Scattering from inhomogeneities

Diffusion par des inhomogénéités

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Ondes élastiques : premier développement de la théorie de la diffusion résonnante par des cavités et inclusions dans la matière solide.

Ondes électromagnétiques : établissement du rapport entre la théorie de la diffusion résonnante et la méthode du développement en singularités (SEM). Théorie de la propagation des ondes radio dans le chenal terre-ionosphère.

Théorie nucléaire : développement de la théorie fondamentale du rayonnement du freinage cohérent (éditeur d'un livre récent). Développement d'un modèle nucléaire de vibrations collectives multipolaires, et applications à la diffusion des électrons (auteur de deux livres). Réactions photopioniques. Modèle collectif de la diffusion ion-ion. Réactions de neutrinos (éditeur d'un livre).

SUMMARY

A microinhomogeneous medium, consisting of randomly distributed cavities or inclusions in a homogeneous elastic matrix, can be represented as a dispersive homogeneous medium with effective material constants (moduli, bulk wave speeds, and absorptions). For wavelengths long compared to the size of the scatterers, Kuster and Toksöz have developed a method (not including rescattering) which obtains these effective material properties by comparing exact and effective monopole, dipole and quadrupole amplitudes. We extend this approach to the case where the wavelength is comparable to the size of the scatterers (assumed spherical); in this case, particle resonances are taken into account and lead to widened resonances in the effective material parameters. The cases of bubbly liquids, of perforated solids, and of solids with solid inclusions (particulate composites) are treated in this fashion. Measurements by Kinra and Anand verify our results. In addition, many previous results for the effective moduli of composites, obtained in the static (i. e., low-frequency) limit, are recovered as particular cases of our approach.

KEY WORDS

Microinhomogeneous medium, effective material constants, resonances, dispersion, bubbly liquids, perforated solids, inclusions.

RÉSUMÉ

Un milieu micro-inhomogène, qui consiste en cavités ou inclusions distribuées dans une matrice élastique homogène de façon aléatoire, peut être considéré comme un milieu homogène et dispersif possédant des constantes de matériaux équivalentes ou effectives (modules, vitesses des ondes de volume et constantes d'absorption). Dans le cas de longueurs d'onde grandes devant la dimension des objets diffuseurs, Kuster et Toksöz ont développé une méthode (ne prenant pas en compte les effets de diffusion multiple) qui donne les propriétés des matériaux équivalents en comparant les amplitudes monopolaire, dipolaire et quadripolaire exactes et effectives. Nous avons étendu cette approche au cas où la longueur d'onde devient comparable aux dimensions des objets diffusants (considérés comme sphériques); dans ce cas, on tient compte des résonances des particules qui causent des résonances élargies dans les paramètres effectifs des matériaux. On traite de cette façon les liquides contenant des bulles, les solides perforés, et les solides contenant des inclusions. Nos résultats sont vérifiés par les mesures de Kinra et Anand. De plus, des résultats antérieurs concernant les modules effectifs de milieux composés, obtenus dans la limite statique (ou de basse fréquence), sont trouvés ici comme cas particuliers de notre approche.

MOTS CLÉS

Milieu micro-inhomogène, constantes de matériaux effectives, résonances, dispersion, liquides à bulles, solides perforés, inclusions.

CONTENTS

1. Introduction

We employ a method first suggested by Ament [1], and later elaborated on and applied to the case of seismic-wave propagation by Kuster and Toksöz [2], in order to obtain effective material constants (moduli, wave speeds, attenuations) for microinhomogeneous media consisting of a random distribution of spherical solid inclusions, fluid-filled cavities or gas bubbles in an otherwise homogeneous solid or fluid host medium. Our method is based on the scattering of waves, and by a comparison of monopole, dipole and quadrupole scattering amplitudes from the micro-inhomogeneities with those from a sphere of the effective material, we were able to obtain frequency-dependent effective moduli, wave speeds and attenuations which, in contrast to previous work, contain the effects of monopole resonances of the individual inclusions, as retained by us in the long-wavelength expansion. Our theory neglects rescattering effects and is thus restricted to small to moderate volume concentrations Φ ($<40\%$ in some experiments). By its dynamic nature, it furnishes dispersion curves for wave propagation constants which reproduce well their acoustical branch, as obtained experimentally for random glass spheres in an epoxy matrix [3], while the optical branch (dipole resonances) is not included in our results due to our restriction to monopole resonances. We also discuss our results for the case of water containing air bubbles, as well as recent experiments on this topic [4]. Our general dynamic results are shown to reduce to previously obtained static limits for the effective material constants.

