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Pricing American-style Options using the Static Hedge Portfolio

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Resumo

A presente tese baseia-se na abordagem proposta por Chung e Shih [13], o *static hedge portfolio* (SHP), para avaliação e cobertura de opções de estilo americano para os modelos de Black-Scholes [4] e *constant elasticity of variance* (CEV) de Cox [15]. O principal objetivo desta tese é estender o *static hedge portfolio* para avaliação de opções de estilo americano quando o ativo subjacente segue o modelo jump-diffusion proposto por Merton [32]. Conforme demonstrado por Chung e Shih [13], o preço de opções de estilo americano pelo método SHP é apurado tendo por base um portfólio de opções de estilo europeu com múltiplos preços de exercicio e maturidades. Este *static hedge portfolio* é assim formulado pela aplicação das condições *value-matching* e *smooth-pasting* na fronteira de exercicio antecipado. A precisão e a eficiência do modelo proposto são testadas através do método *Fourier Space Time-stepping* (método FST) de Jackson et. al [26], que permite avaliar opções *standard* de estilo americano sob processos exponenciais de Lévy. Os resultados indicam que a adição de salto nos processos estocásticos dos ativos introduz limitações ao método em estudo, diminuindo a sua precisão e, consequentemente, excluindo-o do conjunto de ferramentas que permitem avaliar opções de estilo americano. Este artigo sugere ainda alguns tópicos para pesquisas futuras, que podem potencialmente resolver os problemas de precisão identificados.

Palavras-chave: opções de estilo americano; avaliação de opções; Static hedging Portfolio; Merton jump-diffusion model.

Abstract

This thesis uses the static hedge portfolio (SHP) approach proposed by Chung and Shih [13] to price and hedge American-style options under the Black-Scholes [4] model and the constant elasticity of variance (CEV) model of Cox [15]. The main goal of this thesis is to extend the static hedge portfolio approach to price American-style options when the underlying asset follows the jump-diffusion model proposed by Merton [32]. As demonstrated by Chung and Shih [13], the SHP to price American-style options is achieved through a portfolio of European-style options with multiple strikes and maturities and formulated by applying the value-matching and smooth-pasting conditions on the early exercise boundary. The accuracy and efficiency of the proposed pricing model are compared with the Fourier Space Time-stepping method (FST-method) of Jackson et. al [26], which allows pricing American-style standard options under exponential Lévy processes. The results indicate that the jump addition introduces limitations to the model under study, decreasing its accuracy and consequently excluding it from the set of tools that can be used to evaluate American-style options. This paper also suggests some topics for future research, which can potentially solve the accuracy issues identified.

Keywords: American-style options; Option Pricing; Static hedging Portfolio; Merton jump-diffusion model.

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1. Introduction

1973 is marked by two major events in the history of options: the first listed options were traded in the Chicago Board Options Exchange (CBOE); and a valuation model to price European-style options was developed by Merton [31] and Black-Scholes [4] which deeply changed the options' market and how this instrument is used.

Prior to this model, the assessment and measurement of risks was unclear to those who traded options. However, the approach developed by Fisher Black, Myron Scholes and Robert Merton is limited to the assessment of European-style options, which differ from American-style options on the time at which options can be exercised. While European-style options give the right to exercise the option at the maturity date T, American-style options can be exercised at any time up to T. Therefore, one of the greatest difficulties in the valuation of American-style options is to find the date when the option holder will have the greatest benefit in exercising the contingent claim. For this reason, the determination of this ideal time has become an important, as well as difficult, problem in finance research field. Considering that the American-style options are an heavily traded derivative, due to its frequent use in speculation and hedging, they require a fair value recognized by the market, similarly to the one established for European-style options. These factors have been leading to intense study over the last decades, with the objective to develop more robust and efficient models.

In the same article that describes an important study on European-style options valuation models, Merton [31] makes a relevant statement, when it comes to call options: if the underlying asset does not pay dividends, there is no benefit on exercising an American-style call option before its maturity. This assumption allows pricing American-style call contracts as European-style options. Development of papers focused on the case of discrete dividends, for which analytical solutions can be derived, was the next step; Roll [35] and Geske [21] are good examples of these studies, as well as Whaley [41], who proved that if the underlying asset pays discrete dividends over time, it is possible to define an analytical formula for evaluating American-style call options, because the optimal time to exercise the option occurs immediately before dividends payment.

Nevertheless, the mathematical difficulty of finding an analytical formula to price American-style options on dividend-paying asset remains. There are no closed-form solution formulas for pricing American-style options, which is related to the fact that the option price, as well as the early exercise boundary, must be determined simultaneously, as described by Mckean [30] regarding the free boundary problem. This significant financial issue led experts to develop creative solutions for completing the valuation of American-style options, such as numerical solution methods or analytical approximations. All new solutions aim to accurately and efficiently determine the price of complex financial products with early exercise features.

Many of these methods have been successful and are able to evaluate American-style options efficiently. Some of the most commonly used methods are based on Brennan and Schwartz [6], who introduced the finite difference method, and on Cox et al. [16], who developed the binomial method for the valuation of American-style options. Both approaches use a discrete time and a discrete stock price process to approximate the underlying continuous process. These methods may take considerable amount of time to simulate, due to their time-recursive nature, which can generate significant errors in options with long maturities. Besides being an excellent pedagogical tool, these discrete-time models also play an important role in real world practice for valuing most contingent claims, which explains the extensive research of Broadie and Detemple [7], Figlewski and Gao [19], Heston and Zhou [24], Chung and Shih [12]. Monte Carlo simulation is another numerical tool widely used to calculate options prices, whose details can be studied on Glasserman [20]. Recent advanced numerical methods also include the quadrature integral methods of Sullivan [37].

On its turn, Geske and Johnson [22] worked, via Richardson extrapolation, to approximate American-style options prices to an infinite serie of exercisable Bermudan-style options. This approach was well accepted among finance experts such as Bunch and Johnson [8] or Chung and Shackleton [11], who continued to improve the method in order to resolve the lack of uniformity in convergence. Another approach is Carr's [10], who also using Richardson extrapolation, developed a fast and accurate randomization methodology. Despite the efforts, the models implemented are still not as robust as expected, because the use of Richard extrapolation does not allow error determination.

In another way, Kim [28], Jacka [27] and Carr et al. [9] looked for a differentiated solution based on the integral representation. However, approaches presented are based on Black-Scholes model, differing only in the early exercise boundary chosen.

An additional chapter of the American-style option price literature, which will also be studied in this document, is the Static Hedge Portfolio (SHP). This approach was initially developed by Bowie and Carr [5], Derman et al.[17] and Carr et al. [10] for hedging European-style exotic options; its main idea is to create a static portfolio of standard European-style options whose values ensure the matching of the option pay-off at expiration along with the boundary. This matching can be formulated through two different ways: a range of standard European-style options with maturities from time 0 to time T to match the boundary before the exotic option maturity, with the strike equaling the boundary before maturity; or a range of standard European-style options of all strikes, with the maturity date T matching that of the exotic option.

In comparison to dynamic hedge, static hedging is less sensible to model's risks, such as volatility misspecification, as documented by Tompkins [40] and Thomsen [39]. Due to discrete trading, dynamic hedging may also have substantial hedging errors, as described by Primbs and Yamada [34]. Furthermore, when the transaction costs are high, static hedging is significantly cheaper than dynamic hedging.

Based on the static hedge portfolio, Chung and Shih [13] adapted this approach to the pricing of American-style options. In addition to proposing a pricing model for options, the hedge problem is automatically solved, while the static hedge portfolio is found. Moreover, because of the possibility of early exercise in the case of the American-style options, this methodology takes advantage of both approaches by using standard European-style options with multiple strikes and multiple maturities. The free boundary problem presented by static hedge portfolio of American-style options, requires that early exercise boundary to be determined at the same time of hedging. Besides that, Chung and Shih [13] also demonstrated that the static hedge portfolio is efficiently applicable to other stochastic processes beyond the Black-Scholes model [4], proving that their static hedge portfolio approach also works adequately for the constant elasticity of variance (CEV) model.

As mentioned above, the Black-Sholes model is a reference in the theoretical evaluation of options. However, the normal distribution used by geometric Brownian motion to replicate the dynamics of financial assets, does not fit well with empirical data. As a rule, the empirical distribution of asset returns exhibits asymmetric features, fat tails and skewness, as tested by Jarque and Bera [3]. The geometric Brownian motion with drift is not an accurate model for asset pricing. While asset price processes have jumps or spikes, geometric Brownian motion is a diffusion process. This leads to the fact that, under the geometric Brownian motion, the variations of greatest amplitude, commonly defined as jumps in literature, are not likely to happen, i.e., the outliers occur far too infrequently in the Black-Scholes model. Yet, it is crucial for the estimation of the market values of contingent claims to take these events into account. Besides, the geometric Brownian motion, used in the Black-Scholes model, assumes a constant volatility, but the market prices present an implied volatility smile, which is non-constant.

Using the CEV model is another step in the attempt to bring the theoretical models closer to the market prices. This local volatility model tries to incorporate the fact that in equity markets volatility increases when prices decrease, due to investors' risk aversion.

Nevertheless, a set of characteristics of the financial markets are still not considered, such as jumps or spikes, as previously mentioned. One possible solution for this issue is to introduce Lévy processes, namely processes with jumps, which are more realistic because they are based in fat-tailed distributions. In this context, the jump-diffusion models appear as an essential tool to provide an adequate description of stock price fluctuations. Jump processes started to be used as a financial tool to determine stock prices by Merton [32], who made and extension of the Black-Scholes model to create a new method that includes the negative skewness and extra kurtosis of the log stock return density. By using the jump component, Merton [32] aimed to capture several market events, which would make the model more accurate to forecast markets behaviour. The jump component introduced a more realistic market model; thus capturing normal market returns and rare events, i.e., abnormally large returns. For this reason, jump-diffusion models are widely studied, as well as used in the real market options assessments.

Therefore, in order to generalize the static hedge portfolio to other models that better express the market trends, the present thesis will propose a new extension to this approach under the jump-diffusion model developed by Merton [32]. The main objective is to develop and test a simple and efficient method to realistically evaluate the American-style options price.

