SOME REDUCIBILITY PROPERTIES FOR PSEUDOVARIETIES OF THE FORM DRH

CÉLIA BORLIDO

ABSTRACT. Let H be a pseudovariety of groups and DRH be the pseudovariety containing all finite semigroups whose regular $\mathcal{R}$-classes belong to H. We study the relationship between reducibility of H and of DRH with respect to several particular classes of systems of equations. The classes of systems considered (of pointlike, idempotent pointlike and graph equations) are known to play a role in decidability questions concerning pseudovarieties of the forms V $*$ W, V $\vee$ W, and V $\textcircled{m}$ W.

1. INTRODUCTION

The interest in studying pseudovarieties of semigroups is, in part, justified by Eilenberg's correspondence [24], which establishes a bijection between pseudovarieties of finite semigroups and varieties of rational languages. Also, rational languages are a very important object in Theoretical Computer Science, as they correspond to the languages recognized by finite state automata.

In turn, pseudovarieties are quite often described as a result of applying certain operators on pairs of other pseudovarieties, such as the semidirect product $*$, the join $\vee$, and the Mal'cev product $\textcircled{m}$. Therefore, it is a natural question to ask whether pseudovarieties of the form V $*$ W, V $\vee$ W, or V $\textcircled{m}$ W are *decidable* (meaning that they have a decidable *membership problem*). It is known that V and W being decidable is not enough to have decidability of any of those pseudovarieties [1, 30]. It was the search for sufficient conditions to preserve decidability under the operator $*$ that led to the definition of *hyperdecidability*, a stronger notion of decidability [3]. Shortly after, the notion of *tameness* [11, 12] emerged as a method of establishing hyperdecidability of pseudovarieties. Briefly, it may be described in two steps: *decidability of the word problem* and *reducibility*, with respect to some highly computable implicit signature. Some other variants of stronger versions of decidability may be found in the literature (see [5] for an overview).

It is also worth mentioning that a particular instance of hyperdecidability, known as *strong decidability*, was already considered for several years under the name of *computable pointlike sets*.

For instance, in 1988 Henckell [25] proved that aperiodic semigroups have computable pointlike sets or, in other words, that the pseudovariety A of finite aperiodic semigroups is strongly decidable. This study was conducted to produce progress in the question of decidability of the Krohn-Rhodes complexity for semigroups [28]. Along the same line, Ash [19] introduced *inevitable sequences* in a finite monoid (for finite groups) in order to prove the Rhodes type II conjecture [27]. Deciding whether a sequence $(s_1, \ldots, s_n)$ from a finite monoid is inevitable in Ash's sense translates to hyperdecidability of the pseudovariety G of finite groups with respect to the equation $x_1 \cdots x_n = 1$. Also, Pin and Weil [29] described a defining set of identities for a Mal'cev product, which in turn implies that the decidability of idempotent pointlike sets may be used as a sufficient condition for decidability of Mal'cev products of pseudovarieties. The diversity of motivations behind these works somehow indicates that hyperdecidability may lead the way to a better understanding of the structure of finite semigroups. Indeed, many researchers have shown interest in studying strong versions of decidability for pseudovarieties (see, for instance, [4, 8, 9, 10, 16, 18, 26, 32]).

On the other hand, the pseudovarieties of the form DRH have already been considered in the literature. In the mid seventies, Schützenberger [31] identified the associated varieties of rational languages under Eilenberg's correspondence. Also, more recently, a study on the structure of the free pro-DRH semigroup was carried out by Almeida and Weil [15]. Pseudovarieties of the form DRH are the object of our study, in which we answer the following question:

(Q)   Given an implicit signature σ, what conditions on a pseudovariety of groups H guarantee that the pseudovariety DRH is *σ-reducible with respect to a given class* $\mathcal{C}$ *of finite systems of equations* (to be precisely described in Subsection 2.3)?

The classes $\mathcal{C}$ considered are precisely those related with the decidability problems mentioned above. More precisely, we consider systems of *pointlike equations* $(x_1 = \cdots = x_n)$, of *graph equations* (equations arising from finite graphs by assigning to each edge $x \xrightarrow{y} z$ the equation $xy = z$), and of *idempotent pointlike equations* $(x_1 = \cdots = x_n = x_n^2)$.

The paper is organized as follows. We devote Section 2 to an overview of results in the literature that we use in the rest of the paper. In particular, in Subsection 2.3 we expose some concepts and results concerning decidability. The subsequent sections focus on pointlike, graph, and idempotent pointlike equations, in this order. We prove in Section 3 that H being σ-reducible with respect to

systems of pointlike equations, suffices for DRH to enjoy the same property. That result is achieved by considering a certain periodicity phenomenon on the constraints. Then, in Section 4, we study systems of graph equations. We prove that H is σ-reducible with respect to systems of graph equations if and only if so is DRH. For that purpose, we borrow from [7] the notion of *splitting point* considered in the setting of the pseudovariety R. Finally, in the last section, we prove that if H is σ-reducible with respect to systems of graph equations, then DRH is σ-reducible with respect to systems of idempotent pointlike equations. The techniques used are somehow similar to the ones used in Section 3.

2. Preliminaries

We assume that the reader is familiar with the theory of finite and profinite semigroups. We refer to [2, 6] for this topic. For the basic concepts and results on topology, the reader is referred to [33].

2.1. **General definitions and notation.** In the sequel, V and W stand for arbitrary pseudovarieties of semigroups, while H stands for an arbitrary pseudovariety of groups. We list below the pseudovarieties mentioned in this paper.

> S consists of all finite semigroups;
>
> Sl consists of all finite semilattices;
>
> G consists of all finite groups;
>
> Ab consists of all finite Abelian groups;
>
> G_p consists of all finite p-groups (for a prime number p);
>
> G_{sol} consists of all finite solvable groups;
>
> R consists of all finite $\mathcal{R}$-trivial semigroups;
>
> DRH consists of all finite semigroups whose regular $\mathcal{R}$-classes are groups of H;
>
> $\overline{\mathsf{H}}$ consists of all finite semigroups whose subgroups belong to H.

Let A be a finite alphabet. The free A-generated pro-V semigroup is denoted $\overline{\Omega}_A \mathsf{V}$, and its elements are called *pseudowords over* V (or simply *pseudowords*, when V = S). Whenever V is not the trivial pseudovariety, it is usual to identify A with its image under the generating mapping of $\overline{\Omega}_A \mathsf{V}$, so that the free semigroup A^+ is a subsemigroup of $\overline{\Omega}_A \mathsf{V}$. For a subpseudovariety W of V, we represent by ρ_{W} the canonical projection from V onto W, should V be clear from the context. When Sl $\subseteq$ V, we denote ρ_{Sl} by c and call it the *content function*. An *implicit signature*

is a set of pseudowords over A containing the semigroup multiplication, generically denoted σ. Each pseudoword may be naturally seen as an implicit operation [6, Theorem 4.2]. Hence, each profinite semigroup is endowed with a structure of σ-algebra. We denote by $\Omega_A^\sigma \mathsf{V}$ the free A-generated σ-semigroup over V, and we call σ-words over V (or simply σ-words, when $\mathsf{V} = \mathsf{S}$) to its elements. Further, we let $\langle \sigma \rangle$ denote the implicit signature obtained from σ through composition of its elements (see [6, Proposition 4.7]). The implicit operations corresponding to the elements of A^+ are called *explicit operations*. The ω-power is the implicit operation x^ω that assigns to each element s of a finite semigroup the unique idempotent that is a power of s. It plays a distinguished role in this paper.

If S is a semigroup, then we represent by S^I the monoid with subsemigroup S, identity I, and underlying set given by $S \uplus \{I\}$. Based on the identification $A^+ \subseteq \overline{\Omega}_A \mathsf{V}$, we sometimes call *empty word* the identity element $I \in (\overline{\Omega}_A \mathsf{V})^I$. We further set $c(I) = \emptyset$.

Given a formal equality of pseudowords $u = v$, also called *pseudoidentity*, we write $u =_\mathsf{V} v$ if the interpretations of u and v coincide on every semigroup of V. Note that this is equivalent to having $\rho_\mathsf{V}(u) = \rho_\mathsf{V}(v)$. All the expressions $u = v$ *modulo* V, V *satisfies* $u = v$, and $u = v$ *holds in* V mean that $u =_\mathsf{V} v$.

2.2. **The pseudovariety** DRH. For a complete study of pseudovarieties of the form DRH, the reader is referred to [15]. We proceed with the statement of some structural properties of the free pro-DRH semigroup that we use later.

It is well known that for every element u of $\overline{\Omega}_A \mathsf{S}$ (respectively, of $\overline{\Omega}_A \mathsf{DRH}$) there exists a unique factorization $u = u_\ell a u_r$, with u_ℓ and u_r possibly the empty word, such that $c(u_\ell a) = c(u)$ and $a \notin c(u_\ell)$ (see, for instance, [17, Proposition 2.1] and [15, Proposition 2.3.1]). Such a factorization (both over S and over DRH) is called the *left basic factorization of u*.

Let u be either a pseudoword or a pseudoword over DRH. For each $k \geq 1$, we define $\mathsf{lbf}_k(u)$ inductively as follows. We set $u_{0,r} = u$. If $u_{k-1,r} \neq I$, we let $u_{k-1,r} = u_{k,\ell} a_k u_{k,r}$ be the left basic factorization of $u_{k-1,r}$. Then, we set $\mathsf{lbf}_k(u) = I$ if $u_{k-1,r} = I$, and $\mathsf{lbf}_k(u) = u_{k,\ell} a_k$ otherwise. The *cumulative content of u*, denoted $\vec{c}(u)$, is the ultimate value of the sequence $(c(\mathsf{lbf}_k(u)))_{k \geq 1}$. Observe that this sequence indeed stabilizes since it forms a descending chain of subsets of some finite set A.

On the other hand, if we consider the iteration of the left basic factorization to the leftmost factor, then we obtain uniqueness of the so-called *first-occurrences factorization*. We state that fact for later reference.