2. General formalism

In an elastic host medium containing N solid spherical inclusions, the total compressional-wave potential generated by an incident p-wave is:

$$\varphi(\mathbf{r}) = \varphi_{\text{inc}}(\mathbf{r}) + \sum_{k=1}^{N} \varphi_{\text{sc}}^{(k)}(\mathbf{r}, \mathbf{r}_k), \tag{2.1}$$

if one neglects multiple scattering and retains only single-scattering amplitudes $\varphi_{\text{sc}}^{(k)}(\mathbf{r}, \mathbf{r}_k)$ from the N scatterers of radii a_k, their centers being at random positions $\mathbf{r}_k$. Alternately, one may consider a large sphere of radius R, with center at $\mathbf{r}_0$, consisting of an "effective" medium (quantities denoted by a tilde) and imbedded in the same host material; it leads to a total field:

$$\varphi(\mathbf{r}) = \varphi_{\text{inc}}(\mathbf{r}) + \tilde{\varphi}_{\text{sc}}(\mathbf{r}, \mathbf{r}_0). \tag{2.2}$$

We define the effective medium to be such as to produce the same scattered far field as the actual composite, so that at $r \gg \text{R}$,

$$\tilde{\varphi}_{\text{sc}}(\mathbf{r}, \mathbf{r}_0) = \sum_{k=1}^{N} \varphi_{\text{sc}}^{(k)}(\mathbf{r}, \mathbf{r}_k). \tag{2.3}$$

In this latter relation, a comparison of monopole (A_0), dipole (A_2) and quadrupole (A_3) scattering amplitudes (which can be thought of as the leading terms in a long-wavelength expansion of φ_{sc}) is shown to be sufficient to determine the effective material constants [2]. It is necessary, however, to carry out a similar long-wavelength expansion in the known [5, 6] multipole amplitudes A_n (which would appear to be called for here, for the sake of consistency) with a certain amount of care: some terms in A_n contain the square of the frequency, which is normally neglected in the long-wavelength expansion, but multiplied by ρ_1/ρ_2, the ratio of host (1)-to-inclusion (2) densities, which may be large so that the corresponding term is no longer negligible. Retaining such terms in A_0 is seen [7, 8] to effect an inclusion of monopole resonance terms in the scattering amplitude, which indeed appear prominently in such quantities as the dynamic (frequency-dependent) sound speeds and attenuations in bubbly liquids [7], or in the effective wave speeds and moduli of materials containing fluid-filled cavities [8] or solid inclusions [9]. These resonance terms are retained here in A_0, but not in A_1 or A_2. For the case of solid inclusions, this leads to the explanation of the acoustical branches of measured dispersion curves [2] for the effective medium, but not their optical branches; these would have to be explained by retaining resonance terms in A_1.

Denoting, for a sphere of radius R, the normalized frequency $\text{X}_{d1} = k_{d1}\text{R}$ where $k_{d1} = \omega/c_d$ is the p-wave number in medium 1, one has [8]:

$$\text{A}_0 = \frac{\text{X}_{d1}^3}{3i} \frac{3 - \text{X}_{0f}^2 - \text{X}_{0s}^2 + \text{X}_{0i}^2}{\text{X}_{d1}^2(1 + i\text{X}_{d1}) - \text{X}_{0f}^2 - \text{X}_{0s}^2 + \text{X}_{0i}^2}, \tag{2.4 a}$$

$$\text{A}_1 = \frac{\text{X}_{d1}^3}{9i}\left(1 - \frac{\rho_2}{\rho_1}\right), \tag{2.4 b}$$

$$\text{A}_2 = -\frac{\text{X}_{d1}^3}{3i} \times \frac{4\mu_1(\mu_2 - \mu_1)}{6\mu_2(k_{e1} + 2\mu_1) + \mu_1(9k_{e1} + 8\mu_1)}, \tag{2.4 c}$$

where:

(2.5 a) $$X_{0f}^2=(3\rho_2/\rho_1)(c_{d2}/c_{d1})^2$$

is the Minnaert [10] bubble resonance frequency,

(2.5 b) $$X_{0s}^2=(2c_{s1}/c_{d1})^2$$

that of the Meyer-Brendel-Tamm [11] resonance for evacuated cavities in solids, and:

(2.5 c) $$X_{0i}^2=(\rho_2/\rho_1)(2c_{s2}/c_{d1})^2$$

an additional resonance due to shear in the inclusion;

(2.5 d) $$k_{e1}=\lambda_1+\frac{2}{3}\mu_1$$

is the elastic bulk modulus of medium 1 (λ and μ being the Lamé constants).

3. Specific results

We may now compare A_n ($n=0$, 1, 2) separately in the fashion of Equation (2.3). This determines elastic moduli and wave speeds as listed in Reference [9]. One has, e. g., for the (normalized) effective p-wave speed:

(3.1) $$\frac{\tilde{c}_2}{c_{d1}}=\frac{1}{[1-(1-\rho_2/\rho_1)\Phi]^{1/2}}\times\left(\frac{2(U^2+V^2)}{(X^2+Y^2)^{1/2}(U^2+V^2)^{1/2}+(XU+YV)}\right)^{1/2},$$

where:

(3.2 a) $$X=1-\Psi_R,\qquad Y=-\Psi_I,$$

(3.2 b) $$U=1-\frac{4}{3}\left(\frac{c_{s1}}{c_{d1}}\right)^2 X\left(1-\frac{\tilde{\mu}_2}{\mu_1}\right),$$

(3.2 c) $$V=-\frac{4}{3}\left(\frac{c_{s1}}{c_{d1}}\right)^2 Y\left(1-\frac{\tilde{\mu}_2}{\mu_1}\right),$$

(3.3) $$\frac{\tilde{\mu}_2}{\mu_1}=\left[1+\frac{6\mu_2}{\mu_1}\frac{k_{e1}+2\mu_1}{9k_{e1}+8\mu_1}-\left(1-\frac{\mu_2}{\mu_1}\right)\Phi\right]\times\left[1+\frac{6\mu_2}{\mu_1}\frac{k_{e1}+2\mu_1}{9k_{e1}+8\mu_1}+\frac{6(k_{e1}+2\mu_1)}{9k_{e1}+8\mu_1}\times\left(1-\frac{\mu_2}{\mu_1}\right)\Phi\right]^{-1};$$

(3.4 a) $$\Psi_R=\frac{4\pi A}{3}\int_0^\infty\frac{Bg(a)}{B^2+C^2}a^3\,da,$$

(3.4 b) $$\Psi_I=\frac{4\pi A}{3}\int_0^\infty\frac{Cg(a)}{B^2+C^2}a^3\,da,$$

(3.5 a) $$A=1-\frac{\rho_1 c_{d1}^2}{\rho_2 c_{d2}^2}+\frac{4}{3}\frac{\rho_1 c_{s1}^2}{\rho_2 c_{d2}^2}-\frac{4}{3}\frac{\rho_2 c_{s2}^2}{\rho_2 c_{d2}^2},$$

(3.5 b) $$B=1+\frac{4}{3}\frac{\rho_1 c_{s1}^2}{\rho_2 c_{d2}^2}-\frac{1}{3}\frac{\rho_1 c_{d1}^2}{\rho_2 c_{d2}^2}\times\left(\frac{2\pi af}{c_{d1}}\right)^2-\frac{4}{3}\frac{\rho_2 c_{s2}^2}{\rho_2 c_{d2}^2},$$

(3.5 c) $$C=\frac{1}{3}\frac{\rho_1 c_{d1}^2}{\rho_2 c_{d2}^2}\left(\frac{2\pi af}{c_{d1}}\right)^2.$$

Here, the volume concentration is:

(3.6 a) $$\Phi=\frac{4\pi}{3}\int_0^\infty g(a)\,a^3\,da,$$

where $g(a)$ is the radius distribution of the inclusions. From:

(3.7) $$\tilde{\rho}_2=(1-\Phi)\rho_1+\Phi\rho_2$$

and Equation (3.3), one finds the effective s-wave speed:

(3.8) $$\tilde{c}_{s2}=(\tilde{\mu}_2/\tilde{\rho}_2)^{1/2},$$

which is frequency independent.

