On the topology of a boolean representable simplicial complex

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November 30, 2016

2010 Mathematics Subject Classification: 05B35, 05E45, 14F35, 15B34, 55P15, 55U10, 57M05.

Keywords: simplicial complex, matroid, boolean representation, fundamental group, shellability, sequentially Cohen-Macaulay, EL-labeling, homotopy type

ABSTRACT

It is proved that fundamental groups of boolean representable simplicial complexes are free and the rank is determined by the number and nature of the connected components of their graph of flats for dimension ≥ 2. In the case of dimension 2, it is shown that boolean representable simplicial complexes have the homotopy type of a wedge of spheres of dimensions 1 and 2. Also in the case of dimension 2, necessary and sufficient conditions for shellability and being sequentially Cohen-Macaulay are determined. Complexity bounds are provided for all the algorithms involved.

1 Introduction

In a series of three papers [10, 11, 12], Izhakian and Rhodes introduced the concept of boolean representation for various algebraic and combinatorial structures. These ideas were inspired by previous work by Izhakian and Rowen on supertropical matrices (see e.g. [9, 13, 14, 15]), and were subsequently developed by Rhodes and Silva in a recent monograph, devoted to boolean representable simplicial complexes [18].

The original approach was to consider matrix representations over the superboolean semiring $\mathbb{SB}$, using appropriate notions of vector independence and rank. Writing $\mathbb{N} = \{0, 1, 2, \ldots\}$, we can define $\mathbb{SB}$ as the quotient of $(\mathbb{N}, +, \cdot)$ (usual operations) by the congruence which identifies all integers ≥ 2. In this context, boolean representation refers to matrices using only 0 and 1 as entries.

In this paper, we view (finite) simplicial complexes under two perspectives, geometric and combinatorial. The geometric perspective involves subspaces of an euclidean space $\mathbb{R}^n$, while the combinatorial perspective involves collections of subsets of a finite subset. As we recall in Section 2, that each structure determines the other.

As an alternative to matrices, boolean representable simplicial complexes can be characterized by means of their lattice of flats. The lattice of flats plays a fundamental role in matroid theory but is not usually considered for arbitrary simplicial complexes, probably due to the fact that, unlike the matroid case, the structure of a simplicial complex cannot in general be recovered from its lattice of flats. However, this is precisely what happens with boolean representable simplicial complexes. If $\mathcal{H} = (V, H)$ is a simplicial complex and $\mathrm{Fl}\mathcal{H}$ denotes its lattice of flats, then $\mathcal{H}$ is boolean representable if and only if H equals the set of transversals of the successive differences for chains in $\mathrm{Fl}\mathcal{H}$. This implies in particular that all (finite) matroids are boolean representable.

In this paper we begin the study of the topology of boolean representable simplicial complexes (BRSC).

As any finitely presented group can be the fundamental group of a 2-dimensional simplicial complex (see e.g. [19, Theorem 7.45]), the problem of understanding the homotopy type of an arbitrary simplicial complex is hopeless.

However, for matroids, the topology is very restricted. Indeed, it is known that a matroid is pure shellable [2]. This implies that a matroid of rank r has the homotopy type of a wedge of $r-1$ dimensional spheres, the number of which is then the rank of its unique non-trivial homology group. This latter number has a number of combinatorial interpretations [2]. In particular, a matroid of dimension at least 2 has a trivial fundamental group.

One of the main results of this paper is to show that the fundamental group of a BRSC is a free group. We give a precise formula for the rank of this group in terms of the number and nature of the connected components of its graph of flats [18]. In the simple case, this rank is equivalently a function of the number of connected components of the proper part of its lattice of flats.

For 2 dimensional BRSCs, we completely characterize shellable complexes, showing that these are precisely the sequentially Cohen-Macauley complexes [5]. Although not every 2 dimensional BRSC is shellable, we prove that every 2 dimensional BRSC has the homotopy type of a wedge of 1-spheres and 2-spheres.

We consider the connection to EL-labelings [2] of the lattice of flats and give an example of a shellable 2-dimensional complex whose lattice of flats is not EL-labelable.

The paper is organized as follows. In Section 2 we present basic notions and results needed in the paper. In Section 3 we show that the fundamental group of a boolean representable simplicial complex is always free, and provide an exact formula to compute its rank for dimension ≥ 2, using the graph of flats. We also prove that any 2 dimensional BRSC has the homotopy type of a wedge of 1-spheres and 2-spheres.

For higher degree homotopy groups, the situation is of course much harder, and we limit the discussion to shellability in dimension 2. We note that in [18] we had characterized shellability for simple boolean representable complexes of dimension 2. We are now able to deal with the non simple case, and to assist us on this reduction we use the concept of simplification in Section 4. Then Section 5 is devoted to characterizing shellability for boolean representable simplicial complexes of dimension 2. For such complexes, it is also shown that the shellable complexes are precisely the sequentially Cohen-Macaulay complexes.

In Section 6, we consider the concept of the order complex of a lattice L. The vertices of the order complex are the elements of the proper part of L, i.e. $L^* = L \setminus \{0, 1\}$, and its faces are the chains of L^*. We show that, given a boolean representable simplicial complex $\mathcal{H}$, if the order complex of $\mathrm{Fl}\mathcal{H}$ is shellable, so is $\mathcal{H}$. The converse turns out to be false.

In the matroid case, (some) shellings can be obtained from EL-labelings of the lattice of flats (which is always geometric and thus has an EL-labeling by a theorem of Björner [1]). We show that, for arbitrary shellable pure boolean representable simplicial complexes of dimension 2, the lattice of flats does not necessarily admit an EL-labeling.

Finally, Section 7 discusses the complexity of several algorithms designed to compute fundamental groups, decide shellability (for dimension 2) and compute shellings and Betti numbers. Although the number of potential flats in a simplicial complex with n vertices is 2^n and therefore exponential, we achieve polynomial bounds for all algorithms when the dimension of the simplicial complexes is fixed.

2 Preliminaries

All lattices and simplicial complexes in this paper are assumed to be finite. Given a set V and $n \geq 0$, we denote by $P_n(V)$ (respectively $P_{\leq n}(V)$) the set of all subsets of V with precisely (respectively at most) n elements. The *kernel* of a mapping $\varphi : V \to W$ is the relation

$$\mathrm{Ker}\,\varphi = \{(a, b) \in V \times V \mid a\varphi = b\varphi\}.$$

A (finite) simplicial complex is a structure of the form $\mathcal{H} = (V, H)$, where V is a finite nonempty set and $H \subseteq 2^V$ contains $P_1(V)$ and is closed under taking subsets. The elements of V and H are called respectively *vertices* and *faces*. To simplify notation, we shall often denote a face $\{x_1, x_2, \dots, x_n\}$ by $x_1x_2 \dots x_n$.

A face of $\mathcal{H}$ which is maximal with respect to inclusion is called a *facet*. We denote by $\mathrm{fct}\mathcal{H}$ the set of facets of $\mathcal{H}$.

The *dimension* of a face $I \in H$ is $|I| - 1$. An *i-face* (respectively *i-facet*) is a face (respectively facet) of dimension i. We may refer to 0-faces and 1-faces as vertices and edges, respectively.

We say that $\mathcal{H}$ is:

- *simple* if $P_2(V) \subseteq H$;
- *pure* if all the facets of $\mathcal{H}$ have the same dimension.

The dimension of $\mathcal{H}$, denoted by $\dim\mathcal{H}$, is the maximum dimension of a face/facet of $\mathcal{H}$.

A simplicial complex $\mathcal{H} = (V, H)$ is called a *matroid* if it satisfies the *exchange property*:

(EP) For all $I, J \in H$ with $|I| = |J| + 1$, there exists some $i \in I \setminus J$ such that $J \cup \{i\} \in H$.

A simplicial complex $\mathcal{H} = (V, H)$ is *shellable* if we can order its facets as $B_1, \dots, B_t$ so that, for $k = 2, \dots, t$, the following condition is satisfied: if $I(B_k) = (\cup_{i=1}^{k-1} 2^{B_i}) \cap 2^{B_k}$, then

$$(B_k, I(B_k)) \text{ is pure of dimension } |B_k| - 2$$

whenever $|B_k| \geq 2$. Such an ordering is called a *shelling*. In the literature, this is called non-pure shellability and was first defined by Björner and Wachs [3, 4]. For pure complexes, the concept has its roots in the 19th century (see [7]), and was important as means to provide an inductive proof for the Euler-Poincaré formula. In this paper, the concept's most important use is the theorem by Björner which characterizes the homotopy type of a shellable complex [3].

Given an $R \times V$ matrix M and $Y \subseteq R$, $X \subseteq V$, we denote by $M[Y, X]$ the submatrix of M obtained by deleting all rows (respectively columns) of M which are not in Y (respectively X).

A boolean matrix M is *lower unitriangular* if it is of the form

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ ? & 1 & 0 & \dots & 0 \\ ? & ? & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ? & ? & ? & \dots & 1 \end{pmatrix}$$

Two matrices are *congruent* if we can transform one into the other by independently permuting rows/columns. A boolean matrix is *nonsingular* if it is congruent to a lower unitriangular matrix.

Given an $R \times V$ boolean matrix M, we say that the subset of columns $X \subseteq V$ is *M-independent* if there exists some $Y \subseteq R$ such that $M[Y, X]$ is nonsingular.

A simplicial complex $\mathcal{H} = (V, H)$ is *boolean representable* if there exists some boolean matrix M such that H is the set of all M-independent subsets of V.

We denote by $\mathcal{BR}$ the class of all (finite) boolean representable simplicial complexes. All matroids are boolean representable [18, Theorem 5.2.10], but the converse is not true.

We say that $X \subseteq V$ is a *flat* of $\mathcal{H}$ if

$$\forall I \in H \cap 2^X \ \ \forall p \in V \setminus X \quad I \cup \{p\} \in H.$$

The set of all flats of $\mathcal{H}$ is denoted by $\mathrm{Fl}\mathcal{H}$. Note that $V, \emptyset \in \mathrm{Fl}\mathcal{H}$ in all cases.

Clearly, the intersection of any set of flats (including $V = \cap\emptyset$) is still a flat. If we order $\mathrm{Fl}\mathcal{H}$ by inclusion, it is then a $\wedge$-semilattice. Since $\mathrm{Fl}\mathcal{H}$ is finite, it follows that it is indeed a lattice (with the determined join), the *lattice of flats* of $\mathcal{H}$.

We say that X is a *transversal of the successive differences* for a chain of subsets

$$A_0 \subset A_1 \subset \ldots \subset A_k$$

if X admits an enumeration $x_1, \ldots, x_k$ such that $x_i \in A_i \setminus A_{i-1}$ for $i = 1, \ldots, k$.

Let $\mathcal{H} = (V, H)$ be a simplicial complex. If $X \subseteq V$ is a *transversal of the successive differences* for a chain

$$F_0 \subset F_1 \subset \ldots \subset F_k$$

in $\mathrm{Fl}\mathcal{H}$, it follows easily by induction that $x_1x_2 \ldots x_i \in H$ for $i = 0, \ldots, k$. In particular, $X \in H$.

It follows from [18, Corollary 5.2.7] that $\mathcal{H}$ is boolean representable if and only if every $X \in H$ is a transversal of the successive differences for a chain in $\mathrm{Fl}\mathcal{H}$.

The lattice $\mathrm{Fl}\mathcal{H}$ induces a closure operator on 2^V defined by

$$\overline{X} = \cap\{F \in \mathrm{Fl}\mathcal{H} \mid X \subseteq F\}$$

for every $X \subseteq V$.

By [18, Corollary 5.2.7], $\mathcal{H} = (V, H)$ is boolean representable if and only if every $X \in H$ admits an enumeration $x_1, \ldots, x_k$ satisfying

$$\overline{x_1} \subset \overline{x_1x_2} \subset \ldots \subset \overline{x_1 \ldots x_k}. \tag{1}$$

Thus, given $p, q \in V$ distinct, we have

$$pq \notin H \text{ if and only } \overline{p} = \overline{pq} = \overline{q}. \tag{2}$$

This fact will be often used throughout the text with no explicit reference. Note that, in the important particular case of field representable matroids, this equivalence expresses the fact that two nonzero vectors over a field are linearly dependent if and only if they generate the same subspace of dimension 1.

From (2) we can deduce that

$$\overline{p} = \{q \in V \mid \overline{q} = \overline{p}\}. \tag{3}$$

Indeed, let $F = \{q \in V \mid \overline{q} = \overline{p}\}$. Since $p \in F \subseteq \overline{p}$, it suffices to show that $F \in \mathrm{Fl}\mathcal{H}$. Let $I \in H \cap 2^F$ and $a \in V \setminus F$. In view of (2), we may assume that $I = \{q\}$. Since $\overline{a} \neq \overline{q}$, we get $qa \in H$ also by (2). Thus $F \in \mathrm{Fl}\mathcal{H}$ and (3) holds.

We summarize next the geometric perspective of simplicial complexes. For details, the reader is referred to [18, Appendix A.5]. A *geometric simplex* in $\mathbb{R}^n$ is the convex hull C of an affinely independent set of points $X_0, \ldots, X_m$ (i.e. the vectors $X_1 - X_0, \ldots, X_m - X_0$ are linearly independent). The set $\{X_0, \ldots, X_m\}$ is fully determined by the geometric simplex C and may be called the *affine basis* of C. The convex hull of a subset of $\{X_0, \ldots, X_m\}$ is a subsimplex of C. A *geometric simplicial complex* in $\mathbb{R}^n$ is a set of geometric simplexes closed under subsimplexes.

If S is a geometric simplicial complex in $\mathbb{R}^n$, then the set of affine bases of its simplexes defines a simplicial complex $\mathcal{H}(S)$ (in the combinatorial sense). On the other hand, given an (abstract) simplicial complex $\mathcal{H}$, it is possible to construct a geometric simplicial complex $S(\mathcal{H})$ in $\mathbb{R}^n$ (for some n) such that $\mathcal{H} \cong \mathcal{H}(S(\mathcal{H}))$. Moreover, the union of $S(\mathcal{H})$ (a subspace of $\mathbb{R}^n$) is unique up to homeomorphism (for the standard topology of $\mathbb{R}^n$). We denote this subspace by $||\mathcal{H}||$ and call it the *geometric realization* of $\mathcal{H}$.

Let $\mathcal{J} = (V, J)$ be a simplicial complex. We recall the definitions of the (reduced) homology groups of $\mathcal{J}$ (see e.g. [8]).

If $\mathcal{J}$ has s connected components, it is well known that the *0th homology group* $H_0(\mathcal{J})$ is isomorphic to the free abelian group of rank s. For dimension $k \geq 1$, we proceed as follows.

