# A NOTE ON THE FINITE BASIS AND FINITE RANK PROPERTIES FOR PSEUDOVARIETIES OF SEMIGROUPS

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Abstract. The finite basis property is often connected with the finite rank property, which it entails. Many examples have been produced of finite rank varieties which are not finitely based. In this note, we establish a result on nilpotent pseudovarieties which yields many similar examples in the realm of pseudovarieties of semigroups.

Recall that a (pseudo)variety is a class of (finite) algebras of a given type which is closed under the operators of taking homomorphic images, subalgebras and (finitary) direct products. By theorems of Birkhoff [3] and Reiterman [7], we may define a (pseudo)variety by a set of (pseudo)identities, which is called a *basis*. We say that a (pseudo)variety is finitely based if it admits a finite basis of (pseudo)identities. The reader is referred to [1] for more details and basic definitions on this topic.

An algebra $S$ is said to be *$n$-generated* if there is some subset of $S$ with at most $n$ elements that generates $S$. A pseudovariety $\mathsf{V}$ is said to *have rank at most $n$* if, whenever a finite algebra $S$ is such that all its $n$-generated subalgebras belong to $\mathsf{V}$, so does $S$. If there is such an integer $n \geqslant 0$, then $\mathsf{V}$ is said to have *finite rank* and the minimum value of $n$ for which $\mathsf{V}$ has rank at most $n$ is called the *rank* of $\mathsf{V}$. If there is no such $n$, then $\mathsf{V}$ is said to have *infinite rank*.

Similarly, a variety $\mathcal{V}$ of algebras is said to *have rank at most $n$* if, whenever an algebra $S$ is such that all its $n$-generated subalgebras belong to $\mathcal{V}$, so does $S$. The related notions of finite and infinite rank are defined as for pseudovarieties.

By [4, Proposition IV.3.9], we have the variety case of the following proposition whose proof works as well for pseudovarieties.

**Proposition 1.** *A (pseudo)variety has rank at most $n$ if and only if it admits a basis of (pseudo)identities in at most $n$ variables.*

In view of Proposition 1, the rank is also known as *axiomatic rank*. Proposition 1 says, in particular, that if a (pseudo)variety is finitely based then it has finite rank. It is easy to see that the two properties are equivalent for a (pseudo)variety generated by a single finite algebra: indeed, if the rank is $n$ then the diagram of the relatively free algebra on $n$ free generators constitutes a finite basis of identities. This is why most proofs that concrete finite algebras are infinitely based, in the sense that they generate infinitely based (pseudo)varieties, establish directly that the generated (pseudo)varieties have

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infinite rank. A case which has deserved a lot of attention is that of finite semigroups, where the main open question is whether one can effectively decide whether a given finite semigroup is finitely based [10]. Another case of interest is that of varieties generated by a semigroup given by a one-relator presentation, for which again the two properties turn out to be equivalent [9].

The two properties, having finite rank and being finitely based, are not, in general, equivalent. Examples of varieties of semigroups which have finite rank and are infinitely based are well known (see for example Tables 1.3 and 1.4 in [8]). Many such examples, as those in [8, Table 1.3], are actually of non-finitely based systems of balanced identities involving a (small) finite number of variables.

Recall that $\mathsf{N} = [\![x^\omega = 0]\!]$ is the pseudovariety of all finite *nil* (or *power nilpotent*) semigroups. The purpose of this note is to establish the theorem below which shows how to obtain infinitely based pseudovarieties of nil semigroups from infinitely based varieties of semigroups defined by balanced identities.

A major difference between dealing with identities and pseudoidentities is the absence of an analog for pseudoidentities of the completeness theorem for equational logic [1, Theorem 3.8.8], in the sense that there is no complete and sound finite deductive system for pseudoidentities. Instead, we invoke the general theory developed in [1, Section 3.8] for which an alternative has recently been proposed in the form of a deductive system involving infinite proofs [2].

We start by recalling some notation and terminology. We fix a sequence $(x_n)_{n\geqslant 1}$ of distinct variables and we denote by $\overline{\Omega}_n\mathsf{N}$ the pro-$\mathsf{N}$ semigroup freely generated by $\{x_1,\dots,x_n\}$. For $m<n$, we view $\overline{\Omega}_m\mathsf{N}$ as being naturally embedded in $\overline{\Omega}_n\mathsf{N}$ via the unique continuous homomorphism that sends each $x_i$ $(i=1,\dots,m)$ to itself. This leads to a topological semigroup $\widetilde{\Omega}_\omega\mathsf{N}$ which is the inductive limit of the sequence $\overline{\Omega}_1\mathsf{N}\to\overline{\Omega}_2\mathsf{N}\to\overline{\Omega}_3\mathsf{N}\to\cdots$. Note that, since each $\overline{\Omega}_n\mathsf{N}$ is the one-point compactification of the discrete free semigroup $\{x_1,\dots,x_n\}^+$, obtained by adding a zero, $\widetilde{\Omega}_\omega\mathsf{N}$ is also obtained from the discrete free semigroup $\{x_1,x_2,\dots\}^+$ by adding a zero.

A subset $K$ of $\widetilde{\Omega}_\omega\mathsf{N}$ is clopen if and only if $K\cap\overline{\Omega}_n\mathsf{N}$ is clopen for every $n\geqslant 1$, that is, $K\cap\overline{\Omega}_n\mathsf{N}$ is a finite subset of $\{x_1,\dots,x_n\}^+$ or it is the complement in $\overline{\Omega}_n\mathsf{N}$ of such a subset. In other words, $K$ is clopen in $\widetilde{\Omega}_\omega\mathsf{N}$ if and only if $0\notin K$ and each intersection $K\cap\overline{\Omega}_n\mathsf{N}$ is finite, or $0\in K$ and each set $\{x_1,\dots,x_n\}^+\setminus K$ is finite.