In order to achieve this ambitious goal, it is important to start with the detailed investigation of some of the most important concepts of stochastic processes as well as its analysis, which can be found on Chapter 2. Chapter 3 is dedicated to Merton jump-diffusion model, its deltas and how it can be applied to price European-style options. Chapter 4 is focused on the extension of the static hedge portfolio approach to the valuation of American-style options, under the Merton jump-diffusion model. Numerical results of the model developed within the thesis and its potential applications, as well as limitations, are presented in Chapter 5. Finally, conclusions reached and final remarks will be discussed on Chapter 6.

2. Main Assumptions

In this section, mild assumptions are introduced in order to better understand the static hedge portfolio for pricing American-style options, under the Merton jump-diffusion model.

A stochastic process is a mathematical model also known as a random process for the occurrence, at each moment after an initial time, of a random phenomenon. This type of processes are widely used to construct models in which the evolution of the system may have non-deterministic behavior. In finance, these processes are extremely important since they allow modeling of a complete set of stock price path, which due to their random behavior are not deterministic. Formally a stochastic process can be defined as,

Definition 2.1. *A stochastic process is a collection of random variables* $(S_t)_{t\in\mathbb{R}^+}$ *where the randomness is captured by the introduction of a measurable space* $(\Omega, \mathcal{F})$ *which take values in a second measurable space* $(\Omega', \mathcal{F}')$. *The state space* $(\Omega', \mathcal{F}')$ *will be the d-dimensional Euclidean space equipped with the sigma-field of Borel sets. The index* $t \in [0, +\infty)$ *of the random variables* S *is interpreted as the time.*

A filtration $(\mathcal{F}_t)_{t\geq 0}$ in a measurable space $(\Omega, \mathcal{F})$ is a collection of sub-σ-algebra of $\mathcal{F}$ that corresponds to all information generated by the evolution of the spot price of the underlying stock S in the time-interval $[0, t]$. In practice, $\mathcal{F}_t$ can be interpreted as the information known at time t, which increases with time. The measurable concept is important for the study of stochastic processes and jump processes, and it can be defined by,

Definition 2.2. $(\Omega, \mathcal{F}, \mathbb{P})$ *is a probability space, where* Ω *is the set of all possible events, the* σ*-algebra* $\mathcal{F}$ *is a collections of subsets of* Ω *and* $\mathbb{P}$ *represent the "physical probability measure" associated to each measurable event.*

In a stochastic process, the time parameter t may be either discrete or continuous. Sums of independent and identically distributed random variables, provide the simplest examples of stochastic processes in discrete time. On the other hand, in case of continuous processes, which are the object of study of the present thesis, Lévy processes provide key examples of stochastic processes. As per Lévy, a process is essentially characterized by being a sequence of uncorrelated random variables, in other words, the evolution to the next step is not influenced by the past. In addition, the increments that take the process from one point to another, within the ordered set, follow the same distribution. This theory can be mathematically defined as follows,

Definition 2.3. *A stochastic process $(L_t)_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a Lévy process if it respects the following properties:*
1. $L_0 = 0$ almost surely (a.s.);
2. Independent increments: for every increasing sequence of times $0 \leq s < t < s' < t'$, the random variables $L_t - L_s$ and $L_{t'} - L_{s'}$ are independent;
3. Stationary increments: for every increasing sequence of times $0 \leq s < t$ the law of $L_t - L_s$ only depends on the increments $t - s$;
4. Stochastic continuity: $\forall \varepsilon \geq 0$, $\lim_{t\to 0} \mathbb{P}(|L_{s+t} - L_s| \geq \varepsilon) = 0$;
5. The sample paths are right continuous with left limits (a.s).

The geometric Brownian motion, the Poisson processes and also its extension to a compound Poisson process are fundamental examples of Lévy processes.

The Poisson process is a stochastic process with discontinuous trajectories and frequently used as a building block for jump processes. The jump processes are sequences of random variables that create purely discontinuous paths; its definition is given as follows,

Definition 2.4. *Let $T_n = \sum_{i=1}^{n} \tau_i$ and $\{\tau_i\}_{i\geq 1}$ be a sequence of independent exponential random variables. The process,*

$$N_t = \sum_{n\geq 1} 1_{t\geq T_n}, \tag{2.1}$$

is a Poison process. Therefore, a Poisson process is a counting process of a sequence of random variables with jump size 1, where the probability of occurrence of N jumps in the time period $[0,t]$ is,

$$\mathbb{P}[N(t) = n] = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \forall n \in \mathbb{N}, \tag{2.2}$$

where λ is called the intensity and n is the number of events in the time period $[0,t]$ and with independent and stationary increments.

A compound Poisson process is similar to a Poisson process, but distinguishing itself by random size jumps. In this process, the waiting times between jumps are exponential and the jump sizes possess a random distribution. It also models the occurrence of unpredictable events, where the expectation is known and can be mathematically defined as,

Definition 2.5. *A compound Poisson process with intensity* $\lambda > 0$ *is a stochastic process* X_t *defined as,*

$$X_t = \sum_{i=1}^{N_t} Y_i, \tag{2.3}$$

where jumps sizes Y_i *are independent and identically distributed (i.i.d) and* (N_t) *is a Poisson process with intensity* λ*, independent from* $(Y_i)_{i\geq 1}$*.*

Another important stochastic process is the exponential Lévy analogous to the geometric Brownian motion used to model the stock price process. In the Black-Scholes model, the evolution of an asset price is described by the exponential of a Brownian motion with drift and mathematically defined as $S_t = S_0 e^{B_t}$, where $B_t = \mu t + \sigma W_t$ is a Brownian motion with drift, which is the solution of the stochastic differential equation

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t. \tag{2.4}$$

Replacing B_t by a Lévy process, it is possible to obtain the class of exponential Lévy models, defined as,

Definition 2.6. *Suppose a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and a filtration, where* $\mathcal{F}_t$ *is the filtration generated by the Lévy process* $(L_t)_{t\geq 0}$*. A geometric Lévy precess is given by*

$$S_t = S_0 e^{L_t}, \tag{2.5}$$

where L_t *is a Lévy process.*

The geometric Brownian motion is a common case of a diffusion process, which assumes no jump occurs. When we add the jump component we create what is defined as a jump-diffusion process,

Definition 2.7. *Let* $(L_t)_{t\geq 0}$ *be a Lévy process with jump measure* J_t*. Thus, a jump-diffusion is a process of the form,*

$$L_t = L_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s + J_t := L_t^C + J_t, \tag{2.6}$$

where L_t^C *is the continuous component of* L_t *and* $(J_t)_{t\geq 0}$ *is a pure jump process.* $(W_s)_{s\geq 0}$ *is a Brownian motion.*

The probability density function of a Lévy process is generally not known in closed form, thus it is essential to be aware of its characteristic function. The characteristic function of

the probability density can be expressed by elementary terms, for most of the Lévy processes discussed in the literature.
The Lévy-Khintchine representation, described below, is an essential theorem that allows studying distributional properties of the majority of Lévy processes, through the description of its characteristic functions.

Theorem 2.1. *Let $(L_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}$, associated with a triplet (μ, σ^2, ν), where $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}_0^+$ and ν is a measure concentrated on $R\backslash\{0\}$ that satisfies $\int_{\mathbb{R}}(\{1 \wedge x^2\})\nu(dx) < \infty$ then its characteristic function has the form,*

$$\phi_{L_t} = \int_{\mathbb{R}} e^{izL_t}\nu(dx) = e^{t\Psi(z)}, \forall \in \mathbb{R}, \tag{2.7}$$

with,

$$\Psi(z) = i\mu z - \frac{\sigma^2}{2}z^2 + \int_{\mathbb{R}}(e^{izx} - 1 - izx\mathbb{I}_{|x|\leq 1})\nu(dx), \forall z \in \mathbb{R}. \tag{2.8}$$

Proof. See Cont and Tankov [14], page 95

□

Thus, assuming that every Lévy process has a characteristic function of the form of equation (2.7), they can be parametrized using the Lévy triplet (μ, σ^2, ν). The parameter μ is the drift, the σ^2 is the Gaussian variance, since it is associated with the Brownian part of the Lévy process, and $\nu(dx)$ is the so called Lévy measure or jump measure. A pure jump process, which has no diffusion component, can be defined by the triplet form $(\mu, 0, \nu)$. On the other hand, a pure diffusion process, with no jump, can be defined by the form $(\mu, \sigma^2, 0)$.

The Lévy measure $\nu(dx)$, which represents a discontinue component, is present whenever jumps are above normal, creating an irregular path that can be described using the integral of the Lévy density, $\lambda = \int_{-\infty}^{+\infty} \nu(dx)$. These discontinue components can be either finite or infinite: the process will have finite activity, if the integral is also finite. Otherwise, infinite activity will be defined by a process with infinite jumps within a determined time period and the integral must be infinite.

Therefore, the Lévy process activity can be defined as follows,

Theorem 2.2. *Let L_t be a Lévy process with triplet (μ, σ^2, ν).*
(1) The Lévy process has finite activity if $\nu(\mathbb{R}) < \infty$, i.e. if all paths of L_t have a finite number of jumps on every compact interval.
(2) The Léy process has infinite activity if $\nu(\mathbb{R}) = \infty$, i.e. if all paths of L_t have an infinite number of jumps on every compact interval.

For the particular case of a compound Poisson process, the Lévy measure is given by $\nu(dx) = \lambda f(x)dx$, where λ is the intensity and $f(x)$ is the jump size density. Given a finite activity Lévy process, only a finite number of jumps in any finite time interval take place and its Lévy measure is finite, $\int_{-\infty}^{+\infty} \nu(dx) = \lambda < \infty$.

As an attempt to capture jumps occurrence, a new term is added to the traditionally used Itô lemma, which only accepts Brownian motion type components. Since we are analysing processes with jump components, this characteristic must be considered at the extension of Itô lemma. In the revised form, t^- represents the value assumed by the function, immediately before a jump occurrence.

Theorem 2.3. *Let $(L_t)_{t\geq 0}$ be a Lévy process for the type $L_t = L_0 + \int_0^t \mu_s d_s + \int_0^t \sigma_s dW_s + \sum_{i=1}^{N_t} \Delta X_i$ the Itô formula is:*

$$\begin{aligned} df(L_t,t) = {} & \frac{\partial f(L_t,t)}{\partial t}dt + b_t\frac{\partial f(L_t,t)}{\partial x}dt + \frac{\sigma_t^2}{2}\frac{\partial^2 f(L_t,t)}{\partial x^2} \\ & + \sigma_t\frac{\partial f(L_t,t)}{\partial x}dW_t + [f(X_{t^-} + \Delta X_t) - f(X_{t^-})], \end{aligned} \tag{2.9}$$

where $\sum_{i=1}^{N_t} \Delta X_i$ represents the jump component.

