BOUNDING THE GAP BETWEEN A FREE GROUP (OUTER) AUTOMORPHISM AND ITS INVERSE

M. LADRA

Department of Algebra, University of Santiago de Compostela, 15782 Santiago de Compostela, Spain

P.V. SILVA

Centro de Matemática, Faculdade de Ciências, Universidade do Porto,
R. Campo Alegre 687, 4169-007 Porto, Portugal

E. VENTURA

Dept. Mat. Apl. III, Universitat Politècnica de Catalunya, Manresa, Barcelona, Catalunya

Abstract. For any finitely generated group G, two complexity functions α_G and β_G are defined to measure the maximal possible gap between the norm of an automorphism (respectively outer automorphism) of G and the norm of its inverse. Restricting attention to free groups F_r, the exact asymptotic behaviour of α_2 and β_2 is computed. For rank $r \geqslant 3$, polynomial lower bounds are provided for α_r and β_r, and the existence of a polynomial upper bound is proved for β_r.

E-mail addresses: `manuel.ladra@usc.es`, `pvsilva@fc.up.pt`, `enric.ventura@upc.edu`.

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1. Introduction

The goal of this paper is to study automorphisms of groups, specifically to introduce a new technique to measure how easy or difficult is it to invert them. With this in mind, we associate two new functions, $\alpha_G(n)$ and $\beta_G(n)$, to the group G and propose to study its asymptotic behavior.

In the present introduction we define these functions in general, and show they are independent from the set of generators, up to multiplicative constants. Then, for the rest of the paper, we restrict our attention to finitely generated free groups and give several results concerning the asymptotic growth of their corresponding functions. A similar project can be carried out in any other families of groups G; we hope the study of these new functions motivates new interesting results in the near future.

Let G be a finitely generated group, and let us fix a finite set of generators $A = \{a_1, \ldots, a_r\}$.

This naturally gives a metric on G: every element $g \in G$ can be written as a product of the a_i's and their inverses, and one defines $|g|_A$ to be the length of the shortest such expression i.e., $|g|_A \leqslant n$ if and only if $g = a_{i_1}^{\epsilon_1} \cdots a_{i_m}^{\epsilon_m}$ for some $m \leqslant n$, some indices $i_1, \ldots, i_m \in \{1, \ldots, r\}$ and some signs $\epsilon_i = \pm 1$. Of course, $|1|_A = 0$, $|g^n|_A \leqslant |n||g|_A$, and $|gg'|_A \leqslant |g|_A + |g'|_A$ hold for all $g, g' \in G$ and all integer n.

The same can be done with an infinite set of generators. However, $|A| < \infty$ gives us finiteness of balls, $|\{g \in G \mid |g|_A \leqslant n\}| < \infty$, which is a crucial property in many respects; for example, in our definitions below.

Let us consider the group of automorphisms of G, $\operatorname{Aut} G$. We let automorphisms act on the right, so we write $\varphi\colon G \to G$, $g \mapsto g\varphi$. For every $g \in G$, we denote by λ_g the right conjugation by g, namely $x\lambda_g = g^{-1}xg$. Since $\lambda_g\varphi = \varphi\lambda_{g\varphi}$, it follows easily that $\Lambda = \{\lambda_g \mid g \in G\}$ is a normal subgroup of $\operatorname{Aut} G$. Each of the cosets $[\varphi] = \varphi\Lambda$ is said to be an *outer automorphism* of G. We write $\operatorname{Out} G = (\operatorname{Aut} G)/\Lambda$.

Of course, every automorphism $\varphi \in \operatorname{Aut} G$ is determined by the images of the generators $a_1, \ldots, a_r$. And the sum of its lengths is a good measure of the complexity of φ (understood as a rule moving elements of G around). Let us define then the *norm* of φ as

$$\|\varphi\|_A = |a_1\varphi|_A + \cdots + |a_r\varphi|_A.$$

Note that there is no $\varphi \in \operatorname{Aut} G$ with $\|\varphi\|_A \leqslant r-1$, because $a_i\varphi \neq 1$ for all i; the shortest automorphism (among possibly others) is the identity, $\|Id_G\|_A = r$. Note also that, for increasing values of $n \geqslant r$, there is a non-decreasing number of automorphisms $\varphi \in \operatorname{Aut} G$ with $\|\varphi\|_A \leqslant n$, but only finitely many for every fixed n. Observe also that $\|g\varphi\|_A \leqslant |g|_A \cdot \|\varphi\|_A$ for all $g \in G$ and all $\varphi \in \operatorname{Aut} G$.

This measure induces a similar measure on $\operatorname{Out} G$, defined as follows. Given $\Phi \in \operatorname{Out} G$, we define the *norm* of Φ as

$$\|\Phi\|_A = \min\{\|\varphi\|_A \mid \varphi \in \Phi\}.$$

Once again, for every fixed n, there exists a finite number of outer automorphisms $\Phi \in \operatorname{Out} G$ with $\|\Phi\|_A \leqslant n$.

A natural question is to ask about the relation between $\|\varphi\|_A$ and $\|\varphi^{-1}\|_A$ (resp. between $\|\Phi\|_A$ and $\|\Phi^{-1}\|_A$). If one happens to be significantly bigger than the other, then it intuitively means that inverting such an automorphism is hard (just writing down the expression of φ^{-1} as images of the generators will take much longer than doing the same for φ). With the purpose of measuring the (worst case) difference between the complexity of an automorphism φ and that of φ^{-1}, we define the following complexity functions α_A, $\beta_A \colon \mathbb{N} \to \mathbb{N}$:

$$\alpha_A(n) = \max\{\|\varphi^{-1}\|_A \mid \varphi \in \operatorname{Aut} G, \quad \|\varphi\|_A \leqslant n\},$$

$$\beta_A(n) = \max\{\|\Phi^{-1}\|_A \mid \Phi \in \operatorname{Out} G, \quad \|\Phi\|_A \leqslant n\},$$

where, by convention, we take $\max \emptyset = 0$ (i.e. $\alpha_A(n) = \beta_A(n) = 0$ for $n = 0, 1, \ldots, r-1$).

Clearly, $\alpha_A(n) \leqslant \alpha_A(n+1)$ and $\beta_A(n) \leqslant \beta_A(n+1)$ that is, α_A and β_A are non-decreasing functions. Furthermore, it is immediate that $\beta_A(n) = \max\{\|[\varphi^{-1}]\|_A \mid \varphi \in \operatorname{Aut} G, \quad \|\varphi\|_A \leqslant n\}$, hence $\beta_A(n) \leqslant \alpha_A(n)$ for every $n \geqslant 0$.

As we have emphasized in the notation, the values of $|g|_A$, $\|\varphi\|_A$ and $\|[\varphi]\|_A$, as well as the functions α_A and β_A, do depend on the preselected generating set A. However, the asymptotic behaviour of these last two functions do not depend on A and so, they will constitute two invariants of the group G. More precisely, changing to another finite generating system these two functions change only up to multiplicative constants both at the domain and at the range, as proved in the following proposition.

Lemma 1.1. *Let G be a group, and let $A = \{a_1, \ldots, a_r\}$ and $B = \{b_1, \ldots, b_s\}$ be two finite generating sets. Then, there exists a constant*

$C \geqslant 1$ such that, for all $\varphi \in \operatorname{Aut} G$ and $\Phi \in \operatorname{Out} G$, the following inequalities hold:

(i) $\frac{1}{C}\|\varphi\|_B \leqslant \|\varphi\|_A \leqslant C\|\varphi\|_B$,
(ii) $\frac{1}{C}\|\Phi\|_B \leqslant \|\Phi\|_A \leqslant C\|\Phi\|_B$.

Proof. Take $M = \max\{|b_i|_A \mid i = 1, \dots, s\}$, $N = \max\{|a_i|_B \mid i = 1, \dots, r\}$, and let $C = MNrs \geqslant 1$. For every $\varphi \in \operatorname{Aut} G$ we have

$$\begin{array}{rcl} \|\varphi\|_B & = & |b_1\varphi|_B + \cdots + |b_s\varphi|_B \\ & \leqslant & |b_1\varphi|_A N + \cdots + |b_s\varphi|_A N \\ & \leqslant & N\big(|b_1|_A\|\varphi\|_A + \cdots + |b_s|_A\|\varphi\|_A\big) \\ & = & N\big(|b_1|_A + \cdots + |b_s|_A\big)\|\varphi\|_A \\ & \leqslant & NMs\|\varphi\|_A \\ & \leqslant & C\|\varphi\|_A. \end{array}$$

By symmetry, $\|\varphi\|_A \leqslant C\|\varphi\|_B$ and (i) is proved.

To see (ii), given $\Phi \in \operatorname{Out} G$, choose $\varphi \in \Phi$ such that $\|\varphi\|_A = \|\Phi\|_A$ and then

$$\|\Phi\|_B = \min\{\|\theta\|_B \mid \theta \in \Phi\} \leqslant \|\varphi\|_B \leqslant C\|\varphi\|_A = C\|\Phi\|_A.$$

A symmetric argument completes the proof. □

Proposition 1.2. *Let G be a group, and let $A = \{a_1, \dots, a_r\}$ and $B = \{b_1, \dots, b_s\}$ be two finite generating sets. Then, there exists a constant $C \geqslant 1$ such that, for all $n \geqslant 1$, the following inequalities hold:*

(i) $\frac{1}{C} \cdot \alpha_B(\lfloor \frac{n}{C} \rfloor) \leqslant \alpha_A(n) \leqslant C \cdot \alpha_B(Cn)$,
(ii) $\frac{1}{C} \cdot \beta_B(\lfloor \frac{n}{C} \rfloor) \leqslant \beta_A(n) \leqslant C \cdot \beta_B(Cn)$.