Lemma 2.1. *Let u be a pseudoword (respectively, a pseudoword over DRH). Then, there exists a unique factorization $u = a_1 u_1 a_2 u_2 \cdots a_n u_n$ over S (respectively, over DRH) such that $a_i \notin c(a_1 u_1 \cdots a_{i-1} u_{i-1})$, for $i = 2, \ldots, n$, and $c(u) = \{a_1, \ldots, a_n\}$.*

We say that ua is an *end-marked pseudoword* provided $a \notin \vec{c}(u)$. Also, the product uv is *reduced* if v is nonempty and the first letter of v (which is defined, by Lemma 2.1) does not belong to the cumulative content of u. The following result is used later.

Proposition 2.2 ([8, Proposition 4.8]). *The set of all end-marked pseudowords over a finite alphabet constitutes a well-founded forest under the partial order $\leq_{\mathcal{R}}$.*

We end this subsection with a result concerning identities over DRH, whose proof may be found in [21].

Lemma 2.3. *Let $u, v \in \overline{\Omega}_A \mathsf{S}$ and $u_0, v_0 \in (\overline{\Omega}_A \mathsf{S})^I$ be such that $c(u_0) \subseteq \vec{c}(u)$ and $c(v_0) \subseteq \vec{c}(v)$. Then, the pseudovariety DRH satisfies $u u_0 = v v_0$ if and only if it satisfies $u \mathrel{\mathcal{R}} v$ and if, in addition, the pseudovariety H satisfies $u u_0 = v v_0$. In particular, by taking $u_0 = I = v_0$, we get that $u =_{\mathsf{DRH}} v$ if and only if $u \mathrel{\mathcal{R}} v$ modulo DRH and $u =_{\mathsf{H}} v$.*

2.3. **Decidability.** The *membership problem* for a pseudovariety V amounts to determining whether a given finite semigroup belongs to V. If there exists an algorithm to solve this problem, then the pseudovariety V is said to be *decidable*. As we already referred in the Introduction, other stronger notions of decidability have been set up over the years. They are related with so-called systems of pseudoequations.

Let X be a finite set of *variables*. A *pseudoequation* is a formal expression $u = v$ with $u, v \in \overline{\Omega}_X \mathsf{S}$. If $u, v \in \Omega_X^\sigma \mathsf{S}$, then $u = v$ is said to be a σ-*equation*. A *finite system of pseudoequations* (respectively, σ-*equations*) is a finite set

$$(2.1) \qquad \{u_i = v_i \colon i = 1, \ldots, n\},$$

where each $u_i = v_i$ is a pseudoequation (respectively, σ-equation). For each variable $x \in X$, we consider a *constraint* given by a clopen subset K_x of $\overline{\Omega}_A \mathsf{S}$. Then, a *solution modulo* V of the

system (2.1) *satisfying the given constraints* is a continuous homomorphism $\delta : \overline{\Omega}_X S \to \overline{\Omega}_A S$ such that the following conditions are satisfied:

(S.1) $\delta(u_i) =_\mathsf{V} \delta(v_i)$, for $i = 1, \ldots, n$;

(S.2) $\delta(x) \in K_x$, for every variable $x \in X$.

If $\delta(X) \subseteq \Omega_A^\sigma S$, then we say that δ is a solution modulo V of (2.1) *in σ-words*.

Remark 2.4. It follows from Hunter's Lemma that, for each clopen set K_x, there exists a finite semigroup S_x and a continuous homomorphism $\varphi_x : \overline{\Omega}_A S \to S_x$ such that K_x is the preimage of $\varphi_x(K_x)$ under φ_x (see [6, Proposition 3.5], for instance). It is sometimes more convenient to think of the constraints of the variables in terms of a fixed pair (φ, ν), where $\varphi : \overline{\Omega}_A S \to S$ is a continuous homomorphism into a finite semigroup S and $\nu : X \to S$ is a map. In that way, the requirement (S.2) becomes a finite union of requirements of the form "$\varphi(\delta(x)) = \nu_j(x)$, for every variable $x \in X$", for a certain finite family $(\nu_j : X \to S)_j$ of mappings. We may also assume, without loss of generality that S has a content function (see [13, Proposition 2.1]), that is, that the homomorphism $c : \overline{\Omega}_A S \to \overline{\Omega}_A S I$ factors through φ. Moreover, usually, we wish to allow δ to take its values in $(\overline{\Omega}_A S)^I$. For that purpose, we naturally extend the function φ to a continuous homomorphism $\varphi^I : (\overline{\Omega}_A S)^I \to S^I$ by letting $\varphi^I(I) = I$. It is worth noticing that this assumption does not lead to trivial solutions since the constraints must be satisfied. We allow ourselves some flexibility in these points, adopting each approach according to which is the most suitable. In the case where we consider the homomorphism φ^I, we abuse notation and denote it by φ.

Given a class $\mathcal{C}$ of finite systems of σ-equations, we say that a pseudovariety V is *σ-reducible with respect to $\mathcal{C}$* (or simply, *σ-reducible for $\mathcal{C}$*) provided a solution modulo V of a system in $\mathcal{C}$ guarantees the existence of a solution modulo V of that system given by σ-words.

An important instance of a class of systems of equations comes from graphs. Let $\Gamma = V \uplus E$ be a directed graph, where V and E are finite sets, respectively, of *vertices* and *edges*. We consider Γ equipped with two maps $\alpha : E \to V$ and $\omega : E \to V$, such that an edge $e \in E$ goes from the vertex $v_1 \in V$ to the vertex $v_2 \in V$ if and only if $\alpha(e) = v_1$ and $\omega(e) = v_2$. We may associate to each edge $e \in E$, the equation $\alpha(e)e = \omega(e)$. We denote by $\mathcal{S}(\Gamma)$ the finite system of equations obtained in this way from Γ. Whenever $\mathcal{S}$ is a finite system of this form, we say that $\mathcal{S}$ is a *system of graph equations*. We notice that any system of graph equations is of the form $\{x_i y_i = z_i\}_{i=1}^N$, where $y_i \neq y_j$ for $i \neq j$ and $y_i \notin \{x_j, z_j\}$, for all i, j.

When the constraints of the variables $e \in E$ are set to be given by the clopen subset $K_e = \{I\}$, the system $\mathcal{S}(\Gamma)$ is called a *system of pointlike equations.*

Finally, systems of the form $x_1 = \cdots = x_n = x_n^2$ are called *systems of idempotent pointlike equations.*

The pseudovariety V is said to be *σ-reducible* if it is σ-reducible for the class of finite systems of graph equations and it is *completely σ-reducible* if it is σ-reducible for the class of all finite systems of σ-equations. The following result involves the notion of reducibility.

Proposition 2.5 ([5, Proposition 10.2]). *If V is σ-reducible with respect to the equation $x = y$, then V is σ-equational.*

Since we are aiming to achieve decidability results for V, it is reasonable to require that V is recursively enumerable and that σ is *highly computable*, meaning that it is a recursively enumerable set and that all of its elements are computable operations. Henceforth, we make this assumption without further mention. Also, we should be able to decide whether two given σ-words have the same value over V, the so-called *σ-word problem.* When $\sigma = \kappa$ is the canonical implicit signature consisting of the multiplication and of the $(\omega - 1)$-power, it is possible to characterize decidability of the κ-word problem for pseudovarieties of the form DRH in terms of the same property for H.

Theorem 2.6 ([21, Chapter 3]). *Let H be a pseudovariety of groups. Then, the pseudovariety DRH has decidable κ-word problem if and only if so has H.*

We say that V is *σ-tame with respect to $\mathcal{C}$*, for a highly computable implicit signature σ, if it is σ-reducible for $\mathcal{C}$ and has decidable σ-word problem. We say that V is *σ-tame* (respectively, *completely σ-tame*) when it is σ-tame with respect to the class of all finite systems of graph equations (respectively, to the class of all finite systems of σ-equations).

We end this subsection with a list of decidability results concerning some pseudovarieties of groups, to which we refer later.

Theorem 2.7. *We have the following:*

- *the pseudovariety Ab is completely κ-tame ([10]);*
- *the pseudovariety G is κ-tame ([18] and [11, Theorem 4.9]), but it is not completely κ-reducible ([22]);*

- *for every extension closed pseudovariety of groups* H, *there exists an implicit signature* $\sigma(\mathsf{H})$ *such that* H *is* $\sigma(\mathsf{H})$-*reducible (*[4]*);*

- *no proper subpseudovariety of* G *containing a pseudovariety* G_p *(for a certain prime p) is* κ-*reducible (Proposition 2.5 and* [20]*);*

- *no proper non locally finite subpseudovariety of* Ab *is* κ-*reducible (*[23]*).*

3. Pointlike equations

Throughout this section, we shall assume that σ contains a non-explicit operation. In other words, that means that $\langle \sigma \rangle \neq \langle \{\_\cdot\_\} \rangle$. Clearly, that is the case of the canonical implicit signature κ.

In this section, we take for $\mathcal{C}$ in Question (Q) the class of all finite systems of pointlike equations. To guarantee that DRH is σ-reducible for $\mathcal{C}$, it suffices to suppose that H is σ-reducible for $\mathcal{C}$ as well.

Theorem 3.1. *Let σ be an implicit signature containing a non-explicit operation, and assume that* H *is a pseudovariety of groups that is σ-reducible for finite systems of pointlike equations. Then, the pseudovariety* DRH *is also σ-reducible for finite systems of pointlike equations.*

Proof. Let $\mathcal{S} = \{x_{k,1} = \cdots = x_{k,n_k}\}_{k=1}^N$ be a finite system of pointlike equations in the set of variables X with constraints given by the pair (φ, ν). Without loss of generality, we may assume that, for all $k, \ell \in \{1, \ldots, N\}$, with $k \neq \ell$, the subsets of variables $\{x_{k,1}, \ldots, x_{k,n_k}\}$ and $\{x_{\ell,1}, \ldots, x_{\ell,n_\ell}\}$ do not intersect. Further, with this assumption, we may also take $N = 1$. The general case is obtained by treating each system of equations $x_{k,1} = \cdots = x_{k,n_k}$ separately. Write $\mathcal{S} = \{x_1 = \cdots = x_n\}$ and suppose that the continuous homomorphism $\delta : \overline{\Omega}_X \mathsf{S} \to (\overline{\Omega}_A \mathsf{S})^I$ is a solution modulo DRH of $\mathcal{S}$. To prove that $\mathcal{S}$ also has a solution in σ-words we argue by induction on $m = |c(\delta(x_1))|$.

If $m = 0$, then $\delta(x_i) = I$ for every $i = 1, \ldots, n$ and δ is already a solution in σ-words.