Proof. See Cont and Tankov [14], page 279. □

It is worth to note that the lemma presented above can only be used for finite activity processes. Processes with infinite activities, will require a different analysis.

3. Merton Jump-diffusion Model

As previously discussed, empirically, stock returns tend to have fat tails that are inconsistent with the assumptions of the geometric Brownian motion, used by the Black-Scholes model. The idea of a jump diffusion process was developed to try to solve this problem. By adding a jump component to the diffusion part, the model aims for capturing normal market returns and rare events, i.e., abnormally large returns. Thus, in jump-diffusion models the diffusion component is supposed to represent the normal fluctuations in the risky asset's price caused by "*temporary imbalance between supply and demand, changes in capitalization rates, changes in the economic outlook, or other new information that causes marginal changes in the stock's value*", as defined by Merton [32]. This component is modeled by a geometric Brownian motion with drift. In which concerns the non-marginal variations in price, it is expected, by its own nature, that the important information arrives only at discrete time points, so it can be modeled by a jump process.

Before presenting a detailed revision of Merton jump-diffusion model it is essential to better know the continuous diffusion processes with addition of discontinuous jump processes. Thus, this will be main focus of this section.

3.1. Drifted Brownian Motion with a Compound Poisson Process

Consider an exponential Lévy model, with drifted Brownian motion plus a compound Poisson process,

$$L_t = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t + \sum_{i=0}^{N_t} Y_i, \tag{3.1}$$

where $\sum_{i=0}^{N_t} Y_i$ is a compound Poisson process and hence N_t is a Poisson counting process with Y_i, $i \in \{0, ..., N_t\}$, representing the corresponding jump amplitude. The process $(W_t)_{t\geq 0}$ represents a Brownian motion with mean equal to zero and standard deviation $\sqrt{t}$.

In this way, a compound Poisson jump process is added to the Black-Scholes model, yielding a jump-diffusion model.
As described in the previous chapter, a compound Poisson jump process embraces random aspects with two different origins. The Poisson process Nt with intensity λ (defined by the average of jumps within a predefined unit of time) will cause random jumps in the asset price - this process is also known by random timing. The other origin of randomness in the compound Poisson jump process is the random jump size.
The probability of a certain number of jumps occurring in time interval dt, using a Poison process N_t with intensity λ, can be written as follows,

$$\begin{aligned}
\mathbb{P}(dN_t = 1) &= \mathbb{P}\{\text{asset price jumps once in dt}\} \cong \lambda dt, \\
\mathbb{P}(dN_t) \geq 1) &= \mathbb{P}\{\text{asset price jumps more than once in dt}\} \cong 0, \\
\mathbb{P}(dN_t = 0) &= \mathbb{P}\{\text{asset price does not jump in dt}\} \cong 1 - \lambda dt,
\end{aligned}$$

where the parameter $\lambda \in \mathbb{R}^+$ stands for the jump intensity, independent of time t.

Suppose that in an infinitesimal time frame dt the asset price S_t jumps to yS_t. Consequently, the relative price jump size, i.e. the percentage change in the asset price caused by the jump, is given by,

$$\frac{dS_t}{S_t} = \frac{y_t S_t - S_t}{S_t} = y_t - 1. \tag{3.2}$$

The addition of the jump component to the Brownian motion with drift, leads to the existence of more than one equivalent martingale measure, which turns the jump diffusion models into an incomplete model. In terms of hedging, the above definition points to the fact that it is not possible to develop an hedging portfolio with zero risk. Thus, in order to allow a risk neutral valuation, it is required that the numéraire of the economy is taken to be the *money-market account*, i.e. the asset grows at the risk free rate. This change will be enough to evaluate the asset through an equivalent martingale measure $\mathbb{Q} \sim \mathbb{P}$. In this context, Merton [32] introduced a change in the drift of the Brownian motion process, at the same time all the other components remained as before. Additionally, he proved that the Lévy triplet of jump-diffusion processes in a risk-neutral measure $\mathbb{Q}$ is equal to $(r - q - \lambda\zeta, \sigma, \lambda P)$, where $P(x) = f(x)dx$. Further details will be presented in the next section.

3.2. Model Setup

The Merton jump-diffusion model assumes that the underlying price process (S_t) follows the stochastic differential equation, under the risk-neutral measure $\mathbb{Q}$,

$$\frac{dS_t}{S_t} = (r - q - \lambda\zeta)dt + \sigma dW_t^{\mathbb{Q}} + (y_t - 1)dN_t, \tag{3.3}$$

with $S_t > 0$, where μ is the instantaneous expected return and σ is the volatility associated with the Brownian motion. Thus, combining a standard Brownian motion, with drift and a compound Poisson process, with intensity λ, Merton [32] obtained a simple jump-diffusion model. It is assumed that S_t, N_t and y_t are mutually independent.

Merton [32] also considered that the absolute jump size, y_t, is a non-negative random variable with jumps log-normally distributed, i.e. $\ln(y_t) \sim i.i.d.\ \mathcal{N}(\mu_J, \sigma_J)$, where $\mathcal{N}(\mu_J, \sigma_J)$ denotes a Gaussian distribution with mean μ_J and variance σ_J^2. As a result,

$$(y_t) \sim i.i.d. Lognormal(e^{\mu_J + \frac{1}{2}\sigma_J^2}, e^{2\mu_J + \sigma_J^2}(e^{\sigma_J^2} - 1)).$$

In this way, the expected price variation caused by the jump is given by $\mathbb{E}[y_t - 1] = e^{\mu_J + \frac{1}{2}\sigma_J^2} - 1 \equiv \zeta$, with a probability density function of the jump size in S_t described as,

$$f(dx) := \frac{1}{\sigma_J \sqrt{2\pi}} \exp\left[-\frac{(dx - \mu_J)^2}{2\sigma_J^2}\right], \forall x \in \mathbb{R}. \tag{3.4}$$

Thus, the Merton jump-diffusion model tries to capture excess kurtosis and the (negative) skewness of the log return density which is not considered in the Black-Scholes model.

As explained above, Merton [32] introduced an adjustment in the drift, using the term $-\lambda\zeta$, which compensates the jump process, making the asset to grow at the risk-free rate, expressed by $(r - q)dt$. Consequently, the discounted price of the asset becomes a martingale:

$$\begin{aligned} \mathbb{E}\left[\frac{dS_t}{S_t}\middle|\mathcal{F}_t\right] &= \mathbb{E}_{\mathbb{Q}}[(r - q - \lambda\zeta)dt|\mathcal{F}_t] + \mathbb{E}_{\mathbb{Q}}[\sigma dW_t|\mathcal{F}_t] + \mathbb{E}_{\mathbb{Q}}[(y_t - 1)dN_t|\mathcal{F}_t] \quad (3.5) \\ &= (r - q - \lambda\zeta)dt + 0 + \lambda\zeta dt = (r - q)dt. \end{aligned}$$

Figure 3.1. illustrates a simulation of a jump diffusion path of the Merton model. Values of the vertical axis correspond to the price of the asset. In the horizontal axis, the time to maturity is represented in years. The parameters used were: $S = 5$, $r = 5\%$, $\sigma = 30\%$, $q = 0\%$, $\tau = 1$, $\lambda = 0.1$, $\mu_J = -0.92$, $\sigma_J = 0.425$.

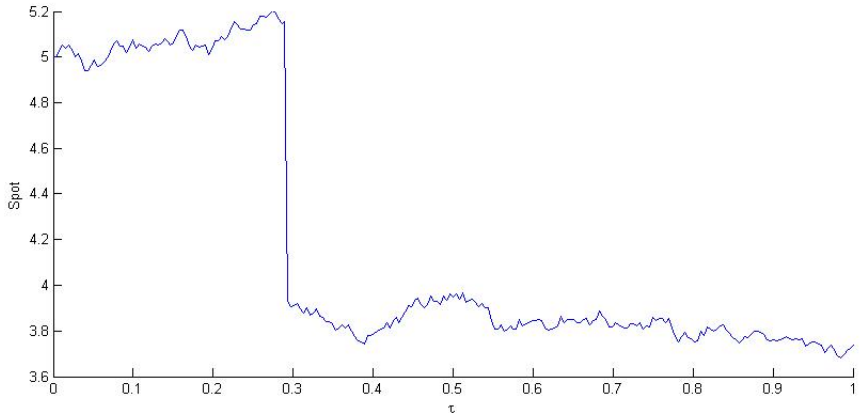

Figure 3.1.: Simulation of a path example of the Merton jump-diffusion model

It is now possible to identify the Lévy triplet $(\mu, \sigma^2, \nu(dx))$ of the Merton jump-diffusion process, where $\mu = r - q - \lambda\zeta$ and $\nu(dx) = \lambda p(dx)$, with $p(dx)$ as defined in equation (3.4). Thus, based on Theorem 2.1, the $\mathbb{Q}$-measure characteristic exponential of this Lévy process is

$$\Psi(z) = \left(r - q - \frac{\sigma^2}{2} - \lambda\left(e^{\mu_J + \frac{\sigma_J^2}{2}} - 1\right)\right) z + \frac{\sigma^2}{2} z^2 + \lambda\left(e^{\mu_J z + \frac{\sigma_J^2}{2} z^2} - 1\right), \forall z \in \mathbb{R}. \quad (3.6)$$

Based on the Lévy density, we can conclude that this is a finite activity process, given,

$$\int_{-\infty}^{+\infty} v(dx) = \lambda.$$

Supposing that in an infinitesimal interval of time no jumps occur, described by $dN_t = 0$, in this case, the jump-diffusion process would be a Brownian motion with drift process:

$$\frac{dS_t}{S_t} = (\mu - \lambda\zeta)dt + \sigma dW_t. \quad (3.7)$$

Theorem 3.1. *To solve the stochastic differential equation (3.3) the Itô formula for jump-diffusion processes (2.9) should be used, yielding the following solution:*

$$S_t = S_0 \exp\left[(\mu - \frac{\sigma^2}{2} - \lambda\zeta)t + \sigma dW_t^{\mathbb{Q}} + \sum_{i=1}^{N_t} Y_i\right].$$

Proof. See Appendix I □

Thus, Merton in [32] assumes that a financial asset, whose price is a stochastic process, can be defined by an exponential Lévy process $S_t = S_0 e^{L_t}$, where L_t is of the form,

$$L_t = \left(\mu - \frac{\sigma^2}{2} - \lambda\zeta\right) t + \sigma dW_t^{\mathbb{Q}} + \sum_{i=1}^{N_t} Y_i, \tag{3.8}$$

meaning that its log-return is modeled as a Lévy process, resulting in,

$$\ln\left(\frac{S_t}{S_0}\right) = L_t.$$

3.3. Pricing Solution for European-style Options

Let $v(S_T)$ be the value of an European-style option. In the particular case of a European-style put option, the payoff at its maturity is $p(S_T) = (K - S_T)^+$. In opposition, a European-style call option will be defined by $c(S_T) = (S_T - K)^+$.