Proof. For $n = 0, 1, \dots, r-1$, the left and middle terms in both inequalities are zeros and the result is trivial. For $n \geqslant r$, and using the contant C from the previous lemma, we have

$$\begin{array}{rcl} \alpha_A(n) & = & \max\{\|\theta^{-1}\|_A \mid \theta \in \operatorname{Aut}(G),\ \|\theta\|_A \leqslant n\} \\ & \leqslant & \max\{\|\theta^{-1}\|_A \mid \theta \in \operatorname{Aut}(G),\ \|\theta\|_B \leqslant Cn\} \\ & \leqslant & \max\{C\|\theta^{-1}\|_B \mid \theta \in \operatorname{Aut}(G),\ \|\theta\|_B \leqslant Cn\} \\ & = & C \cdot \max\{\|\theta^{-1}\|_B \mid \theta \in \operatorname{Aut}(G),\ \|\theta\|_B \leqslant Cn\} \\ & = & C \cdot \alpha_B(Cn). \end{array}$$

By symmetry, $\alpha_B(n) \leqslant C \cdot \alpha_A(Cn)$. Hence, for every $n \geqslant r$,

$$\alpha_B(\left\lfloor \frac{n}{C} \right\rfloor) \leqslant C \cdot \alpha_A\left(C \cdot \left\lfloor \frac{n}{C} \right\rfloor\right) \leqslant C \cdot \alpha_A(n),$$

completing the proof of (i).

The exact same argument changing α to β proves (ii). $\square$

Straightforward computations show that the following is an equivalence relation on the set of non-decreasing functions from $\mathbb{N}$ to $\mathbb{N}$: $f \sim g$ if and only if there exists a constant $C > 0$ such that for all $n \geq 0$, $\frac{1}{C} \cdot g(\lfloor \frac{n}{C} \rfloor) \leqslant f(n) \leqslant C \cdot g(Cn)$. Then, Proposition 1.2 is precisely saying that the equivalence classes of the functions α_A and β_A *do not* depend on the set of generators A chosen, that is, they are invariants of the group G. We shall denote them by α_G and β_G, respectively.

The relevant information about these (equivalence classes of) functions is their asymptotic growth. One says that the equivalence class of f grows *at least polynomially with degree* d if there is a constant $L > 0$ such that $Ln^d \leqslant f(n)$ for all $n \gg 0$ (i.e. for all $n \geqslant n_0$ and certain $n_0 \geqslant 0$); it is usually said *at least linearly, quadratically,* or *cubically* when $d = 1$, $d = 2$, or $d = 3$, respectively. It is also said that f grows *super-polynomially* if it grows at least polynomially with degree d for every $d > 0$. And f grows *exponentially* if there exists constants $L > 0$ and $\lambda > 1$ such that $L\lambda^n \leqslant f(n)$ for all $n \gg 0$. One can also define exact growth: f grows *exactly polynomially with degree* d if there are constants L and M such that $Ln^d \leqslant f(n) \leqslant Mn^d$ for all $n \gg 0$ (which is equivalent to saying $f(n) \sim n^d$). Clearly, all these notions are well defined not just for functions but for equivalence classes of functions.

Accordingly, we shall use the asymptotic behaviour of the functions $\alpha_G(n)$ and $\beta_G(n)$ of a given finitely generated group G to define the gap of G for (outer) automorphism inversion:

Definition 1.3. Let G be a finitely generated group and consider the (equivalence classes of) functions $\alpha_G(n)$ and $\beta_G(n)$. We say that G has *linear* (resp. *quadratic, cubic, polynomial of degree d, super-polynomial, exponential*) *gap for* [resp. *outer*] *automorphism inversion* if the function $\alpha_G(n)$ [resp. $\beta_G(n)$] grows linearly (resp. quadratically, cubically, polynomially of degree d, super-polynomially, exponentially).

This notion opens a new direction of research investigating the gap of groups for (outer) automorphism inversion, by means of analyzing the asymptotic growth of the corresponding functions. It is easy to see that $\alpha_G(n)$ is equivalent to a constant function if and only if $|\operatorname{Aut} G| < \infty$; similarly, $\beta_G(n)$ is equivalent to a constant function if and only if

$|\operatorname{Out} G| < \infty$. So, in this sense, interesting groups are those with infinitely many (outer) automorphisms.

Immediately after giving these notions, one can ask many interesting questions which, as far as we know, are open:

Question 1.4. *Is there a finitely generated group G with super-polynomial gap for (outer) automorphism inversion? And with exponential gap ?*

Question 1.5. *Is there a global upper bound to the gap for (outer) automorphism inversion in the class of finitely generated groups ? In other words, is it true that given a non-decreasing function $f\colon \mathbb{N} \to \mathbb{N}$ there exists a finitely generated group G whose gap for (outer) automorphism inversion grows at least like f ?*

Question 1.6. *Is there a finitely generated group G with $|\operatorname{Out} G| = \infty$ and whose gap for automorphism inversion is strictly bigger than its gap for outer automorphism inversion ?*

The goal of this paper is to investigate the gap for (outer) automorphism inversion in the family of finitely generated free groups. For the free group of rank r, denoted F_r, we shall write $\alpha_r = \alpha_{F_r}$ and $\beta_r = \beta_{F_r}$.

We can complete this project for the rank two case, which is quite special compared with higher ranks. On one hand we shall see that, for every free basis A and every $\Phi \in \operatorname{Out} F_2$, $\|\Phi^{-1}\|_A = \|\Phi\|_A$; hence, $\beta_2(n) = n$, while the same equality in higher rank is far from true. On the other hand, we prove that $\alpha_2(n)$ is bounded above and below by quadratic functions, i.e. F_2 has an exact quadratic gap for automorphism inversion. Collecting Theorems 3.5, 3.6 and 3.7 below, we have

Theorem 1.7.
(i) *For $n \geqslant 4$, $\alpha_2(n) \leqslant \frac{(n-1)^2}{2}$,*
(ii) *for $n \geqslant 10$, $\frac{n^2}{4} - 6n + 42 \leqslant \alpha_2(n)$,*
(iii) *for $n \geqslant 0$, $\beta_2(n) = n$.*

For higher rank, the problem is much more complicated and our results are less precise. We show that $\alpha_r(n)$ grows at least polynomially with degree r, and $\beta_r(n)$ grows between polynomially with degree $r-1$, and polynomially with a big enough degree. Collecting Theorem 4.4 and Corollary 4.6, we have

Theorem 1.8. *For every $r \geqslant 3$, there exist constants $K_r, K'_r, K''_r, M_r > 0$ such that, for every $n \geqslant 0$,*

(i) $K_r n^r \leqslant \alpha_r(n)$,
(ii) $K'_r n^{r-1} \leqslant \beta_r(n) \leqslant K''_r n^{M_r}$.

To our knowledge, nothing else is know about the gap for (outer) automorphism inversion in free groups of rank bigger than two. In particular, we highlight the following interesting open questions:

Question 1.9. *What is the exact gap for (outer) automorphism inversion in free groups F_r, with $r \geqslant 3$?*

Question 1.10. *Is there a polynomial upper bound for the gap for automorphism inversion in free groups F_r, with $r \geqslant 3$?*

2. Free groups

2.1. **Notation.** Let $A_r = \{a_1, \ldots, a_r, a_1^{-1}, \ldots, a_r^{-1}\}$ be an *alphabet* of r symbols together with their formal inverses (a total of $2r$ symbols different from each other). All along the paper we assume $r \geqslant 2$ to avoid trivial cases.

The set of all words on A_r, including the empty one denoted 1, together with the operation of concatenation of words, forms a free monoid denoted A_r^*. For any subset $S \subseteq A_r^*$, the symbol S^* denotes the submonoid generated by S, namely the set of all (arbitrarily long) finite formal products of elements in S. For example, $\{a_1, \ldots, a_r\}^*$ is precisely the set of all *positive* words on the alphabet A_r.

Let $F_r = \langle a_1, \ldots, a_r \rangle$ be the free group (of rank r) on the alphabet A_r, i.e. $A_r^*/\sim$ where $\sim$ is the congruence generated by the elementary reductions $a_i a_i^{-1} \sim a_i^{-1} a_i \sim 1$. A word of A_r^* is said to be *(cyclically) reduced* if it contains no (cyclic) factor of the form $a_i^{\epsilon} a_i^{-\epsilon}$, $\epsilon = \pm 1$. Given a word $w \in A_r^*$, we shall denote by $\overline{w}$ its reduction, namely the unique reduced word representing the same element of F_r as w. We shall do the standard abuse of notation consisting on using words, specially reduced ones, to refer to elements of F_r.

Note that the length $|w|_A$ of an element $w \in F_r$ is precisely the number of letters in $\overline{w}$; we shall simplify notation and just denoted it by $|w|$ (there will be no risk of confusion because, since now on, we shall always work with respect to the preselected generating set A).

Let us consider now automorphisms. Since every $\varphi \in \operatorname{Aut} F_r$ is determined by the images of $a_1, \ldots, a_r$, say $a_1\varphi = u_1, \ldots, a_r\varphi = u_r$, we shall adopt the notation $\varphi = \eta_{u_1,\dots,u_r}$, on occasion. When all of the u_i's are positive words, we say that $\eta_{u_1,\dots,u_r}$ is a *positive* automorphism (also known in the literature as *invertible substitution*, see e.g. [7]). The submonoid of $\operatorname{Aut} F_r$ consisting of all positive automorphisms is denoted by $\operatorname{Aut}^+ F_r$. An automorphism $\eta_{u_1,\dots,u_r}$ is said to be *cyclically reduced* when $u_1, \dots, u_r$ are all cyclically reduced.