Suppose that $m > 0$ and that the statement holds for every system of pointlike equations with a smaller value of the parameter. Whenever the p-th iteration of the left basic factorization of $\delta(x_i)$ is nonempty, we write $\mathsf{lbf}_p(\delta(x_i)) = \delta(x_i)_p a_{i,p}$ and we let $\delta(x_i)'_p$ be such that

$$\delta(x_i) = \mathsf{lbf}_1(\delta(x_i)) \cdots \mathsf{lbf}_p(\delta(x_i))\delta(x_i)'_p.$$

Notice that the uniqueness of left basic factorizations in $\overline{\Omega}_A\mathsf{DRH}$ entails the following properties

$$a_{1,p} = \cdots = a_{n,p};$$

(3.1)
$$\delta(x_1)_p =_{\mathsf{DRH}} \cdots =_{\mathsf{DRH}} \delta(x_n)_p;$$

$$\delta(x_1)'_p =_{\mathsf{DRH}} \cdots =_{\mathsf{DRH}} \delta(x_n)'_p.$$

If $\vec{c}(\delta(x_1)) \neq c(\delta(x_1))$, then we set $\ell = \min\{p \geq 1 : c(\delta(x_1)'_p) \subsetneqq c(\delta(x_1))\}$. Otherwise, since S is finite, there exist indices $k < \ell$ such that, for all $i = 1, \ldots, n$, we have

(3.2)
$$\varphi(\mathsf{lbf}_1(\delta(x_i)) \cdots \mathsf{lbf}_k(\delta(x_i))) = \varphi(\mathsf{lbf}_1(\delta(x_i)) \cdots \mathsf{lbf}_\ell(\delta(x_i))).$$

Let $\eta \in \langle \sigma \rangle$ be a non-explicit operation. Without loss of generality, we may assume that η is a unary operation. In particular, since S is finite, there is an integer M such that $\varphi(\eta(s)) = s^M$ for every $s \in S$. Then, equality (3.2) yields

(3.3)
$$\varphi(\delta(x_i)) = \varphi(\mathsf{lbf}_1(\delta(x_i)) \cdots \mathsf{lbf}_k(\delta(x_i)) \cdot \eta(\mathsf{lbf}_{k+1}(\delta(x_i)) \cdots \mathsf{lbf}_\ell(\delta(x_i)))\delta(x_i)'_k).$$

Now, consider a new set of variables $X' = \{x_{i,p}, x'_i : i = 1, \ldots, n; \ p = 1, \ldots, \ell\}$ and a new system of pointlike equations

(3.4)
$$\mathcal{S}' = \begin{cases} \{x_{1,p} = \cdots = x_{n,p}\}^\ell_{p=1} \cup \{x'_1 = \cdots = x'_n\}, & \text{if } \vec{c}(\delta(x_1)) \neq c(\delta(x_1)) \\ \{x_{1,p} = \cdots = x_{n,p}\}^\ell_{p=1}, & \text{if } \vec{c}(\delta(x_1)) = c(\delta(x_1)) \end{cases}$$

By (3.1), the continuous homomorphism $\delta' : \overline{\Omega}_{X'}\mathsf{S} \to (\overline{\Omega}_A\mathsf{S})^I$ assigning $\delta(x_i)_p$ to each variable $x_{i,p}$ and $\delta(x_i)'_\ell$ to each variable x'_i is a solution modulo DRH of $\mathcal{S}'$, with constraints given by (φ, ν'), where $\nu'(x_{i,p}) = \varphi(\delta(x_i)_p)$, and $\nu'(x'_i) = \varphi(\delta(x_i)'_k)$ $(i = 1, \ldots, n$ and $p = 1, \ldots, \ell)$. Moreover, whatever is the system $\mathcal{S}'$ considered in (3.4), we decreased the induction parameter. By induction hypothesis, there exists a solution modulo DRH of $\mathcal{S}'$ in σ-words, say ε', keeping the values of the variables under φ. We distinguish between the case where $\vec{c}(\delta(x_1)) \neq c(\delta(x_1))$ and the case where $\vec{c}(\delta(x_1)) = c(\delta(x_1))$. In the former, it is easy to check that the continuous homomorphism

$$\varepsilon : \overline{\Omega}_X\mathsf{S} \to (\overline{\Omega}_A\mathsf{S})^I$$

$$x_i \mapsto \varepsilon'(x_{i,1})a_{i,1} \cdots \varepsilon'(x_{i,\ell})a_{i,\ell}\varepsilon'(x'_i)$$

is a solution modulo DRH of $\mathcal{S}$. In the latter case, we consider the system of pointlike equations

$$\mathcal{S}_0 = \{x'_1 = \cdots = x'_n\}.$$

From (3.1), it follows that δ' is a solution modulo H of $\mathcal{S}_0$. As we are taking for H a pseudovariety that is σ-reducible for systems of pointlike equations, there exists a solution modulo H of $\mathcal{S}_0$, say ε'', keeping the values of the variables under φ. Let $\varepsilon : \overline{\Omega}_X \mathsf{S} \to (\overline{\Omega}_A \mathsf{S})^I$ be given by

$$\varepsilon(x_i) = \varepsilon'(x_{i,1})a_{i,1} \cdots \varepsilon'(x_{i,k})a_{i,k} \cdot \eta(\varepsilon'(x_{i,k+1})a_{i,k+1} \cdots \varepsilon'(x_{i,\ell})a_{i,\ell})\varepsilon''(x'_i).$$

Since ε' is a solution modulo DRH of $\mathcal{S}'$, η is non-explicit, and we are assuming that the semigroup S has a content function, it follows that, for all $i, j \in \{1, \ldots, n\}$, the pseudowords $\varepsilon(x_i)$ and $\varepsilon(x_j)$ are $\mathcal{R}$-equivalent modulo DRH. On the other hand, for all $i, j \in \{1, \ldots, n\}$, the following equalities are valid in H:

$$\begin{aligned}
\varepsilon(x_i) &= \varepsilon'(x_{i,1})a_{i,1} \cdots \varepsilon'(x_{i,k})a_{i,k} \cdot \eta(\varepsilon'(x_{i,k+1})a_{i,k+1} \cdots \varepsilon'(x_{i,\ell})a_{i,\ell})\varepsilon''(x'_i) \\
&= \varepsilon'(x_{j,1})a_{j,1} \cdots \varepsilon'(x_{j,k})a_{j,k} \cdot \eta(\varepsilon'(x_{j,k+1})a_{j,k+1} \cdots \varepsilon'(x_{j,\ell})a_{j,\ell})\varepsilon''(x'_j) \\
&= \varepsilon(x_j).
\end{aligned}$$

The second equality holds because ε' and ε'' are solutions modulo H of $\mathcal{S}'$ and $\mathcal{S}_0$, respectively. Therefore, Lemma 2.3 yields that DRH satisfies $\varepsilon(x_i) = \varepsilon(x_j)$. It remains to verify that the given constraints are still satisfied. But that is straightforwardly implied by (3.3). $\square$

Remark 3.2. We observe that the construction performed in the proof of the previous theorem not only gives a solution modulo DRH in σ-terms of the original pointlike system of equations, but it also provides a solution keeping the cumulative content of each variable.

As a consequence of Propositions 2.5 and Theorem 3.1, we have the following.

Corollary 3.3. *If a pseudovariety of groups* H *is* σ*-reducible with respect to the equation* $x = y$, *then* DRH *is* σ*-equational.* $\square$

As far as we are aware, all known examples of pseudovarieties of groups that are σ-reducible with respect to systems of pointlike equations are also σ-reducible. For that reason, for now, we skip such examples, since they illustrate stronger results in the next section. We just point out the case of

the pseudovariety Ab (recall Theorem 2.7). It is interesting to observe that, although $\overline{\mathsf{Ab}}$ is not a κ-equational pseudovariety [14, Theorem 3.1], by Corollary 3.3 the pseudovariety $\mathsf{DRAb} = \mathsf{DRG} \cap \overline{\mathsf{Ab}}$ is.

On the other hand, taking into account Theorem 2.6, we also have the following.

Corollary 3.4. *If* H *is a pseudovariety of groups that is κ-tame with respect to finite systems of pointlike equations, then so is* DRH. □

4. GRAPH EQUATIONS

With the aim of proving tameness, we now let $\mathcal{C}$ be the class of all systems of graph equations. We prove that, for an implicit signature σ containing a non-explicit operation, if H is a σ-reducible pseudovariety of groups, then so is DRH. To this end, we drew inspiration from [7]. Moreover, we assert the converse statement, which holds for every σ.

Henceforth, we fix a finite graph $\Gamma = V \uplus E$ and a solution $\delta : \overline{\Omega}_\Gamma \mathsf{S} \to (\overline{\Omega}_A \mathsf{S})^I$ modulo DRH of $\mathcal{S}(\Gamma)$ such that, for every $x \in \Gamma$ the pseudoword $\delta(x)$ belongs to the clopen subset K_x of $(\overline{\Omega}_A \mathsf{S})^I$. We further denote by 1 the identity element of $\overline{\Omega}_A \mathsf{H}$.

Let y be an edge of Γ, and let $x = \alpha(y)$ and $z = \omega(y)$. If $c(\delta(y)) \nsubseteq \vec{c}(\delta(x))$ then, by Lemma 2.1, we have unique factorizations $\delta(y) = u_y a v_y$ and $\delta(z) = u_z a v_z$ such that $c(u_y) \subseteq \vec{c}(\delta(x))$, $a \notin \vec{c}(\delta(x))$ and the pseudovariety DRH satisfies both $\delta(x)u_y = u_z$ and $v_y = v_z$. We refer to these factorizations as *direct* DRH-*splittings associated with the edge* y and we say that a is the corresponding *marker*. We call *direct* DRH-*splitting points* the triples (u_y, a, v_y) and (u_z, a, v_z).

The first remark spells out the relationship between the notion of a DRH-splitting factorization defined above and the notion of a splitting factorization in the context of [7] (in [7], a splitting factorization is defined as being an R-splitting factorization). It is a consequence of Lemma 2.1 applied to the pseudovariety DRH and to the pseudovariety R.