Then the present value of the European-style option, both for put options or call options, is the discounted value of its expectation under the risk neutral measure:

$$v_t(S_t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[v(S_T)|\mathcal{F}_t], \tag{3.9}$$

where $\mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_t)$ is the expected value of a random variable X, conditional on the σ-algebra $\mathcal{F}_t$ and computed under the equivalent martingale measure $\mathbb{Q} \sim \mathbb{P}$.

Therefore, as proved in Matsuda [29], Merton's pricing formula is:

$$v_t^M(S_t) = \sum_{n\geq 0} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} v^{BS}(S_n \equiv S_t e^{n\mu_J + \frac{n\sigma_J^2}{2} - \lambda(e^{\mu_J + \frac{\sigma_J^2}{2}} - 1)\tau}, \sqrt{\sigma^2 + \frac{n\sigma_J^2}{\tau}}, r - q, \tau). \quad (3.10)$$

Alternatively,

$$\begin{aligned} v_t^M(S_t) &= e^{r\tau} \sum_{n\geq 0} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \mathbb{E}^{\mathbb{Q}}[v(S_t e^{\{r-q-\lambda(e^{\mu_J+\frac{\sigma_J^2}{2}}-1)\}\tau + \frac{n\mu_J + \frac{n\sigma_J^2}{2}}{\tau} - \frac{\sigma_J^2}{2} + \sigma^2 W^{\mathbb{Q}}\tau})] \\ &= \sum_{n\geq 0} \frac{e^{-\widetilde{\lambda}\tau}(\widetilde{\lambda}\tau)^n}{n!} v^{BS}[S_t, \sigma_n, r_n, \tau], \end{aligned} \quad (3.11)$$

where,

$$\sigma_n = \sqrt{\sigma^2 + n\frac{\sigma_J^2}{2}},$$

$$\widetilde{\lambda} = \lambda(1+\zeta) = \lambda(e^{\mu_J + \frac{\sigma_J^2}{2}}),$$

$$r_n = r - \lambda(e^{\mu_J + \frac{\sigma_J^2}{2}} - 1) + \frac{n\mu_J + \frac{n\sigma_J}{2}}{\tau}.$$

When handling the above form, under the Black-Scholes model and in the specific case of a European-style put option, p_t^{BS} is defined as

$$p^{BS} = Ke^{-r(T-t)}\mathcal{N}(-d_2) - S_t e^{-q(T-t)}\mathcal{N}(-d_1). \quad (3.12)$$

On the contrary, if the form is used to deal with European-style call options, c_t^{BS},is represented by

$$c^{BS} = S_t e^{-q(T-t)}\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2), \quad (3.13)$$

where $\mathcal{N}$ represents the cumulative distribution function of the standard normal law and d_1 and d_2 are respectively equal to

$$\begin{aligned} d_1 &= \frac{ln(S_t/K) + (r - q + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= d_1 - \sigma\sqrt{T-t}. \end{aligned}$$

Therefore, the price of a European-style option, under Merton jump-diffusion model, will be defined as a weighted average of standard Black-Scholes prices, conditioned by the number of jumps.

Through the application of equation (3.13), it is possible to obtain the results illustrated in Figures 3.2. and 3.3., which represent European-style options prices, for a strike equal to 100, at each spot price, within a range from 80 to 120. Figures 3.2. and 3.3. show the value of a call and put option, respectively, as a function of the spot, with a variation of the parameters of Merton jump-diffusion model, but maintaining all other parameters fixed.

The parameters used were: $r = 5\%$, $\sigma = 20\%$, $q = 0\%$, $\tau = 0.25$. The jumps were parametrized as: $\lambda \in \{0.1, 0.25, 0.5, 0.75\}$, $\mu_J = -0.92$, $\sigma_J = 0.425$.

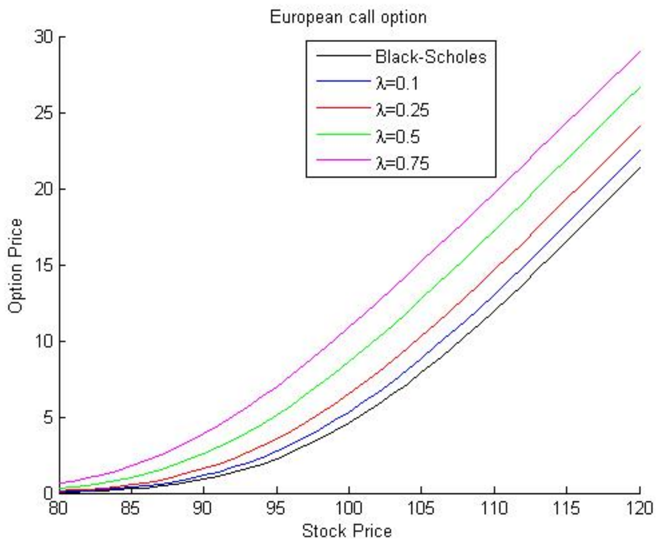

Figure 3.2.: Call option prices with Merton model: Value vs Spot - varying parameter λ

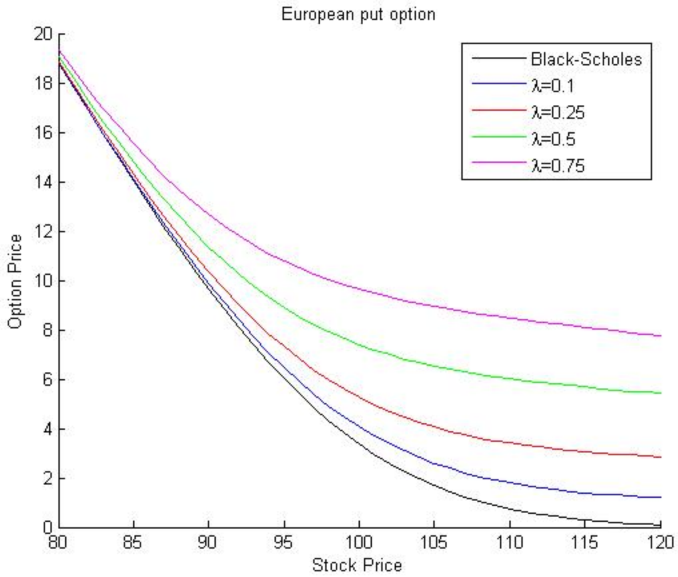

Figure 3.3.: Put option prices with Merton model: Value vs Spot - varying parameters λ

When comparing options values determined using the Merton model versus the ones obtained under the Black-Scholes model, it is possible to notice that they become larger as the arrival of jumps increases. The same situation takes place when the variance of the jump distribution increases. This behavior is expected since more jumps and greater variance, represent more uncertainty in the final payoff and leads to additional potential earnings or loses.

3.4. Delta Hedge of European-style Options

Delta calculation, sometimes also referred to as a Greek, is crucial to formulate a static hedge portfolio for valuing American-style options. This Greek is an important parameter in the pricing and hedging of options, in such an extension that, the construction of a riskless portfolio, is often referred to as delta hedging. Each Greek letter measures a different aspect of the options' risks and corresponds to a parameter that is able to influence the options' value, such as the underlying price, or the interest rate, the volatility, or even the time to maturity.

Considering these characteristics, delta Δ is an essential tool in risk management. The delta Δ of an option is defined as a measure of change in the option price, resulting from a change in the price of the underlying. This measure of sensitivity is given by,

$$\Delta = \frac{\partial H}{\partial S}. \tag{3.14}$$

Any of the Greeks of Merton's jump-diffusion model can be determined by simply replacing the call or put option formula by any of the Black-Scholes Greek formula. This conclusion can be reached because the jump-diffusion is achieved by a weighted average of standard Black-Scholes prices, conditioned by the number of jumps; and the jump components is not dependent from Delta and corresponding derivatives.

Therefore, the delta of an option under the Merton jump-diffusion model can be defined by,

$$\Delta_t^M(S_t) = \sum_{n \geq 0} \frac{e^{-\widetilde{\lambda}\tau}(\widetilde{\lambda}\tau)^n}{n!} \Delta^{BS}[S_t, \sigma_n, r_n, \tau], \tag{3.15}$$

where the delta of a call, $\Delta_t^{c^{BS}}(S_t)$, and put option, $\Delta_t^{p^{BS}}(S_t)$, under the Black-Scholes model, is respectively,

$$\begin{aligned} \Delta_t^{c^{BS}}(S_t) &= e^{-q(T-t)}\Phi(d_1), \\ \Delta_t^{p^{BS}}(S_t) &= -e^{-q(T-t)}\Phi(-d_1). \end{aligned}$$

4. Valuation of American-style Options under Static Hedge Portfolio

In this section, we discuss the extension of the static hedge portfolio approach of Chung and Shih [13] to the Merton jump-diffusion model.

We suppose that trading occurs continuously on the time-interval $T := [t_0, T]$, it is possible for short-sales to take place and the financial market is frictionless and completely liquid. Uncertainty is represented by a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, where the martingale $\mathbb{Q}$ is the risk neutral measure associated to the *money market account* numéraire.

4.1. Standard American-style Options

As stated before, an American-style option on the underlying asset price S, with strike price K and expiration date T, allows the holder to exercise, at any time t during its life, obtaining the payoff $(\phi K - \phi S_\tau)^+$; symbolically written as $\max(0, K - S_\tau)$ for American-style put options (if $\phi = 1$) and written as $\max(0, S_\tau - K)$ for American-style call options (if $\phi = -1$). Hence, the time-$t_0(\leq T)$ price for an American-style option will be denoted by $V_{t_0}(S, K, T)$ and is the solution to an optimal stopping problem:

$$V_{t_0}(S, K, T, \phi) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}}[e^{-r[(T\wedge\tau)-t_0]}(\phi K - \phi S_{T\wedge\tau})^+ | \mathcal{F}_{t_0}], \tag{4.1}$$

where $\mathcal{T}$ represents all stopping times for the filtration $\mathcal{F}$ and taking values in $[t_0, \infty]$.