Fix a total ordering of V. Let $C_k(\mathcal{J})$ denote the free abelian group on $J \cap P_{k+1}(V)$, that is, all the formal sums of the form $\sum_{i \in I} n_i X_i$ with $n_i \in \mathbb{Z}$ and $X_i \in J \cap P_{k+1}(V)$ (distinct). Given $X \in J \cap P_{k+1}(V)$, write $X = x_0 x_1 \ldots x_k$ with $x_0 < \ldots < x_k$. We define

$$X\partial_k = \sum_{i=0}^{k} (-1)^i (X \setminus \{x_i\}) \in C_{k-1}(\mathcal{J})$$

and extend this by linearity to a homomorphism $\partial_k : C_k(\mathcal{J}) \to C_{k-1}(\mathcal{J})$ (the kth *boundary map* of $\mathcal{J}$). Then the *kth homology group* of $\mathcal{J}$ is defined as the quotient

$$H_k(\mathcal{J}) = \mathrm{Ker}\, \partial_k / \mathrm{Im}\, \partial_{k+1}.$$

The 0*th reduced homology group* of $\mathcal{J}$, denoted by $\tilde{H}_0(\mathcal{J})$, is isomorphic to the free abelian group of rank $s-1$, where s denotes the number of connected components of $\mathcal{J}$. For $k \geq 1$, the *kth reduced homology group* of $\mathcal{J}$, denoted by $\tilde{H}_k(\mathcal{J})$ coincides with the kth homology group.

A *wedge* of spheres $S_1, \ldots, S_m$ (of possibly different dimensions) is a topological space obtained by identifying m points $s_i \in S_i$ for $i = 1, \ldots, m$.

Given a group G and $X \subseteq G$, we denote by $\langle X \rangle$ (respectively $\langle\langle X \rangle\rangle$) the subgroup (respectively normal subgroup) of G generated by X.

We denote by F_A the free group on an alphabet A. A *group presentation* is a formal expression of the form $\langle A \mid R \rangle$, where A is an alphabet and $R \subseteq F_A$. It defines the group $F_A / \langle\langle R \rangle\rangle$, and is said to be a presentation for any group isomorphic to this quotient.

Given a (finite) alphabet A, we denote by A^+ the *free semigroup* on A (finite nonempty words on A, under concatenation). Given a partial order on A, we define the *lexicographic order* on A^+ as follows. Given $a_1, \ldots, a_k, a'_1, \ldots, a'_m \in A$, we write $a_1 \ldots a_k < a'_1 \ldots a'_m$ if one of the following conditions holds:

- $k < m$ and $a_i = a'_i$ for $i = 1, \ldots, k$;
- there exists some $i \leq \min\{k, m\}$ such that $a_1 = a'_1, \ \ldots, \ a_{i-1} = a'_{i-1}, \ a_i < a'_i$.

3 The fundamental group

Let $\mathcal{H} = (V, H)$ be a simplicial complex. The *graph* of $\mathcal{H}$ is the truncation $(V, H \cap P_{\leq 2}(V))$. We say that $\mathcal{H}$ is *connected* if its graph is connected. We say that $T \subseteq H \cap P_2(V)$ is a spanning tree of $\mathcal{H}$ if it is a spanning tree of its graph.

Lemma 3.1 *Let $\mathcal{H} = (V, H)$ be a boolean representable simplicial complex. Then $\mathcal{H}$ is connected unless $H = P_1(V)$ and $|V| > 1$.*

Proof. Obviously, $\mathcal{H}$ is disconnected if $H = P_1(V)$ and $|V| > 1$, and connected if $|V| = 1$. Hence we may assume that $pq \in H$ for some distinct $p, q \in V$.

Let M be an $R \times V$ boolean matrix representing $\mathcal{H}$. It follows from $pq \in H$ that $M[R, p] \neq M[R, q]$. Thus, for every $v \in V$, we have either $M[R, v] \neq M[R, p]$ or $M[R, v] \neq M[R, q]$, implying that vp or vq is an edge of H. Therefore $\mathcal{H}$ is connected. $\square$

Note that, if we consider the *geodesic distance* on the graph of a boolean representable simplicial complex of dimension ≥ 2 (the distance between two vertices is the length of the shortest path connecting them), it follows from the above proof that the distance between any two vertices is at most 2. Indeed, given vertices $v, w \in V$, we have two possibilities:

- $M[R, v] \neq M[R, w]$. Then v — w is an edge in $\mathcal{H}$.
- $M[R, v] = M[R, w]$. Then either $M[R, v] \neq M[R, p]$ or $M[R, v] \neq M[R, q]$. Assuming the latter, we get a path v — q — w in $\mathcal{H}$.

We consider now the geometric realization $||\mathcal{H}||$, described in Section 2.

Given a point $v_0 \in ||\mathcal{H}||$, the *fundamental group* $\pi_1(||\mathcal{H}||, v_0)$ is the group having as elements the homotopy equivalence classes of closed paths

the product being determined by the concatenation of paths.

If $\mathcal{H}$ is connected, then $\pi_1(||\mathcal{H}||, v_0) \cong \pi_1(||\mathcal{H}||, w_0)$ for all points v_0, w_0 in $||\mathcal{H}||$, hence we may use the notation $\pi_1(||\mathcal{H}||)$ without ambiguity. We produce now a presentation for $\pi_1(||\mathcal{H}||)$. This combinatorial description is also known as the *edge-path group* of $\mathcal{H}$ (for details on the fundamental group of a simplicial complex, see [20]).

We fix a spanning tree T of $\mathcal{H}$ and we define

$$A = \{a_{pq} \mid pq \in H \cap P_2(V)\},$$

$$R_T = \{a_{qp}a_{pq}^{-1} \mid pq \in H \cap P_2(V)\} \cup \{a_{pq}a_{qr}a_{pr}^{-1} \mid pqr \in H \cap P_3(V)\} \cup \{a_{pq} \mid pq \in T\}.$$

From now on, we view $\pi_1(||\mathcal{H}||)$ as the group defined by the group presentation

$$\langle A \mid R_T \rangle. \tag{4}$$

We denote by $\theta : F_A \to \pi_1(||\mathcal{H}||)$ the canonical homomorphism. We note that the six relators induced by a single 2-face pqr (corresponding to different enumerations of the vertices) are all equivalent to $a_{pq}a_{qr}a_{pr}$: each one of them is a conjugate of either $a_{pq}a_{qr}a_{pr}$ or its inverse.

Given a boolean representable connected simplicial complex $\mathcal{H} = (V, H)$, the *graph of flats* $\Gamma\text{Fl}\mathcal{H}$ has vertex set V and edges p — q whenever $p \neq q$ and $\overline{pq} \subset V$.

Lemma 3.2 *Let $\mathcal{H} = (V, H)$ be a boolean representable connected simplicial complex. Let $u, v \in V$ be distinct non adjacent vertices of $\Gamma\mathrm{Fl}\mathcal{H}$. Then $uv \in H$.*

Proof. Since $|V| > 1$ and $\mathcal{H}$ is connected, there exists some $pq \in H \cap P_2(V)$. Suppose that $uv \notin H$. By (2), we get $\overline{u} = \overline{uv} = \overline{v}$. Since there is no edge u — v in $\Gamma\mathrm{Fl}\mathcal{H}$, we get $\overline{u} = V$. By (3), we get $\overline{p} = \overline{q} = \overline{u} = V$. In view of (2), this contradicts $pq \in H$. $\square$

Let C be a connected component of $\Gamma\mathrm{Fl}\mathcal{H}$. If $H \cap P_2(C) \neq \emptyset$, we shall say that C is *H-nontrivial*. Otherwise, we say that C is *H-trivial*. The *size* of C is its number of vertices.

If $\mathcal{H}$ is a connected simplicial complex of dimension ≤ 1 (i.e. a graph), then (4) is a presentation of a free group, its rank equal to the number of edges of the graph that are not in T.

The next result shows that the graph of flats and the size of its H-disconnected components determines completely the fundamental group for dimension ≥ 2.

Theorem 3.3 *Let $\mathcal{H}$ be a boolean representable simplicial complex of dimension ≥ 2. Assume that $\Gamma\mathrm{Fl}\mathcal{H}$ has s H-nontrivial connected components and r H-trivial connected components of sizes $f_1, \ldots, f_r$. Then $\pi_1(||\mathcal{H}||)$ is a free group of rank*

$$\binom{s + f_1 + \ldots + f_r - 1}{2} - \sum_{i=1}^{r} \binom{f_i}{2},$$

or equivalently,

$$\binom{s-1}{2} + (s-1)(f_1 + \ldots + f_r) + \sum_{1 \leq i < j \leq r} f_i f_j.$$

Proof. Let $\mathcal{H} = (V, H)$ and $\Gamma = \Gamma\mathrm{Fl}\mathcal{H}$. Since $\mathcal{H}$ has dimension ≥ 2, there exists some $xyz \in H \cap P_3(V)$. Since $\mathcal{H}$ is boolean representable, we may assume by (1) that $\overline{yz} \subset V$, hence y — z is an edge of Γ. In view of (2), we may also assume that $y \notin \overline{z}$.

Let

$$Z = \{p \in V \setminus \{z\} \mid pz \in H\}.$$

Note that $y \in Z$. Now let

$$T = \{pz \mid p \in Z\} \cup \{yq \mid q \in V \setminus (Z \cup \{z\})\}.$$

We claim that T is a spanning tree of $\mathcal{H}$.

Indeed, suppose that $q \in V \setminus (Z \cup \{z\})$. Then $qz \notin H$ and so $\overline{q} = \overline{qz} = \overline{z}$. Since $y \notin \overline{z}$, we get $y \notin \overline{q}$, hence $yq \in H$ and so $T \subseteq H \cap P_2(V)$. Now T has precisely $|V| - 1$ edges and every vertex of V occurs in some edge of T. Therefore T is a spanning tree of $\mathcal{H}$.

We consider now the finite presentation (4) of $\pi_1(||\mathcal{H}||)$ induced by the spanning tree T. Our goal is to use a sequence of *Tietze transformations* (see [16]) to obtain a presentation that can be seen to be that of the free group in the statement of the theorem. This requires some preliminary work.

To understand the procedure, it may help to consider two partial colourings of the complete graph K_V on the vertex set V. An edge pq has color red if $pq \in H$, and color blue if $pq \in E(\Gamma)$. Note that an edge may have one color, both or none. Then the H-nontrivial components of Γ are the blue components of K_V which contain a red edge. The strategy for the Tietze transformation reduction consists then in establishing the following facts:

- every blue-and-red edge represents the identity in $\pi_1(||\mathcal{H}||)$ (5);

- if two vertices are red-adjacent and blue-connected, then they can be connected by a path of blue-and-red edges (7);

- every red edge with endpoints connected by a path of blue-and-red edges represents the identity in $\pi_1(||\mathcal{H}||)$ (8).

In terms of $\pi_1(||\mathcal{H}||)$, this will allow huge simplification inside each H-nontrivial component, but no simplification will take place at the H-trivial components. This is the reason for the asymmetry in the statement of the theorem.

Let $\theta : F_A \to \pi_1(||\mathcal{H}||)$ denote the canonical homomorphism. We show that

$$pq \in E(\Gamma) \cap H \Rightarrow a_{pq}\theta = 1. \tag{5}$$

Suppose first that $z \notin \overline{pq}$. Then $pqz \in H$, hence $p, q \in Z$ and we get

$$a_{pq}\theta = (a_{zp}a_{pq}a_{zq}^{-1})\theta = 1.$$

Thus we may assume that $z \in \overline{pq}$.

Suppose that $y \notin \overline{pq}$. Then $pqy \in H$. We claim that

$$a_{yp}\theta = a_{yq}\theta = 1. \tag{6}$$

If $p \in V \setminus Z$, then $yp \in T$ and so $a_{yp}\theta = 1$. If $p \in Z$, then $pz \in T$. Since $\overline{pz} \subseteq \overline{pq}$ yields $y \notin \overline{pz}$, we get $yzp \in H$ and so

$$a_{yp}\theta = (a_{yz}a_{zp})\theta = 1.$$

Similarly, $a_{yq}\theta = 1$ and so (6) holds.

Now $pqy \in H$ yields

$$a_{pq}\theta = (a_{py}a_{yq})\theta = (a_{yp}^{-1}a_{yq})\theta = 1.$$

So finally we may assume that $z, y \in \overline{pq}$. Let $v \in V \setminus \overline{pq}$ (note that $\overline{pq} \subset V$ since $pq \in E(\Gamma)$). We prove that $a_{pv}\theta = 1$ by considering two cases. If $\overline{p} \neq \overline{z}$, then $pzv \in H$ and so $a_{pv}\theta = (a_{pz}a_{zv})\theta = 1$. Hence we assume that $\overline{p} = \overline{z}$. Now $yzv \in H$ yields $a_{yv}\theta = (a_{yz}a_{zv})\theta = 1$, and $pyv \in H$ (which holds since $\overline{p} = \overline{z}$ implies $\overline{p} \neq \overline{y}$) yields $a_{pv}\theta = (a_{py}a_{yv})\theta = 1$ (since $py \in T$).

Hence $a_{pv}\theta = 1$ and by symmetry also $a_{qv}\theta = 1$. Finally, $pqv \in H$ yields $a_{pv}\theta = (a_{pq}a_{qv})\theta$ and thus $a_{pq}\theta = 1$. Therefore (5) holds.

Let $C_1, \ldots, C_s$ (respectively $C'_1, \ldots, C'_r$) denote the H-nontrivial (respectively H-trivial) connected components of Γ. We assume also that C'_i has size f_i for $i = 1, \ldots, r$.

We say that two vertices $p, q \in C_i$ are *H-connected* if there exists a path

$$p = p_0 \text{ --- } p_1 \text{ --- } \ldots \text{ --- } p_n = q$$

in C_i with $n \geq 0$ and $p_{j-1}p_j \in H$ for $j = 1, \ldots, n$.

We claim that

$$pq \in H \cap P_2(C_i) \Rightarrow p \text{ and } q \text{ are } H\text{-connected} \tag{7}$$

holds for $i = 1, \ldots, s$.

Let d denote the geodesic distance on C_i. We show that $p, q \in C_i$ are H-connected using induction on $d(p, q)$.

The case $d(p,q) \leq 1$ is trivial, hence we assume that $d(p,q) = n > 1$ and (8) holds for closer vertices. Take $p', p'' \in C_i$ such that $d(p,p') = n-2$ and $d(p',p'') = d(p'',q) = 1$:

$$p \text{ ——— } p' \text{ —— } p'' \text{ —— } q$$

Suppose that $p''q \notin H$. Then $\overline{p''} = \overline{p''q} = \overline{q}$. It follows that $\overline{p'q} = \overline{p'p''} \subset V$ and so there exists an edge p' —— q in Γ, contradicting $d(p,q) = n$.

Thus $p''q \in H$. Since $d(p,q) > 1$, we have $p \notin \overline{p''q} \subset V$. Hence $pp'' \in H$. But $d(p,p'') = n-1$, so by the induction hypothesis p and p'' are H-connected. Since $p''q \in H$, it follows that p and q are H-connected. Therefore (7) holds.

We show next that

$$pq \in H \cap P_2(C_i) \Rightarrow a_{pq}\theta = 1 \tag{8}$$

holds for $i = 1, \ldots, s$.

We use induction on $d(p,q)$. The case $d(p,q) = 1$ follows from (5), hence we assume that $d(p,q) = n > 1$ and (8) holds for closer vertices. Take $p', p'' \in C_i$ as in the proof of (7). By that same proof, we must have $p''q \in H$. Since $d(p,q) > 1$, we have $p \notin \overline{p''q}$. Hence $pp''q \in H$ and so $pp'', p''q \in H$. By the induction hypothesis, we get $a_{pp''}\theta = a_{p''q}\theta = 1$. But now $pp''q \in H$ yields $a_{pq}\theta = (a_{pp''}a_{p''q})\theta = 1$. Therefore (8) holds.

