$ is the Zeeman splitting and $N_\downarrow$ and $N_\uparrow$ are the total numbers of down and up spins in the system. Very similarly, the term representing the bias voltage (i.e., the difference in electrostatic potential between the layers) is written as

$$H_V=-\sum_X\Delta_V I^z_X=-\frac{1}{2}\Delta_V(N_R-N_L), \tag{5}$$

where Δ_V is the potential difference between the layers, and N_L and N_R are the total number of particles in the left and right wells, respectively. The bias voltage clearly acts as an effective external field that couples to the isospin.

Within the tight-binding approximation,[14] tunneling between the layers can also be expressed as an effective external field coupling to the isospin. When the magnetic field applied to the system is perpendicular to it (so that the electrons tunnel along the direction of the magnetic field and never see any magnetic flux), the tunneling term can be written as

$$H_T^{B_\parallel=0}=-\Delta^0_{\text{SAS}}\sum_X I^x_X \tag{6}$$

$$=-\frac{\Delta^0_{\text{SAS}}}{2}\sum_{X,s}(c^\dagger_{RsX}c_{LsX}+c^\dagger_{LsX}c_{RsX}), \tag{7}$$

i.e., it acts as an external field acting on the isospin in the I^x direction. The coefficient Δ^0_{SAS} is the symmetric-antisymmetric gap induced by the tunneling in the absence of the other external fields and interactions.

When the magnetic field is tilted with respect to the normal to the sample, the electrons tunneling between the layers pick up an Aharonov-Bohm phase $[(e/c)\int\mathbf{A}\cdot\mathbf{dl}]$. As a result, the tunneling term acquires phase factors

$$H_T=-\Delta_{\text{SAS}}\sum_{X,s}[e^{iQ_\parallel X}c^\dagger_{RsX}c_{LsX}+e^{-iQ_\parallel X}c^\dagger_{LsX}c_{RsX}] \tag{8}$$

$$=-\frac{\Delta_{\text{SAS}}}{2}\sum_X[e^{iQ_\parallel X}I^+_X+e^{-iQ_\parallel X}I^-_X], \tag{9}$$

where the operators $I^\pm=I^x\pm iI^y$ are the isospin raising and lowering operators. The wave vector $\vec{Q}_\parallel$ is defined as

$$\vec{Q}_\parallel=\frac{\hat{z}\times\vec{B}_\parallel}{(B_\perp l^2/d)} \tag{10}$$

with d the distance between the two layers[16] (and the in-plane magnetic field, in our gauge, pointing in the $\hat{y}$ direction). Note that $|\vec{Q}_\parallel|$ is proportional to $\tan\theta=B_\parallel/B_\perp$, where θ is the tilt angle. Interference between the electrons tunneling in the presence of an in-plane field results in the reduction of the effective symmetric antisymmetric gap[14]

$$\Delta_{\text{SAS}}=\Delta^0_{\text{SAS}}e^{-Q_\parallel^2 l^2/4}. \tag{11}$$

We note that, had we chosen a different gauge, the Gaussian decay of Δ_{SAS} would have remained although the form of this term would have been different, and the phase factors could disappear from the tunneling term and reappear in other terms in the Hamiltonian (see Sec. V A).

The three terms of the Hamiltonian (H_Z, H_V, and H_t) given above all describe single electrons and comprise the noninteracting part of the Hamiltonian

$$H_0 = H_Z + H_V + H_T = -\sum_X \left[\Delta_Z S_X^z + \Delta_V I_X^z + \frac{\Delta_{\mathrm{SAS}}}{2}(e^{iQ_\parallel X} I_X^+ + e^{-iQ_\parallel X} I_X^-) \right]. \tag{12}$$

The Coulomb interactions between the electrons are taken into account by an additional term

$$H_I = \frac{1}{2\Omega} \sum_{\substack{X_1 X_2 \\ \nu_1,\nu_2 \\ \sigma_1,\sigma_2}} \sum_{\mathbf{q}} e^{iq_x(X_1 - X_2)} e^{-q^2 l^2/2} V_{\mu_1\mu_2}(q) \times c^\dagger_{\mu_1\sigma_1 X_1 + q_y l^2} c^\dagger_{\mu_2\sigma_2 X_2} c_{\mu_2\sigma_2 X_2 + q_y l^2} c_{\mu_1\sigma_1 X_1}. \tag{13}$$

where intralayer and interlayer Coulomb interactions are

$$V_{RR}(q) = \frac{2\pi e^2}{\varepsilon q},\ V_{RL}(q) = \frac{2\pi e^2}{\varepsilon q} e^{-dq}, \tag{14}$$

respectively, d is the distance between the layers, and Ω is the area of the sample. The total Hamiltonian is therefore simply

$$H = H_0 + H_I. \tag{15}$$

III. THE UNRESTRICTED HARTREE-FOCK APPROXIMATION

A. Trial ground state

The Coulomb-interacting Hamiltonian in Eq. (15) is not tractable exactly, and we solve it using the Hartree-Fock approximation. In the usual manner, we assume that the many-body ground state $|G\rangle$ is a Slater determinant of single-particle states and perform a functional minimization of the expectation value $\langle G|H|G\rangle$ with respect to these single-particle states. Under the assumption of translational invariance in the $\hat{y}$ direction, the most general single-particle state is a superposition of all the combinations of spin and isospin degrees of freedom $R\uparrow$, $R\downarrow$, $L\uparrow$, and $L\downarrow$ which can be described as a normalized four-spinor

$$W = (w_{R\uparrow}, w_{R\downarrow}, w_{L\uparrow}, w_{L\downarrow}). \tag{16}$$

Such a normalized complex four dimensional vector transforms under U(4). To make a $\nu=2$ Slater determinant state, we occupy each orbital X by two particles[7]

$$|G\rangle = \prod_X f^\dagger_{1X} f^\dagger_{2X} |0\rangle, \tag{17}$$

where

$$f^\dagger_{nX} = \sum_{\mu\sigma} w^n_{\mu\sigma X} c^\dagger_{\mu\sigma X} \tag{18}$$

creates a particle described by Eq. (16). The requirement that the operators $f^\dagger_{nX}$ obey the fermionic anticommutation relations

$$\{f_{mX}, f^\dagger_{nX'}\} = \delta_{mn}\delta_{XX'} \tag{19}$$

is equivalent to an orthonormality constraint on the W^n

$$\sum_{\mu\sigma} (w^n_{\mu\sigma X})^* w^m_{\mu\sigma X} = \delta_{nm}. \tag{20}$$

Each element of U(4) specifies four orthonormal W^n spinors. When the filling fraction is $\nu=2$, the two states with lowest mean-field energies (which we label 1 and 2 throughout the paper) are occupied.[15]

The unrestricted Hartree-Fock ground state given by Eq. (17) is very general. The freedom provided by the coefficients $w^n_{\mu\sigma X}$ includes a possibility of nonuniform states, such as stripe states. In this paper, we restrict our attention to states with uniform density and uniform spin density, ignoring the possibility of charge density waves or spin density waves. We note that Ref. 16 considered the possibility of nonuniform phases in bilayer quantum Hall systems at $\nu=2$ in the case of equal electron densities in the two layers. The conclusion of that work was that charge or spin density wave phases were never stabilized in the realistic range of parameters. The excitation spectra above the uniform-density states, which we present in Ref. 12) also support the absence of stripe or spin density wave instabilities.