Remark 4.1. Let $y \in E$ be such that $c(\delta(y)) \nsubseteq \vec{c}(\delta(\alpha(y)))$. Consider factorizations $\delta(y) = u_y a v_y$ and $\delta(\omega(y)) = u_z a v_z$, with $c(u_y) \subseteq \vec{c}(\delta(\alpha(y)))$ and $a \notin \vec{c}(\delta(\alpha(y)))$, such that DRH satisfies $\delta(\alpha(y))u_y = u_z$, as above. Then, these factorizations are direct R-splittings (note that δ is also a solution modulo R of $\mathcal{S}(\Gamma)$ and so, it makes sense to refer to R-splitting factorizations) if and only if they are direct DRH-splittings.

We also define the *indirect* DRH-*splitting points* as follows. Let $t \in \Gamma$ and suppose that we have a factorization $\delta(t) = u_t a v_t$, with $a \notin \vec{c}(u_t)$. Then, one of the following three situations may occur.

- If there is an edge $y \in E$ such that $\alpha(y) = t$ and $\omega(y) = z$, then there is also a factorization $\delta(z) = u_z a v_z$ with DRH satisfying $u_t = u_z$ and $v_t \delta(y) = v_z$. In fact, this is a consequence of the pseudoidentity $\delta(t)\delta(y) = \delta(z)$ modulo DRH, which holds for every edge $t \xrightarrow{y} z$ in Γ.

- Similarly, if there is an edge $y \in E$ such that $\alpha(y) = x$ and $\omega(y) = t$ (and so, DRH satisfies $\delta(x)\delta(y) = \delta(t)$), then the factorization of $\delta(t)$ yields either a factorization $\delta(x) = u_x a v_x$ such that DRH satisfies $u_x = u_t$ and $v_x \delta(y) = v_t$, or a factorization $\delta(y) = u_y a v_y$ such that DRH satisfies $\delta(x)u_y = u_t$ and $v_y = v_t$.

- On the other hand, if t is itself an edge, say $\alpha(t) = x$ and $\omega(t) = z$, and if $\delta(x)u_t a$ is an end-marked pseudoword, then the factorization of $\delta(t)$ determines a factorization $\delta(z) = u_z a v_z$, such that DRH satisfies $\delta(x)u_t = u_z$ and $v_z = v_t$.

These considerations make clear the possible propagation of the DRH-direct splitting points. If the mentioned factorization of $\delta(t)$ comes from a DRH-(in)direct splitting factorization obtained through the successive factorization of the values of edges and vertices under δ in the way described above, then we say that each of the triples (u_x, a, x_x), (u_y, a, v_y), and (u_z, a, v_z) is an indirect DRH-splitting point *induced by the (in)direct* DRH-*splitting point* (u_t, a, v_t).

Yet again, we obtain a nice relationship between the indirect DRH-splitting points just defined and the indirect splitting points introduced in [7] (which are the indirect R-splitting points). The reason is precisely the same as in Remark 4.1, together with the definition of indirect splitting points.

Remark 4.2. Let $t_0 \in \Gamma$ and $\delta(t_0) = u_0 a v_0$ be a direct R-splitting factorization and consider $\{(u_i, a, v_i)\}_{i=1}^n \subseteq (\overline{\Omega}_A \mathsf{S})^I \times A \times (\overline{\Omega}_A \mathsf{S})^I$. Then, the following are equivalent:

(a) (u_i, a, v_i) is an indirect R-splitting point induced by (u_{i-1}, a, v_{i-1}), for $i = 1, \ldots, n$;

(b) (u_i, a, v_i) is an indirect DRH-splitting point induced by (u_{i-1}, a, v_{i-1}), for $i = 1, \ldots, n$.

The following lemma ensures that a direct R-splitting point does not propagate infinitely many times.

Lemma 4.3 ([7, Lemma 5.14]). *Given a solution δ over* R *of a system of graph equations, there is only a finite number of splitting points in the values of variables under δ.*

As an immediate consequence of Lemma 4.3 and of the relationship between (in)direct R-splitting points and (in)direct DRH-splitting points made explicit in Remarks 4.1 and 4.2, we have the following:

Corollary 4.4. *Given a solution δ over* DRH *of a system of graph equations, there is only a finite number of splitting points in the values of variables under δ.* $\square$

Taking into account Remarks 4.1 and 4.2, from now on we say (in)direct splitting point (respectively, factorization) instead of (in)direct DRH-splitting point (respectively, factorization).

Let Γ be a finite graph and consider the system of equations $\mathcal{S}(\Gamma)$. For each variable $x \in \Gamma$, let $\{(u_{x,i}, a_{x,i}, v_{x,i})\}_{i=1}^{m_x}$ be the (finite) set of splitting points of $\delta(x)$. By definition, each pseudoword $u_{x,i}a_{x,i}$ is an end-marked prefix of $\delta(x)$. By Proposition 2.2, we may assume, without loss of generality, the following relations:

$$u_{x,1}a_{x,1} >_{\mathcal{R}} u_{x,2}a_{x,2} >_{\mathcal{R}} \cdots >_{\mathcal{R}} u_{x,m_x}a_{x,m_x} >_{\mathcal{R}} \delta(x).$$

Hence, by Lemma 2.1, we have a reduced factorization

(4.1) $$\delta(x) = \delta(x)_0 \cdot \delta(x)_1 \cdots \delta(x)_{m_x},$$

such that $\delta(x)_0 \cdots \delta(x)_{i-1} = u_{x,i}$, for $i = 1, \ldots, m_x$, induced by the splitting points of $\delta(x)$. If $x \in V$, then we write the reduced factorization in (4.1) as $\delta(x) = w_{x,1} \cdot w_{x,2} \cdots w_{x,n_x}$ and, if $y \in E$, then we write that factorization as $\delta(y) = w_{y,0}w_{y,1} \cdots w_{y,n_y}$. Observe that, for $x \in V$, we have $n_x = m_x + 1$, while for $y \in E$, we have $n_y = m_y$. Although this notation may not seem coherent, it is justified by property (c) of Lemma 4.5.

Lemma 4.5. *Let $xy = z$ be an equation of $\mathcal{S}(\Gamma)$. Using the above notation, the following holds:*

(a) $n_x + n_y = n_z$;

(b) DRH *satisfies* $\begin{cases} w_{x,k} = w_{z,k}, & \text{for } k = 1, \ldots, n_x - 1; \\ w_{x,n_x}w_{y,0} = w_{z,n_x}; \\ w_{y,k} = w_{z,n_x+k}, & \text{for } k = 1, \ldots, n_y; \end{cases}$

(c) $c(w_{y,0}) \subseteq \vec{c}(w_{x,n_x})$;

(d) *each of the following products is reduced:*

$$w_{x,k} \cdot w_{x,k+1} \ (k = 1, \ldots, n_x - 1);$$

$$(w_{x,n_x}w_{y,0}) \cdot w_{y,1};$$

$$w_{z,k} \cdot w_{z,k+1} \ (k = 1, \ldots, n_z - 1).$$

Proof. As we already observed, the number of splitting points of $\delta(z)$ is $m_z = n_z - 1$. We distinguish between two situations.

- If $c(\delta(y)) \nsubseteq \vec{c}(\delta(x))$, then there are two direct splitting factorizations given by $\delta(y) = u_y a v_y$ and $\delta(z) = u_z a v_z$. So, by definition, the inclusion $c(u_y) \subseteq \vec{c}(\delta(x))$ holds. We notice that any other splitting point of $\delta(y)$, say (u'_y, b, v'_y), is necessarily induced by a splitting point of $\delta(z)$, say (u'_z, b, v'_z). Moreover, since the product $(\delta(x)u'_y) \cdot bv'_y$ is reduced (because so is $u'_z \cdot (bv'_z)$ and DRH satisfies $\delta(x)u'_y = u'_z$), the pseudoword u_y is a prefix of u'_y. On the other hand, the set of all splitting points of $\delta(z)$ induces a factorization of the pseudoword $\delta(x)\delta(y)$, say

$$(4.2) \qquad \delta(x)\delta(y) = w'_1 \cdot w'_2 \cdots w'_{n_z}.$$

  Of course, for each $k = 1, \ldots, n_x - 1$, the prefix $w'_1 \cdots w'_k$ of $\delta(x)\delta(y)$ corresponds to the first component of one of the splitting points of $\delta(x)$ (which is either induced by one of the splitting points of $\delta(z)$ or it induces a splitting point of $\delta(z)$). More specifically, the pseudoidentity $w_{z,k} = w'_k = w_{x,k}$ is valid in DRH. From the observation above, we also know that the first components of the indirect splitting points of $\delta(y)$ have u_y as a prefix. Therefore, we have $u_y = w_{y,0}$, the factor $w'_{n_x} = w_{z,n_x}$ coincides with $w_{x,n_x} w_{y,0}$ modulo DRH, and $c(w_{y,0}) = c(u_y) \subseteq \vec{c}(\delta(x)) = \vec{c}(w_{x,n_x})$. It also follows that $w'_{n_x+k} = w_{z,n_x+k} = w_{y,k}$ modulo DRH, for $k = 1, \ldots, n_y$. We just proved (b), (c) and (d). Finally, part (a) results from counting the involved factors in both sides of (4.2).

- If $c(\delta(y)) \subseteq \vec{c}(\delta(x))$, then $\delta(y)$ has no direct splitting points. As y is an edge, an indirect splitting point of $\delta(y)$ must be induced by some splitting point of $\delta(z)$. Suppose that (u_z, a, v_z) is a splitting point of $\delta(z)$ that induces a splitting point in $\delta(y)$, say (u_y, a, v_y). Then, we would have a reduced product $(\delta(x)u_y) \cdot (av_y)$, which contradicts the assumption $c(\delta(y)) \subseteq \vec{c}(\delta(x))$. Therefore, the pseudoword $\delta(y)$ has no splitting points at all. With the same kind of argument as the one above, we may derive the claims (a)–(d). $\qquad\square$

Now, write $\mathcal{S}(\Gamma) = \{x_i y_i = z_i\}_{i=1}^N$. Note that $y_j \notin \{x_i, z_i\}$ for all i, j. We let $\mathcal{S}_1$ be the system of equations containing, for each $i = 1, \ldots, N$, the following set of equations:

$$(x_i)_k = (z_i)_k, \ \text{ for } k = 1, \ldots, n_{x_i} - 1;$$

(4.3)
$$(x_i)_{n_{x_i}} y_{i,0} = (z_i)_{n_{x_i}};$$

$$y_{i,k} = (z_i)_{n_{x_i}+k}, \ \text{ for } k = 1, \ldots, n_{y_i}.$$

In the system $\mathcal{S}_1$, we are assuming that $(x_i)_k$ and $(x_j)_k$ (respectively, and $(z_j)_k$) represent the same variable whenever so do x_i and x_j (respectively, and z_j). By Lemma 4.5, it is clear that each solution modulo DRH of $\mathcal{S}_1$ yields a solution modulo DRH of $\mathcal{S}(\Gamma)$ and conversely. We next prove that, for a σ-reducible pseudovariety of groups H, if $\mathcal{S}_1$ has a solution modulo DRH, then it has a solution modulo DRH given by σ-words, thus concluding that the same happens with $\mathcal{S}(\Gamma)$. Before that, we establish the following.