Now we may use (8) to simplify the group presentation $\langle A \mid R_T \rangle$. In view of (8), we start by adding as relators all the $a_{pq} \in A$ such that p, q belong to the same C_i.

For $i = 1, \ldots, s$, we fix some vertex $c_i \in C_i$. We may assume without loss of generality that $c_1 = z$. Given $p \in V$, we write $\widehat{p} = c_i$ if $p \in C_i$. We define

$$\begin{aligned} R' = \; & \{a_{qp}a_{pq}^{-1} \mid pq \in H \cap P_2(V)\} \cup \{a_{pq} \mid pq \in T\} \\ & \cup \{a_{pq} \mid pq \in H \cap P_2(C_i),\ i \in \{1,\ldots,s\}\} \\ & \cup \{a_{pq}a_{\widehat{p}\widehat{q}}^{-1} \mid p \in C_i,\ q \in C_j,\ i,j \in \{1,\ldots,s\},\ i \neq j\} \\ & \cup \{a_{pq}a_{\widehat{p}q}^{-1} \mid p \in C_i,\ q \in C'_j,\ i \in \{1,\ldots,s\},\ j \in \{1,\ldots,r\}\} \\ & \cup \{a_{pq}a_{p\widehat{q}}^{-1} \mid p \in C'_j,\ q \in C_i,\ i \in \{1,\ldots,s\},\ j \in \{1,\ldots,r\}\}. \end{aligned}$$

In view of Lemma 3.2, R' is well defined. We show that $\langle\langle R' \rangle\rangle = \langle\langle R_T \rangle\rangle$.

We show first that $R' \subseteq \langle\langle R_T \rangle\rangle$. In view of (8), we only need to discuss the last three terms of the union.

We start by proving that

$$a_{pq}\theta = a_{\widehat{p}q}\theta \tag{9}$$

whenever $p \in C_i$ and $q \notin C_i$. We may assume that $p \neq \widehat{p}$. By (7), there exists a path

$$p = p_0 \text{ —— } p_1 \text{ —— } \ldots \text{ —— } p_n = \widehat{p}$$

in C_i with $n \geq 1$ and $p_{k-1}p_k \in H$ for $k = 1, \ldots, n$. By Lemma 3.2, we have $p_kq \in H$ for every k. Also $\overline{p_{k-1}p_k} \subset V$ for $k = 1, \ldots, n$. Suppose that $q \in \overline{p_{k-1}p_k}$. Then $\overline{p_kq} \subset V$ and $q \in C_i$, a contradiction. Hence $q \notin \overline{p_{k-1}p_k}$. Since $p_{k-1}p_k \in H$, it follows that $p_{k-1}p_kq \in H$ and in view of (8) we get

$$a_{p_{k-1}q}\theta = (a_{p_{k-1}p_k}a_{p_kq})\theta = a_{p_kq}\theta.$$

Now (9) follows by transitivity.

Similarly,

$$a_{pq}\theta = a_{p\widehat{q}}\theta \tag{10}$$

whenever $q \in C_i$ and $p \notin C_i$.

Finally, if $p \in C_i$ and $q \in C_j \neq C_i$, we may apply (9) and (10) to get $a_{pq}\theta = a_{\widehat{p}q}\theta = a_{\widehat{p}\widehat{q}}\theta$. Therefore $R' \subseteq \langle\langle R_T \rangle\rangle$ and so $\langle\langle R' \rangle\rangle \subseteq \langle\langle R_T \rangle\rangle$.

To prove the opposite inclusion, let $\theta' : F_A \to F_A/\langle\langle R' \rangle\rangle$ denote the canonical homomorphism. It suffices to show that $(a_{pq}a_{qr}a_{pr}^{-1})\theta' = 1$ for every $pqr \in H \cap P_3(V)$.

Since $\mathcal{H}$ is boolean representable and $pqr \in H$, one of the three elements p, q, r is not in the closure of the other two. We remarked before that each one of the six relators of R_T arising from distinct enumerations of the elements of p, q, r is a conjugate of $a_{pq}a_{qr}a_{pr}^{-1}$ or its inverse, hence we may assume that $r \notin \overline{pq}$. Hence there exists an edge p — q in Γ and so $p, q \in C_i$ for some $i \in \{1, \ldots, s\}$.

Suppose that $r \in C_i$. Since $pq, qr, pr \in H$, we get $a_{pq}\theta' = a_{qr}\theta' = a_{pr}\theta' = 1$ and so $(a_{pq}a_{qr}a_{pr}^{-1})\theta' = 1$.

Thus we may assume that $r \notin C_i$. If $r \notin C'_1 \cup \ldots \cup C'_r$, then

$$a_{qr}\theta' = a_{\widehat{q}\widehat{r}}\theta' = a_{\widehat{p}\widehat{r}}\theta' = a_{pr}\theta'.$$

The case $r \in C'_1 \cup \ldots \cup C'_r$ is analogous. Since $pq \in H \cap P_2(C_i)$ yields $a_{pq}\theta' = 1$, we get $(a_{pq}a_{qr}a_{pr}^{-1})\theta' = 1$. Therefore $\langle\langle R' \rangle\rangle = \langle\langle R_T \rangle\rangle$.

Now we simplify the presentation $\langle A \mid R' \rangle$ by means of further Tietze transformations.

The third term of the union in R' ensures that we may omit all generators with both indices in the same connected components, and the three last terms allow us to restrict ourselves to generators with indices in $\{c_1, \ldots, c_s\} \cup C'_1 \cup \ldots \cup C'_r$. Since $y, z \in C_1$, the second term allows us to eliminate all the generators where $c_1 = z$ appears as index, and we may now use the first term relators to remove half of the remaining generators, ending up with the free group on the set

$$\begin{aligned} B = &\{a_{c_ic_j} \mid 2 \leq i < j \leq s\} \\ &\cup \{a_{c_iq} \mid 2 \leq i \leq s,\ q \in C'_1 \cup \ldots \cup C'_r\} \\ &\cup \{a_{pq} \mid p \in C'_i,\ q \in C'_j\ 1 \leq i < j \leq r\} \end{aligned}$$

Now

$$|B| = \binom{s-1}{2} + (s-1)(f_1 + \ldots + f_r) + \sum_{1 \leq i < j \leq r} f_i f_j.$$

On the other hand, we have

$$\begin{aligned} \binom{s+f_1+\ldots+f_r-1}{2} &= \tfrac{(s-1+f_1+\ldots+f_r)(s-2+f_1+\ldots+f_r)}{2} \\ &= \tfrac{(s-1)(s-2)}{2} + (s-1)(f_1 + \ldots + f_r) + \tfrac{(f_1+\ldots+f_r)(f_1+\ldots+f_r-1)}{2} \\ &= \binom{s-1}{2} + (s-1)(f_1 + \ldots + f_r) + \textstyle\sum_{1 \leq i < j \leq r} f_i f_j + \frac{\sum_{i=1}^r (f_i^2 - f_i)}{2} \\ &= \binom{s-1}{2} + (s-1)(f_1 + \ldots + f_r) + \textstyle\sum_{1 \leq i < j \leq r} f_i f_j + \sum_{i=1}^r \binom{f_i}{2}, \end{aligned}$$

proving the theorem. $\square$

Given a lattice L with top element 1 and bottom element 0, write $L^* = L \setminus \{0, 1\}$ (the *proper part* of L) and define a graph $\Delta L^* = (L^*, UH_{L^*})$, where UH_{L^*} denotes the set of undirected edges in the Hasse diagram of L^*. More formally, we can define UH_{L^*} as the set of all edges a — b such that a covers b in L^* (i.e. $a > b$ and there exists no $c \in L^*$ such that $a > c > b$).

Corollary 3.4 *Let $\mathcal{H}$ be a boolean representable simple simplicial complex of dimension ≥ 2. Then $\pi_1(||\mathcal{H}||)$ is a free group of rank $\binom{t-1}{2}$, where t denotes the number of connected components of $\Gamma \mathrm{Fl}\mathcal{H}$. This number is also equal to the number of connected components of $\Delta(\mathrm{Fl}\mathcal{H})^*$.*

Proof. If $\mathcal{H} = (V, H)$ is simple, then each H-trivial connected component of $\Gamma\text{Fl}\mathcal{H}$ has precisely one vertex. Hence, by Theorem 3.3, $\pi_1(||\mathcal{H}||)$ is a free group of rank $\binom{t-1}{2}$.

Note that, since $\mathcal{H}$ is simple, then $P_1(V) \subseteq \text{Fl}\mathcal{H}$ (so all points of $\mathcal{H}$ belong to $(\text{Fl}\mathcal{H})^*$).

Let $p, q \in V$ be adjacent in $\Gamma\text{Fl}\mathcal{H}$. Then $\overline{pq} \subset V$ and so $\overline{pq}$ is the join of p and q in $\Delta(\text{Fl}\mathcal{H})^*$. It follows that each connected component of $\Gamma\text{Fl}\mathcal{H}$ is contained in the union of the points of some connected component of $\Delta(\text{Fl}\mathcal{H})^*$.

On the other hand, if $F — F'$ is an edge of $\Delta(\text{Fl}\mathcal{H})^*$ (say, with $F \subset F'$), then F' is a clique of $\Gamma\text{Fl}\mathcal{H}$ (i.e. induces a complete subgraph). It follows easily that the union of the points of a connected component of $\Delta(\text{Fl}\mathcal{H})^*$ belong to the same connected component of $\Gamma\text{Fl}\mathcal{H}$.

Since every connected component of $\Delta(\text{Fl}\mathcal{H})^*$ contains necessarily a point, the number of connected components must coincide in both graphs. □

We show next that free groups of rank $\binom{n}{2}$ $(n \geq 2)$ occur effectively as fundamental groups of boolean representable simplicial complexes of dimension 2, even in the simple case.

Example 3.5 *Let $t \geq 3$. Let $\mathcal{H} = (V, H)$ be defined by $V = \{a_1, b_1, a_2, b_2, \ldots, a_t, b_t\}$ and*

$$H = P_{\leq 2}(V) \cup \{X \in P_3(V) \mid a_i b_i \subset X \text{ for some } i \in \{1, \ldots, t\}\}.$$

Then $\mathcal{H}$ is a boolean representable simple simplicial complex of dimension 2 and $\pi_1(||\mathcal{H}||) \cong F_{\binom{t-1}{2}}$.

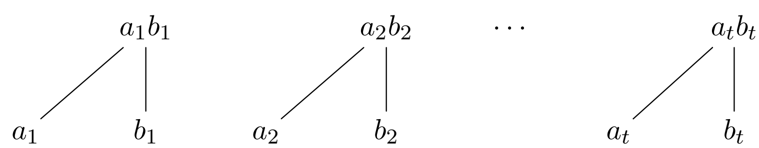
Indeed, it is easy to check that

$$\text{Fl}\mathcal{H} = P_{\leq 1}(V) \cup \{a_1 b_1, a_2 b_2, \ldots, a_t b_t, V\},$$

hence every face of $\mathcal{H}$ is a transversal of the successive differences for some chain in $\text{Fl}\mathcal{H}$. Thus $\mathcal{H}$ is boolean representable. Clearly, the graph of flats of $\mathcal{H}$ is

$$a_1 — b_1, \quad a_2 — b_2, \quad \ldots \quad a_t — b_t,$$

hence it possesses t connected components. Therefore $\pi_1(||\mathcal{H}||) \cong F_{\binom{t-1}{2}}$ by Corollary 3.4. Note also that $\Delta(\text{Fl}\mathcal{H})^*$ is



By shellability of matroids, every matroid $\mathcal{H} = (V, H)$ of dimension $d \geq 2$ has the homotopy type of a wedge of spheres of dimension d. In particular, its fundamental group is trivial. We note that this fact also follows from the preceding theorem, since $\Gamma\text{Fl}\mathcal{H}$ is a complete graph. Indeed, given $p, q \in V$ distinct, it is well known (see e.g. [18, Proposition 4.2.5(ii)]) that

$$\overline{pq} = pq \cup \{r \in V \setminus pq \mid I \cup \{r\} \notin H \text{ for some } I \in H \cap 2^{pq}\}.$$

Since every matroid is pure and $\dim\mathcal{H} \geq 2$, pq cannot be a facet and so $\overline{pq} \subset V$. Thus $\Gamma\text{Fl}\mathcal{H}$ has a single connected component and so $\pi_1(||\mathcal{H}||)$ is trivial by Theorem 3.3.

Theorem 3.3 also yields the following consequence, one of the main theorems of the paper.

Theorem 3.6 *Let $\mathcal{H}$ be a boolean representable simplicial complex of dimension 2. Then:*

(i) the homology groups of $\mathcal{H}$ are free abelian;

(ii) $\mathcal{H}$ has the homotopy type of a wedge of 1-spheres and 2-spheres.

Proof. (i) It follows from Lemma 3.1 that $\mathcal{H}$ is connected. By Hurewicz Theorem (see [8]), the 1st homology group of $\mathcal{H}$ is the abelianization of $\pi_1(||\mathcal{H}||)$, and therefore, in view of Theorem 3.3, a free abelian group of known rank. The second homology group of any 2-dimensional simplicial complex is $\operatorname{Ker} \partial_2 \leq C_2(\mathcal{H})$, that is, a subgroup of a free abelian group. Therefore $H_2(\mathcal{H})$ is itself free abelian.

(ii) By [23, Proposition 3.3], any finite 2-dimensional simplicial complex with free fundamental group has the homotopy type of a wedge of 1-spheres and 2-spheres. $\square$

4 The simplification of a complex

Let $\mathcal{H} = (V, H)$ and $\mathcal{H}' = (V', H')$ be simplicial complexes. A *simplicial map* from $\mathcal{H}$ to $\mathcal{H}'$ is a mapping $\varphi : V \to V'$ such that $X\varphi \in H'$ for every $X \in H$ (that is, φ sends simplices to simplices). This simplicial map is *rank-preserving* if $|X\varphi| = |X|$ for every $X \in H$.

Let $\mathcal{H} = (V, H) \in \mathcal{BR}$. We define an equivalence relation $\eta_{\mathcal{H}}$ on V by

$$a \eta_{\mathcal{H}} b \text{ if } \overline{a} = \overline{b}.$$

If no confusion arises, we omit the index from $\eta_{\mathcal{H}}$.

It follows from (2) that $a\eta b$ if and only if $ab \notin H$. If M is a boolean matrix representation of $\mathcal{H}$, it is easy to see that $a\eta b$ if and only if the column vectors $M[\_, a]$ and $M[\_, b]$ are equal. Indeed, $M[\_, a] = M[\_, b]$ implies $ab \notin H$ trivially and the converse follows from the fact that there exist no zero columns in M (since $P_1(V) \subseteq H$). Note also that (3) implies that $\overline{p} = p\eta$ for every $p \in V$.

The following lemma enhances the role played by η in the context of rank-preserving simplicial maps.

Lemma 4.1 *Let $\mathcal{H} = (V, H) \in \mathcal{BR}$ and let τ be an equivalence relation on V. Then the following conditions are equivalent:*

(i) τ is the kernel of some rank-preserving simplicial map $\varphi : \mathcal{H} \to \mathcal{H}'$ into some simplicial complex $\mathcal{H}'$;

(ii) $\tau \subseteq \eta_{\mathcal{H}}$.