We note that while the three parameter variational ansatz of MRJ (Ref. 11) captures all the physics qualitatively (at least in perpendicular field) it is not sufficiently general to obtain the exact Hartree-Fock ground state (which is crucial for properly obtaining the correct excitation spectra) in the canted phases. In order to have a state of uniform spin density and uniform real density in each layer, the occupation number $N_{\mu\sigma X}$ of each state $\mu\sigma X$

$$N_{\mu\sigma X} = |w^1_{\mu\sigma X}|^2 + |w^2_{\mu\sigma X}|^2 \tag{21}$$

has to be independent of the position X. Thus, we may write[17]

$$w^n_{\mu\sigma X} = e^{i\phi_{\mu\sigma}(X)} z^n_{\mu\sigma}, \tag{22}$$

where $z^n_{\mu\sigma}$'s are independent of position and the only positional dependence arises in the phases. In the case of zero in-plane field ($B_\parallel = Q_\parallel = 0$), it will be very easy to see below that the lowest energy solution should have no positional dependence of the phases, so that $\phi_{\mu\sigma}(X)$ can be taken to be zero. However, in the more general case of nonzero in-plane field a nontrivial positional dependence will be favored.

Following BM,[7] we use a simple trial form for the positional phase dependence

$$\phi_{\mu\sigma}(X) = Q_{\mu\sigma} X \tag{23}$$

or, equivalently,

$$f_{nX} = \sum_{\mu\sigma} (z^n_{\mu\sigma})^* e^{-iQ_{\mu\sigma}X} c_{\mu\sigma X}. \tag{24}$$

A ground state [Eq. (17)] with nonzero $Q_{\mu\sigma}$ possesses spin-isospin-wave order, discussed at length in Sec. V D.

As was mentioned above, the proposed ground state [Eqs. (24) and (17)] is not the most general Slater determinant (Hartree-Fock) state. Indeed, having made a specific choice [Eq. (23)] of the positional dependence of the phase, this trial state is not even the most general state with uniform density and spin density in each layer. However, our analysis of the collective modes[12] around the ground states obtained by the minimization of $\langle G|H|G\rangle$ indicate the stability of these states against second-order transitions that cannot be described within the Hilbert space defined by our ansatz. The possibility of phase transitions into a soliton-lattice state will be discussed in the concluding Sec. VI.

B. Hartree-Fock minimization

We minimize the expectation value of the Hamiltonian (i.e. the trial ground state energy) with respect to the variational parameters $z^n_{\mu\sigma}$ and $Q_{\mu\sigma}$. Since

$$\langle c^\dagger_{\mu sX} c_{\nu s' X}\rangle = e^{i(Q_{\nu s'}-Q_{\mu s})X}\sum_{n=1,2}(z^n_{\mu s})^* z^n_{\nu s'}, \tag{25}$$

the expectation value of the ground state energy per unit flux in our nonuniform ansatz is

$$\frac{1}{g}\langle G|H|G\rangle = -\frac{1}{2}\sum_{\mu\nu ss'}(\Delta_Z\delta_{\mu\nu}\sigma^z_{ss'}+\Delta_V\delta_{ss'}\tau^z_{\mu\nu}+\Delta_{\mathrm{SAS}}\delta_{ss'}\cos[(Q_\parallel-Q_{Rs}+Q_{Ls})X]\tau^x_{\mu\nu}+\sin[(Q_\parallel-Q_{Rs}+Q_{Ls})X]\tau^y_{\mu\nu})\sum_{n=1,2}(z^n_{\mu s})^* z^n_{\nu s'}$$
$$+H_-\sum_{\mu s}\sum_{n=1,2}|z^n_{\mu s}|^2\left[\sum_{s',m=1,2}\left|z^m_{\mu s'}\right|^2-1\right]-\frac{1}{2}\sum_{\mu\nu,ss'}F_{\mu\nu}[-(Q_{\mu s}-Q_{\nu s'})\hat{q}_x]\sum_{m,n=1,2}(z^n_{\mu s})^* z^n_{\nu s'}(z^m_{\nu s'})^* z^m_{\mu s}, \tag{26}$$

where g is the Landau level degeneracy $g=\Omega/2\pi l^2$, and

$$F_{\mu\nu}(\mathbf{q}) = \int\frac{d^2k}{(2\pi)^2}e^{-k^2l^2/2}V_{\mu\nu}(k)e^{i\mathbf{q}\wedge\mathbf{k}l^2}$$
$$= \int\frac{dk}{2\pi}e^{-k^2l^2/2}V_{\mu\nu}(k)kJ_0(kql^2), \tag{27}$$

$$H_-(\mathbf{q}) = \frac{1}{4\pi l^2}[V_{RR}(\mathbf{q})-V_{RL}(\mathbf{q})] = \frac{e^2}{\varepsilon l}\frac{1-e^{-dq}}{2ql}, \tag{28}$$

so that $H_-=H_-(0)=(e^2/\varepsilon l)(d/2l)$. The functions $F_{\mu\nu}$ are monotonically decreasing functions of q and therefore the higher the wave vector of the spin-isospin-wave order, the lower the contribution of exchange to the ground-state energy. The exchange term thus favors a uniform state. The tunneling term, however, does not contribute to the ground-state energy, unless $Q_{Rs}-Q_{Ls}=Q_\parallel$, and therefore favors isospin-wave order. It is this competition that gives rise to the novel first-order transition discovered by BM (Ref. 7) and is presented in more detail in the next section. Using the results of BM,[7] we make a simplifying assumption that $Q_{R\sigma}-Q_{L\sigma}=Q_l$, for both $\sigma=\uparrow$ and $\downarrow$, and that $Q_{\mu\uparrow}-Q_{\mu\downarrow}=Q_S$, for $\mu=R$ and L. In other words, we reduce the number of phase parameters from three to two and write

$$Q_{\mu\sigma} = \frac{\mu}{2}Q_I+\frac{\sigma}{2}Q_S, \tag{29}$$

where a finite Q_I indicates the presence of an isospin-wave order, while a finite Q_S reflects the real spin-wave order.

To find a variational minimum of this Hamiltonian, we differentiate $(1/g)\langle G|H|G\rangle$ with respect to $(z^n_{\mu\sigma})^*$ (Ref. 18) under the orthonormality constraints on W^n [Eq. (20)]. The resulting set of minimization conditions can be arranged in the form of a Schrödinger equation

$$MZ^n = \epsilon_n Z^n, \tag{30}$$

where $Z^n=(z_{R\uparrow},z_{R\downarrow},z_{L\uparrow},z_{L\downarrow})$ and M is a 4×4 matrix, which is just the mean-field single-particle Hartree-Fock Hamiltonian

$$M_{\nu s';\mu s} = -\Delta_Z\delta_{\mu\nu}\sigma^z_{ss'}-\Delta_V\delta_{ss'}\tau^z_{\mu\nu}-\Delta_{\mathrm{SAS}}\delta_{ss'}\left[\frac{1}{g}\sum_X\cos[(Q_\parallel-Q_I)X]\tau^x_{\mu\nu}+\frac{1}{g}\sum_X\sin[(Q_\parallel-Q_I)X]\tau^y_{\mu\nu}\right]$$
$$+2H_-\sum_{\mu s}\delta_{\mu\nu}\delta_{ss'}\left[\sum_{s',m=1,2}|z^m_{\mu s'}|^2-1\right]-F_{\mu\nu}$$
$$-[Q_I/2(\mu-\nu)+Q_S/2(s-s')]\hat{q}_x\sum_{n=1,2}(z^n_{\nu s'})^* z^n_{\mu s}. \tag{31}$$