Proposition 4.6. *Let σ be an implicit signature that contains a non-explicit operation. Let H be a σ-reducible pseudovariety of groups and $\Gamma = V \uplus E$ be a finite graph. Suppose that there exists a solution $\delta : \overline{\Omega}_\Gamma \mathsf{S} \to (\overline{\Omega}_A \mathsf{S})^I$ modulo DRH of $\mathcal{S}(\Gamma)$ such that:*

(a) $\vec{c}(\delta(x)) \neq \emptyset$, for every vertex $x \in V$;

(b) $c(\delta(y)) \subseteq \vec{c}(\delta(\alpha(y)))$, for every edge $y \in E$.

Then, $\mathcal{S}(\Gamma)$ has a solution modulo DRH in σ-words, say ε, such that $\varphi(\varepsilon(x)) = \varphi(\delta(x))$, for all $x \in \Gamma$.

Proof. Without loss of generality, we may assume that Γ has only one connected component (when disregarding the directions of the arrows). Otherwise, we may treat each component separately. Because of the hypothesis (b), the pseudowords $\delta(\alpha(y))$ and $\delta(\omega(y))$ are $\mathcal{R}$-equivalent modulo DRH for every edge $y \in E$. Since we are assuming that all vertices of Γ are in the same connected component, it follows that for all $x, z \in V$, the pseudowords $\delta(x)$ and $\delta(z)$ are $\mathcal{R}$-equivalent modulo DRH. Fix a variable $x_0 \in V$ and let u_0 be an accumulation point of $(\mathsf{lbf}_1(\delta(x_0)) \cdots \mathsf{lbf}_m(\delta(x_0)))_{m \geq 1}$ in $\overline{\Omega}_A \mathsf{S}$. Since, in DRH, the pseudowords u_0 and $\delta(x_0)$ are $\mathcal{R}$-equivalent, for each $x \in V$ there is a factorization $\delta(x) = u_x v_x$ (with v_x possibly empty) such that $c(v_x) \subseteq \vec{c}(u_x)$ and $u_x =_{\mathsf{DRH}} u_0$.

Consider the set $\widehat{V} = \{\widehat{x} \colon x \in V\}$ with $|V|$ distinct variables, disjoint from Γ, the system of equations $\mathcal{S}_0 = \{\widehat{x} = \widehat{z} \colon x, z \in V\}$ with variables in $\widehat{V}$, and let

$$\delta_0 : \overline{\Omega}_{\widehat{V} \uplus \Gamma}\mathsf{S} \to (\overline{\Omega}_A\mathsf{S})^I$$

$$\widehat{x} \mapsto u_x, \quad \text{if } \widehat{x} \in \widehat{V};$$

$$x \mapsto v_x, \quad \text{if } x \in V;$$

$$y \mapsto \delta(y), \quad \text{otherwise.}$$

By construction, the homomorphism δ_0 is a solution modulo DRH of $\mathcal{S}_0$ which is also a solution modulo H of $\mathcal{S}(\Gamma)$. Hence, on the one hand, Theorem 3.1 together with Remark 3.2 yield a solution $\varepsilon_0 : \overline{\Omega}_{\widehat{V}}\mathsf{S} \to \overline{\Omega}_A\mathsf{S}$ modulo DRH in σ-words of $\mathcal{S}_0$ such that

$$\varphi(\varepsilon_0(\widehat{x})) = \varphi(\delta_0(\widehat{x})) = \varphi(u_x),$$

$$\vec{c}(\varepsilon_0(\widehat{x})) = \vec{c}(\delta_0(\widehat{x})),$$

for every $\widehat{x} \in \widehat{V}$. On the other hand, the fact that H is σ-reducible implies that there is a solution $\varepsilon' : \overline{\Omega}_\Gamma\mathsf{S} \to (\overline{\Omega}_A\mathsf{S})^I$ modulo H of $\mathcal{S}(\Gamma)$ given by σ-words satisfying

$$\varphi(\varepsilon'(x)) = \varphi(\delta_0(x)),$$

for every $x \in \Gamma$. Thus, we take $\varepsilon : \overline{\Omega}_\Gamma\mathsf{S} \to (\overline{\Omega}_A\mathsf{S})^I$ to be the continuous homomorphism defined by $\varepsilon(x) = \varepsilon_0(\widehat{x})\varepsilon'(x)$ if $x \in V$, and $\varepsilon(y) = \varepsilon'(y)$ otherwise. Taking into account that S has a content function, we may use Lemma 2.3 to deduce that ε is a solution modulo DRH of $\mathcal{S}(\Gamma)$ in σ-words. It is easy to check that the constraints for the variables of Γ are also satisfied. Therefore, ε is the required homomorphism. $\qquad\square$

Lemma 4.7. *Let $\mathcal{S}_1$ be the system of equations* (4.3) *in the set of variables X_1 and let $\delta_1 : \overline{\Omega}_{X_1}\mathsf{S} \to (\overline{\Omega}_A\mathsf{S})^I$ be its solution modulo DRH. Suppose that the implicit signature σ contains a non-explicit operation. If the pseudovariety H is σ-reducible, then $\mathcal{S}_1$ has a solution modulo DRH in σ-words.*

Proof. Analyzing the equations in (4.3), we easily conclude that there are no variables occurring simultaneously in the first row and in one of the other two rows. Therefore, the system $\mathcal{S}_1$ can be thought as a system of pointlike equations $\mathcal{S}_2$ together with a system of graph equations $\mathcal{S}_3$ such that the condition (b) of Proposition 4.6 holds and none of the variables occurring in $\mathcal{S}_2$ occurs in

S_3. Note that we are also including in S_2 the equations in the last two rows of (4.3) such that the cumulative content of the value of the involved variables under δ_1 is empty.

By Theorem 3.1 the system S_2 has a solution modulo DRH in σ-words, while by Proposition 4.6 the system S_3 has a solution modulo DRH in σ-words. Therefore, the intended solution for S_1 also exists. □

We just proved the announced result.

Theorem 4.8. *When σ is an implicit signature containing a non-explicit operation, the pseudovariety* DRH *is σ-reducible if so is* H. □

We recall that, by Theorem 2.7, for every nontrivial extension closed pseudovariety of groups H, there is an implicit signature $\sigma(H) \supseteq \kappa$ that turns H into a $\sigma(H)$-reducible pseudovariety. For instance, G_p and G_{sol} are both extension closed. Thus, DRG_p and DRG_{sol} are both σ-reducible for suitable signatures σ.

The combination of Theorems 2.6 and 4.8 yields the next result.

Corollary 4.9. *Let* H *be a pseudovariety of groups. If* H *is κ-tame, then so is* DRH.

We further prove that the converse of Theorem 4.8 also holds.

Proposition 4.10. *Let* H *be a pseudovariety of groups such that the pseudovariety* DRH *is σ-reducible. Then, the pseudovariety* H *is also σ-reducible.*

Proof. Let $\Gamma = V \uplus E$ be a graph such that $S(\Gamma)$ admits $\delta : \overline{\Omega}_\Gamma S \to (\overline{\Omega}_A S)^I$ as a solution modulo H. We consider a new graph $\widehat{\Gamma} = \widehat{V} \uplus \widehat{E}$, where $\widehat{V} = \{\widehat{v} \colon v \in V\} \uplus \{v_0\}$ and $\widehat{E} = V \uplus E$. The functions α and ω of $\widehat{\Gamma}$ are given by $\alpha(v) = v_0$ and $\omega(v) = \widehat{v}$, for all $v \in V$ and by $\alpha(e) = \widehat{v}_1$ and $\omega(e) = \widehat{v}_2$ whenever $e \in E$ and $(\alpha(e), \omega(e)) = (v_1, v_2)$. The relationship between the graphs Γ and $\widehat{\Gamma}$ is depicted in Figure 1. Let $u \in \overline{\Omega}_A S$ be a regular pseudoword modulo DRH such that
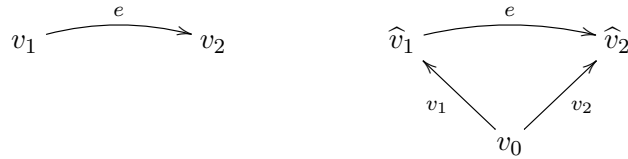

FIGURE 1. On the left, an edge of Γ; on the right, the corresponding edges of $\widehat{\Gamma}$.

$c(\delta(x)) \subseteq \vec{c}(u)$ for all $x \in \Gamma$. We take $\delta' : \overline{\Omega}_{\widehat{\Gamma}} S \to (\overline{\Omega}_A S)^I$ to be the continuous homomorphism

defined by $\delta'(e) = \delta(e)$, if $e \in E$; $\delta'(v) = \delta(v)$ and $\delta'(\widehat{v}) = u\delta(v)$, if $v \in V$; and $\delta'(v_0) = u$. Then, Lemma 2.3 combined with the fact that δ is a solution modulo H of $\mathcal{S}(\Gamma)$ imply that δ' is a solution modulo DRH of $\mathcal{S}(\widehat{\Gamma})$. Thus, since DRH is σ-reducible, there exists a solution in σ-words $\varepsilon : \overline{\Omega}_{\widehat{\Gamma}}\mathsf{S} \to (\overline{\Omega}_A\mathsf{S})^I$ modulo DRH of $\mathcal{S}(\widehat{\Gamma})$. In particular, for each edge $e \in E$ such that $\alpha(e) = v_1$ and $\omega(e) = v_2$, we have that $v_0 v_1 = \widehat{v}_1$, $\widehat{v}_1 e = \widehat{v}_2$, and $v_0 v_2 = \widehat{v}_2$ are equations of $\mathcal{S}(\widehat{\Gamma})$. Therefore, the equalities $\varepsilon(v_0 v_1 e) = \varepsilon(\widehat{v}_1 e) = \varepsilon(\widehat{v}_2) = \varepsilon(v_0 v_2)$ hold in DRH. Hence, H satisfies $\varepsilon(v_1 e) = \varepsilon(v_2)$ and so, we may conclude that the restriction of ε to $\overline{\Omega}_\Gamma\mathsf{S}$ is a solution in σ-words modulo H of $\mathcal{S}(\Gamma)$. $\square$

Combined with Proposition 4.10, the results in the literature supply a family of pseudovarieties DRH that are not κ-reducible. Namely, DRG_p and DRH for every proper non locally finite subpseudovariety H of Ab (recall Theorem 2.7).