Proof. (i) $\Rightarrow$ (ii). Let $a, b \in V$ and suppose that $(a, b) \notin \eta$. Then $\overline{a} \neq \overline{b}$ and so $ab \in H$. Since φ is a rank-preserving simplicial map, it follows that $a\varphi \neq b\varphi$ and so $(a, b) \notin \tau$. Thus $\tau \subseteq \eta$.

(ii) $\Rightarrow$ (i). We define a simplicial complex $\mathcal{H}/\tau = (V/\tau, H/\tau)$, where

$$H/\tau = \{\{a_1\tau, \ldots, a_k\tau\} \mid a_1 \ldots a_k \in H\}.$$

Let $\varphi : V \to V/\tau$ denote the canonical projection. By definition, φ is a simplicial map. We claim that

$$\varphi \text{ is rank-preserving.} \tag{11}$$

Indeed, every (nonempty) $X \in H$ admits an enumeration $x_1, \ldots, x_k$ satisfying (1) and so $\overline{x_i} \neq \overline{x_j}$ whenever $i \neq j$. Thus

$$x_i \tau x_j \Rightarrow x_i \eta x_j \Rightarrow \overline{x_i} = \overline{x_j} \Rightarrow i = j$$

and so $|X\varphi| = |X|$. Thus (11) holds and so τ is the kernel of some rank-preserving simplicial map. $\square$

Note that, if $\tau \subseteq \eta$, it follows from the characterization of H in (1) that

$$\text{if } a_i \tau b_i \text{ for } i = 1, \ldots, k, \text{ then } a_1 \ldots a_k \in H \text{ if and only if } b_1 \ldots b_k \in H. \tag{12}$$

We collect in the next result some of the properties of the simplicial complexes $\mathcal{H}/\tau$ (using the notation introduced in the proof of Lemma 4.1).

Proposition 4.2 *Let $\mathcal{H} = (V, H) \in \mathcal{BR}$ and let $\tau \subseteq \eta$ be an equivalence relation on V. Let $\varphi : V \to V/\tau$ denote the canonical projection. Then:*

(i) $\dim(\mathcal{H}/\tau) = \dim\mathcal{H}$*;*

(ii) $\mathrm{Fl}\mathcal{H} = \{F\varphi^{-1} \mid F \in \mathrm{Fl}(\mathcal{H}/\tau)\}$*;*

(iii) $\mathrm{Fl}\mathcal{H} \cong \mathrm{Fl}(\mathcal{H}/\tau)$*;*

(iv) $\mathcal{H}/\tau$ is boolean representable;

(v) $\mathcal{H}/\tau$ is simple if and only if $\tau = \eta$;

(vi) $\mathcal{H}$ is pure if and only if $\mathcal{H}/\tau$ is pure;

(vii) $\mathcal{H}$ is a matroid if and only if $\mathcal{H}/\tau$ is a matroid;

(viii) if $v, w \in V$ are such that $v\tau \neq w\tau$, then $v — w$ is an edge of $\Gamma\mathrm{Fl}\mathcal{H}$ if and only if $v\tau — w\tau$ is an edge of $\Gamma\mathrm{Fl}(\mathcal{H}/\tau)$;

(ix) for every $X \subseteq V$,

$$X \in \mathrm{fct}\mathcal{H} \textit{ if and only if } (\varphi|_X \textit{ is injective and } X\varphi \in \mathrm{fct}(\mathcal{H}/\tau)).$$

(x) if $\mathcal{H}/\tau$ is shellable, so is $\mathcal{H}$.

Proof. (i) It follows from the definition of H/τ and (11).

(ii) Let $F \in \mathrm{Fl}(\mathcal{H}/\tau)$. Let $X \in H \cap 2^{F\varphi^{-1}}$ and $p \in V \setminus F\varphi^{-1}$. Then $X\varphi \in (H/\tau) \cap 2^F$ and $p\tau \in (V/\tau) \setminus F$, hence $F \in \mathrm{Fl}(\mathcal{H}/\tau)$ yields $X\varphi \cup \{p\tau\} \in H/\tau$. Since the elements of $X\varphi \cup \{p\tau\}$ are all distinct, it follows easily from (12) that $X \cup \{p\} \in H$. Thus $F\varphi^{-1} \in \mathrm{Fl}\mathcal{H}$.

To prove the opposite inclusion, we start by showing that

$$\text{if } Z \in \mathrm{Fl}\mathcal{H}, \text{ then } Z\varphi \in \mathrm{Fl}(\mathcal{H}/\tau). \tag{13}$$

Let $Y \in (H/\tau) \cap 2^{Z\varphi}$ and $p\tau \in (V/\tau) \setminus (Z\varphi)$. We may write $Y = X\varphi$ for some $X \in H$. Since $a\varphi\varphi^{-1} \subseteq \overline{a}$ for every $a \in V$, we have $Z\varphi\varphi^{-1} \subseteq Z$. Hence $X \in H \cap 2^Z$. On the other hand, $p\tau \in (V/\tau) \setminus (Z\varphi)$ implies $p \in V \setminus Z$. Since $Z \in \mathrm{Fl}\mathcal{H}$, we get $X \cup \{p\} \in H$ and so $Y \cup \{p\tau\} \in H/\tau$. Therefore $Z\varphi \in \mathrm{Fl}(\mathcal{H}/\tau)$ and so (13) holds.

Let $Z \in \mathrm{Fl}\mathcal{H}$. Since we have already remarked that $Z\varphi\varphi^{-1} \subseteq Z$ and the opposite inclusion holds trivially, we get $Z = Z\varphi\varphi^{-1} \in \{F\varphi^{-1} \mid F \in \mathrm{Fl}(\mathcal{H}/\tau)\}$.

(iii) By part (ii), the mapping

$$\begin{aligned} \mathrm{Fl}(\mathcal{H}/\tau) &\to \mathrm{Fl}\mathcal{H} \\ F &\mapsto F\varphi^{-1} \end{aligned}$$

is bijective, and is clearly a poset isomorphism. Therefore it is a lattice isomorphism.

(iv) Let $X \in H$ so that $X\varphi \in H/\tau$. In view of (11) and part (ii), there exists some enumeration $x_1, \ldots, x_k$ of the elements of X and some $F_0, \ldots, F_k \in \mathrm{Fl}(H/\tau)$ such that

$$F_0\varphi^{-1} \subset F_1\varphi^{-1} \subset \ldots \subset F_k\varphi^{-1}$$

and $x_i \in (F_i\varphi^{-1}) \setminus (F_{i-1}\varphi^{-1})$ for $i = 1, \ldots, k$. It follows that $F_0 \subset \ldots \subset F_k$ and $x_i\varphi \in F_i \setminus F_{i-1}$ for every i, hence $X\varphi$ is a transversal of the successive differences for a chain in $\mathrm{Fl}(\mathcal{H}/\tau)$. Therefore $\mathcal{H}/\tau$ is boolean representable.

(v) Given $X \subseteq V$, let $\mathrm{Cl}_\tau(X\varphi)$ denote the closure of $X\varphi$ in $\mathcal{H}/\tau$. We show that

$$\mathrm{Cl}_\tau(X\varphi) = \overline{X}\varphi. \tag{14}$$

Indeed, by (13) we have $\overline{X}\varphi \in \mathrm{Fl}(H/\tau)$, and trivially $X\varphi \subseteq \overline{X}\varphi$. Suppose now that $F \in \mathrm{Fl}(H/\tau)$ contains $X\varphi$. By part (ii), we have $X \subseteq F\varphi^{-1} \in \mathrm{Fl}\mathcal{H}$, hence $\overline{X} \subseteq F\varphi^{-1}$ by minimality and so $\overline{X}\varphi \subseteq F$. Therefore (14) holds.

Suppose now that $(a, b) \in \eta \setminus \tau$. Then (14) yields $\mathrm{Cl}_\tau(a\varphi) = \overline{a}\varphi = \overline{b}\varphi = \mathrm{Cl}_\tau(b\varphi)$ and so $\{a\tau, b\tau\} \notin H\tau$ by (2). Therefore $\mathcal{H}/\tau$ is not simple.

Finally, assume that $\tau = \eta$. Let $a, b \in V$ be such that $a\eta \neq b\eta$. Then $\overline{a} \neq \overline{b}$ and by (2) we get $ab \in H$. Hence $\{a\eta, b\eta\} \in H/\eta$ and so $\mathcal{H}/\eta$ is simple.

(vi) Considering transversals of successive differences, it is immediate that a boolean representable simplicial complex is pure if and only if its lattice of flats satisfies the Jordan-Dedekind condition (all the maximal chains have the same length). Now we use part (iii).

(vii) It is well known that $\mathcal{H}$ is a matroid if and only if $\mathrm{Fl}\mathcal{H}$ is geometric [17, Theorem 1.7.5]. Now we use part (iii).

(viii) Assume that v — w is an edge of $\Gamma\mathrm{Fl}\mathcal{H}$. By part (ii), there exists some $F \in \mathrm{Fl}(\mathcal{H}/\tau)$ such that $vw \subseteq F\varphi^{-1} \subset V$. It follows that $\{v\tau, w\tau\} \subseteq F \subset V/\tau$, hence $v\tau$ — $w\tau$ is an edge of $\Gamma\mathrm{Fl}(\mathcal{H}/\tau)$.

Conversely, assume that $v\tau$ — $w\tau$ is an edge of $\Gamma\mathrm{Fl}(\mathcal{H}/\tau)$. Then there exists some $F \in \mathrm{Fl}(\mathcal{H}/\tau)$ such that $\{v\tau, w\tau\} \subseteq F \subset V/\tau$. Hence $vw \subseteq F\varphi^{-1} \subset V$. Since $F\varphi^{-1} \in \mathrm{Fl}\mathcal{H}$ by part (ii), it follows that v — w is an edge of $\Gamma\mathrm{Fl}\mathcal{H}$.

(ix) Let $X \in \mathrm{fct}\mathcal{H}$. Then $X\varphi \in H/\tau$ and $\varphi|_X$ is injective by (11). Suppose that $X\varphi \subset Y$ for some $Y \in H/\tau$. We may write $Y = X\varphi \cup Z\varphi$ with Z minimal. It follows from the minimality of Z that $\varphi|_{X\cup Z}$ is injective, hence $X \cup Z \in H$ in view of (12), contradicting $X \in \mathrm{fct}\mathcal{H}$. Therefore $X\varphi \in \mathrm{fct}(\mathcal{H}/\tau)$.

Conversely, assume that $\varphi|_X$ is injective and $X\varphi \in \mathrm{fct}(\mathcal{H}/\tau)$. In view of (12), we have $X \in H$. Suppose that $X \cup \{p\} \in H$ with $p \in V \setminus X$. By (11), $\varphi|_{X\cup\{p\}}$ is injective and $(X \cup \{p\})\varphi \in H/\tau$, hence $X\varphi \subset (X \cup \{p\})\varphi \in H$, contradicting $X\varphi \in \mathrm{fct}(\mathcal{H}/\tau)$. Therefore $X \in \mathrm{fct}\mathcal{H}$ and the equivalence holds.

(x) We may assume that $|V| = |V/\tau| + 1$, and then apply this case successively. Assume that $\{a_1, a_2\}$ is the only nonsingular τ-class.

Let $B_1, \ldots, B_t$ be a shelling of $\mathcal{H}/\tau$. For $k = 1, 2$, let $\psi_k : V/\tau \to V$ be defined by

$$x\varphi\psi_k = \begin{cases} a_k & \text{if } x \in \{a_1, a_2\} \\ x & \text{otherwise} \end{cases}$$

Consider the sequence

$$B_1\psi_1, B_1\psi_2, B_2\psi_1, B_2\psi_2, \ldots, B_t\psi_1, B_t\psi_2. \tag{15}$$

We have $B_i\psi_1 = B_i\psi_2$ if and only if $a_1\varphi \notin B_i$. To avoid repetitions, we remove from (15) all the entries $B_i\psi_2$ such that $a\varphi \notin B_i$. We refer to this sequence as *trimmed* (15).

It follows from part (ix) that trimmed (15) is an enumeration of the facets of $\mathcal{H}$. We prove it is a shelling.

Let $i \in \{2, \ldots, t\}$ and assume that $|B_i| \geq 2$. Write

$$I(B_i) = (\cup_{j=1}^{i-1} 2^{B_j}) \cap 2^{B_i}, \quad I'(B_i\psi_1) = ((\cup_{j=1}^{i-1} 2^{B_j\psi_1}) \cup (\cup_{j=1}^{i-1} 2^{B_j\psi_2})) \cap 2^{B_i\psi_1}.$$

It is immediate that $I'(B_i\psi_1) = (I(B_i))\psi_1$. Since $B_1, \ldots, B_t$ is a shelling of $\mathcal{H}/\tau$, then $(B_i, I(B_i))$ is pure of dimension $|B_i| - 2$. Thus $(B_i\psi_1, I'(B_i\psi_1))$ is pure of dimension $|B_i\psi_1| - 2$.

Assume now that $i \in \{1, \ldots, t\}$, $a_1\varphi \in B_i$ and $|B_i| \geq 2$. Write

$$I'(B_i\psi_2) = ((\cup_{j=1}^{i} 2^{B_j\psi_1}) \cup (\cup_{j=1}^{i-1} 2^{B_j\psi_2})) \cap 2^{B_i\psi_2}.$$

Assume first that $i = 1$. Then

$$I'(B_1\psi_2) = 2^{B_1 \setminus \{a_2\}},$$

hence $(B_1\psi_2, I'(B_1\psi_2))$ is pure of dimension $|B_1\psi_2| - 2$.

Thus we may assume that $i > 1$. It is easy to check that

$$I'(B_i\psi_2) = (I(B_i) \cup 2^{B_i \setminus \{a_1\varphi\}})\psi_2. \tag{16}$$

Since $(B_i, I(B_i))$ is pure of dimension $|B_i| - 2$, it follows that $(B_i, I(B_i) \cup 2^{B_i \setminus \{a_1\varphi\}})$ has also dimension $|B_i| - 2$. Since the only new facet with respect to $(B_i, I(B_i))$ is possibly $B_i \setminus \{a_1\varphi\}$, then $(B_i, I(B_i) \cup 2^{B_i \setminus \{a_1\varphi\}})$ is also pure. In view of (16), $(B_i\psi_2, I'(B_i\psi_2))$ is pure of dimension $|B_i\psi_2| - 2$. Therefore trimmed (15) is a shelling of $\mathcal{H}$ and we are done. $\square$

Part (ii) implies that the maps φ constitute a particular case of maps known in matroid theory as *strong maps* [24, Chapter 8].

We could not prove so far the converse of Proposition 4.2(x), which remains an open problem. However, it follows from Theorem 5.2 that it holds for the particular case of η and dimension 2.

From now on, and in view of part (v), we shall refer to $\mathcal{H}_S = \mathcal{H}/\eta$ as the *simplification* of $\mathcal{H}$.

The next result shows how we can produce a boolean representation for $\mathcal{H}_S$ from a boolean representation of $\mathcal{H}$.