The Schrödinger equation (30) is solved iteratively.[9] At each iteration, the two eigenstates corresponding to the lowest eigenvalues are filled (i.e., chosen to be states 1 and 2). These lowest-energy eigenstates Z^1 and Z^2 are then used to obtain the matrix M for the next iteration. The procedure is repeated until a self-consistent solution is achieved. This solution—a set of eigenspinors Z^n sorted according to their eigenvalues—defines the lowest energy trial state among the Slater determinants defined by Eqs. (17) and (24) subject to fixed values of the $Q_{\mu s}$'s. The eigenvalues ε_n give the binding energy of a particle in the subband n, i.e., the energy lost

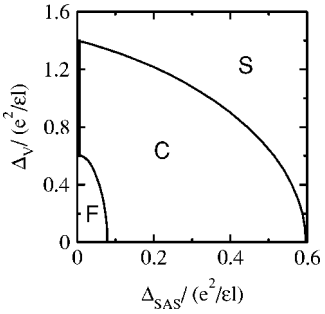

FIG. 1. Global phase diagram for a $\nu=2$ bilayer sample in perpendicular magnetic field. The Zeeman energy is $\Delta_Z^0=0.01(e^2/\varepsilon l)$ and the interlayer spacing is $d=l$. The tunneling strength Δ_{SAS}^0 and the bias voltage Δ_V are given in units of $(e^2/\varepsilon l)$. The phase S is the spin-singlet phase, F is the ferromagnetic phase, and C is the canted phase. The thick line along $\Delta_{\mathrm{SAS}}^0=0$ represents the I phase.

when the particle is taken out of the system. The sum of individual binding energies does not give the groundstate energy; the ground state energy is evaluated using Eq. (27). The minimization of the energy of the ground state over the $Q_{\mu s}$'s is done last.[7] Thus, we find the Hartree-Fock ground state in two steps: First, for fixed values of Q_I and Q_S, we minimize the expectation value of the ground-state energy with respect to $(z_{\mu\sigma}^n)^*$. Then, we minimize the ground-state energy with respect to Q_I and Q_S. The results are summarized in the following sections.

IV. GLOBAL PHASE DIAGRAM OF THE $\nu=2$ BILAYERS IN ZERO IN-PLANE FIELD

We start by considering the zero in-plane field case. As was mentioned in the introductory section, the phase diagram of the $\nu=2$ bilayers in perpendicular field has been studied in some detail.[8–11] In this section, we highlight the properties of the zero in-plane field phase diagram that help elucidate the physics of the $\nu=2$ bilayers in tilted field and discuss a number of issues, to which little attention has been paid to date.

The Zeeman splitting for a typical bilayer GaAs sample[21] with filling fraction $\nu=2$ in perpendicular field is $\Delta_Z^0 \approx 0.01(e^2/\varepsilon l)$; it is set by the properties of the material (the g factor and the effective electron mass) and the density of the 2DEG's. We assume that, in the absence of an external bias voltage, the density of the 2DEG's in the two layers is the same. Applying a finite bias voltage Δ_V perpendicularly to the layers, one can induce a charge imbalance between the layers. We assume that the charge imbalance can be created while keeping both the filling fraction and the magnetic field constant. The tunneling strength Δ_{SAS}^0 is assumed to be unaffected by the induced charge imbalance.

It is apparent from Eq. (27) that, when the magnetic field is oriented perpendicularly to the plane of the bilayer sample, i.e., $Q_\parallel=0$, spin- or isospin-wave order is not favored. Throughout this section, we therefore assume that $Q_I=Q_S=0$.

Under these conditions, we obtain the global phase diagram for the $\nu=2$ bilayers. A cross section of the phase diagram for $\Delta_Z^0=0.01(e^2/\varepsilon l)$ is presented in Fig. 1.[10] The phase diagram exhibits four phases: ferromagnetic F, spin-singlet S, canted C, and the spin-isospin-entangled noncanted phase I (represented by the thick line along $\Delta_{\mathrm{SAS}}^0=0$). As expected, when both the bias voltage Δ_V and the tunneling strength Δ_{SAS}^0 are small, the Zeeman energy dominates and gives rise to the ferromagnetic phase. In the opposite limit, of large Δ_{SAS}^0 and Δ_V, the spin-singlet phase is stabilized. In the intermediate regime, the Coulomb interactions give rise to the canted phase when $\Delta_{\mathrm{SAS}}^0\neq 0$ and to the I phase when $\Delta_{\mathrm{SAS}}^0=0$.

The topology of the phase diagram is the same for all other finite values of Δ_Z^0. For larger values of Δ_Z^0, the phase space volume of the ferromagnetic phase increases, and the spin-singlet phase is shifted to higher values of Δ_{SAS}^0 and Δ_V; the width of the canted phase (slowly) decreases. For smaller values of Δ_Z^0, the opposite effect takes place: the volume of the ferromagnetic phase decreases and the canted phase becomes wider. The ferromagnetic phase does not disappear from the phase diagram until $\Delta_Z^0=0$, when, within the Hartree-Fock approximation, a many-body phase (the $\Delta_Z^0 \to 0$ limit of the canted phase) fills the low-$\Delta_{\mathrm{SAS}}^0-\Delta_V$ region completely.[19] In real bilayer samples Δ_Z^0 is always finite, while $0<\Delta_{\mathrm{SAS}}^0\lesssim 0.08(e^2/\varepsilon l)$. As was proposed by Brey, Demler, and Das Sarma,[10] and is clear form Fig. 1, by sweeping the external bias voltage one can probe the three phases of the $\nu=2$ bilayer system that occur in the presence of finite tunneling: ferromagnetic, canted, and spin singlet. In tilted fields, the I phase can also be attained in the limit of a large tilt angle (see Sec. V D).

A. Ferromagnetic phase

The simplest phase in the phase diagram is the ferromagnetic phase. The ferromagnetic ground state has the simple form $|F\rangle=\Pi_X c_{R\uparrow X}^\dagger c_{L\uparrow X}^\dagger|0\rangle$. It is effectively a single-particle state, which could occur in the absence of interactions. The $\nu=2$ system in the ferromagnetic state can be viewed as two decoupled spin-polarized $\nu=1$ monolayers. The state is clearly not interlayer phase coherent, therefore, an in-plane field would not affect it.

B. Spin-singlet phase

The spin-singlet state is also a single-particle state. It is stabilized by large Δ_{SAS}^0 and/or Δ_V. The spin-singlet state $|S\rangle=\Pi_X(z_R c_{R\uparrow X}^\dagger+z_L c_{L\uparrow X}^\dagger)(z_R c_{R\downarrow X}^\dagger+z_L c_{L\downarrow X}^\dagger)|0\rangle$ is interlayer phase coherent unless $\Delta_{\mathrm{SAS}}^0=0$. When $\Delta_{\mathrm{SAS}}^0=0$, the spin-singlet state is simply $|S\rangle=\Pi_X c_{R\uparrow X}^\dagger c_{R\downarrow X}^\dagger|0\rangle$, a $\nu=2$ monolayer quantum Hall state. In the presence of interlayer tunneling, the $\nu=2$ bilayer system in the spin-singlet state can be viewed as two oppositely spin-polarized $\nu=1$ bilayer systems that possess interlayer phase coherence. The main difference between the $\nu=2$ bilayer system in the spin-singlet state and a set of two $\nu=1$ bilayer systems is that in the latter case the interactions play an important role alongside tunneling in creating the interlayer phase coherence; in the spin-singlet state of the $\nu=2$ bilayers interactions are not important. The phase space region where the interactions play an active role in $\nu=2$ bilayers is the region of stability of the canted phase.