As above, we shall also omit the reference to A from the notation for the norm of an automorphism $\varphi \in \operatorname{Aut} F_r$, the norm of an outer automorphism $\Phi \in \operatorname{Out} F_r$, and also from the gap functions:

$$\begin{aligned}
\|\varphi\| &= |a_1\varphi| + \cdots + |a_r\varphi|, \\
\|\Phi\| &= \min\{\|\varphi\| \mid \varphi \in \Phi\}, \\
\alpha_r(n) &= \max\{\|\varphi^{-1}\| \mid \varphi \in \operatorname{Aut} F_r, \quad \|\varphi\| \leqslant n\}, \\
\beta_r(n) &= \max\{\|\Phi^{-1}\| \mid \Phi \in \operatorname{Out} F_r, \quad \|\Phi\| \leqslant n\}.
\end{aligned}$$

Note that there are exactly $r!2^r$ automorphisms with $\|\varphi\| = r$, namely those of the form $a_1 \mapsto a_{1\pi}^{\epsilon_1}, \dots, a_r \mapsto a_{r\pi}^{\epsilon_r}$, where $\pi \in S_r$ is a permutation of $\{a_1, \dots, a_r\}$ and $\epsilon_i = \pm 1$. These automorphisms are the simplest ones and are called *letter permutation* automorphisms of F_r. They will be useful to reduce the number of cases in our arguments below.

Observe also that the natural inclusion $\operatorname{Aut} F_r \hookrightarrow \operatorname{Aut} F_{r+1}$ defined by fixing the last generator, gives the inequality $\alpha_{r+1}(n+1) \geqslant 1 + \alpha_r(n)$.

The following proposition is another reason for omitting the reference to A from the notation. It presents a stronger form of Proposition 1.2 when restricting our attention to free generating sets: given two bases A and B of F_r, the functions α_A and α_B are not only equivalent but exactly equal i.e., $\alpha_A(n) = \alpha_B(n)$ for all $n \geqslant 0$. The same is true for the β functions.

Proposition 2.1. *Let A and B be two bases of F_r. Then, $\alpha_A(n) = \alpha_B(n)$ and $\beta_A(n) = \beta_B(n)$, for all $n \geqslant 0$.*

Proof. Let $\psi\colon F_r \to F_r$ be the automorphism defined by $b_i\psi = a_i$, $i = 1, \dots, r$. It is clear that, for every $w \in F_r$, $|w|_B = |w\psi|_A$. Now, for every $\varphi \in \operatorname{Aut} F_r$, we have

$$\begin{aligned}
\|\varphi\|_B &= |b_1\varphi|_B + \cdots + |b_r\varphi|_B \\
&= |a_1\psi^{-1}\varphi|_B + \cdots + |a_r\psi^{-1}\varphi|_B \\
&= |a_1\psi^{-1}\varphi\psi|_A + \cdots + |a_r\psi^{-1}\varphi\psi|_A \\
&= \|\psi^{-1}\varphi\psi\|_A.
\end{aligned}$$

Furthermore, for every $\Phi \in \operatorname{Out} F_r$, we also have

$$\begin{aligned} \|\Phi\|_B &= \min\{\|\varphi\|_B \mid \varphi \in \Phi\} \\ &= \min\{\|\psi^{-1}\varphi\psi\|_A \mid \varphi \in \Phi\} \\ &= \min\{\|\nu\|_A \mid \nu \in \Psi^{-1}\Phi\Psi\} \\ &= \|\Psi^{-1}\Phi\Psi\|_A, \end{aligned}$$

where $\Psi = [\psi] \in \operatorname{Out} F_r$. And from these equalities we deduce that, for every $n \geqslant 0$,

$$\begin{aligned} \alpha_B(n) &= \max\{\|\varphi^{-1}\|_B \mid \varphi \in \operatorname{Aut} F_r, \quad \|\varphi\|_B \leqslant n\} \\ &= \max\{\|\psi^{-1}\varphi^{-1}\psi\|_A \mid \varphi \in \operatorname{Aut} F_r, \quad \|\psi^{-1}\varphi\psi\|_A \leqslant n\} \\ &= \max\{\|\nu^{-1}\|_A \mid \nu \in \operatorname{Aut} F_r, \quad \|\nu\|_A \leqslant n\} \\ &= \alpha_A(n). \end{aligned}$$

A similar argument shows that $\beta_B(n) = \beta_A(n)$. □

2.2. **The p-norm of an automorphism.** To prove the main results in the paper, we need to introduce a technical generalization of the notion of norm for an (outer) automorphism (and its corresponding gap functions). We shall use standard facts about norms on real (or complex) vectors and matrices. Recall that the maps $\|\cdot\|_p \colon \mathbb{R}^k \to \mathbb{R}$, $\|(x_1, \ldots, x_k)\|_p = (|x_1|^p + \cdots + |x_k|^p)^{1/p}$ (for $p \in \mathbb{R}^+$) and $\|\cdot\|_\infty \colon \mathbb{R}^k \to \mathbb{R}$, $\|(x_1, \ldots, x_k)\|_\infty = \max\{|x_1|, \ldots, |x_k|\}$ are *vector norms*, i.e. they satisfy the following axioms: (1) $\|\mathbf{x}\|_p \geqslant 0$ with equality if and only if $\mathbf{x} = \mathbf{0}$; (2) $\|\mu\mathbf{x}\|_p = |\mu|\|\mathbf{x}\|_p$; and (3) $\|\mathbf{x}+\mathbf{y}\|_p \leqslant \|\mathbf{x}\|_p + \|\mathbf{y}\|_p$.

Let us extend these notions to the non-abelian context, via the length function. For $p \in \overline{\mathbb{R}}^+ = \mathbb{R}^+ \cup \{\infty\}$ and $\mathbf{w} = (w_1, \ldots, w_k) \in F_r^k$, we define

$$\|\mathbf{w}\|_p = \|(w_1, \ldots, w_k)\|_p = (|w_1|^p + \cdots + |w_k|^p)^{1/p}$$

for $p \in \mathbb{R}^+$, and

$$\|\mathbf{w}\|_\infty = \|(w_1, \ldots, w_k)\|_\infty = \max\{|w_1|, \ldots, |w_k|\}$$

for $p = \infty$. Note that the notation is coherent with the fact $\|\mathbf{w}\|_\infty = \lim_{p\to\infty} \|\mathbf{w}\|_p$.

Observe that this map $F_r^k \to \mathbb{R}$ can be expressed in terms of the corresponding vector norm, $\|(w_1, \ldots, w_k)\|_p = \|(|w_1|, \ldots, |w_k|)\|_p$. Hence, it satisfies the following properties:

1) (positivity) $\|\mathbf{w}\|_p \geqslant 0$ with equality if and only if $\mathbf{w} = (1, \ldots, 1)$;
2) (powers) $\|(w_1^n, \ldots, w_k^n)\|_p \leqslant |n|\|(w_1, \ldots, w_k)\|_p$;
3) (triangular inequality) $\|(v_1w_1, \ldots, v_kw_k)\|_p \leqslant \|(v_1, \ldots, v_k)\|_p + \|(w_1, \ldots, w_k)\|_p$.

By analogy, we shall refer to these three properties by naming $\|\cdot\|_p$ as the *p-norm* in F_r^k.

Let us move now to morphisms. Thinking of endomorphisms of F_r (and, in particular, automorphisms) as r-tuples of elements, $\varphi \leftrightarrow (a_1\varphi, \ldots, a_r\varphi)$, we define the *p-norm* of an endomorphism $\varphi \in \operatorname{End} F_r$, $p \in \overline{\mathbb{R}}^+$, as

$$\|\varphi\|_p = \|(a_1\varphi, \ldots, a_r\varphi)\|_p.$$

Given $\Phi \in \operatorname{Out} F_r$, define also

$$\|\Phi\|_p = \min\{\|\varphi\|_p \mid \varphi \in \Phi\}.$$

Of course, $\|\varphi\|_1$ and $\|\Phi\|_1$ equal, respectively, the values $\|\varphi\|$ and $\|\Phi\|$ defined in the previous section.

Further, we define the corresponding gap functions α_r^p and β_r^p in the natural way:

$$\begin{aligned} \alpha_r^p(n) =& \ \max\{\|\varphi^{-1}\|_p \mid \varphi \in \operatorname{Aut} F_r, \quad \|\varphi\|_p \leqslant n\}, \\ \beta_r^p(n) =& \ \max\{\|\Phi^{-1}\|_p \mid \Phi \in \operatorname{Out} F_r, \quad \|\Phi\|_p \leqslant n\}. \end{aligned}$$

Clearly, these are non-decreasing functions from $\mathbb{N}$ to $\mathbb{R}$. Again, α_r and β_r from the previous section are just α_r^1 and β_r^1, respectively. Furthermore, the following proposition states that the functions α_r^p belong to the same equivalence class for all different values of $p \in \overline{\mathbb{R}}^+$; the same happens for the functions β_r^p (note that the equivalence relation defined above for functions from $\mathbb{N}$ to $\mathbb{N}$ can naturally be extended to functions from $\mathbb{N}$ to $\mathbb{R}$). For this reason, we shall restrict our attention to the case $p = 1$ (with occasional references to the ∞-norm for some technical arguments).