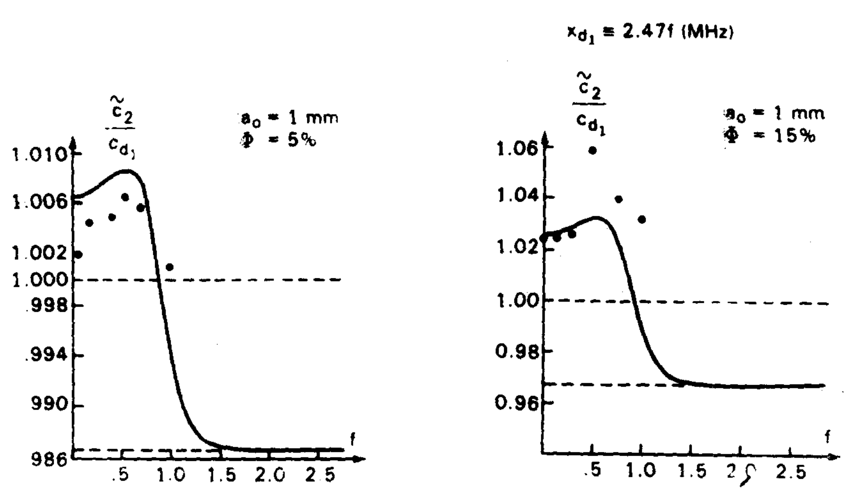

Fig. 3.1. — Effective sound speed $\tilde{c}_2$ vs. frequency (0–2.5 MHz) for $a=1$ mm glass spheres in epoxy (concentrations $\Phi=5\%$ and 15%).

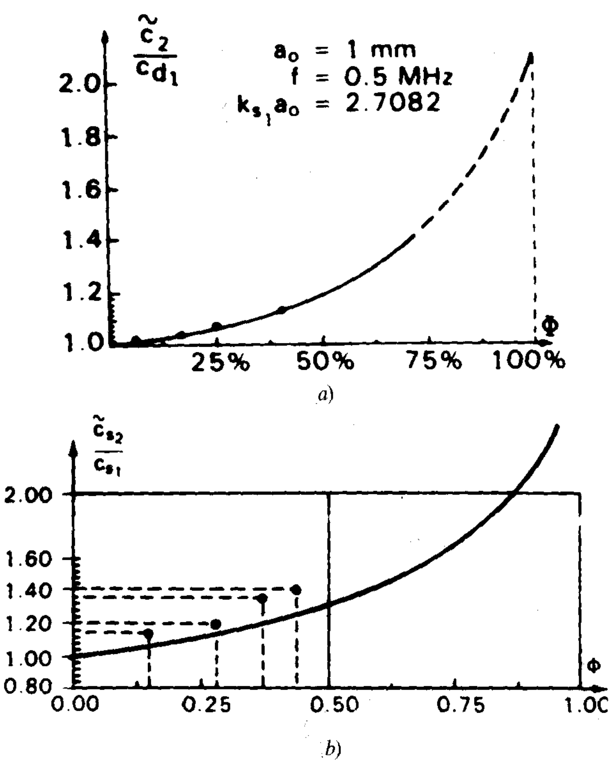

Fig. 3.2. — Effective wave speeds for $a=1$ mm glass spheres in epoxy, vs. concentration Φ: (a) effective sound speed $\tilde{c}_2$ at $f=0.25$ MHz, (b) effective shear wave speed.

Figure 3.1 shows theoretical [9] and measured [3] results for the frequency dependence of the effective p-wave speed in a medium of glass spheres (average radius 1 mm) in epoxy, at concentrations $\Phi=5\%$ and 15%. Figure 3.2 *a* shows this speed at $f=0.5$ MHz plotted vs. Φ, and Figure 3.2 *b* displays the shear speed vs. Φ; the agreement is good up to $\Phi=40\%$.

4. Particular cases

The static effective material constants are obtained in the limit $f=0$. Our above results then agree with the previous static results of Mal and Bose [12], Kerner [13], and Hashin [14]. For rigid inclusions (λ_2, $\mu_2 \to \infty$) of dilute concentration ($\Phi \ll 1$) we recover the results of Moon and Mao [15], and if the host medium is additionally assumed incompressible (Poisson ratio $\nu \cong 1/2$), one has:

$$\tilde{\mu}_2/\mu_1 = 1 + \frac{5}{2}\Phi, \quad (4.1)$$

a result obtained by Einstein [16].