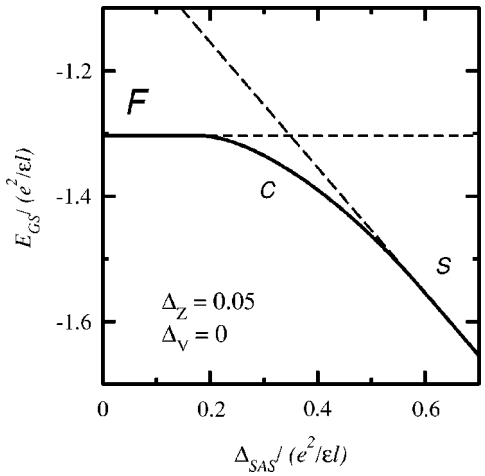

FIG. 2. Energy profile of charge-balanced $\nu=2$ bilayers with Zeeman energy $\Delta_Z=0.05(e^2/\varepsilon l)$, interlayer spacing $d=l$, and a range of tunneling amplitudes. The dashed line represents the energy of the ferromagnetic state. The long dashed line is the energy of the spin-singlet state. In the absence of interactions, there would be a first order phase transition at the intersection of the two dashed lines. The interactions effectively "smooth out" the profile by stabilizing the canted phase.

The boundaries of the canted phase therefore mark the boundaries of the influence of the interactions. Moreover, unlike in the $\nu=1$ bilayers where tunneling and exchange work cooperatively, in the $\nu=2$ bilayers tunneling and exchange are in competition, since the Coulomb interactions favor the ferromagnetic state (due to their intralayer-interlayer anistrophy).

C. Canted phase

While the ferromagnetic and the spin-singlet phases are essentially single-particle phases, stabilized by single-particle fields, the canted phase is a many-body phase stabilized by the interactions. As was mentioned in the Introduction, in the absence of the interactions there would be a first-order phase transition between the spin-singlet and the ferromagnetic phases. The interactions can lower the energy of the system by creating the canted phase (Fig. 2), and the ground-state energy and width of the canted phase depend on the strength of the interactions.[20] As will be shown in Sec. V, the importance of the interactions in the canted phase implies that the phase can be nontrivially affected by the in-plane component of the magnetic field.

In the canted phase, the interactions effectively mix the ferromagnetic and the spin-singlet states[22] giving rise to a finite magnetization $\langle S^z\rangle$ and antiferromagnetic spin correlations $|\langle\hat{S}_R^z\times\vec{S}_R\rangle-\langle\hat{S}_L^z\times\vec{S}_L\rangle|\neq\vec{0}$, where $\hat{S}_\mu^z$ is the unit vector in the spin-up direction in the layer μ.[9] (One often defines an antiferromagnetic order parameter $\mathcal{O}_{xz}$,[7,11] where $\mathcal{O}_{\alpha\beta}=\langle S^\alpha\otimes I^\beta\rangle$, which is finite only in the canted phase.) The U(1) symmetry associated with rotations around S^z is spontaneously broken. The canted ground state $|C\rangle$ has the most general form, Eq. (17), and is spin-isospin entangled—i.e., it cannot be decoupled into two independent spin or isospin channels such as the ferromagnetic or the spin-singlet ground states. The canted ground state is therefore interlayer phase coherent, which is confirmed by a nonzero value of $\langle(I^x)^2+(I^y)^2\rangle$.

D. *I* phase

While the canted phase has attracted the most attention,[8–11,22] the many-body phase that occurs in charge-unbalanced systems in the absence of interlayer tunneling is no less interesting. In the absence of tunneling, the interactions give rise to a *spontaneously* interlayer phase coherent *I* state (Refs. 10 and 11) (with the antiferromagnetic order parameter $\mathcal{O}_{zx}=0$). The interlayer phase coherence of the *I* state is spontaneous since the single-particle fields Δ_Z^0 and Δ_V do not stabilize interlayer phase coherent states. The *I* ground state has a simple form $|I\rangle=\Pi_X c_{R\uparrow X}^\dagger(z_{R\downarrow}c_{R\downarrow X}^\dagger+z_{L\uparrow}c_{L\uparrow X}^\dagger)|0\rangle$, which manifests the interlayer phase coherence of the state, its spin-isospin entanglement, and the U(1) spontaneous symmetry breaking. Simply speaking, the U(1) symmetry of the *I* state is the freedom to choose the relative phase of $z_{R\downarrow}$ and $z_{L\uparrow}$ in the expression for $|I\rangle$. One can also formally consider the U(1) symmetry associated with rotation around I^z or around S^z (applying $e^{i\theta I^z}$ or $e^{i\theta S^z}$ clearly gives the desired effect). It is important to note that the state $|I\rangle$ does not break the $U(1)\times U(1)$ symmetry of spin and isospin rotations completely.[9,11] This state is an eigenstate of I^z+S^z with an eigenvalue g (the degeneracy of the Landau Level), but it mixes states with different I^z-S^z quantum numbers. So the state $|I\rangle$ breaks the $U(1)\times U(1)$ symmetry down to the diagonal U(1).

The feature of the state $|I\rangle$ to be an eigenstate of I^z+S^z will prove useful in the following discussion of the many-body phases in the presense of an in-plane field. In fact, all the possible ground states of the $\nu=2$ bilayers in the absence of tunneling are eigenstates of I^z+S^z. If the bias voltage and the Zeeman energy are positive, all the possible zero-tunneling ground states are (locally—for a given orbital quantum number *X*) linear combinations of a $S_X^z=+1$ spin triplet and an $I_X^z=+1$ isospin triplet (the ferromagnetic and the spin-singlet states being the two extremes with only the spin triplet or isospin triplet contributing, respectively). In this case, all the zero-tunneling ground states are eigenstates of I^z+S^z with an eigenvalue g, where g is the Landau level degeneracy.

Even though I^z+S^z is a formal construction, whose expectation value cannot be directly measured, it can help advance our understanding of the physics of the *I* phase and of the canted phase in the presence of small interlayer tunneling. Thus, since applying a rotation $e^{i\theta(I^z+S^z)}$ to the state $|I\rangle$ generates a trivial phase factor, I^z+S^z can be treated as the effective direction in which the *I* state "points." In contrast, if tunneling is added to the system, I^z+S^z does not commute with the Hamiltonian. The operator $e^{i\theta(I^z+S^z)}$ then generates a nontrivial rotation of the canted state around I^z+S^z. When interlayer tunneling is small, the canted ground state has a large overlap with an *I* state, and can be visualized in its Hilbert space as pointing slightly away from the I^z+S^z direction. The relevance of this picture to the $\nu=2$ bilayer physics

will become more clear in the next section (Sec. V D).

The relation between the canted phase and the *I* phase is reminiscent of the relation between phases of the $\nu=1$ bilayers in the presence and absence of tunneling. The main similarity is that, when there is no tunneling between the layers, both the $\nu=1$ and the $\nu=2$ bilayers support a spontaneously interlayer phase coherent phase. There are, however, marked differences. The main difference is that the symmetry properties of the canted phase and the *I* phase are the same, while in the $\nu=1$ bilayers spontaneous symmetry-breaking occurs only in the absence of tunneling. Moreover, unlike the zero-tunneling phase of the $\nu=1$ bilayers, the *I* phase is spin-isospin entangled. As we will show in the next section, because of their similar nature, both the $\nu=1$ and $\nu=2$ systems undergo essentially similar phase transitions in the tilted magnetic field; the systems' differences, however, lead to surprisingly different behaviors of the $\nu=1$ and $\nu=2$ bilayers in the vicinity of the phase transitions.