Proposition 2.2. *For all $p, q \in \overline{\mathbb{R}}^+$ there exists a natural number $C = C_{p,q,r} > 0$ such that*

$$\frac{1}{C}\|\varphi\|_q \leqslant \|\varphi\|_p \leqslant C\|\varphi\|_q \quad \text{and} \quad \frac{1}{C}\|\Phi\|_q \leqslant \|\Phi\|_p \leqslant C\|\Phi\|_q$$

hold for all $\varphi \in \operatorname{End} F_r$ and $\Phi \in \operatorname{Out} F_r$. Furthermore, for all $n \geqslant 0$,

$$\frac{1}{C}\alpha_r^p\left(\left\lfloor \frac{n}{C} \right\rfloor\right) \leqslant \alpha_r^q(n) \leqslant C\alpha_r^p(Cn),$$

$$\frac{1}{C}\beta_r^p\left(\left\lfloor \frac{n}{C} \right\rfloor\right) \leqslant \beta_r^q(n) \leqslant C\beta_r^p(Cn).$$

Proof. It is well-known (see [4, Corollary 5.4.5]) that the exact similar fact holds for the corresponding vector norms: there exists a positive constant, and so a natural number $C = C_{p,q,r}$ such that

$$\frac{1}{C}\|\mathbf{x}\|_q \leqslant \|\mathbf{x}\|_p \leqslant C\|\mathbf{x}\|_q$$

for every $\mathbf{x} \in \mathbb{R}^r$. Now $\frac{1}{C}\|\varphi\|_q \leqslant \|\varphi\|_p \leqslant C\|\varphi\|_q$ follows immediately from the equality

$$\|\varphi\|_p = \|(a_1\varphi, \dots, a_r\varphi)\|_p = \|(|a_1\varphi|, \dots, |a_r\varphi|)\|_p.$$

On the other hand, since $\|\Phi\|_q = \|\theta\|_q$ for some $\theta \in \Phi$, we get

$$\|\Phi\|_p = \min\{\|\varphi\|_p \mid \varphi \in \Phi\} \leqslant \|\theta\|_p \leqslant C\|\theta\|_q = C\|\Phi\|_q$$

and $\frac{1}{C}\|\Phi\|_q \leqslant \|\Phi\|_p \leqslant C\|\Phi\|_q$ follows by symmetry.

For the second part of the statement, we have

$$\begin{aligned} \alpha_r^q(n) = &\ \max\{\|\varphi^{-1}\|_q \mid \varphi \in \operatorname{Aut} F_r, \quad \|\varphi\|_q \leqslant n\} \\ \leqslant &\ \max\{\|\varphi^{-1}\|_q \mid \varphi \in \operatorname{Aut} F_r, \quad \|\varphi\|_p \leqslant Cn\} \\ \leqslant &\ C\max\{\|\varphi^{-1}\|_p \mid \varphi \in \operatorname{Aut} F_r, \quad \|\varphi\|_p \leqslant Cn\} \\ = &\ C\alpha_r^p(Cn) \end{aligned}$$

for all n. Symmetrically, $\alpha_r^p(n) \leqslant C\alpha_r^q(Cn)$. Now, for every natural number n, write $C\lfloor \frac{n}{C} \rfloor \leqslant n$ and we have $\alpha_r^p\left(\lfloor \frac{n}{C} \rfloor\right) \leqslant C\alpha_r^q\left(C\lfloor \frac{n}{C} \rfloor\right) \leqslant C\alpha_r^q(n)$ and so, $\frac{1}{C}\alpha_r^p\left(\lfloor \frac{n}{C} \rfloor\right) \leqslant \alpha_r^q(n)$.

The same argument gives the corresponding inequalities for the β functions. $\square$

The following lemmas state some basic properties of norms of automorphisms and outer automorphisms of free groups, that will be useful later.

Lemma 2.3. *Let φ, θ, ψ_1, $\psi_2 \in \operatorname{Aut} F_r$ with ψ_1 and ψ_2 letter permuting, and let $w \in F_r \setminus \{1\}$. Then:*

(i) $\frac{\|\varphi\|_1}{r} \leqslant \|\varphi\|_\infty < \|\varphi\|_1$,
(ii) $\|\psi_1\varphi\psi_2\|_p = \|\varphi\|_p$ *for all* $p \in \overline{\mathbb{R}}^+$,
(iii) $||\varphi\theta||_1 \leqslant ||\varphi||_1 \cdot ||\theta||_\infty < ||\varphi||_1 \cdot ||\theta||_1$,
(iv) $||\lambda_w\varphi||_1 \leqslant (2r|w| + r - 2)||\varphi||_\infty < (2r|w| + r - 2)||\varphi||_1$.

Proof. (i) and (ii) are clear from the definitions.

(iii) For every $a \in A_r$, we have $|a\varphi\theta| \leqslant |a\varphi| \cdot ||\theta||_\infty$ and so

$$||\varphi\theta||_1 = \sum_{i=1}^{r} |a_i\varphi\theta| \leqslant \sum_{i=1}^{r} |a_i\varphi| \cdot ||\theta||_\infty = ||\varphi||_1 \cdot ||\theta||_\infty < ||\varphi||_1 \cdot ||\theta||_1.$$

(iv) Since $w \neq 1$, exactly one of the words $w^{-1}a_iw$ is non reduced, and so

$$\begin{aligned} ||\lambda_w\varphi||_1 = &\sum_{i=1}^{r} |(\overline{w^{-1}a_iw})\varphi| \leqslant (r-1)(2|w|+1)||\varphi||_\infty + (2|w|-1)||\varphi||_\infty \\ = &\ (2r|w|+r-2)||\varphi||_\infty < (2r|w|+r-2)||\varphi||_1. \end{aligned}$$ □

Lemma 2.4. *Let* Φ, $\Theta \in \operatorname{Out} F_r$ *and let* ψ_1, $\psi_2 \in \operatorname{Aut} F_r$ *be letter permuting. Then:*

(i) $\|[\psi_1]\Phi[\psi_2]\|_1 = \|\Phi\|_1$,
(ii) $\|\Phi\Theta\|_1 \leqslant \|\Phi\|_1\|\Theta\|_1$.

Proof. We have $[\psi_1]\Phi[\psi_2] = \psi_1\Lambda_r\Phi\psi_2\Lambda_r = \psi_1\Lambda_r\Phi\Lambda_r\psi_2 = \psi_1\Phi\psi_2$. Now Lemma 2.3(ii) yields

$$\|[\psi_1]\Phi[\psi_2]\|_1 = \min\{\|\psi_1\varphi\psi_2\|_1 \mid \varphi \in \Phi\} = \min\{\|\varphi\|_1 \mid \varphi \in \Phi\} = \|\Phi\|_1$$

and so (i) holds.

For (ii), we use Lemma 2.3(iii) to get

$$\begin{aligned} \|\Phi\Theta\|_1 = &\ \min\{\|\psi\|_1 \mid \psi \in \Phi\Theta\} = \min\{\|\varphi\theta\|_1 \mid \varphi \in \Phi,\ \theta \in \Theta\} \\ \leqslant &\ \min\{\|\varphi\|_1\|\theta\|_1 \mid \varphi \in \Phi,\ \theta \in \Theta\} \\ = &\ (\min\{\|\varphi\|_1 \mid \varphi \in \Phi\})(\min\{\|\theta\|_1 \mid \theta \in \Theta\}) = \|\Phi\|_1\|\Theta\|_1 . \end{aligned}$$ □

Lemma 2.5. *Let* $\varphi \in \operatorname{Aut} F_r$ *be cyclically reduced. Then* $\|[\varphi]\|_1 = \|\varphi\|_1$.

2.3. **Abelianization.** Abelianization will be a valuable tool to derive lower bounds for $\|\varphi\|_1$ and $\|\Phi\|_1$.

The 1-norm for vectors $\|(x_1, \ldots, x_r)\|_1 = |x_1| + \cdots + |x_r|$ gives rise to the 1-norm for matrices, namely

$$\|M\|_1 = \sum_{i,j} |m_{i,j}|,$$

where $M = (m_{i,j}) \in \mathrm{GL}_r(\mathbb{Z})$. It is straightforward to verify that, for all $\mathbf{x}$, $\mathbf{y} \in \mathbb{Z}^r$ and $M, N \in \mathrm{GL}_r(\mathbb{Z})$, we have the inequalities $\|\mathbf{x}+\mathbf{y}\|_1 \leqslant \|\mathbf{x}\|_1 + \|\mathbf{y}\|_1$, $\|\mathbf{x}M\|_1 \leqslant \|\mathbf{x}\|_1 \cdot \|M\|_1$, $\|M+N\|_1 \leqslant \|M\|_1 + \|N\|_1$, and $\|MN\|_1 \leqslant \|M\|_1\|N\|_1$.

Let us denote the abelianization map by $(\cdot)^{\mathrm{ab}}\colon F_r \twoheadrightarrow \mathbb{Z}^r$, $w \mapsto w^{\mathrm{ab}} = ([w]_{a_1}, \ldots, [w]_{a_r})$. Here, $[w]_{a_i}$ is the total exponent of a_i in w, i.e. the total number of times the letter a_i occurs in $\overline{w}$, taking into account the exponents' signs (for example, $[a_1a_2a_1^{-2}]_{a_1} = -1$ and $[a_1a_1^{-1}a_2]_{a_1} = [a_2]_{a_1} = 0$).