Proposition 4.3 *Let M be an $R \times V$ boolean matrix representation of the simplicial complex $\mathcal{H} = (V, H)$. Let M' be the matrix obtained from M by removing repeated columns. Then M' is a boolean matrix representation of $\mathcal{H}_S$.*

Proof. By the remark following the definition of η, we have $a\eta b$ if and only if $M[\_, a] = M[\_, b]$. Hence we may view the column space of M' as V/η. Let $\varphi : V \to V/\eta$ denote the canonical projection.

Let $X \in H$ so that $X\varphi \in H/\eta$. Then there exists some $Y \subseteq R$ such that $M[Y, X]$ is nonsingular. Then $M[Y, X]$ has no repeated columns and so $M'[Y, X\varphi]$ is nonsingular. Thus $X\varphi$ is M'-independent.

Conversely, assume that $X' \subseteq V/\eta$ is M'-independent. Write $X' = X\varphi$ with $|X|$ minimum. Then there exists some $Y \subseteq R$ such that $M'[Y, X']$ is nonsingular. Since $|X| = |X'|$ by minimality, it follows easily that $M[Y, X]$ and $M'[Y, X']$ have the same structure, hence $M[Y, X]$ is nonsingular. Therefore $X \in H$ and so $X' = X\varphi \in H/\eta$ as required. □

We end this section by discussing how the fundamental groups of $\mathcal{H}$ and $\mathcal{H}_S$ are related.

Proposition 4.4 *Let $\mathcal{H}$ be a boolean representable simplicial complex of dimension ≥ 2. Then the following conditions are equivalent:*

(i) $\pi_1(||\mathcal{H}||) \cong \pi_1(||\mathcal{H}_S||)$*;*

(ii) every H-trivial connected components of $\Gamma\mathrm{Fl}\mathcal{H}$ *has size 1.*

Proof. We show that

$$\Gamma\mathrm{Fl}\mathcal{H} \text{ and } \Gamma\mathrm{Fl}\mathcal{H}_S \text{ have the same number of connected components.} \tag{17}$$

Let $\mathcal{H} = (V, H)$ and denote by $\varphi : V \to V/\eta$ the canonical projection. Let $C_1, \ldots, C_m \subseteq V$ denote the connected components of $\Gamma\mathrm{Fl}\mathcal{H}$ and let $C'_1, \ldots, C'_n \subseteq V/\eta$ denote the connected components of $\Gamma\mathrm{Fl}\mathcal{H}_S$.

Given $i \in \{1, \ldots, m\}$, it follows easily from Proposition 4.2(viii) that $C_i\varphi \subseteq C'_{k_i}$ for some $k_i \in \{1, \ldots, n\}$. Since $V/\eta = C_1\varphi \cup \ldots \cup C_m\varphi$, it follows that $m \geq n$.

Suppose now that $k_i = k_j$ for some distinct $i, j \in \{1, \ldots, m\}$. Take vertices v_i and v_j in C_i and C_j, respectively. If $v_i\eta \neq v_j\eta$, it follows easily from Proposition 4.2(vi) that v_i, v_j are connected by some path, a contradiction. Hence we may assume that $v_i\eta v_j$ and so $\overline{v_i} = \overline{v_j}$ in $\mathcal{H}$.

But $\mathcal{H}_S$ is simple, hence $\{v_i\eta\} \in \mathrm{Fl}\mathcal{H}_S$ and so $v_i\varphi\varphi^{-1} \in \mathrm{Fl}\mathcal{H}$ by Proposition 4.2(ii). Since $\{v_i\eta\}$ and V/η are distinct flats of $\mathcal{H}_S$, it also follows from Proposition 4.2(ii) that $v_i\varphi\varphi^{-1} \neq (V/\eta)\varphi^{-1} = V$, hence v_i — v_j should be an edge of $\Gamma\mathrm{Fl}\mathcal{H}$, a contradiction. Thus the correspondence $i \mapsto k_i$ is injective and so $m = n$.

Therefore $\Gamma\mathrm{Fl}\mathcal{H}$ and $\Gamma\mathrm{Fl}\mathcal{H}_S$ have the same number of connected components.

Assume that $\Gamma\mathrm{Fl}\mathcal{H}$ has s H-nontrivial connected components and r H-trivial connected components of sizes $f_1, \ldots, f_r$. By Theorem 3.3, $\pi_1(||\mathcal{H}||)$ is a free group of rank

$$\binom{s-1}{2} + (s-1)(f_1 + \ldots + f_r) + \sum_{1 \leq i < j \leq r} f_i f_j.$$

On the other hand, in view of (17) and Corollary 3.4, $\pi_1(||\mathcal{H}_S||)$ is a free group of rank

$$\binom{s+r-1}{2} = \frac{(s+r-1)(s+r-2)}{2} = \frac{(s-1)(s-2) + (2s-3)r + r^2}{2} = \binom{s-1}{2} + (s-1)r + \binom{r}{2}.$$

Now $(s-1)(f_1 + \ldots + f_r) \geq$ and $\sum_{1 \leq i < j \leq r} f_i f_j \geq \binom{r}{2}$, and both equalities hold if and only if $f_1 = \ldots = f_r = 1$. □

The following is one of the simplest examples with $\pi_1(||\mathcal{H}||) \neq \pi_1(||\mathcal{H}_S||)$.

Example 4.5 *Let $V = 12345$ and $H = (P_{\leq 2}(V) \setminus 45) \cup \{123, 124, 125\}$. Then $\mathcal{H} = (V, H)$ is a boolean representable simplicial complex of dimension ≥ 2 suvh that $\pi_1(||\mathcal{H}||) \not\cong \pi_1(||\mathcal{H}_S||)$.*

Indeed, it is easy to check that

$$\mathrm{Fl}\mathcal{H} = \{\emptyset, 1, 2, 3, 12, 45, V\}$$

and $\mathcal{H}$ is a boolean representable. Its graph of flats is

$$1 — 2 \qquad 3 \qquad 4 — 5,$$

hence the H-trivial connected components of $\Gamma\mathrm{Fl}\mathcal{H}$ have size 1 and 2, respectively. Now the claim follows from Proposition 4.4.

Note that there is a natural embedding of $\pi_1(||\mathcal{H}_S||)$ into $\pi_1(||\mathcal{H}||)$ (since $\mathcal{H}_S$ is isomorphic to a restriction of $\mathcal{H}$ to a cross-section of η) and this embedding splits since $\pi_1(||\mathcal{H}_S||)$ is a free factor of $\pi_1(||\mathcal{H}||)$.

5 Shellability and sequentially Cohen-Macaulay in dimension 2

We discuss in this section shellability for boolean representable simplicial complexes of dimension 2. The simple case was completely solved in [18, Theorem 7.2.8], now we generalize this theorem to arbitrary boolean representable simplicial complexes of dimension 2.

We consider also another property of topological significance, sequentially Cohen-Macaulay. It is often associated with shellability since a shellable complex is necessarily sequentially Cohen-Macaulay [5, 21]. We need to introduce a few concepts and notation before defining it.

Assume that $\dim\mathcal{H} = d$. For $m = 0, \ldots, d$, we define the complex $\mathrm{pure}_m(\mathcal{H}) = (V_m, H_m)$ to be the subcomplex of $\mathcal{H}$ generated by all the faces of $\mathcal{H}$ of dimension m. Clearly, $\mathrm{pure}_m(\mathcal{H})$ is the largest pure subcomplex of $\mathcal{H}$ of dimension m.

Write $\mathcal{H} = (V, H)$. Given $Q \in H \setminus \{V\}$, we define the *link* $\mathrm{lk}(Q)$ to be the simplicial complex $(V/Q, H/Q)$, where

$$H/Q = \{X \subseteq V \setminus Q \mid X \cup Q \in H\} \quad \text{and} \quad V/Q = \bigcup_{X \in H/Q} 2^X.$$

Here it is convenient to allow a simplicial complex to have an empty set of vertices.

In view of [6, Theorem 3.3], we say that $\mathcal{H}$ is *sequentially Cohen-Macaulay* if

$$\tilde{H}_k(\mathrm{pure}_m(\mathrm{lk}(X))) = 0$$

for all $X \in H$ and $k < m \leq d$.

We start with the following lemma.

Lemma 5.1 *Let $\mathcal{H}$ be a sequentially Cohen-Macaulay simplicial complex of dimension 2. Then the simplification $\mathcal{H}_S$ is sequentially Cohen-Macaulay.*

Proof. Write $\mathcal{H} = (V, H)$. Since $\dim \mathcal{H}_S = 2$ by Proposition 4.2(i), we have to prove the following facts:

(1) $\mathrm{pure}_2(\mathcal{H}_S)$ is connected;

(2) $\mathrm{pure}_1(\mathcal{H}_S)$ is connected;

(3) $\mathrm{pure}_1(\mathrm{lk}(v\eta))$ is connected for every $v \in V$;

(4) $\tilde{H}_1(\mathrm{pure}_2(\mathcal{H}_S)) = 0$.

We assume of course the similar statements for $\mathcal{H}$.

(1) Let $a\eta, b\eta$ denote two distinct vertices from $\mathrm{pure}_2(\mathcal{H}_S)$. Then there exist $\{a\eta, a'\eta, a''\eta\}$, $\{b\eta, b'\eta, b''\eta\} \in (H/\eta) \cap P_3(V/\eta)$. In view of (12), we have $aa'a'', bb'b'' \in H \cap P_3(V)$, hence a, b are two distinct vertices from $\mathrm{pure}_2(\mathcal{H})$. Since $\mathrm{pure}_2(\mathcal{H})$ is connected, there exists in $\mathrm{pure}_2(\mathcal{H})$ a path of the form

$$a = c_0 — c_1 — \ldots — c_n = b$$

for some $n \geq 1$. Let $i \in \{1, \ldots, n\}$. Since $c_{i-1}c_i$ is an edge of $\mathrm{pure}_2(\mathcal{H})$, there exists some c_i' such that $c_{i-1}c_ic_i' \in H \cap P_3(V)$. In view of (11), we get $\{c_{i-1}\eta, c_i\eta, c_i'\eta\} \in (H/\eta) \cap P_3(V/\eta)$. It follows that

$$a\eta = c_0\eta — c_1\eta — \ldots — c_n\eta = b\eta$$

is a path in $\mathrm{pure}_2(\mathcal{H}_S)$ and so $\mathrm{pure}_2(\mathcal{H}_S)$ is connected.

(2) Similar to (1).

(3) Let $a\eta, b\eta$ denote two distinct vertices from $\mathrm{pure}_1(\mathrm{lk}(v\eta))$. Then there exist some edges $a\eta — a'\eta$, $b\eta — b'\eta$ in $\mathrm{lk}(v\eta)$. Hence $\{a\eta, a'\eta, v\eta\}, \{b\eta, b'\eta, v\eta\} \in (H/\eta) \cap P_3(V/\eta)$. By (12), we get $aa'v, bb'v \in H \cap P_3(V)$, hence $a — a'$ and $b — b'$ are edges in $\mathrm{lk}(v)$ and so a, b are two distinct vertices from $\mathrm{pure}_1(\mathrm{lk}(v))$. Since $\mathrm{pure}_1(\mathrm{lk}(v))$ is connected, there exists in $\mathrm{pure}_1(\mathrm{lk}(v))$ a path of the form

$$a = c_0 — c_1 — \ldots — c_n = b$$

for some $n \geq 1$. Let $i \in \{1, \ldots, n\}$. Since $c_{i-1}c_i$ is an edge of $\mathrm{pure}_1(\mathrm{lk}(v))$, we have $c_{i-1}c_iv \in H \cap P_3(V)$ and so (11) yields $\{c_{i-1}\eta, c_i\eta, v\eta\} \in (H/\eta) \cap P_3(V/\eta)$. It follows that

$$a\eta = c_0\eta — c_1\eta — \ldots — c_n\eta = b\eta$$

is a path in $\mathrm{pure}_1(\mathrm{lk}(v\eta))$ and so $\mathrm{pure}_1(\mathrm{lk}(v\eta))$ is connected.

(4) Fix a cross section $V_0 \subseteq V$ for η. We consider the ordering of V/η induced by the restriction of the ordering of V to V_0.

Suppose that $\tilde{H}_1(\mathrm{pure}_2(\mathcal{H}_S)) \neq 0$. Let ∂_k (respectively ∂_k') denote the kth boundary map of $\mathrm{pure}_2(\mathcal{H})$ (respectively $\mathrm{pure}_2(\mathcal{H}_S)$). Since $\mathrm{Ker}\, \partial_1' / \mathrm{Im}\, \partial_2' = \tilde{H}_1(\mathrm{pure}_2(\mathcal{H}_S)) \neq 0$, there exist some distinct edges $X_1, \ldots, X_m$ in $\mathrm{pure}_2(\mathcal{H}_S)$ and some $n_1, \ldots, n_m \in \mathbb{Z}$ such that $\sum_{i=1}^m n_iX_i \in \mathrm{Ker}\, \partial_1' \setminus \mathrm{Im}\, \partial_2'$. Write $X_i = \{a_i\eta, b_i\eta\}$ with $a_i, b_i \in V_0$ and $a_i < b_i$. By definition of $\mathrm{pure}_2(\mathcal{H}_S)$, there exists some $c_i \in V_0$ such that $\{a_i\eta, b_i\eta, c_i\eta\} \in (H/\eta) \cap P_3(V/\eta)$. In view of (12), we have $a_ib_ic_i \in H \cap P_3(V_0)$, hence a_ib_i is an edge from $\mathrm{pure}_2(\mathcal{H})$. Now

$$0 = (\sum_{i=1}^m n_iX_i)\partial_1' = \sum_{i=1}^m n_i(b_i\eta - a_i\eta)$$

yields $\sum_{i=1}^m n_i(b_i - a_i) = 0$ since V_0 is a cross-section for η and so $\sum_{i=1}^m n_i(a_ib_i) \in \mathrm{Ker}\,\partial_1$.

Since $0 = \tilde{H}_1(\mathrm{pure}_2(\mathcal{H})) = \mathrm{Ker}\,\partial_1/\mathrm{Im}\,\partial_2$, we must have

$$\sum_{i=1}^m n_i(a_ib_i) = (\sum_{j=1}^r k_j(x_jy_jz_j))\partial_2 \tag{18}$$

for some distinct triangles $x_jy_jz_j$ in $\mathrm{pure}_2(\mathcal{H})$ and $k_j \in \mathbb{Z}$. Since $a_i, b_i \in V_0$ for every i, we may assume that $x_j < y_j < z_j$ and $x_j, y_j, z_j \in V_0$ for every j: indeed, we may replace each letter in $V \setminus V_0$ by its representative in V_0, and remain inside $\mathrm{pure}_2(\mathcal{H})$ by (12). In view of (11), $\{x_j\eta, y_j\eta, z_j\eta\}$ is a triangle in $\mathcal{H}_S$ (and therefore in $\mathrm{pure}_2(\mathcal{H}_S)$) for $j = 1, \ldots, r$. Now (18) yields

$$\sum_{i=1}^m n_i(a_ib_i) = \sum_{j=1}^r k_j(y_jz_j - x_jz_j + x_jy_j)$$

and consequently

$$\sum_{i=1}^m n_i\{a_i\eta, b_i\eta\} = \sum_{j=1}^r k_j(\{y_j\eta, z_j\eta\} - \{x_j\eta, z_j\eta\} + \{x_j\eta, y_j\eta\}).$$

Since $x_j\eta < y_j\eta < z_j\eta$, we get

$$\sum_{i=1}^m n_iX_i = (\sum_{j=1}^r k_j\{x_j\eta, y_j\eta, z_j\eta\})\partial_2' \in \mathrm{Im}\,\partial_2',$$

a contradiction. Therefore $\tilde{H}_1(\mathrm{pure}_2(\mathcal{H}_S)) = 0$ as required. $\square$

We may now prove one of our main theorems. The simple case (for dimension 2) had been established in [18, Corollary 7.2.9].