V. GLOBAL PHASE DIAGRAM OF THE $\nu=2$ BILAYERS IN TILTED FIELD

As was discussed in the previous section, the physics of the $\nu=2$ bilayers is very rich. The $\nu=2$ bilayers exhibit a host of many-body phenomena, such as spontaneous symmetry breaking, spontaneous interlayer phase coherence, and spin-isospin entanglement. To further explore the many-body nature of the phases and phase transitions of the $\nu=2$ bilayer system, in this section we study the behavior of the system in the presence of a tilted magnetic field.

Tilted magnetic fields have been successfully used to study the role of both spin and layer degrees of freedom in quantum Hall physics. In thin monolayer systems, the tilted field technique was used to investigate the spin-unpolarized fractional quantum Hall ground states. The technique is based on the fact that, in the infinitely thin limit, the orbital motion in a 2DEG depends solely on the perpendicular component of the magnetic field $B_\perp$, while the Zeeman energy is proportional to the total field *B*. The Zeeman energy can therefore be increased independently of the effective interactions in the quantum Hall monolayer by adding an in-plane field $B_\parallel$. In bilayer systems, the presence of an in-plane field affects not only the Zeeman energy but also the *relative* orbital motion in the two layers and, therefore, the tunneling between the layers and interlayer interactions.

A. Commensurate-incommensurate transition in the $\nu=1$ bilayers: An overview

The coupling of the tilted field to the layer degree of freedom is easier to demonstrate using the thoroughly studied $\nu=1$ bilayers[2–6] as an example. In $\nu=1$ bilayers the spin degree of freedom is frozen out by the ferromagnetic exchange[8] and, in the absence of an external bias voltage, the physics of the system is fully determined by the interplay of tunneling and Coulomb interactions. As was discussed above, tunneling [Eq. (9)] can be considered as an external field that couples to I^x; Coulomb interactions give rise to a charging energy (from the direct term) and an anisotropic isospin stiffness, both of which favor the isospin-*xy* plane.[4] When the layers are separated by a distance comparable to the distance between the electrons in a single layer, the anisotropic Coulomb interactions support a spontaneously interlayer phase coherent ground state even in the absence of tunneling.[2] The U(1) symmetry associated with rotations around I^z is spontaneously broken. Tunneling, always present to some degree in real samples, breaks this symmetry, but it does not destroy the interlayer phase coherence. Instead, in the absence of an in-plane field, tunneling acts cooperatively with the interactions to stabilize the interlayer phase coherent state.[3,4]

A finite in-plane field introduces a competition between tunneling and the Coulomb interactions in $\nu=1$ bilayers.[4,6] In the presence of an in-plane field, the tunneling term [Eq. (9)] now favors an isospin-wave ground state, in which the isospin twists around the I^z direction with the wave vector $Q_\parallel$. Exchange interactions, on the other hand, favor a uniform configuration. The competition between tunneling and exchange results in the commensurate-incommensurate transition between the isospin-wave commensurate and uniform incommensurate phases.

The picture of the commensurate-incommensurate transition outlined above is somewhat simplistic.[4,6] The transition, in fact, does not happen directly from the commensurate to the incommensurate state, but instead occurs through formation of a soliton-lattice phase. When, as the in-plane field is increased, the losses in exchange energy due to twisting become approximately equal to the expectation value of the tunneling term, the system can optimize its ground-state energy by forming a soliton in an otherwise commensurate state.[4,6,23,24] By forming a soliton the system recovers some exchange energy while saving most of the tunneling energy. The commensurate-incommensurate transition by means of soliton-formation is the Talapov-Pokrovsky commensurate-incommensurate transition.[4]

The microscopic description of the physics of a bilayer system in an in-plane field depends on a particular choice of the gauge, but the underlying picture of an induced competition between tunneling and exchange is valid for any gauge.[6] Thus, for example, in the gauge $\vec{A}=(0,B_\perp x-B_\parallel z,0)$, the tunneling electrons acquire no Aharonov-Bohm phase and the tunneling term in the microscopic Hamiltonian is unaffected.[3,14] However, in this gauge it is the interlayer exchange that acquires an additional phase in this case, so that it is the interaction term that now favors an isospin wave.

B. Possibility of commensurate-incommensurate transition in $\nu=2$ bilayers

To create a framework for the interpretation of our numerical results, we start by a qualitative discussion of the behavior of the $\nu=2$ bilayers in tilted field. We consider the theoretical possibility of the commensurate-incommensurate transition in the $\nu=2$ bilayers, much as has been done for the $\nu=1$ bilayers: We assume for simplicity that the commensurate-incommensurate transition happens between a commensurate state that maximizes tunneling at the ex-

pense of exchange (i.e., $Q_I=Q_\parallel$ and $Q_S=0$) and an incommensurate state that maximizes exchange while losing all the tunneling energy ($Q_I=Q_S=0$). This is the scenario that we called simplistic in our discussion of the $\nu=1$ bilayers since better mathematical models indicate that the commensurate-incommensurate transition occurs through a succession of soliton lattice phases rather than directly. Nevertheless, the "naive" commensurate-incommensurate transition serves as a good simple approximation that allows one to make experimentally relevant predictions and explanations without any numerical calculations. We therefore expect that the naive commensurate-incommensurate approximation will capture some of the physics of the $\nu=2$ bilayers in tilted fields as well.

The naive commensurate-incommensurate transition can happen only in the canted phase, since it is the only phase in which tunneling and exchange are comparable to each other. The other two phases—the ferromagnetic phase and spin-singlet phase—essentially mark the regions in the phase space where interactions are less important than the single particle fields. We therefore expect the spin-singlet phase to be always commensurate. The ferromagnetic state is not interlayer phase coherent and therefore the in-plane field cannot affect it. The canted phase, however, can be commensurate at lower in-plane fields, but it may turn incommensurate when the in-plane fields are so high that the losses in exchange become greater than the contribution of the tunneling term.

The naive commensurate-incommensurate transition, however, cannot happen in charge-balanced $\nu=2$ bilayers. In order for the commensurate-incommensurate transition to happen, an incommensurate state should exist, which would become lower in energy than the corresponding commensurate state as the tilt angle is increased. Since the tunneling contribution to the energy of the naive incommensurate state is zero, the incommensurate state is equivalent to the state of the system with no interlayer tunneling (all other parameters unchanged). In charge-balanced $\nu=2$ bilayers, the zero-tunneling ground state is always ferromagnetic. As is clear from Fig. 2, the energy of a ferromagnetic state created in a system with no tunneling and given intralayer interactions is always higher than that of a canted state created by adding a finite amount of tunneling to this system (all other things being equal). Therefore, there can be no naive commensurate-incommensurate transition in a charge-balanced $\nu=2$ bilayer system.

The situation changes if a finite bias voltage can be applied to the system. As was mentioned in Sec. IV, in the presence of finite bias voltage one can obtain an interlayer phase coherent phase—the I phase—even in the absence of tunneling. The ground-state energy of the I state depends not only on the relative strength of the Zeeman energy and the external bias voltage, but also on the interlayer Coulomb interactions. Therefore, it is straightforward to argue that, in charge-unbalanced samples with small tunneling amplitudes, a commensurate-incommensurate transition may be possible: Let us consider such a sample. In a perpendicular field, the many-body state of the sample is a canted state. Because the tunneling amplitude is small, this canted state is just a slight perturbation of the corresponding zero-tunneling I state. The interaction energies of the two states are very close; the energy of the canted phase is lower largely due to the tunneling term. As the magnetic field is tilted, the isospin starts twisting commensurately with the in-plane field, thereby losing some interlayer exchange. When the losses in exchange energy become equal to the tunneling energy, the energies of the canted commensurate state and the corresponding zero-tunneling I state become approximately equal. At slightly higher tilt angles, a transition to the I state clearly would occur. In anticipation of this discussion, at the start of the chapter, we named the zero-tunneling many-body phase the I phase, where I stands for "incommensurate."