Every automorphism $\varphi \in \operatorname{Aut} F_r$ abelianizes to an automorphism φ^{ab} of $\mathbb{Z}^r$ which we shall represent by its $r \times r$ (invertible) matrix over $\mathbb{Z}$. We want automorphisms to act on the right, and so we write matrices by rows, i.e. with the i-th row describing the image of the i-th generator:

$$\varphi^{\mathrm{ab}} = \begin{pmatrix} [a_1\varphi]_{a_1} & \cdots & [a_1\varphi]_{a_r} \\ \cdots & \cdots & \cdots \\ [a_r\varphi]_{a_1} & \cdots & [a_r\varphi]_{a_r} \end{pmatrix} \in \mathrm{GL}_r(\mathbb{Z}).$$

This way, for every $w \in F_r$, $(w\varphi)^{\mathrm{ab}} = w^{\mathrm{ab}}\varphi^{\mathrm{ab}}$. Furthermore, $(\varphi\theta)^{\mathrm{ab}} = \varphi^{\mathrm{ab}}\theta^{\mathrm{ab}}$, and $(\varphi^{-1})^{\mathrm{ab}} = (\varphi^{\mathrm{ab}})^{-1}$.

Observe that, for every $w \in F_r$, $|w| \geqslant \|w^{\mathrm{ab}}\|_1 = |[w]_{a_1}| + \cdots + |[w]_{a_r}|$ with equality if and only if no letter occurs in $\overline{w}$ with the two opposite signs. This can be expressed in the following useful way:

Lemma 2.6. *For every $\varphi \in \operatorname{Aut} F_r$, $\|\varphi\|_1 \geqslant \|[\varphi]\|_1 \geqslant \|\varphi^{\mathrm{ab}}\|_1$, with equalities if and only if, for every $i = 1, \ldots, r$, no letter occurs in $\overline{a_i\varphi}$ with the two opposite signs. In particular, $\|\varphi\|_1 = \|\varphi^{\mathrm{ab}}\|_1$ for positive automorphisms.*

Proof. Clearly, $\|\varphi\|_1 \geqslant \|[\varphi]\|_1$. We may write $\|[\varphi]\|_1 = \|\varphi\lambda_w\|_1$ for some $w \in F_r$. Then

$$\begin{aligned} \|\varphi\|_1 \geqslant \|[\varphi]\|_1 = \|\varphi\lambda_w\|_1 = \sum_{i=1}^{r} |a_i\varphi\lambda_w| \geqslant \sum_{i=1}^{r} \|(a_i\varphi)^{\mathrm{ab}}\|_1 \\ = \sum_{i=1}^{r} \|a_i^{\mathrm{ab}}\varphi^{\mathrm{ab}}\|_1 = \sum_{i=1}^{r}\sum_{j=1}^{r} |[a_i\varphi]_{a_j}| = \|\varphi^{\mathrm{ab}}\|_1, \end{aligned}$$

where a_i^{ab} is the i-th canonical vector and so, $a_i^{\mathrm{ab}}\varphi^{\mathrm{ab}}$ is the i-th row in φ^{ab}. It is immediate that the inequality $\|\varphi\|_1 \geqslant \|\varphi^{\mathrm{ab}}\|_1$ becomes an equality if and only if, for every $i = 1, \ldots, r$, no letter occurs in $\overline{a_i\varphi}$ with the two opposite signs. This is the case when $\varphi \in \operatorname{Aut}^+ F_r$. □

3. The rank two case

In this section we shall deal with the rank 2 case. For the duration of this section, we simplify our notation to $A = A_2 = \{a, b, a^{-1}, b^{-1}\}$.

We start by proving that inversion preserves the norm in the case of positive automorphisms. It is known that positive automorphisms of F_2 are generated as a monoid by $\Delta = \{\eta_{b,a}, \eta_{a,ab}, \eta_{a,ba}\}$, that is, they all can be obtained as a composition of these elementary ones, i.e. $\operatorname{Aut}^+ F_2 = \Delta^*$ (see [7]).

Lemma 3.1. *Let $\varphi \in \operatorname{Aut}^+ F_2$ and write $\varphi^{-1} = \eta_{u,v}$. Then either $u \in \{a, b^{-1}\}^*$ and $v \in \{a^{-1}, b\}^*$, or $u \in \{a^{-1}, b\}^*$ and $v \in \{a, b^{-1}\}^*$. In particular, φ^{-1} is cyclically reduced.*

Proof. The result is clear for the three elementary positive automorphisms, $\eta_{b,a}^{-1} = \eta_{b,a}$, $\eta_{a,ab}^{-1} = \eta_{a,a^{-1}b}$, $\eta_{a,ba}^{-1} = \eta_{a,ba^{-1}}$. Since all positive automorphisms are compositions of elements from Δ, it is sufficient to show that, given a positive automorphism φ and $\theta \in \Delta$, the lemma holds for $\varphi\theta$ whenever it holds for φ. To see this, write $\varphi^{-1} = \eta_{u,v}$ and assume u and v are as in the statement. Then we get

$$\begin{aligned}(\varphi\eta_{b,a})^{-1} &= \eta_{b,a}\eta_{u,v} = \eta_{v,u},\\ (\varphi\eta_{a,ab})^{-1} &= \eta_{a,a^{-1}b}\eta_{u,v} = \eta_{u,u^{-1}v},\\ (\varphi\eta_{a,ba})^{-1} &= \eta_{a,ba^{-1}}\eta_{u,v} = \eta_{u,vu^{-1}},\end{aligned}$$

completing the proof. □

Proposition 3.2. *Let $\varphi \in \operatorname{Aut}^+ F_2$. Then $\|\varphi^{-1}\|_1 = \|\varphi\|_1$.*

Proof. Abelianizing, we have

$$\varphi^{\mathrm{ab}} = \begin{pmatrix} [a\varphi]_a & [a\varphi]_b \\ [b\varphi]_a & [b\varphi]_b \end{pmatrix} \quad \text{and} \quad (\varphi^{-1})^{\mathrm{ab}} = \pm \begin{pmatrix} [b\varphi]_b & -[a\varphi]_b \\ -[b\varphi]_a & [a\varphi]_a \end{pmatrix};$$

hence, $\|(\varphi^{-1})^{\mathrm{ab}}\|_1 = \|\varphi^{\mathrm{ab}}\|_1$. Also, $\|\varphi^{\mathrm{ab}}\|_1 = \|\varphi\|_1$ since φ is positive (see Lemma 2.6). Now, write $\varphi^{-1} = \eta_{u,v}$. By Lemma 3.1 no letter occurs with both signs in neither u nor v so, again by Lemma 2.6, $\|(\varphi^{-1})^{\mathrm{ab}}\|_1 = \|\varphi^{-1}\|_1$, concluding the proof. □

From positive automorphisms we can gain control of all cyclically reduced ones.

Lemma 3.3. *For every cyclically reduced $\varphi \in \operatorname{Aut} F_2$, there exist two letter permuting automorphisms $\psi_1, \psi_2 \in \operatorname{Aut} F_2$ and $\theta \in \operatorname{Aut}^+ F_2$ such that $\varphi = \psi_1\theta\psi_2$.*

Proof. Write $\varphi = \eta_{u,v}$. Since both u and v are cyclically reduced, the main result in [2] tells us that at most two letters of A occur in u, and at most two of them (not necessarily the same ones) occur in v.

Without loss of generality, we may assume that two different letters occur in either u or v, say in u. Inverting all possibly negative letters in u, we can write $\eta_{u,v} = \eta_{u',v'}\eta_{a^\epsilon,b^\delta}$ with ϵ, $\delta = \pm 1$, $u' \in \{a,b\}^*$ and $|u'| = |u|$ and $|v'| = |v|$.

If $v' \in \{a,\, b\}^*$, i.e. is a positive word, then $\eta_{u',v'} \in \operatorname{Aut}^+ F_2$ and we are done. If $v' \in \{a^{-1},\, b^{-1}\}^*$, take $\eta_{u,v} = \eta_{a,b^{-1}}\eta_{u',v'^{-1}}\eta_{a^\epsilon,b^\delta}$ and we are also done. The remaining cases to consider are $v' \in \{a^{-1},\, b\}^*$ or $v' \in \{a,\, b^{-1}\}^*$ with exactly two letters occurring in v'; they will lead us to contradiction. Indeed, abelianizing we get $u'^{\mathrm{ab}} = ([u]_a,\, [u]_b) = (p,q)$ with $p,q > 0$, and $v'^{\mathrm{ab}} = ([v]_a,\, [v]_b) = (r,s)$ with $rs < 0$. This contradicts $ps - qr = \pm 1$ coming from the fact that $\eta_{u',v'}$ is an automorphism of F_2. $\square$

And from those, we can reach the general case:

Lemma 3.4. *For every $\varphi \in \operatorname{Aut} F_2$, there exist two letter permuting automorphisms ψ_1, $\psi_2 \in \operatorname{Aut} F_2$, $\theta \in \operatorname{Aut}^+ F_2$, and an element $g \in F_2$ such that $\varphi = \psi_1\theta\psi_2\lambda_g$ and $\|\theta\|_1 + 2|g| \leqslant \|\varphi\|_1$.*

Proof. Note that, by Lemmas 2.3(ii) and 3.3, it suffices to show that there exists a cyclically reduced $\varphi' \in \operatorname{Aut} F_2$ and $g \in F_2$, such that $\varphi = \varphi'\lambda_g$ and $\|\varphi'\|_1 + 2|g| \leqslant \|\varphi\|_1$. Let us prove this claim by induction on $\|\varphi\|_1$.