Proof. See for example Karatzas [25] $\square$

Since American-style options can be exercised at any time up to maturity, this lead us with uncertainty regarding when the holder should exercise his right over the underlying asset. It is the choice of the optimal time to exercise that makes the analysis of these contracts more complex, when compared to European-style options. This problem is difficult to solve using analytical methods, leading to the use of *free boundary problems*, as described by McKean in [30].

Pham [33] demonstrated that, for each time $t \in \mathcal{T}$, it is possible to establish the *critical asset price* E_t, which will match its intrinsic value, allowing to determine the optimal early exercise time. In other words, the optimal time to exercise an American-style option, corresponds to the first passage time the asset price achieves its critical level. The first passage time is defined as

$$\tau_e := \inf\{t \geq t_0 : S_t = E_t\}. \tag{4.2}$$

Therefore, it can be assumed that,

$$V_{t_0}(S, K, T, \phi) = \mathbb{E}_{\mathbb{Q}}[e^{-r[(T \wedge \tau_e) - t_0]}(\phi K - \phi S_{T \wedge \tau_e})^+ | \mathcal{F}_{t_0}], \tag{4.3}$$

where τ_e is the first time the underlying asset price process crosses the early exercise boundary.

As Carr presented in [9] the early exercise boundary is defined as,

Definition 4.1. *The exercise boundary is the time path of critical stock prices, E_t, $t \in [0, T]$. This boundary is independent of the current stock price S_t and is a smooth function of time t terminating in the strike price, i.e. $E_T = K$.*

This exercise boundary, E_t, $t \in [0, T]$, divides the into a continuation and stopping regions of the American-style options.

Using the result that proofs the existence of a unique and continuous early exercise boundary, under a single-factor and time-homogeneous diffusion model, Pham [33] extended the same conclusion to the jump-diffusion setup. Pham [33] conclusions are limited by the positiveness of risk-free interest rate already adjusted by the jump risk.

Based on the early exercised boundary, associated to the definition of static hedge portfolio (SHP), as used in the evaluation of European-style exotic options, Chung and Shih [13] were able to present a static hedge portfolio specific for the assessment of American-style options. This approach will be presented in detail in the present chapter and afterwards it will be extended to the model of Merton jump-diffusion.

4.2. Static Hedge Portfolio Approach

Several financial publications contribute to the definition of the static hedge portfolio. There are two main approaches: the one defined by Bowie and Carr [5] and Carr et al. [10] and the theory developed by Derman et al. [17]. Thus, the definition of static hedge portfolios of American-style options benefits from both approaches through the use of European-style options with multiple strikes and maturities. The use of options with these characteristics can be justified by the fact that the early exercise boundary of an American-style option is time variant.

The unknown boundary nature of American-style options is a complex problem, which contrasts greatly with the static hedge of exotic options with optimal exercise boundary, where the optimal exercise time is most frequently known ex-ante. Thus, the difficulties related with the free boundary persist on the model of SHP of American-style options. This issue leads the early exercise boundary to be determined simultaneously with hedging, through the use of two well-known conditions: value-matching and smooth-pasting conditions.

If the American-style option is not exercised before the maturity date, its terminal condition will be exactly the same as in the corresponding European-style option. Therefore, as proposed by Chung and Shih [13], the static hedge portfolio defined in this thesis will start with one unit of the European-style option, with strike K and maturity date at time T. Thus, the maturity date will be the starting point to evaluate the American-style option and it will allow working backwards until the valuation date, such as it is done with the binomial option pricing models for American-style contracts.

Assuming this static hedge portfolio also matches the American-style option boundary conditions before achieving maturity, we divide the time to expiry date of the option contract into n evenly-spaced time-steps, i.e. $t_0 = 0, t_1, ..., t_{n-1} = T - \Delta t$, where $\theta t = (T - t_0)/n$. To match the unknown early exercise boundary E_i at each time, $t_i := t_0 + i\theta t$ (with $i = n-1, ..., 1, 0$), it is required to add w_i units of a standard European-style option with strike equal to E_i

and maturity at time t_{i+1}, solving E_i and w_i through the *value-matching* and *smooth-pasting* conditions.

Figure 4.1, expresses an example of the early exercise boundary determination, using the static hedge portfolio approach.

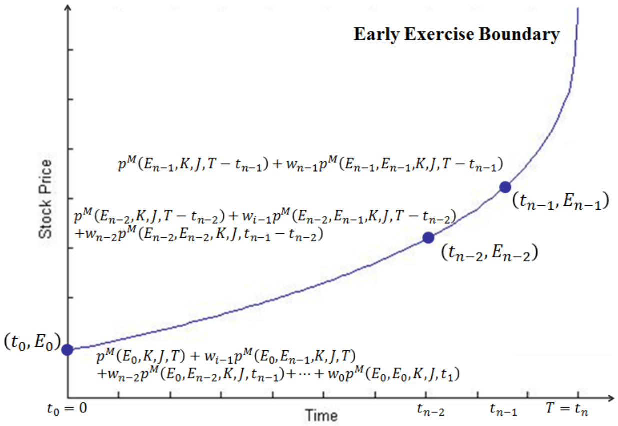

Figure 4.1.: Valuation form of n-point SHP at its early exercise boundary

Therefore, to extend the SHP approach proposed by Chung and Shih [13] to the pricing of American-style options under the Merton jump-diffusion model, we consider that the underlying price satisfies the jump-diffusion process presented in equation (3.3). Hence, the price for a European-style option, under the Merton jump-diffusion model will be denoted by $v^M = (S, K, J, \phi, \tau)$ and its delta by $\Delta^{v^M}(S, X, J, \phi, \tau)$, where $\phi = 1$ for American-style put options and $\phi = -1$ for American-style call options.

Thus, at time t_{n-1}, to match the stock price and the critical price E_{n-1} through the *value-matching* and *smooth-pasting* conditions, we must have:

$$\phi K - \phi E_{n-1} = v^M(E_{n-1}, K, J, \phi, T - t_{n-1}) + w_{n-1} v^M(E_{n-1}, E_{n-1}, J, \phi, T - t_{n-1}) \quad (4.4)$$

and

$$-\phi = \Delta^{v^M}(E_{n-1}, K, J, \phi, T - t_{n-1}) + w_{n-1} \Delta^{v^M}(E_{n-1}, E_{n-1}, J, \phi, T - t_{n-1}). \quad (4.5)$$

As a widespread strategy, the unknowns E_i and w_i existent at every time unit can be determined by simultaneously solving the following recurrence conditions:

$$\phi K - \phi E_{n-i} = v_t^M(E_{n-i}, K, J, \phi, T - t_{n-i}) + \sum_{j=1}^{i} w_{n-j} v_{t_{n-i}}^{M_0}(E_{n-i}, E_{n-j}, J, \phi, t_{n-j+1} - t_{n-i}) \tag{4.6}$$

and

$$-\phi = \Delta^{v_t^M}(E_{n-i}, K, J, \phi, T - t_{n-i}) + \sum_{j=1}^{i} w_{n-j} \Delta^{v_{t_{n-i}}^{M_0}}(E_{n-i}, E_{n-j}, J, \phi, t_{n-j+1} - t_{n-i}), \tag{4.7}$$

for $i = 1, 2, ..., n$.

After solving all the unknowns w_i and E_i (with $i = n-1, ..., 1, 0$), the value of the n-points static hedge portfolio price of an American-style option, under the Merton jump-diffusion model, V^{shp}, at time-t_0 is given by,

$$\begin{aligned} V_{t_0}^{shp}(S_{t_0}, K, J, \phi, T) &:= v^M(S_{t_0}, K, J, \phi, T - t_0) + w_{n-1} v^M(S_{t_0}, E_{n-1}, J, \phi, T - t_0) \\ &+ w_{n-2} v^M(S_{t_0}, E_{n-2}, J, \phi, t_{n-1} - t_0) + ... + w_0 v^M(S_{t_0}, E_0, J, \phi, t_1 - t_0), \end{aligned} \tag{4.8}$$

and it can also be expressed by,

$$\begin{aligned} V_{t_0}^{shp}(S_{t_0}, K, J, \phi, T) &:= v^M(S_{t_0}, K, J, \phi, T - t_0) \\ &+ \sum_{j=1}^{n} w_{n-j} v^M(S_{t_0}, E_{n-j}, J, \phi, t_{n-j+1} - t_0). \end{aligned} \tag{4.9}$$

4.3. Hedge Ratios

As mentioned earlier, knowing the delta calculation is crucial to formulate a static hedge portfolio for valuing American-style options. In the models used by Chung and Shih [13], namely Black-Scholes or CEV, the market can be classified as complete. In a complete market, as observed by Glasserman [20], any option can be perfectly replicated through a trading strategy, i.e. the price of the target option is equivalent to the price of the exact hedge, and the martingale measure associated with a numéraire is unique. All risks can be perfectly hedged through a delta-hedging strategy, which consists in a continuous update of the amount of the asset equal to $\Delta(t) = \frac{\partial H}{\partial S}(S,t)$, at all times.

Thus, assuming that the hedge portfolio also contains an amount $B(0)$ at time 0 , invested at the risk-free rate in b bonds and that trading in the underlying asset may be performed in infinitesimally small time increments until maturity T, the portfolio's value is

$$\Pi(S,t) = \Delta(t)S(t) + b(t)B(t) - H(S,t), \tag{4.10}$$

which is zero, indicating that the option is perfectly hedged.
On the contrary, in an incomplete market, such as in jumps models, it is not possible to determine a perfect replication of an option through other instruments. Jumps occurrence in asset prices makes it almost impossible to hedge the effect of discontinuous movements. Therefore, in Merton's jump-diffusion model the delta-hedging is not optimal and has a significant probability of failing, due to the presence of jumps in the prices.