Theorem 5.2 *Let $\mathcal{H}$ be a boolean representable simplicial complex of dimension 2. Then the following conditions are equivalent:*

(i) $\mathcal{H}$ is shellable;

(ii) $\mathcal{H}$ is sequentially Cohen-Macaulay;

(iii) $\Gamma\mathrm{Fl}\mathcal{H}_S$ contains at most two connected components or contains exactly one nontrivial connected component.

Proof. (i) $\Rightarrow$ (ii). By [5, 21].

(ii) $\Rightarrow$ (i). By Lemma 5.1, $\mathcal{H}_S$ is sequentially Cohen-Macaulay. It follows from [18, Corollary 7.2.9] that $\mathcal{H}_S$ shellable. Therefore $\mathcal{H}$ is shellable by Proposition 4.2(x).

(i) $\Rightarrow$ (iii). We adapt the proof of [18, Lemma 7.2.7].

Let $C_1, \ldots, C_m$ denote the connected components of $\Gamma\mathrm{Fl}\mathcal{H}_S$. We suppose that $m \geq 3$ and at least C_1, C_2 are nontrivial. Since $\mathcal{H}_S$ is simple of dimension 2, we know by [18, Lemma 6.4.3] that

$$\text{if } pqr \in P_3(V/\eta),\ p \text{ --- } q \text{ is an edge of } \Gamma\mathrm{Fl}\mathcal{H}_S \text{ but } p \text{ --- } r \text{ is not, then } pqr \in H/\eta. \tag{19}$$

Let $\varphi : V \to V/\eta$ be the canonical projection. For $i = 1, \ldots, m$, let $V_i = \{v \in V \mid v\varphi \in C_i\}$. It follows that $V = V_1 \cup \ldots \cup V_m$ constitutes a partition of V. We show that

$$\text{if } pqr \in H \cap P_3(V), \text{ then } p, q, r \text{ belong to at most two distinct } V_i. \tag{20}$$

It follows from (11) that $\{p\varphi, q\varphi, r\varphi\} \in (H/\eta) \cap P_3(V/\tau)$. By Proposition 4.2, $\mathcal{H}_S$ is a simple boolean representable simplicial complex of dimension 2, so it follows from [18, Lemma 6.4.4] that the three vertices $p\varphi, q\varphi, r\varphi$ belong to at most two connected components of $\Gamma\mathrm{Fl}\mathcal{H}_S$. Therefore (20) holds.

We split now the discussion into two cases. Suppose first that $\Gamma\mathrm{Fl}\mathcal{H}_S$ has a trivial connected component C_k. Let v be its single vertex. We consider the link $\mathrm{lk}(v)$. By [3] (see also [18, Proposition 7.1.5]), $\mathcal{H}$ shellable implies $\mathrm{lk}(v)$ shellable. Let $p_i\eta$ — $q_i\eta$ be an edge of C_i for $i = 1, 2$. By (19), we have $\{p_i\eta, q_i\eta, v\eta\} \in H/\tau$. By (12), we get $p_iq_iv \in H$, hence $p_iq_i \in H/v$ and so $\mathrm{lk}(v)$ has dimension 1.

The facets of a complex of dimension 1 are the edges and the isolated vertices. It is immediate that such a complex is shellable if and only the complex has a unique nontrivial connected component. Therefore, since $\mathrm{lk}(v)$ is shellable of dimension 1, the edges $p_1q_1, p_2q_2 \in H/v$ must belong to the same connected component of $\mathrm{lk}(v)$. Hence there exist distinct $r_0, \ldots, r_n \in V \setminus \{v\}$ such that $r_0 \in p_1q_1$, $r_n \in p_2q_2$ and $r_{j-1}r_j \in H/v$ for $j = 1, \ldots, n$.

Now we have $r_{j-1}r_jv \in H$. Since v is an isolated vertex of $\Gamma\mathrm{Fl}\mathcal{H}_S$, then $H \cap P_2(V_k) = \emptyset$ by (11). Hence (20) yields $r_{j-1}, r_j \in V_i$ for some $i \in \{1, \ldots, m\} \setminus \{k\}$. Thus $r_0, r_n \in V_i$. But $r_0 \in p_1q_1$ and $r_n \in p_2q_2$ imply $r_0 \in C_1$ and $r_n \in C_2$, a contradiction.

Therefore we may assume that all the connected components $C_1, \ldots, C_m$ of $\Gamma\mathrm{Fl}\mathcal{H}_S$ are nontrivial.

Suppose that $pq \in H \cap P_2(V)$. By (2), we have $p\eta \neq q\eta$. If $p\eta$ — $q\eta$ is an edge of $\Gamma\mathrm{Fl}\mathcal{H}_S$, let $r \in V$ be such that $r\eta \notin \overline{\{p\eta, q\eta\}}$. Then $\{p\eta, q\eta, r\eta\} \in H/\eta$ and in view of (12) we get $pqr \in H \cap P_3(V)$. Thus $\mathcal{H}$ has no 1-facets.

On the other hand, given $p \in V$, we may take $q \in V \setminus p\varphi^{-1}$. Since $\mathcal{H}_S$ is simple, we have $\{p\eta, q\eta\} \in H/\eta$, yielding $pq \in H$ in view of (12). Therefore every facet of $\mathcal{H}$ has dimension 2.

Let $B_1, \ldots, B_t$ be a shelling of $\mathcal{H}$. For $k = 1, \ldots, t$, define a graph $\Gamma_k = (W_k, E_k)$ by

$$W_k = \cup_{j=1}^k B_j, \quad E_k = \cup_{j=1}^k P_2(B_j).$$

It follows easily from the definition of shelling that each Γ_k is connected.

We say that $p, q \in W_k$ have the same color if $p, q \in V_i$ for some $i \in \{1, \ldots, m\}$. We write $p\gamma_kq$ if there exists a monochromatic path of the form

$$p = r_0 — r_1 — \ldots — r_n = q$$

in Γ_k for some $n \geq 0$. It is immediate that γ_k is an equivalence relation on W_k. We define a graph $\overline{\Gamma_k} = (\overline{W_k}, \overline{E_k})$ by taking $\overline{W_k} = \{p\gamma_k \mid p \in W_k\}$ and

$$\overline{E_k} = \{\{p\gamma_k, q\gamma_k\} \mid p\gamma_k \neq q\gamma_k \text{ and } pq \in E_k\}.$$

We prove that

$$\overline{\Gamma_k} \text{ is a tree for } k = 1, \ldots, t \tag{21}$$

by induction on k.

In view of (20), $\overline{\Gamma_1}$ has at most two vertices, hence a tree. Assume now that $k > 1$ and $\overline{\Gamma_{k-1}}$ is a tree. We consider several cases and subcases:

Case 1: $B_k \not\subseteq W_{k-1}$.

Since B_k has dimension 2, then $(B_k, I(B_k))$ is pure of dimension 1, hence we may write $B_k = pqr$ with $pq \in E_{k-1}$ and $r \notin W_{k-1}$. By (20), the vertices p, q, r have at most two different colors.

Subcase 1.1: r has the same color as p or q.

Then $\overline{\Gamma_k} = \overline{\Gamma_{k-1}}$, hence a tree by the induction hypothesis.

Subcase 1.2: r has a different color from p and q.

Then $p\gamma_{k-1}q$ and so $\overline{\Gamma_k}$ is obtained from $\overline{\Gamma_{k-1}}$ by adjoining the edge $p\gamma_{k-1} = p\gamma_k$ — $r\gamma_k$. Since $\overline{\Gamma_{k-1}}$ is a tree, $\overline{\Gamma_k}$ is a tree as well.

Case 2: $B_k \subseteq W_{k-1}$.

We may assume that $E_{k-1} \subset E_k$. Since B_k has dimension 2, then $(B_k, I(B_k))$ is pure of dimension 1, hence we may write $B_k = pqr$ with $pq, qr \in E_{k-1}$ and $pr \notin E_{k-1}$. By (20), the vertices p, q, r have at most two different colors.

Subcase 2.1: q has the same color as p or r.

Then $\overline{\Gamma_k} = \overline{\Gamma_{k-1}}$, hence a tree by the induction hypothesis.

Subcase 2.2: q has a different color from p and r.

Then p and r have the same color. If $p\gamma_{k-1}r$, then $\overline{\Gamma_k} = \overline{\Gamma_{k-1}}$, hence we may assume that $(p, r) \notin \gamma_{k-1}$. It follows that $\overline{\Gamma_k}$ is obtained from $\overline{\Gamma_{k-1}}$ by identifying the (non adjacent) vertices $p\gamma_{k-1}$ and $q\gamma_{k-1}$. It is well known that folding such a pair of adjacent edges in a tree still yields a tree.

Therefore $\overline{\Gamma_k}$ is a tree in all cases and so (21) holds.

Let p_i — q_i be an edge in C_i for $i = 1, 2$ and let v be a vertex in C_3. By (19), we have $p_1q_1p_2, p_1q_1v, p_2q_2v \in H$. Since all the facets in $\mathcal{H}$ have dimension 2, we have $E_t = H \cap P_2(V)$, hence

$$\begin{tikzcd} & v\gamma_t \arrow[dl, dash] \arrow[dr, dash] & \\ p_1\gamma_t \arrow[rr, dash] & & p_2\gamma_t \end{tikzcd}$$

is a triangle in $\overline{\Gamma_t}$, contradicting (21). Therefore condition (ii) must hold.

(iii) $\Rightarrow$ (i). By [18, Theorem 7.2.8], $\mathcal{H}_S$ is shellable, which implies $\mathcal{H}$ shellable by Proposition 4.2(x). $\square$

It is well known that a shellable simplicial complex has the homotopy type of a wedge of spheres [3]. But in the case of BRSCs of dimension 2, we already know from Theorem 3.6(ii) that this is always the case, despite there being such complexes that are not shellable (see e.g. Example 3.5 for $t \geq 3$).

6 The order complex of a lattice and EL-labelings

Given a lattice L, let C_{L^*} denote the set of totally ordered subsets of $L^* = L \setminus \{0, 1\}$ (chains). The *order complex* of L is the simplicial complex $\mathrm{Ord}(L) = (L^*, C_{L^*})$.

The concept of EL-labeling provides a famous sufficient condition for shellability of the order complex of a lattice. Let L be a lattice and let EH_L denote the set of edges in the Hasse diagram of L. More formally, we can define EH_L as the set of all ordered pairs $(a, b) \in L \times L$ such that b covers a in L. Let P be a poset and let $\xi : EH_L \to P$ be a mapping. Given a maximal chain $\gamma : \ell_0 < \ell_1 < \ldots < \ell_n$ in L (so that $(\ell_{i-1}, \ell_i) \in EH_L$ for $i = 1, \ldots, n$), we define a word $\gamma\xi$ on the alphabet P by $\gamma\xi = (\ell_0, \ell_1)\xi \ldots (\ell_{n-1}, \ell_n)\xi$. The chain γ is *increasing* if $(\ell_0, \ell_1)\xi < \ldots < (\ell_{n-1}, \ell_n)\xi$. Given $a, b \in L$ with $a < b$, we denote by $[a, b]$ the subsemilattice of L consisting of all $c \in L$ satisfying $a \leq c \leq b$. Clearly, $\xi : EH_L \to P$ induces also a mapping on the maximal chains of $[a, b]$. Consider the lexicographic ordering on P^+. We say that $\xi : EH_L \to P$ is an *EL-labeling* of L if, for all $a, b \in L$ such that $a < b$:

- there exists a unique maximal chain γ_0 in $[a, b]$ such that $\gamma\xi$ is increasing;
- $\gamma_0\xi < \gamma\xi$ for every other maximal chain γ in $[a, b]$.

A fundamental theorem of Björner [2] states that if a lattice L admits an EL-labeling, then $\mathrm{Ord}(L)$ is shellable. Moreover, it is known that every semimodular lattice admits an EL-labeling [22, Exercise 3.2.14(d)]. In the case of boolean representable simplicial complexes, the lattice of flats is semimodular if and only if the complex is a matroid [17, Theorem 1.7.5].

The next result shows how a shelling of the order complex can provide a shelling of the original complex itself.

Theorem 6.1 *Let $\mathcal{H}$ be a boolean representable simplicial complex. If the order complex of* $\mathrm{Fl}\mathcal{H}$ *is shellable, so is $\mathcal{H}$.*

Proof. Write $L = \mathrm{Fl}\mathcal{H}$ and let $d = \dim\mathcal{H} = \dim(\mathrm{Ord}(L)) + 1$. The facets of $\mathrm{Ord}(L)$ can be identified (recall that we are looking at chains in L^* in $\mathrm{Ord}(L)$) with the maximal chains in L, i.e. subsets of L of the form $B = \{F_0, \ldots, F_n\}$ with

$$\emptyset = F_0 \subset F_1 \subset \ldots \subset F_n = V \tag{22}$$

and no intermediate flat $F_{i-1} \subset F' \subset F_i$ for $i = 1, \ldots, n$. Note that $n \leq d+1$. We define $B\tau$ to be the set of transversals of the maximal chain (22), i.e. $B\tau$ consists of all the subsets $\{a_1, \ldots, a_n\} \in P_n(V)$ such that $a_i \in F_i \setminus F_{i-1}$ for $i = 1, \ldots, n$. Note that $F_i = \overline{F_{i-1} \cup \{a_i\}}$ by maximality of (22).

Assume that $B_1, \ldots, B_t$ is a shelling of $\mathrm{Ord}(L)$. Then

$$\mathrm{fct}\mathcal{H} = \bigcup_{i=1}^{t} B_i\tau.$$

We intend to concatenate successive enumerations of $B_1\tau, \ldots, B_t\tau$ so that, after removing repetitions, we get a shelling of $\mathcal{H}$.

We start with $B_1\tau$. Assuming that $B_1\tau$ is the set of transversals of the chain (22), we fix a total ordering $<_1$ of V such that $a <_1 b$ whenever $a \in F_i \setminus F_{i-1}$, $b \in F_j \setminus F_{j-1}$ and $i < j$. We may associate to each $B'_k \in B_1\tau$ a (unique) word $a_1 \ldots a_n \in V^n$ such that $B'_k = \{a_1, \ldots, a_n\}$ and $a_i \in F_i \setminus F_{i-1}$ for $i = 1, \ldots, n$. Then we order the elements of $B_1\tau$ according to the lexicographical ordering of the associated words.