If $\|\varphi\|_1 = 2$ the claim is trivial since φ is already cyclically reduced. So, suppose $\varphi = \eta_{u,v} \in \operatorname{Aut} F_2$ is given with $\|\eta_{u,v}\|_1 > 2$, and let us assume the claim holds for all automorphisms of smaller 1-norm. Again, if u and v are cyclically reduced the claim is trivial so, by symmetry, we can assume that u is not cyclically reduced, say $\overline{u} = c^{-1}u'c$ for some $c \in A$ and $u' \in F_2$. If $\overline{v}$ neither begins with c^{-1} nor ends with c then it could be easily seen that c would not be contained in $\langle u, v\rangle$ contradicting the fact that $\{u, v\}$ generates F_2. Hence, $v \in c^{-1}A^* \cup A^*c$, and so $|\overline{cvc^{-1}}| \leqslant |v|$. Now, factoring $\eta_{u,v}$ as $\eta_{u,v} = \eta_{u',\overline{cvc^{-1}}}\lambda_c$, we have

$$\|\eta_{u',\overline{cvc^{-1}}}\|_1 = |u'| + |\overline{cvc^{-1}}| \leqslant |u| - 2 + |v| = \|\eta_{u,v}\|_1 - 2,$$

and we can apply the induction hypothesis to get a factorization $\eta_{u',\overline{cvc^{-1}}} = \varphi'\lambda_h$ with φ' cyclically reduced and $\|\varphi'\|_1 + 2|h| \leqslant \|\eta_{u',\overline{cvc^{-1}}}\|_1$. Thus, we have $\eta_{u,v} = \eta_{u',\overline{cvc^{-1}}}\lambda_c = \varphi'\lambda_h\lambda_c = \varphi'\lambda_{hc}$ with

$$\|\varphi'\|_1 + 2|hc| \leqslant \|\varphi'\|_1 + 2|h| + 2 \leqslant \|\eta_{u',\overline{cvc^{-1}}}\|_1 + 2 \leqslant \|\eta_{u,v}\|_1 = \|\varphi\|_1.$$

This completes the proof of the claim and so, of the lemma. $\square$

Theorem 3.5. *For every $n \geqslant 4$, we have $\alpha_2(n) \leqslant \frac{(n-1)^2}{2}$.*

Proof. Let $\varphi \in \operatorname{Aut} F_2$ with $\|\varphi\|_1 \leqslant n$, and let us prove that $\|\varphi^{-1}\|_1 \leqslant \frac{(n-1)^2}{2}$. Consider the decomposition given in Lemma 3.4, $\varphi = \psi_1 \theta \psi_2 \lambda_g$ for some letter permuting $\psi_1, \psi_2 \in \operatorname{Aut} F_2$, some $\theta \in \operatorname{Aut}^+ F_2$, and some $g \in F_2$ such that $\|\theta\|_1 + 2|g| \leqslant \|\varphi\|_1$.

If $g = 1$ then

$$\|\varphi^{-1}\|_1 = \|\psi_2^{-1}\theta^{-1}\psi_1^{-1}\|_1 = \|\theta^{-1}\|_1 = \|\theta\|_1 = \|\varphi\|_1 \leqslant n \leqslant \frac{(n-1)^2}{2},$$

by Lemma 2.3(ii) and Proposition 3.2 (and using in the last step that $n \geqslant 4$).

So, let us assume $g \neq 1$ in which case we have $\varphi^{-1} = \lambda_{g^{-1}}\psi_2^{-1}\theta^{-1}\psi_1^{-1}$. By Lemma 2.3 and Proposition 3.2,

$$\begin{aligned}\|\varphi^{-1}\|_1 &\leqslant 4|g| \cdot \|\psi_2^{-1}\theta^{-1}\psi_1^{-1}\|_\infty = 4|g| \cdot \|\theta^{-1}\|_\infty \leqslant 4|g|(\|\theta^{-1}\|_1 - 1) \\ &= 4|g|(\|\theta\|_1 - 1).\end{aligned}$$

Since we also have $\|\theta\|_1 + 2|g| \leqslant \|\varphi\|_1 \leqslant n$, we deduce $|g| \leqslant \frac{n - \|\theta\|_1}{2}$ and so,

$$\|\varphi^{-1}\|_1 \leqslant 2(n - \|\theta\|_1)(\|\theta\|_1 - 1).$$

Finally, since the parabola $f(x) = 2(n-x)(x-1)$ has its absolute maximum in the point $x = \frac{n+1}{2}$, we conclude

$$\|\varphi^{-1}\|_1 \leqslant 2(n - \|\theta\|_1)(\|\theta\|_1 - 1) \leqslant 2\Big(n - \frac{n+1}{2}\Big)\Big(\frac{n+1}{2} - 1\Big) = \frac{(n-1)^2}{2}. \qquad \square$$

In order to establish lower bounds for $\alpha_2(n)$, we need to construct explicit automorphisms of F_2 having inverses with 1-norm much bigger than that of themselves.

Theorem 3.6. *For $n \geqslant 10$, we have $\alpha_2(n) \geqslant \frac{n^2}{4} - 6n + 42$.*

Proof. For $k \geqslant 0$ consider the automorphisms

$$\psi_k = \eta_{ab^{2k},\, ab^{2k+1}} \lambda_{a^{-k}b} = \eta_{b^{-1}a^{k+1}b^{2k}a^{-k}b,\, b^{-1}a^{k+1}b^{2k+1}a^{-k}b}.$$

We have $\|\psi_k\|_1 = 8k + 7$. For the inverse, we have

$$\psi_k^{-1} = \lambda_{b^{-1}a^k}\eta^{-1}_{ab^{2k},\, ab^{2k+1}} = \lambda_{b^{-1}a^k}\eta_{a(b^{-1}a)^{2k},\, a^{-1}b} = \eta_{u,\, v},$$

where u and v are the two words

$$u = ((a^{-1}b)^{2k}a^{-1})^k a^{-1}ba(b^{-1}a)^{2k}b^{-1}a(a(b^{-1}a)^{2k})^k,$$
$$v = ((a^{-1}b)^{2k}a^{-1})^k a^{-1}b(a(b^{-1}a)^{2k})^k.$$

Hence, $\|\psi_k^{-1}\|_1 = 4(4k+1)k + 4k + 7 = 16k^2 + 8k + 7$.

Writing $n = \|\psi_k\|_1 = 8k + 7$, we have $k = \frac{n-7}{8}$ and then

$$\|\psi_k^{-1}\|_1 = 16\frac{(n-7)^2}{64} + n - 7 + 7 = \frac{n^2 - 10n + 49}{4}.$$

Thus, for $n \equiv 7 \mod 8$, we have $\alpha_2(n) \geqslant \frac{n^2-10n+49}{4}$.

Finally, for every $n \geqslant 7$, let n' be the unique integer congruent with 7 modulo 8 in the set $\{n-7, \dots, n-1, n\}$. We have

$$\alpha_2(n) \geqslant \alpha_2(n') \geqslant \frac{n'^2 - 10n' + 49}{4} \geqslant \frac{(n-7)^2 - 10(n-7) + 49}{4}$$
$$= \frac{n^2}{4} - 6n + 42,$$

where the last inequality uses $n \geqslant 10$ since the parabola $f(x) = \frac{x^2-10x+49}{4}$ has its minimum at $x = 5$. □

The outer automorphism case turns out to be simpler:

Theorem 3.7. *For every* $\Phi \in \operatorname{Out} F_2$, $\|\Phi^{-1}\|_1 = \|\Phi\|_1$. *Consequently,* $\beta_2(n) = n$.

Proof. Take $\varphi \in \Phi$. By Lemma 3.4, $\varphi = \psi_1\theta\psi_2\lambda_g$ for some letter permuting automorphisms $\psi_1, \psi_2 \in \operatorname{Aut} F_2$, some $\theta \in \operatorname{Aut}^+ F_2$ and some element $g \in F_2$. Then Lemmas 2.4(i) and 2.5 yield

$$\|\Phi\|_1 = \|[\varphi]\|_1 = \|[\psi_1\theta\psi_2\lambda_g]\|_1 = \|[\psi_1\theta\psi_2]\|_1 = \|[\theta]\|_1 = \|\theta\|_1.$$

Also by Lemma 2.4(i), we get

$$\|\Phi^{-1}\|_1 = \|[\varphi^{-1}]\|_1 = \|[\lambda_{g^{-1}}\psi_2^{-1}\theta^{-1}\psi_1^{-1}]\|_1 = \|[\psi_2^{-1}\theta^{-1}\psi_1^{-1}]\|_1 = \|[\theta^{-1}]\|_1.$$

Since θ^{-1} is cyclically reduced by Lemma 3.1, we may use Lemma 2.5 to get $\|\Phi^{-1}\|_1 = \|[\theta^{-1}]\|_1 = \|\theta^{-1}\|_1$. Since $\|\theta\|_1 = \|\theta^{-1}\|_1$ by Proposition 3.2, we get $\|\Phi^{-1}\|_1 = \|\Phi\|_1$. Therefore $\beta_2(n) = n$. □

4. Higher rank

In this section, we consider arbitrary rank $r \geqslant 2$, compute polynomial lower bounds for both $\alpha_r(n)$ and $\beta_r(n)$, and show that $\beta_r(n)$ admits a polynomial upper bound.

The polynomial lower bounds for $\alpha_r(n)$ and $\beta_r(n)$ have degrees r and $r-1$, respectively. In particular, this separates the asymptotic behavior of the rank two case from all other ranks, with respect to both complexity functions. That is, $\xi_2(n)$ grows more slowly than $\xi_r(n)$ for all $r \geqslant 3$ and $\xi \in \{\alpha, \beta\}$, which agrees with the intuitive fact that $\mathrm{Aut}\, F_r$ is a much easier group to deal with for $r = 2$ than for higher rank.

Finally, the polinomial upper bound for $\beta_r(n)$ is established with the help of the theory of Outer space.

We assume the rank r fixed throughout the whole section.