Delta hedging will not be the optimal solution to hedge options in a world with jumps, unless the jumps are diversifiable. Merton [31] defends that in the case jumps only affect a restricted number of assets at the same period of time, it is possible to create a balanced portfolio, which decreases jump risks. In the case jumps affect the market globally, they cannot be considered diversifiable and, in this circumstance, the delta hedging will not be successful. While there is not a perfect method for replicating a portfolio exposed to random jumps amplitudes, it is possible to use a reasonable method, as proven by Bates [2].

Since it is not possible to determine up front the jump size, it is also not possible to completely hedge the risk associated with the jump. Thus, in the context of incomplete markets, hedging becomes an approximation issue; instead of replicating options, it looks for minimizing the error of residual hedge. Empirical studies proved that strategies only using the delta hedging, conduct to high levels of residual risk, which can be mitigated through the addition of liquid options to the hedging portfolio.

Thus, considering that for each jump there is a finite number of possible amplitudes, defined by N_J, a target option can easily be hedged by using N_J different options, in addition to the underlying and the bonds. On its turn, if there are an infinite number of possible jumps amplitudes, the optimal replication will involve an equally infinite number of hedging instruments, similarly to options at a continuum or strikes. By using hedging strategies, it is possible to ensure that the hedging technic in use will embrace all the space of possible target option values, for a movement in the underlying S.

In order to better describe these concepts, it will be assumed that the period of time in study is defined by $[t_0, t_0 + dt]$. During this period there will be one jump amplitude Y and the following will be linearly independent functions of S: the price of the target option, the price of the hedging instrument with the value I, a constant payoff and underlying.

At a determined time t, the asset price will be S and the portfolio value is $\Pi(S,t)$, which can defined as,

$$\Pi(S,t) = \Delta(t)S(t) + b(t)B(t) + \phi(t)I(t) - H(S,t). \tag{4.11}$$

Therefore, accordingly to Bates [2] conclusions, for each additional possible jump amplitude, a new hedging instrument should be added to the model. This will prevent the impact of the occurrence of jumps with different amplitudes, while at the same time the delta hedging will neutralize the risk introduced by the Brownian motion.

Tankov [38] also showed how to compute optimal hedging strategies, when jumps are present, by demonstrating that in Merton's model, the best alternative is to minimize the risk of the hedging activities, recurring to a finite number of short-dated hedging options. Thus, Tankov [38] determined that *"the optimal quadratic hedging strategy is a weighted sum of two terms: the sensitivity of option price to infinitesimal stock movements, and the average sensitivity to finitely-sized jumps."*

5. Numerical Results

This section, contains the outputs of the implementation of numerical simulations done with the objective of extending the Static Hedge Portfolio approach for American-style options, under the Merton jump-diffusion model described in Chapter 4. The accuracy and efficiency of the solution developed will also be critically analysed.

In order to test the accuracy, the pricing solutions of the static hedge portfolio approach were compared with reference values, calculated using the Fourier Space Time-stepping method (FST-method) of Jackson, Jaimungal and Surkov [26], which allows the pricing American-style standard options under exponential Lévy processes. This method uses a backward induction approach and takes the Partial Integro-Differential Equation (PIDE) satisfied by the option price. With this approach, the PIDE can be converted into a much simpler system of ordinary differential equations, eliminating the need to administrate more complex non-local term of PIDE equations.

On the other hand, efficiency is determined by CPU time (in seconds) consumed to value the full set of contracts under analysis. It is worth to notice that all numeric results presented in this thesis were obtained employing Matlab ($R2013a$) programs running on an Intel Core i5-3337U, 1.80 GHz processor with 6.00 GB of RAM.

The baseline parameters are adopted from Chung and Shih [13]: $K = 100$, $r = 5\%$, $q = 0\%$, $\sigma = 30\%$ and $\tau = 1$. These parameters have a changing variable S equal to either 80, 90, 100, 110 or 120. Afterwards, keeping all other parameters unchanged, single variations are introduced, first with r, then q, after σ and finally the parameter τ.

For the parametrization of the compound Poisson jump process, the values of He et al. [23] were adopted. As described by He et al. [23] these values were obtained by calibration techniques applied to the market options prices for a stock index; the obtained values can, therefore, be considered realistic. Andersen and Andreasen [1] also employed similar parameter values. The jumps were parametrized as

$$\lambda = 0.1, \qquad \mu_J = -0.92, \qquad \sigma_J = 0.425.$$

Table 5.1. displays the accuracy and efficiency of the static hedge portfolio approach for valuing standard American-style put options under the Merton jump-diffusion model for different parameters.

			American-style option			
Parameters	Spot	European-style	Benchmark	SHP 12	SHP 24	SHP 52
	80	20.833	22.257	22.555	22.541	22.527
$r = 5\% \mid q = 0\%$	90	15.503	16.363	16.670	16.683	16.666
$\sigma = 30\%$	100	11.575	12.130	12.458	12.440	12.424
$t = 1$	110	8.783	9.170	9.472	9.454	9.439
	120	6.839	7.130	7.401	7.384	7.369
	80	19.475	21.633	22.035	22.015	21.999
$r = 7\% \mid q = 0\%$	90	14.400	15.665	16.160	16.135	16.115
$\sigma = 30\%$	100	10.712	11.508	12.006	11.979	11.958
$t = 1$	110	8.124	8.669	9.132	9.106	9.085
	120	6.338	6.745	7.161	7.137	7.117
	80	21.323	22.528	22.785	22.772	22.761
$r = 5\% \mid q = 1\%$	90	15.920	16.656	16.939	16.924	16.911
$\sigma = 30\%$	100	11.910	12.391	12.664	12.648	12.635
$t = 1$	110	9.042	9.382	9.631	9.615	9.603
	120	7.036	7.294	7.515	7.501	7.489
	80	17.947	20.166	20.451	20.434	20.419
$r = 5\% \mid q = 0\%$	90	12.067	13.055	13.641	13.609	13.579
$\sigma = 20\%$	100	8.264	8.766	9.397	9.360	9.326
$t = 1$	110	6.035	6.349	6.944	6.906	6.873
	120	4.778	5.018	5.555	5.519	5.489
	80	19.701	20.644	20.751	20.745	20.746
$r = 5\% \mid q = 0\%$	90	13.057	13.514	13.657	13.647	13.648
$\sigma = 30\%$	100	8.398	8.629	8.770	8.758	8.759
$t = 0.5$	110	5.455	5.583	5.713	5.702	5.703
	120	3.732	3.815	3.933	3.922	3.922
CPU (time)			350.14	22.218	36.895	72.908

Table 5.1.: Prices of American-style put options under the Merton jump-diffusion model

The third column of Table 5.1. presents the prices of European-style put options that were determined based on the equation (3.12). Table 5.1. also displays the analysis of

American-style options, presenting its prices determined using 2 alternative methods: FST-method, which corresponds to the Benchmark, and the SHP model, which is the approach under study. Benchmark values on column 4 were computed using 15000-time intervals and 10000 space steps. The last three column reveal the American-style put option prices obtained by the Static Hedge Portfolio approach under the Merton jump-diffusion model, with $n \in \{12, 24, 52\}$.

As it can be observed in Table 5.1, the model under study is quite efficient when compared to the benchmark used. Similar conclusions were previously demonstrated in SHP under Black-Scholes model and CEV model. However, the prices of American-style put options calculated through the SHP approach under Merton jump-diffusion do not match the ones calculated with the Benchmark model; the deviation identified reveals an accuracy noncompliance. In the example under study, the SHP portfolio value is overrated when compared to the benchmark.

The analysis of the boundary determined by each model, also reveals the lack of accuracy of the model under study, as it is illustrated in Figure 5.1.

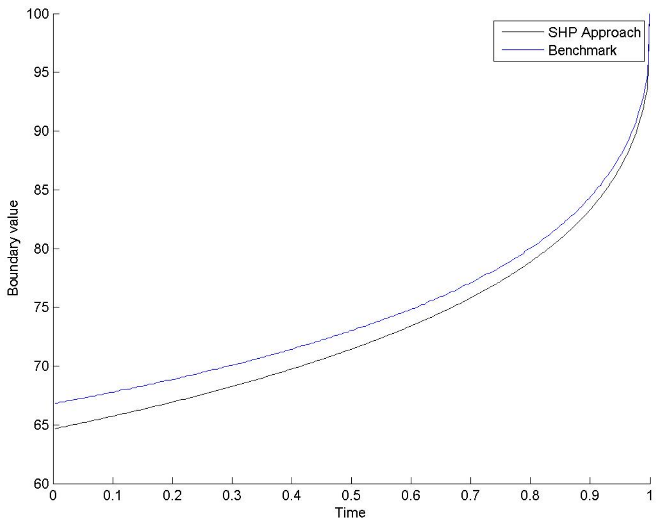

Figure 5.1.: Boundaries of American-style put option using the FST-Method versus the SHP approach

In Figure 5.1. the baseline parameters used are: $S = 100$, $K = 100$, $r = 5\%$, $q = 0\%$, $\sigma = 30\%$ and $\tau = 1$, with the jumps parametrized as: $\lambda = 0.1$, $\mu_J = -0.92$ and $\sigma_J = 0.425$.

When comparing the two boundaries it is easily understood that there is a gap between them, where the SHP model boundary assumes inferior values, leading to a superior premium of the early exercise and consequently higher prices, which is in line with the data available on Table 5.1.

By analysing the accuracy of the model, it was also concluded that option prices obtained through the SHP approach are very similar to the final ones, even if using the SHP pricing procedure with only 12 evenly-spaced time points n. Thus, the results obtained with a superior n do not significantly diverge from results obtained for 12 evenly-spaced time points, as it can be verified in Figure 5.2., which displays price's accuracy with n increments.

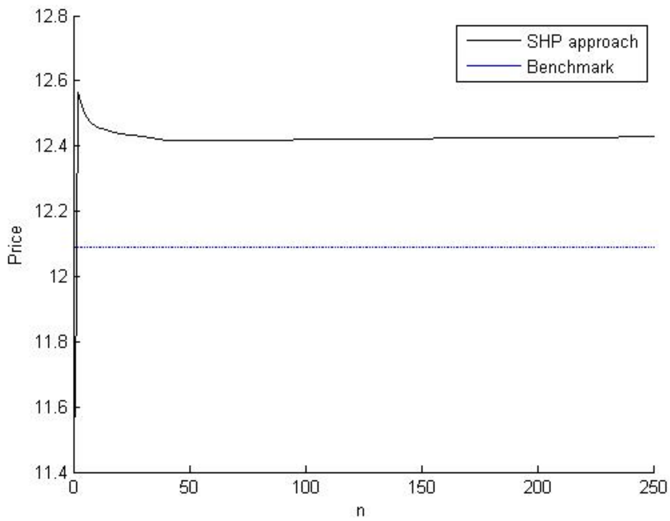

Figure 5.2.: The convergence of the SHP prices of American-style put to the benchmark price.