Let us check the shelling condition for the facets in $B_1\tau$, enumerated as $B'_1, \ldots, B'_p$. Let $k \in \{2, \ldots, p\}$. Let $A \in I(B'_k)$. Then B'_k is not the minimum facet (for the lexicographic order) containing

A. Hence there exists some $i \in \{1, \ldots, n\}$ and some letters $b, c \in F_i \setminus F_{i-1}$ such that $b <_1 c \in B'_k \setminus A$. It follows that $(B'_k \setminus \{c\}) \cup \{b\} = B'_j$ for some $j < k$ and so $A \subseteq B'_k \setminus \{c\} \in I(B'_k)$. Thus $(B'_k, I(B'_k))$ is pure of dimension $n - 2$.

Assume now that $j \in \{2, \ldots, t\}$ and we have already defined enumerations for the facets in $B_1\tau \cup \ldots \cup B_{j-1}\tau$ so that the shelling condition is satisfied. We may assume that $B_j\tau$ is the set of transversals of the chain (22). We fix a total ordering $<_j$ of V such that $a <_j b$ whenever $a \in F_i \setminus F_{i-1}$, $b \in F_r \setminus F_{r-1}$ and $i < r$. Similarly to the case $j = 1$, we associate to each $B'_k \in B_j\tau$ a (unique) word $a_1 \ldots a_n \in V^n$ such that $B'_k = \{a_1, \ldots, a_n\}$ and $a_i \in F_i \setminus F_{i-1}$ for $i = 1, \ldots, n$. Then we order the elements of $B_j\tau$ according to the lexicographical ordering of the associated words, and we concatenate the new elements, say $B'_1, \ldots, B'_p$, to the enumeration of the elements of $B_1\tau \cup \ldots \cup B_{j-1}\tau$ previously defined.

Assume that $q \in \{1, \ldots, p\}$ and $B'_q = \{a_1, \ldots, a_n\}$, where $a_i \in F_i \setminus F_{i-1}$ for $i = 1, \ldots, n$. Let $A \in I(B'_q)$, say $A = \{a_{u_1}, \ldots, a_{u_s}\}$. Let $\widetilde{A} = \{F_{u_1}, \ldots, F_{u_s}\} \in I(B_j)$. Since $(B_j, I(B_j))$ is pure of dimension $n - 2$, there exists some $j' < j$ such that $\widetilde{A} \subseteq B_{j'}$ and $B_{j'}$ contains all the elements of B_j but one, say F_i. We may then assume that $B_{j'}$ originates from the chain

$$\emptyset = F_0 \subset \ldots \subset F_{i-1} \subset G_1 \subset \ldots \subset G_w \subset F_{i+1} \subset \ldots \subset F_n = V \tag{23}$$

in L. Note that the G_i must appear consecutively as a replacement of the missing F_i by maximality of (22). We claim that $B'_q \setminus \{a_i\}$ is a partial transversal of (23) containing A.

Suppose that $a_i \in A$. Then $F_i \in \widetilde{A} \subseteq B_{j'}$, a contradiction since (22) is maximal and different from (23). Hence $a_i \notin A$ and so $A \subseteq B'_q \setminus \{a_i\}$. To show that $B'_q \setminus \{a_i\}$ is a partial transversal of (23), it is enough to note that

$$a_{i+1} \in F_{i+1} \setminus F_i \subseteq (F_{i+1} \setminus G_w) \cup \ldots \cup (G_2 \setminus G_1) \cup (G_1 \setminus F_{i-1}).$$

Thus $A \subseteq B'_q \setminus \{a_i\} \in I(B'_q)$ and so $(B'_q, I(B'_q))$ is pure of dimension $n - 2$. By double induction on q and j, this validates our construction of a shelling of $\mathcal{H}$. $\square$

The next example shows that the converse of Theorem 6.1 does not hold.

Example 6.2 *Let $V = \{1, \ldots, 6\}$ and let Γ be the graph*

$$1 \text{——} 2 \text{——} 3 \text{——} 4 \qquad 5 \text{——} 6$$

Let

$$H = P_{\leq 2}(V) \cup \{X \in P_3(V) \mid \text{ at least two vertices in } X \text{ are adjacent in } \Gamma\}$$

and $\mathcal{H} = (V, H)$. Then $\mathcal{H}$ is a shellable pure boolean representable simplicial complex but the order complex of $\mathrm{Fl}\mathcal{H}$ *is not shellable.*

Since there exist no isolated vertices in Γ, $\mathcal{H}$ is pure. It is easy to compute the flats of $\mathcal{H}$, we have

$$\mathrm{Fl}\mathcal{H} = P_{\leq 1}(V) \cup \{12, 23, 34, 56, V\}.$$

It follows easily that $\mathcal{H}$ is boolean representable. Moreover, Γ is indeed the graph of flats of $\mathcal{H}$, hence $\mathcal{H}$ is shellable by Theorem 5.2. A possible shelling is

$$123, 124, 125, 126, 134, 156, 234, 235, 236, 256, 345, 346, 356, 456.$$

Now the facets of $\mathrm{Ord}(\mathrm{Fl}\mathcal{H})$ are

$$\{1,12\},\{2,12\},\{3,34\},\{4,34\},\{5,56\},\{6,56\}.$$

It is well known that a graph is shellable if and only if has at most one nontrivial connected component: a shelling of a graph must be an enumeration of its edges and isolated vertices where each edge (except the first) shares an endpoint with some previous edge. Hence $\mathrm{Ord}(\mathrm{Fl}\mathcal{H})$ is not shellable.

In the matroid case, we can combine Theorem 6.1 with the aforementioned results of Björner on EL-labelings to produce shellings for matroids (see [2]). Example 6.2 provides an example of a shellable pure boolean representable simplicial complex which admits no EL-labeling of the lattice of flats (otherwise $\mathrm{Ord}(\mathrm{Fl}\mathcal{H})$ would be shellable). Of course, this simplicial complex is not a matroid. The next example shows that the existence of EL-labelings is not exclusive of matroids.

Example 6.3 *Let* $V = \{1,\ldots,7\}$ *and let* Γ *be the graph*

$$1 \text{ --- } 2 \text{ --- } 3 \text{ --- } 4 \text{ --- } 5 \text{ --- } 6 \text{ --- } 7$$

Let

$$H = P_{\leq 2}(V) \cup \{X \in P_3(V) \mid \text{ at least two vertices in } X \text{ are adjacent in } \Gamma\}$$

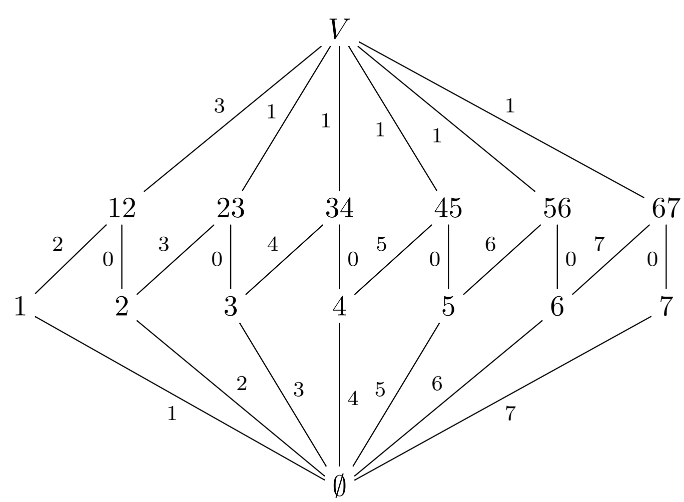
and $\mathcal{H} = (V,H)$. *Then* $\mathcal{H}$ *is a shellable pure boolean representable simplicial complex which is not a matroid and* $\mathrm{Fl}\mathcal{H}$ *admits an EL-labeling.*

Since there exist no isolated vertices in Γ, $\mathcal{H}$ is pure. It is easy to compute the flats of $\mathcal{H}$, we have

$$\mathrm{Fl}\mathcal{H} = P_{\leq 1}(V) \cup \{12,23,34,45,56,67,V\}.$$

It is easy to check now that $\mathcal{H}$ is boolean representable and Γ is indeed the graph of flats of $\mathcal{H}$. Thus $\mathcal{H}$ is shellable by Theorem 5.2.

The exchange property fails for 123 and 57, hence $\mathcal{H}$ is not a matroid. The following diagram describes an EL-labeling $\xi : EH_{\mathrm{Fl}\mathcal{H}} \to \mathbb{N}$. where the naturals are endowed with the usual ordering.



7 Computing the flats

In this section, we discuss the computation of the flats for a boolean representable simplicial complex of fixed dimension d, and relate these computations to the main results of the paper. The case $d \leq 1$ is straightforward and shall be omitted in most results.

We recall the O notation from complexity theory. Let P be an algorithm defined for instances depending on parameters $n_1, \ldots, n_k$. If $\varphi : \mathbb{N}^k \to \mathbb{N}$ is a function, we write $P \in O((n_1, \ldots, n_k)\varphi)$ if there exist constants $K, L > 0$ such that P processes each instance of type $(n_1, \ldots, n_k)$ in time $\leq K((n_1, \ldots, n_k)\varphi) + L$ (where time is measured as the number of elementary operations performed).

Clearly, boolean matrices provide the most natural means of defining a boolean representable simplicial complex $\mathcal{H} = (V, H)$. We may assume that a boolean representation M of $\mathcal{H}$ is *reduced*, i.e. all the rows of M are distinct and nonzero. Note that we are assuming that $P_1(V) \subseteq H$ in all circumstances, hence all columns must be nonzero as well.

Lemma 7.1 *It is decidable in time $O(n!m)$ whether or not the set of columns of an arbitrary $m \times n$ boolean matrix is independent.*

Proof. We use induction on n to show that independence can be checked in at most $n!m \sum_{i=0}^{n-1} \frac{1}{i!}$ elementary steps.

Assume that $n = 1$. Let M denote an $m \times 1$ boolean matrix. Then the single column of M is independent if and only if M is nonzero. Clearly, we may check if M is nonzero in $m = 1!m \sum_{i=0}^{1-1} \frac{1}{i!}$ elementary steps.

Assume now that $n > 1$ and the claim holds for $n - 1$. Let M denote an $m \times n$ boolean matrix. A necessary condition for the columns of M to be independent is existence of a *marker* of type $j \in \{1, \ldots, n\}$: a row having a 1 at column j and zeroes anywhere else. This follows from the fact that a lower unitriangular matrix has a marker and the existence of a marker is preserved by congruence.b We need at most mn elementary steps to determine all $j \in \{1, \ldots, n\}$ admitting a marker of type j. For each such j (and there are at most n), we must check if the columns of the $(m-1) \times (n-1)$ matrix obtained by removing the marker and the jth column from M are independent. Applying the induction hypothesis, we deduce that independence of the columns of M can be checked in at most

$$mn + n(n-1)!(m-1) \sum_{i=0}^{n-2} \frac{1}{i!} = \frac{n!m}{(n-1)!} + n!(m-1) \sum_{i=0}^{n-2} \frac{1}{i!} \leq n!m \sum_{i=0}^{n-1} \frac{1}{i!}$$

elementary steps, completing the induction.

Since $\sum_{i=0}^{n-1} \frac{1}{i!} \leq e$, it follows that independence can be checked on at most $en!m$ steps, hence in time $O(n!m)$. $\square$

Let $\mathcal{H} = (V, H)$ be a boolean representable simplicial complex defined by an $R \times V$ boolean matrix $M = (m_{rv})$. We assume M to be reduced.

For each $r \in R$, let

$$Z_r = \{v \in V \mid m_{rv} = 0\}.$$

By [18, Lemma 5.2.1], we have $Z_r \in \mathrm{Fl}\mathcal{H}$ for every $r \in R$.

If $2 \leq |Z_r| < |V|$, then Z_r is said to be a *line* of M. We denote by $\mathcal{L}_M$ the set of all lines of M. Now every element of $\mathrm{Fl}\mathcal{H}$ is of the form $\overline{X}$ for some $X \in H$ by [18, Proposition 4.2.4]. On the other hand, $\overline{X} = V \notin \mathcal{L}_M$ whenever X is a facet of $\mathcal{H}$ by [18, Proposition 4.2.4]. It follows that

$$|R| \leq |\mathrm{Fl}\mathcal{H}| - 1 \leq |H \setminus \mathrm{fct}\mathcal{H}| \leq \sum_{i=0}^{d} \binom{n}{i} \leq (d+1)n^d. \tag{24}$$

We consider next the problem of recognizing a boolean representation of a simplicial complex of dimension $d \geq 0$. Note that we view d as a fixed constant.

Lemma 7.2 *Let $d \geq 0$. It is decidable in time $O(n^{2d+3})$ whether a reduced boolean matrix with n columns defines a simplicial complex of dimension d.*

Proof. Let M be such a matrix. By (24), M must have at most $(d+1)n^d$ rows and we can check this necessary condition in time $O(n^d)$, hence we may assume that M has $O(n^d)$ rows. On the other hand, M has $\binom{n}{d+1}$ subsets of $d+1$ columns. By Lemma 7.1, we can decide in time $O(n^d)$ whether each such subset is a face of $\mathcal{H}$. Hence we can decide in time $\binom{n}{d+1}O(n^d)$, thus $O(n^{2d+1})$, whether or not $\dim\mathcal{H} \geq d$.

Since $\dim\mathcal{H} = d$ if and only if $\dim\mathcal{H} \geq d$ and $\dim\mathcal{H} \not\geq d+1$, we may decide $\dim\mathcal{H} = d$ in time $O(n^{2d+1}) + O(n^{2d+3})$, hence $O(n^{2d+3})$. $\square$

We present next a complexity bound for the computation of faces.

Theorem 7.3 *Let $d \geq 0$. It is possible to compute in time $O(n^{2d+1})$ the list of faces of a simplicial complex of dimension d defined by a reduced boolean matrix with n columns. Moreover, facets can be marked in this list in time $O(n^{2d+2})$.*

Proof. Note that, by Lemma 7.2, given a reduced boolean matrix M, we can decide in time $O(n^{2d+3})$ whether M defines a simplicial complex of dimension d.

By (24), M has $O(n^d)$ rows. On the other hand, M has $\binom{n}{i}$ subsets of i columns for $i = 0, \ldots, d+1$. In view of Lemma 7.1, we can decide in time $O(n^d)$ whether each such subset is a face. Hence we can enumerate all the faces of $\mathcal{H}$ in time $\sum_{i=0}^{d+1}\binom{n}{i}O(n^d)$, thus $O(n^{2d+1})$.

For each face I of dimension $< d$ and each $p \in V \setminus I$, we can check in time $O(n^d)$ whether $I \cup \{p\}$ is still a face (if I has dimension d, is certainly a facet). Hence we may check whether I is a facet in time $O(n^{d+1})$, and so we may mark all facets (among the $O(n^{d+1})$ faces) in time $O(n^{2d+2})$. $\square$

We discuss now the computation of flats.

Theorem 7.4 *Let $d \geq 2$. It is possible to compute in time $O(n^{3d+3})$ the list of flats of a simplicial complex of dimension d defined by a reduced boolean matrix with n columns.*

Proof. By Theorem 7.3, we may enumerate the list of faces $X_1, \ldots, X_m$ of $\mathcal{H}$ in time $O(n^{2d+1})$. Note that $m \leq \sum_{i=0}^{d+1}\binom{n}{i}$, hence m is $O(n^{d+1})$.