Note that the space $\widetilde{\Omega}_\omega\mathsf{N}$ is not compact: for instance, the sequence $(x_n)_n$ has no convergent subsequence. In fact, a sequence of words $(u_n)_n$ converges in $\widetilde{\Omega}_\omega\mathsf{N}$ if and only if it is eventually constant, or there is $N$ such that $c(u_n)\subseteq\{x_1,\dots,x_N\}$ for all $n$ and $|u_n|\to\infty$, in which case the limit is zero. Indeed, for any sequence $(x_n)_n$ in $\widetilde{\Omega}_\omega\mathsf{N}$ with only a finite number of terms in each $\overline{\Omega}_n\mathsf{N}$, its complement is an open set in $\widetilde{\Omega}_\omega\mathsf{N}$. As another example, the sequence $(x_1\cdots x_n)_n$ has no convergent subsequence in $\widetilde{\Omega}_\omega\mathsf{N}$ even though the length of its terms tends to infinity.

As in [1, Section 3.8], we consider a subset $\Lambda$ of $\widetilde{\Omega}_\omega\mathsf{N}\times\widetilde{\Omega}_\omega\mathsf{N}$, which is viewed as an arbitrary set of pseudoidentities for nil semigroups. We say that $\Lambda$ is *strongly closed* if $\Lambda$ is a fully invariant congruence on the semigroup $\widetilde{\Omega}_\omega \mathsf{N}$ and the clopen unions of $\Lambda$-classes separate the classes of $\Lambda$. By [1, Corollary 3.8.4], the strongly closed sets of nil pseudoidentities are precisely the sets of all pseudoidentities that are valid in a pseudovariety of nil semigroups.

**Theorem 2.** *Let $\Sigma$ be a set of balanced semigroup identities such that the variety $[\Sigma]$ is infinitely based. Then, the pseudovariety $\mathsf{N} \cap [\![\Sigma]\!]$ is infinitely based.*

*Proof.* Suppose that $\mathsf{W} = \mathsf{N} \cap [\![\Sigma]\!]$ admits a finite basis $\Sigma_0$ of pseudoidentities. We may as well assume that the pseudoidentity $x^\omega = 0$ belongs to $\Sigma_0$. In the presence of that pseudoidentity, every pseudoidentity that is not its consequence is equivalent to either an identity $u = v$ or to a pseudoidentity of the form $u = 0$, where $u$ is a word. The latter case is excluded for pseudoidentities in $\Sigma_0$ because no such pseudoidentity is valid in $\mathsf{W}$ since $\mathsf{W}$ contains all monogenic aperiodic semigroups. In the former case, if the identity $u = v$ is not balanced then, together with the pseudoidentity $x^\omega = 0$, it entails a pseudoidentity of the form $w = 0$, for some word $w$, which has already been excluded. Thus, we may as well assume that $\Sigma_0$ consists of the pseudoidentity $x^\omega = 0$ together with a finite set $\Sigma_1$ of balanced identities.

By the above cited results from [1, Section 3.8], an identity $u = v$ is valid in $\mathsf{W} = \mathsf{N} \cap [\![\Sigma_1]\!]$, if and only if it belongs to every strongly closed set of nil pseudoidentities containing $\Sigma_1$. We claim that the smallest strongly closed set of nil pseudoidentities containing $\Sigma_1$ is the fully invariant congruence $\Lambda$ on $\{x_1, x_2, \ldots\}^+$ generated by $\Sigma_1$ together with the trivial pseudoidentity $0 = 0$. Indeed, since the set $\theta = \Lambda \cup \{0 = 0\}$ is certainly contained in every fully invariant congruence on $\widetilde{\Omega}_\omega \mathsf{N}$ containing $\Sigma_1$, it suffices to show that $\theta$ is strongly closed. For this purpose, we start by noting that the $\theta$-classes are the $\Lambda$-classes, which are finite sets since they consist of words of equal length involving the same letters, together with the singleton set $\{0\}$. Since finite subsets of $\{x_1, x_2, \ldots\}^+$ are clopen in $\widetilde{\Omega}_\omega \mathsf{N}$, to separate a class $C$ different from $\{0\}$ from any other class by a clopen union of classes, it suffices to take $C$ itself.

We thus conclude that an identity $u = v$ is valid in the pseudovariety $\mathsf{N} \cap [\![\Sigma_1]\!]$ if and only if it is valid in the variety $[\Sigma_1]$. In particular, we obtain the equality $[\Sigma_1] = [\Sigma]$, which contradicts the assumption that the variety $[\Sigma]$ is infinitely based. Hence, $\mathsf{W}$ is infinitely based. $\square$

Note that the rank of $\mathsf{N} \cap [\![\Sigma]\!]$ is at most $\max\{2, n\}$, where $n$ is the rank of the variety $[\Sigma]$. Hence, combining Theorem 2 with examples of infinitely based varieties of finite rank, we obtain the following result, which is the main purpose of this note.

**Corollary 3.** *There are pseudovarieties of (nil) semigroups that have finite rank and are infinitely based.*

For instance, since the varieties $[yx^n y = xyx^{n-2}yx \ (n \geqslant 3)]$ and $[xyx^n y = yx^n yx \ (n \geqslant 2)]$ are infinitely based (see, respectively, [6] and [5]), their intersections with $\mathsf{N}$ are also infinitely based but both have rank two.

Note that the examples of infinitely based finite rank pseudovarieties resulting from Theorem 2 are not locally finite. We do not know if there are any locally finite infinitely based finite rank pseudovarieties or even varieties with the same properties.

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