We also can expect that the in-plane field will not only induce a transition, but also affect the location of the second-order phase transitions between the spin-singlet, canted, and ferromagnetic phases in the phase space. The reduced effective interlayer interactions in the commensurate states will destabilize the canted commensurate phase, and its two phase boundaries will move toward each other as the in-plane field is increased. The relative position of the boundaries of the incommensurate phase will be nearly constant with the increasing tilt angle. At larger in-plane fields another effect, which we so far have ignored, will become important—the dependence of the Zeeman energy and the tunneling amplitude on the in-plane field. The increasing Zeeman energy and the decreasing tunneling will eventually drive the system into a ferromagnetic state as the tilt angle becomes very large. (At very large in-plane fields, however, our infinitely-thin layer approximation loses its validity completely, and therefore this discussion becomes purely theoretical.)

In the next subsection we explain that the commensurate-incommensurate transition at $\nu=2$ is sufficiently different from the simple picture presented above. Hence, the arguments presented here should be considered only as a motivation for a more detailed discussion in subsecuent sections.

C. Global phase diagram of $\nu=2$ bilayers in tilted magnetic field

The Hartree-Fock phase diagrams are presented in Fig. 3. The axes on the phase diagrams are the bias voltage Δ_V and the in-plane field wave vector $Q_\parallel$. We find this choice of axes convenient, since current experimental techniques allow one to tune both the bias voltage and the in-plane field *in situ* across a wide range. The other parameters of a bilayer sample—the perpendicular-field Zeeman splitting Δ_Z^0, the perpendicular-field tunneling amplitude Δ_{SAS}^0, and the distance between the layers d—are, to a good approximation, intrinsic to a given sample. For each phase diagram, we fix these parameters at values typical of real samples $\Delta_Z^0=0.01(e^2/\varepsilon l)$, $\Delta_{\mathrm{SAS}}^0\leqslant 0.08(e^2/\varepsilon l)$, and $d=l$. Each phase diagram in Fig. 3 therefore corresponds to a single sample with $\Delta_Z^0=0.01(e^2/\varepsilon l)$, $d=l$, and the value of Δ_{SAS}^0 given in the lower right corner of the phase diagram.

The unrestricted Hartree-Fock calculation does provide evidence for a phase transition[7] (the dashed line in Fig. 3) which possesses the properties we qualitatively predicted for the naive commensurate-incommensurate transition: The

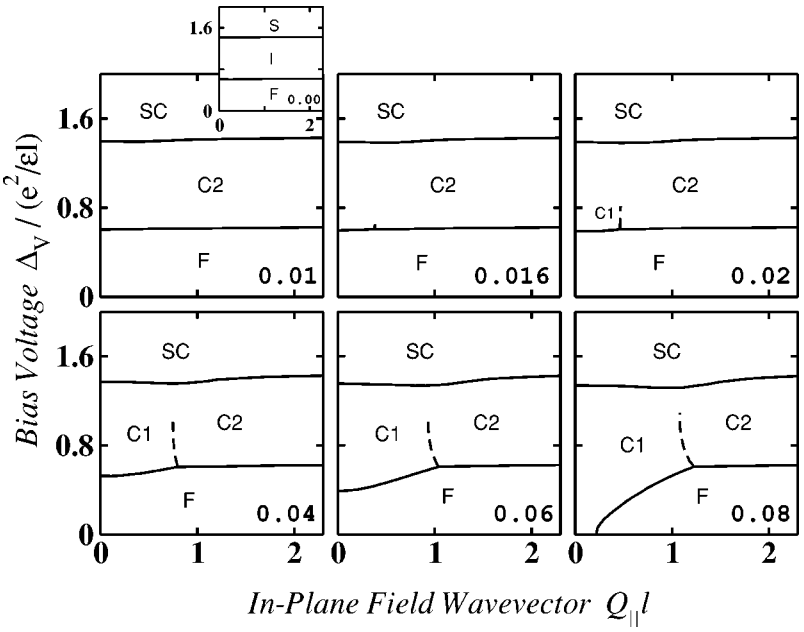

FIG. 3. Global phase diagrams for $\nu=2$ bilayer samples of different tunneling strengths in tilted magnetic field. The tunneling amplitudes Δ_{SAS}^0 are given in the lower right corner of each panel. The Zeeman energy $\Delta_Z^0=0.01(e^2/\varepsilon l)$ and the distance between the layers $d=l$, are the same for all the phase diagrams. The solid lines indicate second-order phase transitions and the dashed lines indicate first order. The phases are *S*: spin-singlet, *F*: ferromagnetic, *I*: incommensurate, SC: spin-singlet commensurate, *C*1: canted commensurate, and *C*2: canted spin-isospin commensurate.

first-order transition occurs only within the canted phase and only in the presence of a finite bias voltage. Moreover, the canted commensurate phase shrinks as the in-plane field is increased, but the decrease in the width of the phase stops after the first-order transition. However, instead of the incommensurate phase, we find an interesting seemingly doubly commensurate phase, in which both the isospin and the spin components are commensurate with the in-plane field.[7] That is to say, throughout the interlayer phase coherent region—in the phases SC, *C*1, and *C*2 in Fig. 3—the wave vector of the isospin-wave $Q_I=Q_{||}$. In the canted phases *C*1 and *C*2, however, the spin-wave wave vector is also nonzero. It is almost zero in the *C*1 phase, except near the phase transition boundary, but it is close to $Q_S=Q_{||}$ in the *C*2 phase (Fig. 4 and Ref. 7). The phase transition between the two canted phases is first order, terminating at a critical point. The onset of the first-order transition occurs at a critical tunneling amplitude $\Delta_{SAS}^0 \approx 0.015(e^2/\varepsilon l)$. As the tunneling amplitude Δ_{SAS}^0 is increased, the *C*1-*C*2 transition becomes more prominent and a higher in-plane field is needed to induce it. The presence of the in-plane component of the magnetic field thus leads to a phase transition that is clearly related to the commensurate-incommensurate transition, but possesses some unexpected properties that invite a physical explanation.

We emphasize that a simple picture of the commensurate-incommensurate transtion should not be carried directly from $\nu=1$ to $\nu=2$. In the former case it appears as a result of the competition between the single particle tunneling energy and the exchange part of the Coulomb interaction. At $\nu=2$ we have $Q_I=Q_{||}$ in all of the canted phase (both *C*1 and *C*2), which optimizes the tunneling term. The origin of the *C*1 -*C*2 transtions is the competition of the exchange terms in Eq. (31). The wave vectors of the exchange terms in this equation are given by Q_s, $Q_{||}-Q_s$, and $Q_{||}+Q_s$ (we used $Q_I=Q_{||}$). So different exchange terms would be minimized for different values of Q_s. It is useful to consider the variational energy of the ground state in Eq. (31) as a function of Q_s for different points in the phase diagram $E(Q_s)$. When we start near the base of the first order transition and far from the critical point (see Fig. 3), the function has two local minima: one for Q_s close to zero and the other for Q_s close to $Q_{||}$. When we are on the *C*1 side of the transition, the former is the global minimum, and when we are on the *C*2 side, the latter corresponds to the true ground state. As we move along the first order line toward the critical point (by increasing the gate voltage), the positions of the two local minima move together until the critical point where they merge into a single minimum. Anywhere above the critical point the system has only one local minimum in $E(Q_s)$. It is also useful to