In Figure 5.2. the baseline parameters are defined by: $S = 100$, $K = 100$, $r = 5\%$, $q = 0\%$, $\sigma = 30\%$ and $\tau = 1$, with the jumps parametrized as: $\lambda = 0.1$, $\mu_J = -0.92$ and $\sigma_J = 0.425$.

Therefore, it is proven that the problem remains unsolved, independently from the n used. For this reason, the use of the SHP approach under the Merton jump-diffusion to price American-style call options was also analysed in order to confirm if the problems are also present or not.

It is important to highlight the following Merton conclusion: if the underlying asset does not pay dividends, there is no benefit in exercising an American-style call before its maturity. This assumption allows assessing American-style call contracts as European-style call option, as shown in Table 5.2, where the prices of both contracts are equal.

			American-style option			
Parameters	Spot	European-style	Benchmark	SHP 12	SHP 24	SHP 52
	80	5.711	5.711	5.711	5.711	5.711
$r = 5\% \mid q = 0\%$	90	10.380	10.380	10.380	10.380	10.380
$\sigma = 30\%$	100	16.452	16.452	16.452	16.452	16.452
$t = 1$	110	23.661	23.661	23.661	23.661	23.661
	120	31.717	31.717	31.717	31.717	31.717

Table 5.2.: Prices of American-style call options under the Merton jump-diffusion model with $q = 0\%$

Based on the conclusions presented by Merton, the only parameter adjusted to analyse American-style calls was the dividend yield (q value increased from 0% to 7%), so that there might be a benefit in exercising an American-style option before maturity.

Thus, the baseline parameters presented in Table 5.1 will be used, except for the dividend yield. The parameters are: $K = 100$, $r = 5\%$, $q = 7\%$, $\sigma = 30\%$, $\tau = 1$ and $S \in \{80, 90, 100, 110, 120\}$.
The jumps were parametrized as

$$\lambda = 0.1, \qquad \mu_J = -0.92, \qquad \sigma_J = 0.425.$$

Afterwards, keeping all other parameters unchanged, single variations were introduced, first with r, then q, after σ and finally the parameter τ, similarly to the setup done in the previous analysis for the put options.

As observed in Table 5.1., the SHP approach under the Merton jump-diffusion model does not match the prices calculated with the Benchmark model for the put case. However, on Table 5.3. that displays the results for the American-style call options, the model under study is highly precise and it reveals accuracy compliance when valuing American-style call options. Furthermore, the speed-accuracy demonstrates that the numerical efficiency of the

SHP method against the FST method is notorious.

Parameters	Spot	European-style	American-style option			
			Benchmark	SHP 12	SHP 24	SHP 52
	80	3.826	3.839	3.839	3.839	3.839
$r = 5\% \mid q = 7\%$	90	7.359	7.398	7.399	7.398	7.398
$\sigma = 30\%$	100	12.205	12.302	12.303	12.303	12.302
$t = 1$	110	18.204	18.410	18.411	18.410	18.410
	120	25.125	25.514	25.516	25.515	25.514
	80	4.060	4.065	4.065	4.065	4.065
$r = 7\% \mid q = 7\%$	90	7.745	7.762	7,762	7.762	7.762
$\sigma = 30\%$	100	12.760	12.805	12.805	12.805	12.805
$t = 1$	110	18.928	19.029	19.030	19.030	19.029
	120	26.010	26.212	26.213	26.213	26.212
	80	4.019	4.027	4.027	4.027	4.027
$r = 5\% \mid q = 6\%$	90	7.668	7.692	7.694	7.692	7.692
$\sigma = 30\%$	100	12.633	12.696	12.697	12.696	12.696
$t = 1$	110	18.739	18.880	18.881	18.880	18.880
	120	25.751	26.025	26.027	26.026	26.025
	80	1.488	1.489	1.489	1.489	1.489
$r = 5\% \mid q = 7\%$	90	4.175	4.181	4.181	4.181	4.181
$\sigma = 20\%$	100	8.743	8.768	8.768	8.768	8.768
$t = 1$	110	15.009	15.092	15.093	15.093	15.093
	120	22.504	22.728	22.729	22.729	22.729
	80	1.579	1.580	1.580	1.580	1.580
$r = 5\% \mid q = 7\%$	90	4.274	4.277	4.277	4.277	4.277
$\sigma = 30\%$	100	8.815	8.829	8.829	8.829	8.829
$t = 0.5$	110	15.082	15.125	15.125	15.125	15.125
	120	22.669	22.779	22.780	22.779	22.779
CPU (time)			350.14	22.218	36.895	72.908

Table 5.3.: Prices of American-style call options under the Merton jump-diffusion model

In order to reinforce the previous conclusions, new tests were performed for the baseline parameters: $K = 100$, $r = 5\%$, $\sigma = 30\%$, $\tau = 1$ and $S \in \{80, 90, 100, 110, 120\}$ and with jump parametrized by: $\lambda = 0.1$, $\sigma_J = 0.425$ and varying $\mu_J \in \{-0,5, -0,25, 0, 0.25, 0,5\}$.

On Table 5.4. it is possible to find call's prices and put's prices, using the parameters previously established. The parameter will assume different values depending on the type of option under analysis; in the case of put options $q = 0\%$, however for call options $q = 7\%$. The difference can be understood by restoring to the data presented in Table 5.2

		American-style option			
Parameters	Spot	Benchmark-Call	Benchmark-Put	SHP 52-Call	SHP 52-Put
$\mu_J = 0.5$	80	6.029	23.176	5.928	23.177
	90	9.232	16.916	10.394	16.917
	100	13.558	12.025	16.041	12.026
	110	19.007	8.352	22.626	8.352
	120	25.542	5.689	29.918	5.689
$\mu_J = 0.25$	80	4,552	22.231	4.335	22.237
	90	7,750	15.866	8.080	15.874
	100	12,186	11.043	13.116	11.046
	110	17,850	7.526	19.284	7.530
	120	24,632	5.046	26.374	5.048
$\mu_J = 0$	80	3.772	21.823	3.741	21.853
	90	6.988	15.419	7.032	15.450
	100	11.520	10.663	11.719	10.689
	110	17.321	7.256	17.480	7.276
	120	24.250	4.889	24.354	4.904
$\mu_J =$ - 0.25	80	3.501	21.750	3.514	21.824
	90	6.771	15.405	6.770	15.485
	100	11.390	10.748	11.385	10.819
	110	17.282	7.443	17.273	7.501
	120	24.284	5.156	24.274	5.202
$\mu_J =$ - 0.5	80	3.533	21.874	3.533	22.011
	90	6.894	15.671	6.894	15.823
	100	11.609	11.155	11.608	11.294
	110	17.577	7.964	17.576	8.084
	120	24.617	5.757	24.617	5.857

Table 5.4.: Prices of American-style call/put options under the Merton jump-diffusion model

Table 5.4 also reveals that the jump parameterization has an impact on the results taken from the study approach, and additionally the proximity of the μ_j to zero makes the prices of both calls and puts inaccurate.

Additionally, taking into account the put-call relationship for American-style options, it is possible to calculate the price of a put option through a suitable change in the arguments used to determine the price of a call option and vice versa. Additional details on put-call duality relationship can be found in Appendix II.

Schroder [36] suggests that is possible to change the roles of the argument pairs (S_t, K) and (r, q) and calculate a new λ e μ_J, as described in Appendix II. The changes proposed above are only possible because the arguments used are interchanged in the two markets. The results from pricing American-style options using the put-call relationship are presented in Appendix III.

Overall the results allow us to conclude that the SHP approach has a positive outcome when used to determine put options with an underlying with jumps parametrized using $\mu_J > 0$ and call options with an underlying represented by a jump with $\mu_J < 0$ can also be priced using the model developed, as presented on Table 5.3. and reinforced in Table 5.4. On the other hand, the symmetric case leads to the poor results.

Considering that the Merton jump-diffusion represents the underlying price through the addition of one compound Poisson jump process to the Black-Scholes model, thus the model problem seems to be related with jump addition. This leads to the conclusion that the problem of the SHP approach under the Merton jump-diffusion is associated to the fact that it cannot accurately foreseen the occurrence of jumps with different amplitudes.

The fact that when the underlying of the put (resp., call) option has jumps with positive (negative) μ_J allows the SHP approach to price American-style put (call) options, reinforces the theory above because the existence of jumps symmetric to the boundary will have no impact in the anticipated exercise of the option. Additionally, as discussed on Chapter 4, jumps occurrence in asset prices of the Merton jump-diffusion model turns the delta-hedging into a non-optimal method to determine a perfect replication. Thus, the pricing of the American-style options by the SHP procedure through the delta-hedging sensitivity may also be the root cause for the limitations already discovered.

Therefore, in order to validate this conclusion, it was decided to analyse the impact of a jump amplitude increase, λ, in the interval between the SHP approach and benchmark. The previous baseline parameters were use: $S = 100$, $K = 100$, $r = 5$, $\sigma = 30$, $q = 0$, $\tau = 1$. The jumps were parametrized as: $\mu_J = -0.92$, $\sigma_J = 0.425$ and varying λ.