5. IDEMPOTENT POINTLIKE EQUATIONS

We now take for $\mathcal{C}$ the class of all systems of idempotent pointlike equations. In the two preceding situations, the answers to Question (Q) were of the form "it is enough to assume that H is σ-reducible with respect to $\mathcal{C}$". When considering systems of idempotent pointlike equations, we have been unable to give such an answer. However, we prove that assuming σ-reducibility of H with respect to a still "satisfactory" class of systems serves our purpose. More precisely, we prove that, for an implicit signature σ satisfying certain conditions, if H is a σ-reducible pseudovariety of groups, then DRH is σ-reducible with respect to systems of idempotent pointlike equations.

In order to make the expression "reducible for systems of graph equations" more embracing, we first introduce a definition.

Definition 5.1. Let V be a pseudovariety and $\mathcal{S}$ a finite system of equations in the set of variables X with certain constraints. We say that $\mathcal{S}$ is V-*equivalent to a system of graph equations* if there exists a graph Γ such that $X \subseteq \Gamma$ and such that every solution modulo V of $\mathcal{S}$ can be extended to a solution modulo V of $\mathcal{S}(\Gamma)$ (the constraints for the variables of $X \subseteq \Gamma$ are those given by the system $\mathcal{S}$). Moreover, whenever δ is a solution modulo V of $\mathcal{S}(\Gamma)$, the restriction $\delta|_{\overline{\Omega}_X\mathsf{S}}$ is a solution modulo V of $\mathcal{S}$. Each graph Γ with that property is said to be an $\mathcal{S}$-*graph* and we say that $\mathcal{S}$ is V-*equivalent to* $\mathcal{S}(\Gamma)$ for every $\mathcal{S}$-graph Γ.

It is immediate from the definition that any σ-reducible pseudovariety V is σ-reducible for systems of equations that are V-equivalent to a system of graph equations. In the next few results we exhibit

some systems of equations that are H-equivalent to a system of graph equations (for a pseudovariety of groups H). Instead of giving complete proofs, we identify on each situation what graph should be considered and leave the details to the reader.

Lemma 5.2. *Let* $\mathcal{S} = \{x_1 w_1 \cdots x_n w_n x_{n+1} = 1\}$ *be a system consisting of a single equation, where* x_i *is a variable with* $x_i \neq x_j$ *whenever* $i \neq j$, $\{w_i\}_{i=1}^n \subseteq A^*$, *and the constraint of the variable* x_i *is given by the clopen subset* $K_i \subseteq (\overline{\Omega}_A \mathsf{S})^I$. *Then, for every pseudovariety of groups* H, *the system* $\mathcal{S}$ *is* H-*equivalent to a system of graph equations.*

Proof. Let $\Gamma = V \uplus E$ be the finite graph with the sets of vertices and edges given by $V = \{y_i, z_i \colon i = 1, \ldots, n+1\}$ and $E = \{x_0\} \uplus \{x_i \colon i = 1, \ldots, n+1\} \uplus \{w_i \colon i = 1, \ldots, n\}$, respectively. To define the mappings α and ω, we take

$$(\alpha(x_0), \omega(x_0)) = (y_1, z_{n+1});$$

$$(\alpha(x_i), \omega(x_i)) = (y_i, z_i), \quad \text{for } i = 1, \ldots, n+1;$$

$$(\alpha(w_i), \omega(w_i)) = (z_i, y_{i+1}), \quad \text{for } i = 1, \ldots, n ;$$

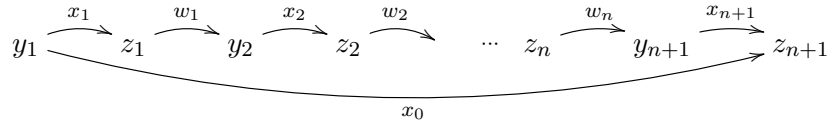as shown in Figure 2. Let us denote by K_x the clopen set that defines the constraint of $x \in \Gamma$. We

FIGURE 2. The graph Γ.

set $K_{x_i} = K_i$, $K_{w_i} = \{w_i\}$, $K_{x_0} = \{I\}$, and $K_{y_i} = (\overline{\Omega}_A \mathsf{S})^I = K_{z_i}$ (for every i such that each of the variables is defined). Then, Γ is an $\mathcal{S}$-graph. $\qquad \square$

Lemma 5.3. *Let* H *be a pseudovariety of groups. If* $\mathcal{S}$ *is* H-*equivalent to a system of graph equations,* x *is a variable occurring in* $\mathcal{S}$, *and* $\mathcal{S}_0 = \{x = x_1 = \cdots = x_n\}$, *where* $x_1, \ldots, x_n$ *are new variables, then* $\mathcal{S} \cup \mathcal{S}_0$ *is also* H-*equivalent to a system of graph equations.*

Proof. Let $\Gamma = V \uplus E$ be an $\mathcal{S}$-graph. We construct a new graph Γ' as follows. If $x \in V$, then we consider new variables $x_0, x_1, \ldots, x_n$ and Γ' is obtained by adding to Γ the edges represented in Figure 3 on the left. Otherwise, if $x \in E$, then n new edges are added to Γ as depicted in Figure 3 on the right, resulting the graph Γ'. We do not explicit the constraints on the new variables, since

FIGURE 3. The piece of the graph Γ' where it differs from Γ, when $x \in V$ (left) and when $x \in E$ (right).

they may be taken to be given by the clopen set $\overline{\Omega}_A \mathsf{S}$. In any case, it is a routine matter to verify that the system $\mathcal{S} \cup \mathcal{S}_0$ is H-equivalent to $\mathcal{S}(\Gamma')$. $\qquad \square$

Lemma 5.4. *Let* H *be a pseudovariety of groups,* $\mathcal{S}$ *be a system of equations with variables in X that is* H*-equivalent to a system of graph equations and* $\mathcal{S}_0 = \{x = x_1 w_1 \cdots x_n w_n x_{n+1}\}$*, where $x \in X$, $x_1, \ldots, x_n$ are new variables, x_{n+1} is either the empty word or a new variable, and $\{w_i\}_{i=1}^n \subseteq A^*$. Then, $\mathcal{S} \cup \mathcal{S}_0$ is also* H*-equivalent to a system of graph equations.*

Proof. We start by observing that it really does not matter whether x_{n+1} is the empty word or a new variable. Indeed, if it is the empty word, then we just need to set a constraint $K_{x_{n+1}} = \{I\}$ for x_{n+1} and we may treat it as a variable.

Let $\Gamma = V \uplus E$ be an $\mathcal{S}$-graph. We construct a new graph Γ' depending on whether x is a vertex or an edge. If x is a vertex, then we add to Γ a new path going from a new vertex y_1 to x, whose edges are labeled by $x_1, w_1, \ldots, x_n, w_n, x_{n+1}$ in this order, as depicted in Figure 4. We further take
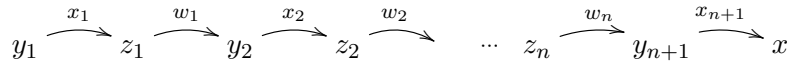

FIGURE 4. The new path in Γ if x is a vertex.

$K_{y_1} = \{I\}$, $K_{y_{i+1}} = (\overline{\Omega}_A \mathsf{S})^I = K_{z_i}$, and $K_{w_i} = \{w_i\}$ as the clopen sets defining the constraints for the new variables y_{i+1}, z_i, and w_i, respectively ($i = 1, \ldots, n$). On the other hand, when x is an edge, we simply obtain Γ' by adding a path in Γ from $\alpha(x)$ to $\omega(x)$ with edges labeled by $x_1, w_1, \ldots, x_n, w_n, x_{n+1}$ (see Figure 5). We leave it to the reader to verify that, in both situations,
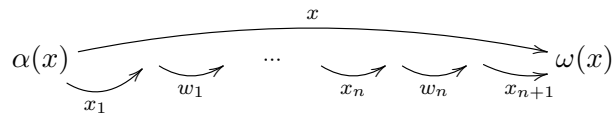

FIGURE 5. The added path to Γ if x is an edge.

$\mathcal{S} \cup \mathcal{S}_0$ is H-equivalent to $\mathcal{S}(\Gamma')$. $\qquad \square$

Corollary 5.5. *Let* H *be a pseudovariety of groups and let* S *be a system of equations* H*-equivalent to a system of graph equations and suppose that* $x_1, \ldots, x_N$ *are variables occurring in* S. *Also, suppose that the variables* $y_{i,1}, \ldots, y_{i,k_i}$ *and* $z_{i,1}, \ldots, z_{i,n_i}$ $(i = 1, \ldots, N)$ *are all distinct and do not occur in* S, *and let* $\{w_{i,p} : i = 1, \ldots, N; \ p = 1, \ldots, k_i\} \subseteq A^*$. *We make each* t_i *be either the empty word or another different variable. Then, the system of equations*

$$S' = S \cup \{x_i = y_{i,1}w_{i,1} \cdots y_{i,k_i}w_{i,k_i}t_i\}_{i=1}^N$$

$$\cup \{t_i = z_{i,1} = \cdots = z_{i,n_i} : i = 1, \ldots, N \ and \ t_i \ is \ not \ the \ empty \ word\}$$

is also H*-equivalent to a system of graph equations.*

Proof. The result follows immediately by successively applying Lemmas 5.3 and 5.4. □

The next statement consists of a general scenario that is instrumental for establishing the claimed answer to Question (Q) mentioned in the beginning of this section.