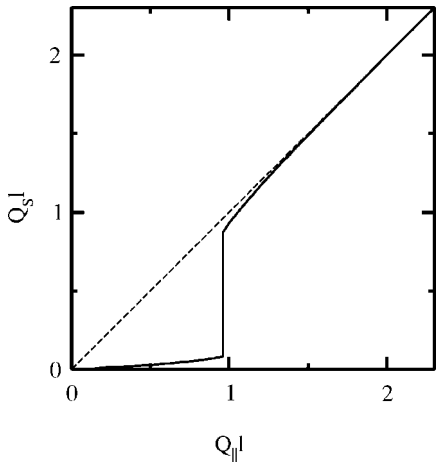

FIG. 4. Evolution of the spin-wave wave vector Q_S (solid line) as a function of the in-plane field wave vector $Q_{||}$ (given in units of $1/l$). The dashed line is $Q_I=Q_{||}$, given for comparison. The wave vector Q_S is small in the phase *C*1 (low $Q_{||}$) and abruptly jumps to $Q_S \approx Q_{||}$ at the phase transition to *C*2. The Zeeman energy in this figure is $\Delta_Z^0=0.01(e^2/\varepsilon l)$, the interlayer spacing is $d=l$, the tunneling constant is $\Delta_{SAS}^0=0.06(e^2/\varepsilon l)$, and the external bias voltage is $\Delta_V=0.8(e^2/\varepsilon l)$.

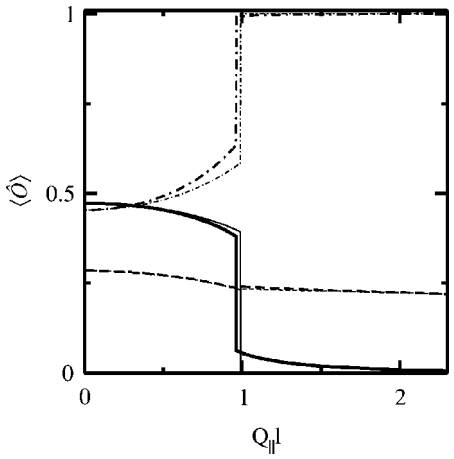

FIG. 5. Expectation values $\langle I_Q^x\rangle$ (solid lines), $\langle I^z\rangle$ (dashed lines), and $\langle I^z+S^z\rangle$ (dash-dotted lines), per flux quantum, across $C1$-$C2$ (thick lines) and the naive commensurate-incommensurate (thin lines) phase transitions. The Zeeman energy is $\Delta_Z^0=0.01(e^2/\varepsilon l)$, the interlayer spacing is $d=l$, the tunneling constant is $\Delta_{\mathrm{SAS}}^0=0.06(e^2/\varepsilon l)$, and the external bias voltage is $\Delta_V=0.8(e^2/\varepsilon l)$.

point out that there is a region of metastability of $C1$ and $C2$ phases around the first-order line separating them. We expect interesting hysteresis effects to occur in this region.

D. Canted commensurate phases in $\nu=2$ bilayers

The most surprising part of the phase diagram is the $C2$ phase, in which both the spin and the isospin degrees of freedom are commensurate with the in-plane field.[7] The location of the $C2$ phase on the phase diagram, so similar to that expected of the incommensurate phase, strongly suggests that, despite its apparent complexity, the $C2$ phase is simply related to the naive incommensurate phase. The close relationship between the $C2$ phase and the I phase becomes more clear if one considers the expectation values of spin and isospin operators. In Fig. 5, we plot three expectation values: $\langle I_Q^x\rangle=\Sigma_X\langle e^{iQ_\parallel X}I_X^+ + e^{-iQ_\parallel X}I_X\rangle$, $\langle I^z\rangle$, $\rangle I^z+S^z\rangle$ for a sample with $\Delta_Z^0=0.01(e^2/\varepsilon l)$, $\Delta_{\mathrm{SAS}}^0=0.06(e^2/\varepsilon l)$, and the bias voltage held at $\Delta_V=0.8(e^2/\varepsilon l)$ as we move across the first-order transition by increasing $Q_\parallel$. The operator I_Q^x is the tunneling operator in the Hamiltonian of the system, and therefore its expectation value indicates if tunneling contributes to the energy of a particular state. The expectation value $\langle I_Q^x\rangle$ is zero in the naive incommensurate state. The naive incommensurate state, the I state, is an eigenstate of the operator I^z+S^z with the eigenvalue g. Thus, $(1/g)\langle I^z+S^z\rangle=1$ in this state. To distinguish the I state from the ferromagnetic state and the fully charge-unbalanced spin-singlet state, which are also eigenstates of I^z+S^z, we also plot $\langle I^z\rangle$, which satisfies $0<(1/g)\langle I^z\rangle<1$ in the I state. The thinner lines in Fig. 5 represent the expectation values obtained under the assumption that the system undergoes the naive commensurate-incommensurate transition (the spin-wave wave vector Q_S is held at 0). The discontinuity in the expectation values marks the commensurate-incommensurate transition. As expected, after the transition into the incommensurate phase $(1/g)\times\langle I_Q^x\rangle=0$ and $(1/g)\langle I^z+S^z\rangle=1$. The thick lines represent the expectation values that we obtain for the system allowed to undergo the $C1$-$C2$ transition. The expectation values obtained from the full unrestricted Hartree-Fock solution and those obtained under the assumption of the naive commensurate-incommensurate transition exhibit strikingly similar behaviors. The main difference is that, unlike in the I phase, the expectation value of the I_Q^x operator in the $C2$ phase is small but finite. This means that there is a contribution from the interlayer tunneling to the ground state energy in the $C2$ phase. It is therefore clear that the $C2$ phase is the optimized version of the I phase much as the soliton-lattice phase of the $\nu=1$ bilayers is the optimized $\nu=1$ incommensurate state.

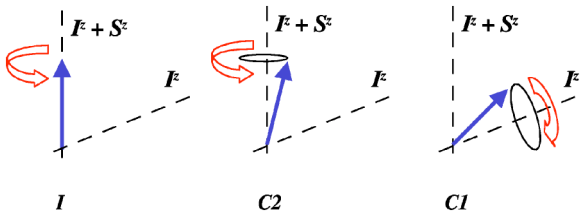

FIG. 6. Schmatic representation of the comensurate $C1$ and $C2$ states. In the left figure, an I state is represented. The I state is shown in Sec. IV D to be an eigenstate of the operator I^z+S^z. Thus, the I state is represented as a vector pointing in the I^z+S^z direction. In the presence of tunneling, the system can satisfy the Aharonov-Bohm phases in the tunneling term by winding around either the I^z+S^z axis, yielding the $C2$-commensurate state shown above, or by winding around the I^z axis yielding the $C1$-commensurate state. The $C2$ state asymptotically approaches the I state in large in-plane field where the winding becomes very fast and therefore must be very tight (due to "spin stiffness").