5. Gas bubbles in fluids

Here one has $c_{s1}=c_{s2}=0$, further $U=1$, $V=0$. The theory then reduces to a previously treated case [7], and Figure 5.1 *a* presents numerical results for water containing $a=1$ mm air bubbles at $\Phi=0.000377$

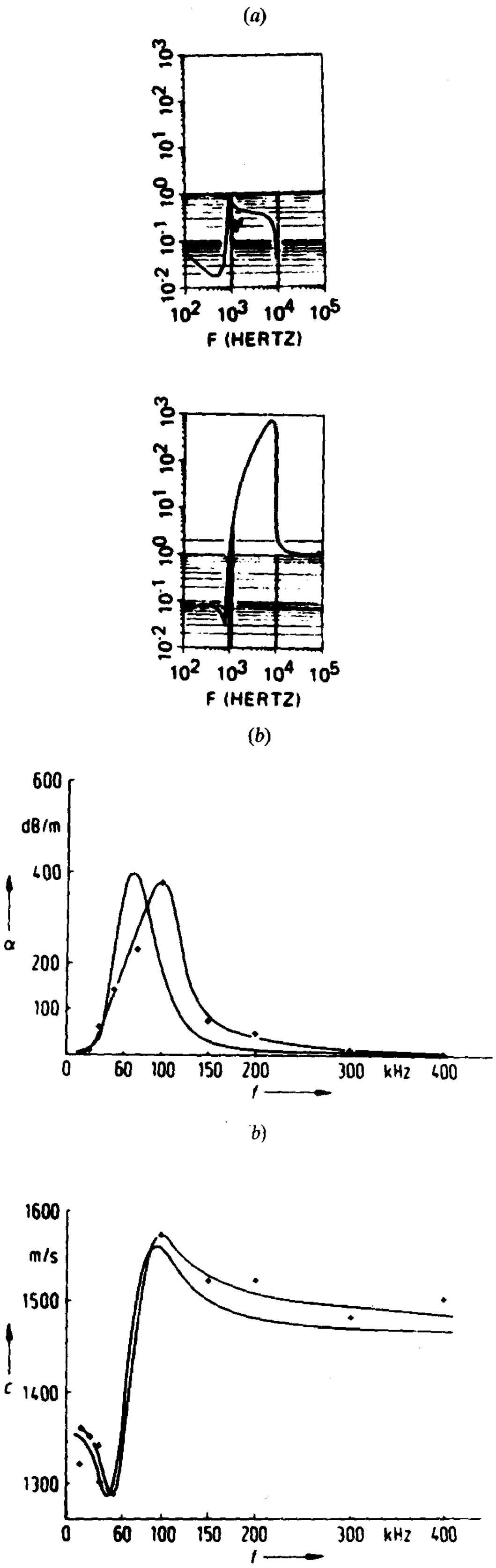

Fig. 5.1. — Attenuation (top) and effective sound speed (bottom) in bubbly water at (a) $a=1$ mm, $\Phi=0.000377$, (b) $\bar{a}=45$ μm, $\Phi=0.0000267$.

(their resonance frequency being 3.285 kHz) compared to experimental results of Silberman [17]: the top portion shows absorption, the botton portion the effective sound speed. Figure 5.1 *b* gives more recent measurements (crosses) at $\Phi = 0.000\,026\,7$ for a bubble distribution centered on a $=45$ μm, compared to theory (solid curves) [4]. In this latter work, it is also shown how, with one measurement of *c* at low frequency, and two at high frequency, the first three moments of the bubble size distributions can be experimentally determined.

6. Conclusion

The foregoing theory permits the determination of dynamic effective material constants for microinhomogeneous media containing gas bubbles, fluid-filled cavities or solid inclusions (treated as spherical), for low to moderate concentrations of the inhomogeneities, and taking their resonances (as functions of frequency) into account. For higher concentrations, self-consistent methods have to be invoked such as that of Chaban [18]; this latter method, however, treats resonances in an ad-hoc fashion only.

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