Let $X \in H$. We claim that we can compute $\overline{X}$ in time $O(n^{2d+3})$. Note that if X is a facet, then we have $\overline{X} = V$ by [18, Proposition 4.2.4].

Indeed, let $Y = X$. By Theorem 7.3, we may check whether Y contains a facet in time $O(n^{2d+2})$, yielding $\overline{Y} = V$. Hence we may assume that Y contains no facet. For every non-facet X_i and $p \in V \setminus Y$, we may check whether $X_i \subseteq Y$ and $X_i \cup \{p\} \notin H$ hold simultaneously. There exist $O(n^d)$ non-facets X_i, hence we have $O(n^{d+1})$ choices for both i and p. Since m is $O(n^{d+1})$ we may check if $X_i \cup \{p\} \notin H$ in time $O(n^{d+1})$. If this happens, we replace Y by $Y \cup \{p\}$ and we restart the process. Eventually, we reach a point where Y contains a facet or there are no more p's to add. In view of [18, Proposition 4.2.5], we may then deduce that $Y = \overline{X}$.

Now each cycle $Y \longrightarrow Y \cup \{p\}$ can be performed in time $O(n^{2d+2})$ and there are at most n cycles to be performed, hence $\overline{X}$ can be computed in time $O(n^{2d+3})$. Since the number of non-facets X_i is $O(n^d)$, we can compute their closures (and consequently all flats) in time $O(n^{3d+3})$. $\square$

Corollary 7.5 *Let $d \geq 2$. Let $\mathcal{H}$ denote an arbitrary simplicial complex of dimension d represented by a reduced boolean matrix M with n columns. Then:*

(i) $\Gamma\mathrm{Fl}\mathcal{H}$ can be computed in time $O(n^{2d+5})$;

(ii) $\pi_1(||\mathcal{H}||)$ can be computed in time $O(n^{2d+5})$.

Proof. (i) We have $\binom{n}{2}$ potential edges a — b in $\Gamma\mathrm{Fl}\mathcal{H}$. By the proof of Theorem 7.4, we may compute $\overline{ab}$ in time $O(n^{2d+3})$, and check whether or not $\overline{ab} = V$. Thus we reach a global complexity bound of $O(n^{2d+5})$.

(ii) By Theorem 3.3, we need to compute the number of connected components of $\Gamma\mathrm{Fl}\mathcal{H}$ (a graph with n vertices and at most $\binom{n}{2}$ edges) and to identify the H-trivial components. It is easy to see by induction that the number of connected components can be computed in time $O(n^2)$. In view of Theorem 7.3, we can identify the H-trivial connected components in time $O(n^{2d+3})$. Therefore $\pi_1(||\mathcal{H}||)$ can be computed in time $O(n^{2d+5}) + O(n^2) + O(n^{2d+3}) = O(n^{2d+5})$. $\square$

We show next how these complexity bounds can be improved in the case of dimension 2.

Let $\Gamma = (V, E)$ be a graph. Given $v \in V$, we write $\mathrm{nbh}(v) = \{w \in V \mid vw \in E\}$. We say that $A \subseteq V$ is a *superanticlique* if $|A| > 1$ and

$$\mathrm{nbh}(a) \cup \mathrm{nbh}(b) = V \setminus A$$

holds for all $a, b \in A$ distinct. In particular, the superanticlique A is a maximal anticlique (i.e. maximal with respect to $P_2(A) \cap E = \emptyset$).

Superanticliques play a major role in the theory of boolean representable simple simplicial complexes of dimension 2. Let M be a boolean matrix representation of such a complex, say $\mathcal{H} = (V, H)$. We denote by ΓM the graph with vertex set V and edges of the form p — q whenever pq is a 2-subset of a line of M. By [18, Theorem 6.3.6], $\mathrm{Fl}\mathcal{H}$ is the union of $P_{\leq 1}(V) \cup \{V\} \cup \mathcal{L}_M$ with the set of all superanticliques of ΓM.

Given two graphs $\Gamma = (V, E)$ and $\Gamma' = (V', E')$, assumed to be disjoint, we define their *join* to be the graph $\Gamma + \Gamma' = (V \cup V', E \cup E' \cup E'')$, where $E'' = \{vv' \mid v \in V, v' \in V'\}$. Their *coproduct* is the graph $\Gamma \sqcup \Gamma' = (V \cup V', E \cup E')$.

Given $n \geq 1$, we denote by K_n the complete graph on n vertices. We denote by $\overline{K_n}$ the complement graph of K_n, so that $\overline{K_n}$ has n vertices and no edges.

We define now two classes of graphs as follows. Let Ω_1 be the class of all graphs of the form $(\overline{K_n} + \Delta) \sqcup K_1$, where $n \geq 1$ and Δ is any finite graph. Let Ω_2 be the class of all graphs of the form $(K_1 + \Delta) \sqcup (K_1 + \Delta')$, where Δ and Δ' are any finite graphs.

Theorem 7.6 *Let M be a boolean matrix representation of a simple simplicial complex $\mathcal{H}$ of dimension 2. Then:*

(i) if ΓM is connected or belongs to $\Omega_1 \cup \Omega_2$, then $\Gamma\mathrm{Fl}\mathcal{H}$ is connected;

(ii) in all other cases, $\Gamma\mathrm{Fl}\mathcal{H} = \Gamma M$.

Proof. (i) Since $\mathcal{L}_M \subseteq \mathrm{Fl}\mathcal{H}$ by [18, Lemma 5.2.1], then ΓM is a subgraph of $\Gamma\mathrm{Fl}\mathcal{H}$ with the same vertex set. Therefore ΓM connected implies $\Gamma\mathrm{Fl}\mathcal{H}$ connected.

Assume next that $\Gamma M \in \Omega_1$, say of the form $(\overline{K_n}+\Delta)\sqcup K_1$. Let A be the union of the n vertices of $\overline{K_n}$ and the single vertex of K_1. Given $a, b \in A$, then $\mathrm{nbh}(a) \cup \mathrm{nbh}(b)$ are the vertices of Δ, i.e. $V \setminus A$. Thus A is a superanticlique of ΓM and so $A \in \mathrm{Fl}\mathcal{H} \setminus \{V\}$ by [18, Theorem 6.3.6]. Since A intersects the two connected components of ΓM, it follows that $\Gamma\mathrm{Fl}\mathcal{H}$ is connected.

Assume now that $\Gamma M \in \Omega_2$, say of the form $(K_1 + \Delta) \sqcup (K_1 + \Delta')$. Let A consists of the two vertices in both copies of K_1, say a, b. Then $\mathrm{nbh}(a)\cup\mathrm{nbh}(b)$ are the vertices of Δ and Δ', i.e. $V \setminus A$. Thus A is a superanticlique of ΓM and so $A \in \mathrm{Fl}\mathcal{H} \setminus \{V\}$. Since A intersects the two connected components of ΓM, it follows that $\Gamma\mathrm{Fl}\mathcal{H}$ is connected.

(ii) Suppose that ΓM is disconnected and $\Gamma\mathrm{Fl}\mathcal{H} \neq \Gamma M$. We must show that $\Gamma M \in \Omega_1 \cup \Omega_2$. In view of [18, Theorem 6.3.6], there exists some superanticlique A of ΓM. It follows from the definition that A must intersect all the connected components of ΓM.

Suppose that ΓM has more than two connected components. Since $\mathcal{H}$ has dimension 2, one of the connected components, say C, must be nontrivial. Let $a, b \in A \setminus C$. Then $(\mathrm{nbh}(a)\cup\mathrm{nbh}(b))\cap C = \emptyset$. Since $C\setminus A \neq \emptyset$, this contradicts $\mathrm{nbh}(a)\cup\mathrm{nbh}(b) = V\setminus A$. Therefore ΓM has precisely two connected components, and we may write $\Gamma M = \Gamma \sqcup \Gamma'$ with Γ and Γ' connected.

Suppose that Γ and Γ' are both nontrivial. The same argument used above implies that A has one element a in Γ and another b in Γ'. Since $\mathrm{nbh}(a)\cup\mathrm{nbh}(b) = V \setminus \{a, b\}$, it follows that $\Gamma M \in \Omega_2$.

Thus we may assume that Γ' is trivial. Let $a \in A$ be a vertex of Γ and let b be the unique vertex of Γ' (which is in A). Let Δ (respectively Δ') be the subgraph of Γ induced by $\mathrm{nbh}(a)$ (respectively the remaining vertices of Γ). Since $A = V \setminus (\mathrm{nbh}(a) \cup \mathrm{nbh}(b))$, then Δ' is an edgeless graph. Let c be a vertex of Δ'. Since $\mathrm{nbh}(c) \cup \mathrm{nbh}(b) = V \setminus A = \mathrm{nbh}(a) \cup \mathrm{nbh}(b)$, it follows that $\Gamma = \Delta + \Delta'$. Therefore $\Gamma M \in \Omega_1$. $\square$

Now we can provide complexity bounds for both fundamental group and decidability of shellability in dimension 2.

Theorem 7.7 *Let $\mathcal{H}$ denote an arbitrary simplicial complex of dimension 2 represented by a reduced boolean matrix M with n columns. Then:*

(i) if $\mathcal{H}$ is simple, then $\pi_1(||\mathcal{H}||)$ can be computed in time $O(n^4)$;

(ii) it can be determined in time $O(n^4)$ whether or not $\mathcal{H}$ is shellable.

Proof. (i) Since $\mathcal{H}$ is connected by Lemma 3.1, then $\pi_1(||\mathcal{H}||)$ is well defined. Since M is reduced, it has at most $3n^2$ rows by (24).

By Corollary 3.4, it suffices to compute the number of connected components of $\Gamma\mathrm{Fl}\mathcal{H}$.

Since M has at most $3n^2$ rows, we may compute ΓM in time $O(n^4)$ (there are $\binom{n}{2}$ pairs of vertices to check, and each pair can be checked in time $O(n^2)$).

We claim that we can check whether or not ΓM is connected in time $O(n^4)$. Indeed, let r be the number of rows of M and let M_i be the submatrix of M defined by the first i rows ($i = 1, \ldots, r$). Obviously, we can compute the connected components of ΓM_1 in time $O(n)$. Assume now that $1 < i \leq r$ and the connected components of ΓM_{i-1} were computed in time $O(in^2)$. We can mark the zero entries of the ith row with the connected components of ΓM_{i-1} in time $O(n^2)$ and merge distinct connected components arising this way in time $O(n^2)$, and the complexity constants for these two procedures do not depend on i. Since $r \leq 3n^2$, it follows by induction that the connected components of $\Gamma M_r = \Gamma M$ can be computed in time $O(n^4)$. Therefore we can check whether or not ΓM is connected in time $O(n^4)$.

We claim that we can also decide whether or not $\Gamma M \in \Omega_1 \cup \Omega_2$ in time $O(n^4)$. Since the connected components of ΓM were already computed in time $O(n^4)$, it suffices to show that it is decidable in time $O(n^4)$ whether or not a connected graph with at most n vertices is of the form $K_1 + \Delta$ or $\overline{K_m} + \Delta$. The first case is obvious since we have at most n potential choices for the vertex playing the K_1 role. For the case $\overline{K_m} + \Delta$, we note that we need at most n tries to pick a vertex v in $\overline{K_m}$, and for each such v the vertices of Δ (if it exists) would be necessarily $\mathrm{nbh}(v)$, hence the vertices in both $\overline{K_m}$ and Δ would be fully determined by v. We would be able to mark them as such in time $O(n)$. Finally, we may decide whether $\mathrm{nbh}(v)$ is an anticlique in time $O(n^2)$, and we can check whether a — b is an edge for all $a \in \mathrm{nbh}(v)$ and $b \notin \mathrm{nbh}(v) \cup \{v\}$ in time $O(n^2)$, proving our claim.

Now it follows from Theorem 7.6 that we may compute the number of connected components of $\Gamma \mathrm{Fl}\mathcal{H}$ in time $O(n^4)$, and we apply Theorem 3.3.

(ii) By Proposition 4.3, we can produce a submatrix M' of M representing $\mathcal{H}_S$ by removing repeated columns. We may do it by comparing pairs of columns. There are $\binom{n}{2}$ pairs to compare, and each pair can be compared in time $O(n^2)$, hence we can compute M' in time $O(n^4)$.

In view of Theorem 5.2, we can assume that $\mathcal{H}$ is simple, and use the proof of part (i). $\square$

Note that the quartic bound in part (i) is much better than the $O(n^9)$ bound provided by Corollary 7.5(i).

We remark also that, once shellability is ensured, an actual shelling can be produced in the simple case using the algorithms described in [18, Lemma 7.2.1] and [18, Lemma 7.2.5] within the same quartic complexity bounds. The extension to the general case follows then from Proposition 4.2(x) and Theorem 5.2. Therefore we obtain the following corollary.

Corollary 7.8 *Let $\mathcal{H}$ denote an arbitrary shellable simplicial complex of dimension 2 represented by a reduced boolean matrix M with n columns. Then a shelling of $\mathcal{H}$ can be actually computed in time $O(n^4)$.*

The i-th Betti number $w_i(\mathcal{H})$ is defined as the rank of the ith homology group of $||\mathcal{H}||$. If $\mathcal{H}$ is shellable, then by [3] $w_i(H)$ is the number of homology facets in a shelling $B_1, \ldots, B_t$ of $\mathcal{H}$. We say that B_k $(k > 1)$ is a *homology facet* in this shelling if $2^{B_k} \setminus \{B_k\} \subseteq \cup_{i=1}^{k-1} 2^{B_i}$.

Assume that $\mathcal{H}$ satisfies the conditions of Corollary 7.8. Then we can construct a shelling $B_1, \ldots, B_t$ in time $O(n^4)$. Now we can build a sequence $\Delta_1, \ldots, \Delta_t$ of graphs with vertex set $V(\Delta_k) = \cup_{i=1}^{k} B_i$ and edge set $E(\Delta_k) = \cup_{i=1}^{k} P_2(B_i)$ to help us keep track of homology facets: indeed, if $k > 1$, then B_k is a homology facet if and only if ($|B_k| = 2$ and $B_k \subseteq V(\Delta_{k-1})$) or ($|B_k| = 3$ and $P_2(B_k) \subseteq E(\Delta_{k-1})$). Since $t \in O(n^3)$, this provides a proof for the following result.

Corollary 7.9 *Let $\mathcal{H}$ denote an arbitrary shellable simplicial complex of dimension 2 represented by a reduced boolean matrix M with n columns. Then the Betti numbers of $\mathcal{H}$ can be computed in time $O(n^4)$.*

8 Open problems

The problem of determining the homotopy type for BRSCs of dimension ≥ 3 remains open, as are the problems of identifying the shellable and the sequentially Cohen-Macauley BRSCs for dimension ≥ 3.

Acknowledgments

We would like to thank the referee for a very thorough reading of the paper and for making a number of comments that improved its presentation.

The first author was partially supported by Binational Science Foundation grant number 2012080.

The second author thanks the Simons Foundation-Collaboration Grants for Mathematicians for travel grant #313548.

The third author was partially supported by CNPq (Brazil) through a BJT-A grant (process 313768/2013-7) and CMUP (UID/MAT/00144/2013), which is funded by FCT (Portugal) with national (MEC) and European structural funds through the programs FEDER, under the partnership agreement PT2020.

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