Proposition 5.6. *Let* H *be a* σ-*reducible pseudovariety of groups, where* σ *is an implicit signature such that* $\langle \sigma \rangle$ *contains a non-explicit operation* η *such that* $\eta = 1$ *in* H. *Let* S_1 *and* S_2 *be finite systems of equations, such that* S_1 *contains only pointlike equations, and both* $S_1 \cup S_2$ *and* S_2 *are* H-*equivalent to systems of graph equations. Further assume that, if* X *is the set of variables occurring in* $S_1 \cup S_2$, *then the constraint on each variable* $x \in X$ *is given by a clopen subset* $K_x \subseteq (\overline{\Omega}_A S)^I$. *Then, the existence of a continuous homomorphism that is simultaneously a solution modulo* DRH *of* S_1 *and a solution modulo* H *of* S_2 *entails the existence of a continuous homomorphism in* σ-*words with the same property.*

Proof. Without loss of generality, we assume that η is a unary implicit operation. Let $S_1 = \{x_{i,1} = \cdots = x_{i,n_i}\}_{i=1}^N$, with $x_{i,p} \neq x_{j,q}$, for all $i \neq j$. We consider a continuous homomorphism $\varphi : (\overline{\Omega}_A S)^I \to S^I$ such that each clopen set K_x is the preimage of a finite subset of S^I under φ (recall Remark 2.4). We argue by induction on the parameter

$$M = \max\{|c(\delta(x_{i,p}))| : i = 1, \ldots, N, \ p = 1, \ldots, n_i\}.$$

If $M = 0$, then $\delta(x_{i,p}) = I$ for all $i = 1, \ldots, N$ and $p = 1, \ldots, n_i$ and therefore, every solution ε modulo H of S_2 such that $\varepsilon(x_{i,p}) = I$ (for $i = 1, \ldots, N$, and $p = 1, \ldots, n_i$) is trivially a solution

modulo DRH of $\mathcal{S}_1$. Since we are assuming that $\mathcal{S}_2$ is H-equivalent to a system of graph equations and we are taking for H a σ-reducible pseudovariety, there exists such an ε given by σ-words.

Suppose that $M > 0$ and that the result holds for any smaller parameter. If $\delta(x_{i,p})$ has empty cumulative content, then we let k_i be the maximum integer such that $\mathsf{lbf}_{k_i}(\delta(x_{i,p}))$ is nonempty and we write $\mathsf{lbf}_m(\delta(x_{i,p})) = \delta(x_{i,p})_m a_{i,m}$, for $m = 1, \ldots, k_i$. Otherwise, for each $m \geq 1$, we consider the m-th iteration of the left basic factorization to the right of $\delta(x_{i,p})$, say $\delta(x_{i,p}) = \delta(x_{i,p})_1 a_{i,1} \cdots \delta(x_{i,p})_m a_{i,m} \delta(x_{i,p})'_m$. Since S and A are both finite, there are integers $1 \leq k < \ell$ such that, for all i, p satisfying $\vec{c}(\delta(x_{i,p})) \neq \emptyset$, we have

$$\vec{c}(\delta(x_{i,p})) = c(\delta(x_{i,p})_{k+1} a_{i,k+1});$$

$$\varphi(\delta(x_{i,p})) = \varphi(\delta(x_{i,p})_1 a_{i,1} \cdots \delta(x_{i,p})_k a_{i,k} \cdot \eta(\delta(x_{i,p})_{k+1} a_{i,k+1} \cdots \delta(x_{i,p})_\ell a_{i,\ell}) \delta(x_{i,p})'_k).$$

Now, consider a new set of variables X' given by the union

$$X \uplus \{x_{i,p;m}, a_{i,m} : i = 1, \ldots, N; \ p = 1, \ldots, n_i; \ \vec{c}(\delta(x_{i,p})) = \emptyset; \ m = 1, \ldots, k_i\}$$

$$\uplus \{x_{i,p;m}, a_{i,m}, x'_{i,p} : i = 1, \ldots, N; \ p = 1, \ldots, n_i; \ \vec{c}(\delta(x_{i,p})) \neq \emptyset; \ m = 1, \ldots, \ell\},$$

where all the introduced variables are distinct. In order to simplify the notation, we set $\ell_i = 0$ if $\vec{c}(\delta(x_{i,p})) = \emptyset$, and $k_i = k$ and $\ell_i = \ell$, otherwise. We further take the constraints on X' to be given by K_x if $x \in X$, and by the clopen sets $K_{x_{i,p;m}} = \varphi^{-1}(\varphi(\delta(x_{i,p})_m))$, $K_{a_{i,m}} = \{a_{i,m}\}$, and $K_{x'_{i,p}} = \varphi^{-1}(\varphi(\delta(x_{i,p})'_k))$ for the remaining cases.

Consider the system

$$\mathcal{S}'_1 = \{x_{i,1;m} = \cdots = x_{i,n_i;m} : i = 1, \ldots, N; \ m = 1, \ldots, \max\{k_i, \ell_i\}\}.$$

A new system $\mathcal{S}'_2$ is obtained from the system $\mathcal{S}_1 \cup \mathcal{S}_2$ (which is H-equivalent to a system of graph equations, by hypothesis) by adding two sets of equations:

- for each $i = 1, \ldots, N$, if there exists an index p such that $x_{i,p}$ is a variable occurring in $\mathcal{S}_2$, then we choose such an index, say p_i. Then, we add the equation

$$x_{i,p_i} = x_{i,p_i;1} a_{i,1} \cdots x_{i,p_i;k_i} a_{i,k_i} z_{i,p_i},$$

  where z_{i,p_i} stands for the empty word if $\ell_i = 0$, and for x'_{i,p_i} otherwise;

- and we add the set of equations

$$\{x'_{i,1} = \cdots = x'_{i,n_i} : i = 1, \ldots, N, \ \ell_i \neq 0\}.$$

By Corollary 5.5, the new system $\mathcal{S}'_2$ is still H-equivalent to a system of graph equations. Moreover, if we denote by X'_j the set of variables occurring in $\mathcal{S}'_j$ ($j = 1, 2$), then the following equality holds:

$$X'_1 \cap X'_2 = \{x_{i,p_i;m} : i = 1, \ldots, N; \ p_i \text{ is defined; and } m = 1, \ldots, \max\{k_i, \ell_i\}\}.$$

Thus, Lemma 5.3 yields that the system $\mathcal{S}'_1 \cup \mathcal{S}'_2$ is H-equivalent to a system of graph equations as well. Let $\delta' : \overline{\Omega}_{X'}\mathsf{S} \to (\overline{\Omega}_A\mathsf{S})^I$ be the continuous homomorphism defined by

$$\delta'(x_{i,p;m}) = \delta(x_{i,p})_m, \quad \text{if } i = 1, \ldots, N; \ p = 1, \ldots, n_i; \ m = 1, \ldots, \max\{k_i, \ell_i\};$$

$$\delta'(x'_{i,p}) = \delta(x_{i,p})'_k, \quad \text{if } i = 1, \ldots, N; \ p = 1, \ldots, n_i;$$

$$\delta'(x) = \delta(x), \quad \text{otherwise.}$$

It follows from its definition that δ' is a solution modulo DRH of $\mathcal{S}'_1$ which is also a solution modulo H of $\mathcal{S}'_2$. Since the induction parameter corresponding to the triple $(\mathcal{S}'_1, \mathcal{S}'_2, \delta')$ is smaller than the one corresponding to the triple $(\mathcal{S}_1, \mathcal{S}_2, \delta)$, we may use the induction hypothesis to deduce the existence of a continuous homomorphism $\varepsilon' : \overline{\Omega}_{X'}\mathsf{S} \to (\overline{\Omega}_A\mathsf{S})^I$ in σ-words that is both a solution modulo DRH of $\mathcal{S}'_1$ and a solution modulo H of $\mathcal{S}'_2$.

Finally, we define ε as follows:

$$\varepsilon : \overline{\Omega}_X\mathsf{S} \to (\overline{\Omega}_A\mathsf{S})^I$$

$$x_{i,p} \mapsto \varepsilon'(x_{i,p;1})a_{i,1} \cdots \varepsilon'(x_{i,p;k_i})a_{i,k_i}, \quad \text{if } \ell_i = 0;$$

$$x_{i,p} \mapsto \varepsilon'(x_{i,p;1})a_{i,1} \cdots \varepsilon'(x_{i,p;k})a_{i,k}$$

$$\cdot \eta(\varepsilon'(x_{i,p;k+1})a_{i,k+1} \cdots \varepsilon'(x_{i,p;\ell})a_{i,\ell}) \cdot \varepsilon'(x'_{i,p}), \quad \text{if } \ell_i \neq 0;$$

$$x \mapsto \varepsilon'(x), \quad \text{otherwise.}$$

Then, a straightforward computation shows that ε plays the desired role. $\square$

We now state and prove the result claimed at the beginning of the section.