Unlike the $\nu=1$ bilayer system, the $\nu=2$ bilayer system possesses the spin degrees of freedom which it can use to optimize its ground-state energy around the commensurate-incommensurate transition. The system can satisfy the Aharonov-Bohm phases in the tunneling term by winding either around I^z or I^z+S^z. Winding around I^z+S^z clearly does not affect the Zeeman and the bias voltage terms and in some circumstances can cost less exchange energy: The I state is an eigenstate of the I^z+S^z operator and, as we argued in Sec. IV D, is invariant under rotation around I^z+S^z (see Fig. 6). If the I state is somehow perturbed so that it becomes slightly canted, for example, because a small amount of tunneling is present, the state is no longer invariant under rotations around I^z+S^z, but a precession around I^z+S^z does not cost much exchange energy. (A useful analogy is a Heisenberg ferromagnet with all spins pointing in the same direction $\langle S^z\rangle=+1/2$. If the spins are made to tilt away from the positive S^z direction and precess around it, very little spin-stiffness energy is lost.) As the in-plane field is increased, the winding around I^z+S^z becomes faster and tighter. This causes the phase $C2$ to asymptotically approach I.

The $C1$ and $C2$ phases are both canted and have the same symmetry properties. It is therefore not surprising that they are connected on the phase diagram, i.e., they are essentially the same phase. The qualitative difference between $C1$ and $C2$ is in the involvement of the spin degree of freedom in the

quenching of the in-plane magnetic field. In our mean-field solution the wave vector of the spin wave does not always jump between the qualitatively understood cases of $Q_S=0$ and $Q_S=Q_\parallel$, but it can change gradually. The wave vector Q_S changes gradually when $C1$ turns into $C2$ via a crossover, at larger values of Δ_V, when the canted state has a large overlap with the I phase [$(1/g)\langle I^z+S^z\rangle$ close to 1]. In fact, when Δ^0_{SAS} is very small, $\Delta^0_{\mathrm{SAS}}\leqslant 0.015(e^2/\varepsilon l)$, the first-order transition disappears altogether, since the canted phase is so close to the I phase that the canted phase always has a $C2$ flavor to it.

VI. CONCLUSIONS

To summarize, we have obtained the global phase diagram of the $\nu=2$ bilayers in tilted magnetic field (Fig. 3). We found that, in charge-unbalanced $\nu=2$ bilayers, a finite in-plane component of the magnetic field can induce a first-order phase transition between two commensurate canted phases $C1$ and $C2$. The phase $C1$ possesses isospin-wave order, commensurate with the in-plane field; in the phase $C2$ commensurate spin-wave order is induced alongside with the isospin-wave order. Both $C1$ and $C2$ phases spontaneously break a global U(1) symmetry, and are technically the same phase. Indeed, in the phase diagrams in Fig. 3, phases $C1$ and $C2$ are topologically connected, and the first-order transition between them terminates at a critical end point.

The physics of the commensurate canted phases was discussed in detail in this paper. The behavior of the $\nu=2$ bilayers in tilted magnetic fields was compared to that of their $\nu=1$ counterparts. The phase $C1$ was found to be analogous to the commensurate phase of $\nu=1$ bilayers, while the phase $C2$ was linked to the incommensurate phase. As was predicted by MRJ,[11] the U(1)-symmetry-broken I phase, which had been predicted to exist in the absence of tunneling in charge-unbalanced $\nu=2$ bilayer systems, was found to play the role of a "naive" incommensurate phase in $\nu=2$ bilayers [akin to the "naive"—translationally invariant—incommensurate phase of $\nu=1$ bilayers (see Sec. V A)]. In this paper, the $C2$ phase was argued to be an optimization of the naive incommensurate phase, much as the soliton lattice phase in $\nu=1$ bilayers in tilted fields is an optimization of the naive incommensurate phase in this system.

The rapid convergence of the spin-isospin commensurate canted phase to the I phase can be used to study the I phase. Very similar to the canted phase, the I phase also possesses a number of intriguing many-body properties. However, the I phase can occur only in the absence of interlayer tunneling—a condition impossible in a typical bilayer sample. Tilting the magnetic field allows one to access the I phase in an experimental setting and study its properties.

The possibility of the formation of an interim soliton phase around the $C1$-$C2$ first-order phase transition cannot be ruled out with certainty in our approximation. However, the energy of a soliton phase in $\nu=1$ bilayers converges to the energy of the corresponding naive (translationally invariant) incommensurate phase,[23,24] much more rapidly than the energy of $C2$ converges to the energy of the corresponding I state. We may therefore conclude that, even if the soliton phase in $\nu=2$ bilayers is possible, it will not occupy a significant amount of phase space.

In this paper we present a heuristic argument and numerical evidence that no new transition occurs in *charge-balanced* $\nu=2$ bilayers. This is indeed consistent with the inelastic light-scattering results by Pellegrini *et al.*[21,25] In their experiments, Pellegrini *et al.* used the tilted-field technique to sweep over a range of Zeeman energy *in situ*. No perpendicular bias voltage was applied to the bilayer system; the maximum tilt angle was $\theta=45°$. Pellegrini *et al.* obtained encouraging evidence of the existence of the expected phase transitions between the spin-singlet and canted phases, as well as between canted and ferromagnetic phases. No other transitions have been reported.

The Hartree-Fock approximation, which we used to obtain our results, had been shown to be robust for the $\nu=2$ bilayers in perpendicular fields.[11,26] The phase diagrams obtained in the Hartree-Fock approximation (HFA) closely match those obtained using exact diagonalization.[26] While the Hartree-Fock approximation overestimates the size of the canted region on the spin-singlet side, it reproduces the boundary between the canted and the ferromagnetic phases essentially *exactly* (within the numerical accuracy of the calculations).[26] Since the phase transition occurs closer to the ferromagnetic side of the canted phase, it is reasonable to assume that the quantum fluctuations, not taken into account in the HFA, will not wash it out. The quantum fluctuations will probably effectively renormalize the canted phase and make the first order-phase transition terminate closer to the ferromagnetic-canted line. Indeed, in considering effects beyond HFA, it is reasonable to assume that the gapped phases (the singlet and ferromagnetic phases) will be relatively robust when going beyond HFA, whereas the gapless phases ($C1$, $C2$, and I) could be strongly renormalized, or even changed qualitatively by disorder.[27]

One might also consider the effects of finite temperatures. In the context of the Hartree-Fock approximation, one need only think about thermally exciting electrons from the occupied basis states $f^\dagger_{1X}$ and $f^\dagger_{2X}$ to the unoccupied basis states $f^\dagger_{3X}$ and $f^\dagger_{4X}$ [see Eq. (17) and thereafter]. Indeed, one could easily generalize the current work to treat finite temperature in this way. Without doing this work explicitly one can qualitatively guess many of the results. In the gapped (ferromagnetic and singlet) phases, so long as the gap is larger than the temperature, there are no excitations and the state remains completely unchanged. In the gapped phases at higher temperatures, or in the gapless phases at any finite temperature quasiparticle excitations are thermally excited (see Ref. 12 where we discuss the zero-temperature excitation spectra in great detail). At high enough temperatures the proliferation (and interaction) of multiquasiparticle excitations should self-consistently change the energetics and the nature of the state. In particular, in the gapped phases, we expect the gap to be self-consistently destroyed at high enough temperature. In the canted phase we have linearly dispersing spin-wave excitations, which can be excited at finite temperature. This should lead to a suppression in spin stiffness that is linear in temperature.[28]

Finally, we turn to the issue of whether some of these effects could be seen in other material systems with similarities to bilayers—such as a two-subband systems. While, it

certainly seems likely that such other systems could display analogs of ferromagnetic, spin-singlet, and canted phases (indeed, such phases are often observed in a variety of spin systems, see Ref. 29 for a review), a unique property of the bilayer state is the spatial separation between the two layers that allows us to thread a flux between and thus associate an Aharonov-Bohm phase with the tunneling processes. It is the accommodation of this phase that creates the $C1$ and $C2$ phases and the rich physics of the transition between the two.

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