4.1. **Lower bounds.** Our lower bound for $\beta_r(n)$ is obtained by abelianization of positive automorphisms. The extra unit in the degree of the lower bounds from $\beta_r(n)$ to $\alpha_r(n)$ will be achieved by additionally composing the positive automorphisms with a suitable conjugation that increases in size when inverting. We thank Warren Dicks for suggesting us to use the following automorphisms; this significantly simplified our previous proof of the lower bounds for $\alpha_r(n)$ and $\beta_r(n)$.

We start by defining, for every $p \in \mathbb{Z}$, a matrix $M^{(p)} = (m^{(p)}_{i,j}) \in \mathrm{GL}_r(\mathbb{Z}) = \mathrm{Aut}\, \mathbb{Z}^r$ given by

$$m^{(p)}_{i,j} = \begin{cases} 1, & \text{if } i = j; \\ p, & \text{if } j = i + 1; \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\det M^{(p)} = 1$ and so $M^{(p)}$ is indeed invertible.

Lemma 4.1. *For all $r \geqslant 2$ and $p \in \mathbb{Z}$, let $N^{(p)} = (n^{(p)}_{i,j}) \in \mathrm{GL}_r(\mathbb{Z})$ be defined by*

$$n^{(p)}_{i,j} = \begin{cases} 1, & \textit{if } i = j; \\ (-p)^{j-i}, & \textit{if } i < j; \\ 0, & \textit{otherwise.} \end{cases}$$

Then $N^{(p)} = (M^{(p)})^{-1}$.

Proof. It suffices to show that $M^{(p)}N^{(p)}$ is the identity matrix. Indeed, the (i, j)-th entry of the product matrix is $\sum_{k=1}^{r} m^{(p)}_{i,k} n^{(p)}_{k,j} = \sum_{k=i}^{\min\{i+1,j\}} m^{(p)}_{i,k} n^{(p)}_{k,j}$ which is 0 if $j < i$ and 1 if $j = i$. If $j > i$, we get $m^{(p)}_{i,i} n^{(p)}_{i,j} + m^{(p)}_{i,i+1} n^{(p)}_{i+1,j} = (-p)^{j-i} + p(-p)^{j-i-1} = 0$ and the lemma is proved. □

We immediately obtain:

Lemma 4.2. *For all $r \geqslant 2$ and $p \in \mathbb{Z}$, we have $\|M^{(p)}\|_1 = r + (r-1)p$ and $\|(M^{(p)})^{-1}\|_1 \geq p^{r-1}$.* □

For every integer $p \geqslant 2$, define $\varphi_p \in \mathrm{Aut}^+ F_r$ by

$$a_i\varphi_p = \begin{cases} a_i a_{i+1}^p, & \text{if } 1 \leq i < r; \\ a_r, & \text{if } i = r. \end{cases}$$

Note that φ_p is clearly onto and therefore an automorphism since free groups of finite rank are hopfian [5].

Lemma 4.3. *For all $r \geqslant 2$ and $p \geqslant 2$:*

(i) $\varphi_p^{\mathrm{ab}} = M^{(p)}$,

(ii) $a_r\varphi_p^{-1} = a_r$ *and* $a_i\varphi_p^{-1} = a_i(a_{i+1}\varphi_p^{-1})^{-p}$ *for* $i = 1, \dots, r-1$,

(iii) $\overline{a_i\varphi_p^{-1}} \in a_i A_r^* a_{i+1}^{-1}$ *for* $i = 1, \dots, r-1$,

(iv) $||\varphi_p^{-1}||_1 < 2|a_1\varphi_p^{-1}|$.

Proof. (i) is clear.

To get (ii), it suffices to compute $(a_i(a_{i+1}\varphi_p^{-1})^{-p})\varphi_p = (a_i\varphi_p)a_{i+1}^{-p} = a_i$ for $i < r$. Then (iii) follows from (ii) by reverse induction.

Finally, to see (iv) observe that by (iii) the product $a_i(a_{i+1}\varphi_p^{-1})^{-p}$ is reduced and so $|a_i\varphi_p^{-1}| > p|a_{i+1}\varphi_p^{-1}|$ for every $i < r$. Hence $|a_i\varphi_p^{-1}| < \frac{1}{p^{i-1}}|a_1\varphi_p^{-1}|$ for $i = 2, \dots, r$ and so

$$||\varphi_p^{-1}||_1 = \sum_{i=1}^{r} |a_i\varphi_p^{-1}| < (1 + \frac{1}{p} + \dots + \frac{1}{p^{r-1}})|a_1\varphi_p^{-1}| < 2|a_1\varphi_p^{-1}|. \quad \square$$

Now we are ready to state and prove the lower bounds for our complexity functions.

Theorem 4.4. *For every $r \geqslant 2$, there exists constants $K_r, K_r' > 0$ such that, for every $n \geqslant 1$:*

(i) $K_r n^r \leqslant \alpha_r(n)$,

(ii) $K_r' n^{r-1} \leqslant \beta_r(n)$.

Proof. Let $p \geqslant r$. By Lemmas 2.6, 4.2 and 4.3(i), we have

$$\|\varphi_p\|_1 = \|[\varphi_p]\|_1 = \|\varphi_p^{\mathrm{ab}}\|_1 = \|M^{(p)}\|_1 = r + (r-1)p \leqslant rp. \quad (1)$$

On the other hand, the same results yield

$$\|\varphi_p^{-1}\|_1 \geqslant \|[\varphi_p^{-1}]\|_1 \geqslant \|(\varphi_p^{-1})^{\mathrm{ab}}\|_1 = \|(\varphi_p^{\mathrm{ab}})^{-1}\|_1 = \|(M^{(p)})^{-1}\|_1 \geqslant p^{r-1}. \quad (2)$$

Let $n_0 = \max\left\{r^2, \frac{(r-1)2^{\frac{1}{r-1}}}{2^{\frac{1}{r-1}}-1}\right\}$ and consider $n \geqslant n_0$. Take the integer $p = \lfloor \frac{n}{r} \rfloor \geqslant r$, which satisfies $\frac{n-(r-1)}{r} \leqslant p \leqslant \frac{n}{r}$ and so $rp \in \{n-(r-1), \dots, n\}$. The outer automorphism $[\varphi_p] \in \mathrm{Out}(F_r)$ satisfies $\|[\varphi_p]\|_1 \leqslant$

$rp \leqslant n$; and, on the other hand, $\|[\varphi_p^{-1}]\|_1 \geqslant p^{r-1} \geqslant (\frac{n-(r-1)}{r})^{r-1} = \frac{(n-(r-1))^{r-1}}{r^{r-1}}$. Now it is straightforward to check that

$$(n-a)^s \geqslant \frac{n^s}{2} \iff n \geqslant \frac{a2^{\frac{1}{s}}}{2^{\frac{1}{s}}-1}$$

holds for all positive integers s, a, n. Hence, we deduce that

$$\|[\varphi_p^{-1}]\|_1 \geqslant \frac{1}{2r^{r-1}} n^{r-1}$$

(using that $n \geqslant \frac{(r-1)2^{\frac{1}{r-1}}}{2^{\frac{1}{r-1}}-1}$). We conclude that $\beta_r(n) \geqslant \frac{1}{2r^{r-1}} n^{r-1}$ for $n \geqslant n_0$. Adjusting the value of the constant $\frac{1}{2r^{r-1}}$ to cover the finitely many missing values of n, (ii) holds.

To prove (i) let us restrict ourselves to the case $r \geqslant 3$ (Theorem 3.6 already deals with the case $r = 2$). Fix $p \geqslant r$ and let $\psi_p = \varphi_p \lambda_{a_1^p}$. Then (1) yields

$$\|\psi_p\|_1 = \sum_{i=1}^{r} |a_1^{-p}(a_i\varphi_p)a_1^p| \leqslant 2rp + \|\varphi_p\|_1 \leqslant 3rp.$$

On the other hand,

$$\|\psi_p^{-1}\|_1 = \|\lambda_{a_1^{-p}}\varphi_p^{-1}\|_1 > \sum_{i=3}^{r} |(a_1^p a_i a_1^{-p})\varphi_p^{-1}|.$$

Since the products $(a_1\varphi_p^{-1})^p(a_i\varphi_p^{-1})(a_1^{-1}\varphi_p^{-1})^p$ are reduced by Lemma 4.3(iii), it follows that $\|\psi_p^{-1}\|_1 > 2(r-2)p|a_1\varphi_p^{-1}| > (r-2)p\|\varphi_p^{-1}\|_1 \geqslant (r-2)p^r$, by Lemma 4.3(iv) and (2).

This shows that, for $n = 3rp$ and $p \geqslant r$, we have $\alpha_r(n) > (r-2)p^r = \frac{r-2}{(3r)^r} n^r$ i.e., (i) is proven for all such values of n. Finally, the extension of this inequality to all values of n (after adjusting properly the multiplicative constant) proceeds similarly to part (ii). $\square$

As a final remark for this section, it seems clear that this exhausts the potential of abelianization techniques to provide lower bounds. If the growths of our complexity functions are strictly bigger than what we have proven here, this will have to be obtained by more intricate counting techniques working above the abelian level.

4.2. **Upper bounds.** We can present a polynomial upper bound for $\beta_r(n)$ using Outer space techniques. We thank M. Bestvina for suggesting a simplification of our initial arguments, which leads to a very easy and elegant proof of such a polynomial upper bound, now essentially a corollary of a recent result about the asymmetry of the Lipschitz metric in Outer space.

Let us briefly recall what Outer space $\mathcal{X}_r$ is, $r \geqslant 2$, following the notation from [1] (see [6] for more details).