Theorem 5.7. *Let σ be an implicit operation such that there exists $\eta \in \langle \sigma \rangle$ non-explicit, with $\eta = 1$ in H. If H is a σ-reducible pseudovariety of groups, then DRH is σ-reducible for idempotent pointlike systems of equations.*

Proof. Let $\mathcal{S} = \{x_1 = \cdots = x_n = x_n^2\}$ be an idempotent pointlike system of equations with constraints on the variables given by the pair (φ, ν), and let $\delta : \overline{\Omega}_{\{x_1,\ldots,x_n\}}\mathsf{S} \to \overline{\Omega}_A\mathsf{S}$ be a solution modulo DRH of $\mathcal{S}$. Suppose that $\delta(x_i) = u_i$. Since idempotents over DRH are precisely the pseudowords with cumulative content coinciding with the content and with value 1 over H (cf. [15, Corollary 6.1.5]), DRH satisfies $u_1 = \cdots = u_n = u_n^2$ if and only if the following conditions hold:

$$c(u_n) = \vec{c}(u_n);$$

(5.1)
$$u_n =_{\mathsf{H}} 1;$$

$$u_1 =_{\mathsf{DRH}} \cdots =_{\mathsf{DRH}} u_n.$$

For each $i \in \{1, \ldots, n\}$ and $m \geq 1$, let $u_i = \mathsf{lbf}_1(u_i) \cdots \mathsf{lbf}_m(u_i)u'_{i,m}$ and $\mathsf{lbf}_m(u_i) = u_{i,m}a_m$. Since S is finite, there are positive integers $k < \ell$ such that for all $i = 1, \ldots, n$ the equality

$$\varphi(\mathsf{lbf}_1(u_i) \cdots \mathsf{lbf}_k(u_i)) = \varphi(\mathsf{lbf}_1(u_i) \cdots \mathsf{lbf}_\ell(u_i))$$

holds. Take the set of variables

$$X = \{x_{i,p} \colon i = 1, \ldots, n; \; p = 1, \ldots, \ell\} \uplus \{x'_i \colon i = 1, \ldots, n\},$$

with constraints given by (φ, ν'), where $\nu'(x) = \varphi(u_{i,p})$ if $x = x_{i,p}$, and $\nu'(x) = \varphi(u'_{i,k})$ if $x = x'_i$. We consider the systems of equations $\mathcal{S}_1 = \{x_{1,p} = \cdots = x_{n,p}\}_{p=1}^\ell$ and $\mathcal{S}_2 = \{x_{n,1}a_1 \cdots x_{n,k}a_k x'_n = 1, \; x'_1 = \cdots = x'_n\}$. Then, the homomorphism

$$\delta' : \overline{\Omega}_X\mathsf{S} \to (\overline{\Omega}_A\mathsf{S})^I$$

$$x_{i,p} \mapsto u_{i,p}, \quad \text{for } i = 1, \ldots, n; \; p = 1, \ldots, \ell;$$

$$x'_i \mapsto u'_{i,k}, \quad \text{for } i = 1, \ldots, n;$$

is a solution modulo DRH of $\mathcal{S}_1$ that is also a solution modulo H of $\mathcal{S}_2$. Besides that, since by Lemma 5.2 the system $\{x_{n,1}a_1 \cdots x_{n,k}a_k x'_n = 1\}$ is H-equivalent to a system of graph equations, Lemma 5.3 yields that so is $\mathcal{S}_2$. In turn, again Lemma 5.3 implies that $\mathcal{S}_1 \cup \mathcal{S}_2$ is H-equivalent to

a system of graph equations. Thus, we may invoke Proposition 5.6 to derive the existence of a continuous homomorphism $\varepsilon' : \overline{\Omega}_X S \to (\overline{\Omega}_A S)^I$ in σ-words that is a solution modulo DRH of S_1, and a solution modulo H of S_2.

Now, assuming that η is unary, we let $\varepsilon : \overline{\Omega}_{\{x_1,\ldots,x_n\}} S \to \overline{\Omega}_A S$ be given by

$$\varepsilon(x_i) = \varepsilon'(x_{i,1})a_1 \cdots \varepsilon'(x_{i,k})a_k \cdot \eta(\varepsilon'(x_{i,k+1})a_{k+1} \cdots \varepsilon'(x_{i,\ell})a_\ell)\varepsilon'(x_i').$$

It is easily checked that DRH satisfies $\varepsilon(x_1) = \cdots = \varepsilon(x_n)$, and H satisfies $\varepsilon(x_n) = 1$. Furthermore, by the choice of k and ℓ, we also know that $\varphi(\varepsilon(x_i)) = \varphi(\delta(x_i))$ and, as we are assuming that η is non-explicit and S has a content function, the equality $\vec{c}(\varepsilon(x_i)) = c(\varepsilon(x_i))$ holds. So, by (5.1), we may conclude that ε is a solution modulo DRH of S in σ-words that keeps the values under φ. $\square$

We observe that, whenever the ω-power belongs to $\langle \sigma \rangle$, the hypothesis of Theorem 5.7 concerning the implicit signature σ is satisfied. That is the case of the canonical implicit signature κ. Hence, using again Theorem 2.6, we may conclude the following.

Corollary 5.8. *If H is a κ-tame pseudovariety of groups, then DRH is κ-tame with respect to finite systems of idempotent pointlike equations.* $\square$

In particular, the pseudovarieties DRG and DRAb are both κ-tame with respect to finite systems of idempotent pointlike equations and DRG_p and DRG_{sol} are σ-reducible with respect to the same class (recall Theorem 2.7).

References

[1] D. Albert, R. Baldinger, and J. Rhodes, *Undecidability of the identity problem for finite semigroups*, J. Symbolic Logic **57** (1992), no. 1, 179–192.

[2] J. Almeida, *Finite semigroups and universal algebra*, World Scientific Publishing Co. Inc., River Edge, NJ, 1994, Translated from the 1992 Portuguese original and revised by the author.

[3] ———, *Hyperdecidable pseudovarieties and the calculation of semidirect products*, Internat. J. Algebra Comput. **9** (1999), no. 3-4, 241–261.

[4] ———, *Dynamics of implicit operations and tameness of pseudovarieties of groups*, Trans. Amer. Math. Soc. **354** (2002), no. 1, 387–411 (electronic).

[5] ———, *Finite semigroups: an introduction to a unified theory of pseudovarieties*, Semigroups, algorithms, automata and languages (Coimbra, 2001), World Sci. Publ., River Edge, NJ, 2002, pp. 3–64.

[6] ———, *Profinite semigroups and applications*, Structural theory of automata, semigroups, and universal algebra, NATO Sci. Ser. II Math. Phys. Chem., vol. 207, Springer, Dordrecht, 2005, pp. 1–45.

[7] J. Almeida, J. C. Costa, and M. Zeitoun, *Tameness of pseudovariety joins involving R*, Monatsh. Math. **146** (2005), no. 2, 89–111.

[8] ———, *Complete reducibility of systems of equations with respect to* R, Port. Math. (N.S.) **64** (2007), no. 4, 445–508.

[9] ———, *Reducibility of pointlike problems*, Semigroup Forum **94** (2017), no. 2, 325–335.

[10] J. Almeida and M. Delgado, *Tameness of the pseudovariety of abelian groups*, Internat. J. Algebra Comput. **15** (2005), no. 2, 327–338.

[11] J. Almeida and B. Steinberg, *On the decidability of iterated semidirect products with applications to complexity*, Proc. London Math. Soc. (3) **80** (2000), no. 1, 50–74.

[12] ———, *Syntactic and global semigroup theory: a synthesis approach*, Algorithmic problems in groups and semigroups (Lincoln, NE, 1998), Trends Math., Birkhäuser Boston, Boston, MA, 2000, pp. 1–23.

[13] J. Almeida and P. G. Trotter, *The pseudoidentity problem and reducibility for completely regular semigroups*, Bull. Austral. Math. Soc. **63** (2001), no. 3, 407–433.

[14] J. Almeida and M. V. Volkov, *Profinite identities for finite semigroups whose subgroups belong to a given pseudovariety*, J. Algebra Appl. **2** (2003), no. 2, 137–163.

[15] J. Almeida and P. Weil, *Free profinite R-trivial monoids*, Internat. J. Algebra Comput. **7** (1997), no. 5, 625–671.

[16] J. Almeida and M. Zeitoun, *The pseudovariety* **J** *is hyperdecidable*, RAIRO Inform. Théor. Appl. **31** (1997), no. 5, 457–482.

[17] ———, *An automata-theoretic approach to the word problem for ω-terms over* R, Theoret. Comput. Sci. **370** (2007), no. 1-3, 131–169.

[18] C. J. Ash, *Inevitable graphs: a proof of the type* II *conjecture and some related decision procedures*, Internat. J. Algebra Comput. **1** (1991), no. 1, 127–146.

[19] _____, *Inevitable sequences and a proof of the "type* II *conjecture"*, Monash Conference on Semigroup Theory (Melbourne, 1990), World Sci. Publ., River Edge, NJ, 1991, pp. 31–42.

[20] G. Baumslag, *Residual nilpotence and relations in free groups*, J. Algebra **2** (1965), 271–282.

[21] C. Borlido, *The word problem and some reducibility properties for pseudovarieties of the form* DRH, Ph.D. thesis, University of Porto, 2016.

[22] T. Coulbois and A. Khelif, *Equations in free groups are not finitely approximable*, Proc. Amer. Math. Soc. **127** (1999), no. 4, 963–965.

[23] M. Delgado, A. Masuda, and B. Steinberg, *Solving systems of equations modulo pseudovarieties of abelian groups and hyperdecidability*, Semigroups and formal languages, World Sci. Publ., Hackensack, NJ, 2007, pp. 57–65.

[24] S. Eilenberg, *Automata, languages, and machines. Vol. B*, Academic Press, New York-London, 1976.

[25] K. Henckell, *Pointlike sets: the finest aperiodic cover of a finite semigroup*, J. Pure Appl. Algebra **55** (1988), no. 1-2, 85–126.

[26] K. Henckell, J. Rhodes, and B. Steinberg, *Aperiodic Pointlikes and Beyond*, ArXiv e-prints (2007).

[27] J. Karnofsky and J. Rhodes, *Decidability of complexity one-half for finite semigroups*, Semigroup Forum **24** (1982), no. 1, 55–66.

[28] K. Krohn and J. Rhodes, *Complexity of finite semigroups*, Ann. of Math. (2) **88** (1968), 128–160.

[29] J.-E. Pin and P. Weil, *Profinite semigroups, Mal′cev products, and identities*, J. Algebra **182** (1996), no. 3, 604–626.

[30] J. Rhodes, *Undecidability, automata, and pseudovarieties of finite semigroups*, Internat. J. Algebra Comput. **9** (1999), no. 3-4, 455–473.

[31] M. P. Schützenberger, *Sur le produit de concaténation non ambigu*, Semigroup Forum **13** (1976/77), no. 1, 47–75.

[32] B. Steinberg, *Inevitable graphs and profinite topologies: some solutions to algorithmic problems in monoid and automata theory, stemming from group theory*, Internat. J. Algebra Comput. **11** (2001), no. 1, 25–71.

[33] S. Willard, *General topology*, Dover Publications, Inc., Mineola, NY, 2004, Reprint of the 1970 original [Addison-Wesley, Reading, MA].

Centro de Matemática e Departamento de Matemática, Faculdade de Ciências, Universidade do Porto, Rua do Campo Alegre, 687, 4169-007 Porto, Portugal

E-mail address: `cborlido@fc.up.pt`