By the term *graph* we mean a finite graph Γ of rank r, all whose vertices have degree at least three. A *metric* on Γ is a function $\ell \colon E\Gamma \to [0,1]$ defined on the set of edges of Γ such that $\sum_{e\in E\Gamma} \ell(e) = 1$ and the set of length zero edges forms a forest. Let us denote by Σ_Γ the space of all such metrics ℓ on Γ, viewed as a "simplex with missing faces" (corresponding to degenerate metrics that vanish on a subgraph which is not a forest). If Γ' is obtained from Γ by collapsing a forest, then we will naturally consider $\Sigma_{\Gamma'}$ as a subset of Σ_Γ along the inclusion given by assigning length zero to the collapsed edges.

Fix the rose graph R_r with one vertex (denoted o) and r edges, and identify the free group F_r with the fundamental group $\pi_1(R_r, o)$ in such a way that each generator a_i corresponds to a single oriented edge of R_r. Under this identification, each reduced word in F_r corresponds to a reduced edge-path loop starting and ending at the basepoint o in R_r.

A *marked graph* is a pair (Γ, f) where f is a *marking*, i.e. a homotopy equivalence from the rose R_r to Γ. It is standard to consider the set of marked graphs modulo the following equivalence relation: $(\Gamma, f) \sim (\Gamma', f')$ if and only if there is a homeomorphism $\mu \colon \Gamma \to \Gamma'$ such that $f\mu$ is homotopic to f'. Denote it by $\mathcal{MG}/\sim$.

Noting that all representatives of a given class $[(\Gamma, f)] \in \mathcal{MG}/\sim$ share a common underlying graph, we can consider the space of metrics on Γ and denote it $\Sigma_{[(\Gamma,f)]}$. Now, the *Outer Space* $\mathcal{X}_r$ is obtained from the disjoint union

$$\bigsqcup_{[(\Gamma,f)]\in\mathcal{MG}/\sim} \Sigma_{[(\Gamma,f)]}$$

by identifying the faces of the simplices along the above natural inclusions. Thus, a point in $\mathcal{X}_r$ is represented by a triple of the form (Γ, f, ℓ).

There is a natural action of $\operatorname{Aut} F_r$ on $\mathcal{X}_r$. Given $\varphi \in \operatorname{Aut} F_r$, realize it on the rose, say $\varphi \colon R_r \to R_r$, and for every point $x = (\Gamma, f, \ell) \in \mathcal{X}_r$

define $\varphi \cdot x$ to be $(\Gamma, \varphi f, \ell)$. It is easy to see that this is well defined and gives an action of $\operatorname{Aut} F_r$ on $\mathcal{X}_r$. Notice that, by construction, inner automorphisms act trivially; so, what we have is in fact an action of $\operatorname{Out} F_r$ on $\mathcal{X}_r$.

Recently, the Lipschitz metric for $\mathcal{X}_r$ has been introduced and initially studied in [3], followed by other authors (see, for example, [1]). This metric can be defined as follows.

Let $x, x' \in \mathcal{X}_r$ be two points in the Outer space; take representatives, say (Γ, f, ℓ) and (Γ', f', ℓ'), respectively. A *difference of markings* is a map $\mu\colon \Gamma \to \Gamma'$ which is linear on edges, and such that $f\mu$ is homotopic to f'. For such a difference of markings one can define $\sigma(\mu)$ to be the largest slope of μ over all edges $e \in E\Gamma$. Then define the distance from x to x' as

$$d(x, x') = \min_{\mu} \{\log \sigma(\mu)\},$$

where the minimum is taken over all possible differences of markings (and achieved by Arzela-Ascoli's Theorem).

The basic properties of this "distance" are the following: (1) $d(x, y) \geqslant 0$, with equality if and only if $x = y$; (2) $d(x, z) \leqslant d(x, y) + d(y, z)$ for all $x, y, z \in \mathcal{X}_r$; (3) $\operatorname{Out} F_r$ acts by isometries, i.e. $d([\varphi] \cdot x, [\varphi] \cdot y) = d(\varphi \cdot x, \varphi \cdot y) = d(x, y)$ for all $x, y \in \mathcal{X}_r$ and $\varphi \in \operatorname{Aut} F_r$; but (4) $d(x, y) \neq d(y, x)$ in general. See [3] and [1] for details.

For $\epsilon > 0$, define the ϵ-*thick part* of $\mathcal{X}_r$ as

$$\mathcal{X}_r(\epsilon) = \{(\Gamma, f, \ell) \in \mathcal{X}_r \mid \ell(p) \geqslant \epsilon \;\; \forall p \text{ nontrivial closed path in } \Gamma\}.$$

The following is an interesting result from Y. Algom-Kfir and M. Bestvina (see [1, Theorem 23]):

Theorem 4.5 (Algom-Kfir, Bestvina)**.** *Let $r \geqslant 2$. For any $\epsilon > 0$ there is a constant $M = M(r, \epsilon) > 0$ such that, for all $x, y \in \mathcal{X}_r(\epsilon)$,*

$$d(x, y) \leqslant M \cdot d(y, x).$$

As an easy corollary, we obtain our polynomial upper bound for $\beta_r(n)$:

Corollary 4.6. *For every $r \geqslant 2$, there exist constants $K_r, M_r > 0$ such that $\beta_r(n) \leqslant K_r n^{M_r}$ for every $n \geqslant 1$.*

Proof. Fix an automorphism $\varphi \in \operatorname{Aut} F_r$.

Consider the point of the Outer space $x \in \mathcal{X}_r$ represented by the triple (R_r, id, ℓ_0), i.e. by the identity marking over the balanced rose (here, ℓ_0 assigns constant length $1/r$ to each petal). Now consider the point $[\varphi] \cdot x = (R_r, \varphi, \ell_0) \in \mathcal{X}_r$. From the definitions, $\mu : R_r \to R_r$ is a difference of markings if and only if μ is homotopic to φ; and it is straightforward to see that this happens if and only if $\mu = \varphi \lambda_w \lambda_p$ for some $w \in F_r$ and some path p travelling linearly from the basepoint o to an internal point of a petal and with $\ell(p) \leqslant \frac{1}{2r}$ (if μ fixes the basepoint then p can be taken to be trivial; otherwise, it can always be taken to be the shortest path from o to $o\mu$). Moreover, μ maps each edge a_i linearly to a path of length $\ell(p) + |a_i \varphi \lambda_w| \frac{1}{r} + \ell(p)$ so, $\sigma(\mu) = \sigma(\varphi \lambda_w \lambda_p) = \|\varphi \lambda_w\|_\infty + 2r\ell(p)$. It follows that

$$\begin{aligned} d(x, [\varphi] \cdot x) &= \min_{w,p} \{\log(\sigma(\varphi \lambda_w \lambda_p))\} \\ &= \log(\min_{w,p} (\|\varphi \lambda_w\|_\infty + 2r\ell(p))) \\ &= \log(\|[\varphi]\|_\infty). \end{aligned}$$

Hence, by property (3) above,

$$d([\varphi] \cdot x, x) = d(x, [\varphi^{-1}] \cdot x) = \log(\|[\varphi^{-1}]\|_\infty).$$

But, since all the involved points belong to the $(1/r)$-thick part $\mathcal{X}_r(\frac{1}{r})$, we can take the constant $M_r = M(r, \frac{1}{r})$ from Theorem 4.5 to get $\log(\|[\varphi^{-1}]\|_\infty) \leqslant M_r \log(\|[\varphi]\|_\infty)$ and so, $\|[\varphi^{-1}]\|_\infty \leqslant \|[\varphi]\|_\infty^{M_r}$. Bringing in the constant $C_r = C_{\infty,1,r}$ from Proposition 2.2, we obtain

$$\|[\varphi]^{-1}\|_1 \leqslant C_r \|[\varphi]^{-1}\|_\infty \leqslant C_r \|[\varphi]\|_\infty^{M_r} \leqslant C_r^{M_r+1} \|[\varphi]\|_1^{M_r}.$$

Hence $\beta_r(n) \leqslant K_r n^{M_r}$ holds for $K_r = C_r^{M_r+1}$. □

Remark 4.7. Theorems 4.4(ii) and Corollary 4.6 bound the gap for outer automorphism inversion in free groups F_r or rank $r \geqslant 3$ between polynomial with degree $r - 1$ and polynomial with degree M for a big enough M. This is all the information known at the moment about Question 1.9. These two bounds are far from each other and, intuitively, both of them far from sharp. The proof for the lower bound uses only information coming from the abelianization so, it seems plausible that, playing with more sophisticated automorphisms of F_r than the φ_p's constructed above, one could improve the degree of the lower bound. On the other hand, the proof of Algom-Kfir-Bestvina's theorem is indirect and the actual constant provided there is quite big, indicating that maybe the degree of the upper bound provided for $\beta_r(n)$ is also improvable.

Remark 4.8. We also remark that getting a polynomial upper bound for $\alpha_r(n)$ seems to be more complicated (see Question 1.10). On the one hand, the geometric techniques coming from Outer space do not provide control on the length of possible conjugators showing up when computing the pre-image of the generators a_i by a (even cyclically reduced) given automorphism of F_r. A possibility here could be to try translating the argument above from the Outer space to the Auter space concerning real automorphisms (not just outer ones); unfortunately, the theory for the Auter space is much less developed and, for example, there is no known metric and so no analog to Algom-Kfir-Bestvina's theorem, yet. On the other hand, and oppositely to the much easier case $r = 2$, these conjugators cannot be avoided in general by just composing with an appropriate inner automorphism because they can affect differently the various generators.

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