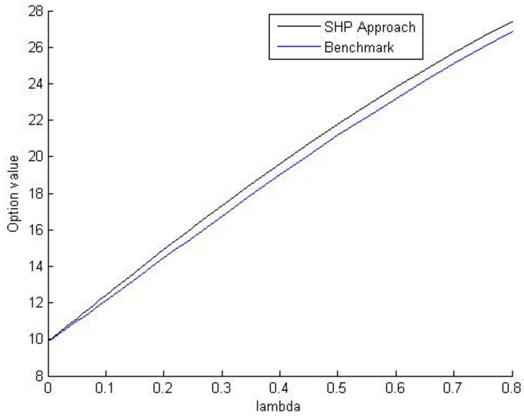

Figure 5.3.: American-style put option price varying λ within a range from 0 to 0.8

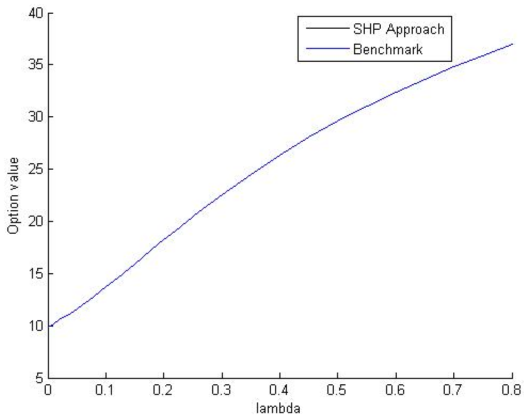

Figure 5.4.: American-style call option price varying the λ within a range from 0 to 0.8

Figure 5.3. represents the impact of underlying's jump amplitude increase in an American-style put option. Figure 5.4 represents the same analysis, but to an American-style call option. It is possible to conclude that, as previously identified, the introduction of jumps in the underlying asset pricing processes creates divergences from the benchmark model, but only in the situations discussed above. Additionally, it is demonstrated that for every situation where the model presents low accuracy, lambda increases will result in the increase of the deviation between the Benchmark and the model under study. In other scenarios the model under study is always accurate.

Thus, the Static Hedge Portfolio approach under the Merton jump-diffusion option pricing model does not correctly account for the potential over shooting of the asset price over the early exercise boundary. It was verified that the SHP approach retrieves accurate results for put options (resp. call options) with an underlying asset parametrized with a positive (negative) average size of the jumps. On the other hand, when the put options (resp. call) with an underlying asset parametrized with a negative (positive) average size of the jumps the model retrieves no accurate results. Critical analyse of this conclusions and final remarks are presented with more detail in the next section.

6. Conclusions

A realistic model that can be widely used to evaluate American-style options, must be both accurate and efficient, in order to respond to market demands. Therefore, the quality of the method under study (SHP) was tested againts the Fourier Space Time-stepping method (FST-method) of Jackson et al. [26], which allows the pricing American-style standard options under exponential Lévy processes. After an extensive numerical study and critical analysis, it is now clear that the jump addition introduces limitations to the static hedge portfolio, reducing model's accuracy and consequently preventing it from becoming another tool to evaluate American-style options.

The static hedge portfolio approach under the Merton jump-diffusion option pricing model does not correctly account for the potential over shooting of the asset price over the early exercise boundary. By evaluating a call and a put through the same parameters, this evidence becomes even clearer: in the case of underlying assets with jumps leading to price decreases (average of the jumps size is negative), the SHP approach retrieves no accurate results for puts, but it works in the case of calls; on the other hand, if jumps lead to an increase of asset prices (average of the jumps size is positive), the model under study presents a positive outcome when used to determine puts, but the same does not verify for calls. These conclusions are reinforced by using the put-call duality relationship for American-style options, where the price of a put option was calculated through a suitable change in the arguments used to determine the price of a call option and vice-versa.

Despite the limitations already identified, the method under study assumes special relevance due to its efficiency and its realistic approach to the dynamics of market models. Jumps occurrence in underlying asset prices makes the delta-hedging a non-optimal measure to hedge the effect of discontinuous movements. For this reason, the use of delta-hedging as a measure of sensitivity in the boundary calculation can create some limitations. The pointed restrictions are the base to understand that the SHP method is not able to accurately account for the possible over shooting of the asset price over the early exercise boundary. Thus, in order to become an accurate method, some adjustments may have to be made in future research.

One possible solution requiring deep study and testing could be the use of a sensitivity measure that includes the two dynamic components of the asset: the sensitivity of option price to infinitesimal continuous stock movements; and the average sensitivity to finitely-sized jumps. This would allow developing a three-dimensional early exercise boundary, instead of the two-dimensional presented by Chung and Shih [13]. The use of three-dimensional boundaries goes far and beyond the standard European-style options with a different strike for each maturity, to hedge the continuous movements. The three-dimensional boundary would also require the use of a finite number of strikes for each maturity, which represents a finite number of possible jump amplitudes and thus captures the finitely-sized jumps.

The resolution of this difficulty on the extension of the static hedge portfolio to the Merton jump diffusion model can be the key to allow a generalization of the proposed approach to incomplete markets model. Thus, future research on this approach has the potential to establish a new method for pricing and hedging American-style options in a much more efficient way.

A. Appendix

A.1. Appendix I

Given the Itô formula presented in Theorem 2.3 for the jump-diffusion process $L_t = L_0 + \int_0^t \mu_s d_s + \int_0^t \sigma_s dW_s + \sum_{i=1}^{N_t} \Delta X_i$,

$$df(L_t,t) = \frac{\partial f(L_t,t)}{\partial t}dt + b_t\frac{\partial f(L_t,t)}{\partial x}dt + \frac{\sigma_t^2}{2}\frac{\partial^2 f(L_t,t)}{\partial x^2}$$
$$+\sigma_t\frac{\partial f(L_t,t)}{\partial x}dW_t + [f(X_{t^-}+\Delta X_t) - f(X_{t^-})],$$

to solve the stochastic differential equation,

$$dS_t = (\mu - \lambda\kappa)S_t dt + \sigma S_t dW_t + (y_t - 1)S_t dN_t.$$

Then,

$$\begin{aligned}
d\ln S_t &= \frac{\partial \ln S_t}{\partial t}dt + (\mu-\lambda\zeta)S_t\frac{\partial \ln S_t}{\partial t}dt + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 \ln S_t}{\partial S_t}dt + \sigma S_t\frac{\partial \ln S_t}{\partial S_t}dW_t + [\ln y_t S_t - \ln S_t] \\
d\ln S_t &= (\mu-\lambda\zeta)S_t\frac{1}{S_t} + \frac{\sigma^2 S_t^2}{2}\left(-\frac{1}{S_t^2}\right)dt + \sigma S_t\frac{1}{S_t}dW_t + [\ln y_t + \ln S_t - \ln S_t] \\
d\ln S_t &= (\mu-\lambda\zeta)dt - \frac{\sigma^2}{2}dt + \sigma S_t dW_t + \ln y_t \\
\ln S_t &= \ln S_0 + (\mu - \frac{\sigma^2}{2} - \lambda\zeta)t + \sigma_t W_t + \sum_{i=1}^{N_t}\ln y_t \\
S_t &= exp\left[\ln S_0 + (\mu - \frac{\sigma^2}{2}\lambda\zeta)t + \sigma_t W_t + \sum_{i=1}^{N_t}\ln y_t\right],
\end{aligned}$$

allowing us to reach the following solution for the stochastic differential equation,

$$S_t = S_0 exp\left[(\mu - \frac{\sigma^2}{2}\lambda\zeta)t + \sigma_t W_t + \sum_{i=1}^{N_t} \ln y_t\right].$$

A.2. Appendix II

The put-call relationship for American-style options allows the price of a put option to be calculated through a suitable modification of the technical features used to determine the price of a call option and vice versa. Based on the change of numéraire technique, Schroder [36] demonstrated the applications of the put-call duality relations for jump-diffusion models. As in the Merton jump-diffusion model, the asset price follows a compound Poisson process with jump intensity, λ under the risk-neutral measure $\mathbb{Q}$, where between jumps the stocks price satisfies,

$$\frac{dS_t}{S_t^-} = (r - q - \lambda(e^{\mu_J + \frac{\sigma_J^2}{2}} - 1)dt + \sigma dW_t. \tag{A.1}$$

The change of probability measure is defined by the following Radon-Nikodým derivative will be used:

$$\frac{\partial\overline{\mathbb{Q}}}{\partial\mathbb{Q}}\Bigg|\mathcal{F}_t = \frac{\frac{e^{q(t-t_0)}S_t}{S_{t_0}}}{\frac{B_t}{B_{t_0}}} = \frac{S_t}{S_{t_0}}e^{-(r-q)(t-t_0)}. \tag{A.2}$$

The jump process intensity, $\mu_J\lambda$ under $\overline{\mathbb{Q}}$ can be obtained using the martingale property of S under $\mathbb{Q}$:

$$\overline{\mathbb{Q}}(\tau_1 > t|\mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(1_{\tau_1 > t}S_t/S_{t_0}|\mathcal{F}_t) = e^{(1-\mu)\lambda t}\mathbb{Q}(\tau_1 > t). \tag{A.3}$$

Then, the jump intensity under $\overline{\mathbb{Q}}$ is equal to the product of the intensity under $\mathbb{Q}$ and the expected jump size is $\overline{\lambda} = \lambda e^{\mu_j + \frac{1}{2}\sigma_J^2}$ and where the distribution functions under $\overline{\mathbb{Q}}$ and $\mathbb{Q}$ satisfy

$$\overline{\Psi}(dy) = \Psi(dy)e^{(-\mu_J - \frac{\sigma_J^2}{2})}y, \tag{A.4}$$

where the $\Psi(dy)$ is the distribution functions under $\mathbb{Q}$ and $\overline{\Psi}(dy)$ is still normally distributed under $\overline{\mathbb{Q}}$ with variance σ_J^2, but with mean $\mu_J + \sigma J$

Thus, the dynamics of $\overline{S_t} \equiv K S_{t_0}/S_t$ is then,

$$\frac{d\overline{S_t}}{\overline{S_t^-}} = (q - r - \overline{\lambda}(e^{-\mu_J - \frac{\sigma_J^2}{2}} - 1)dt + \sigma d\overline{W_t}. \tag{A.5}$$

A.3. Appendix III

Using the put-call relationship for American-style options, it is possible to calculate the price of a put option through a suitable change in the arguments used to determine the price of a call option and vice versa.

Thus, as presented in Chapter 5, Tables A.1. and A.2. display the accuracy and efficiency of the static hedge portfolio approach for valuing standard American-style options under the Merton jump-diffusion model for different parameters.

Table A.1. shows the baseline parameters to pricing the American-style call options: $S = 100$, $r = 0\%$, $q = 5\%$, $\sigma = 30\%$, $\tau = 1$ and $K \in \{80, 90, 100, 110, 120\}$. These parameters were computed through put-call duality and based on the parameters of the put options of the Table 5.1. The jumps were parametrized as,

$$\lambda = 0.043619, \qquad \mu_J = 0.739375, \qquad \sigma_J = 0.425.$$

Afterwards, keeping all other parameters unchanged, single variations are introduced, first with r, then q, after σ and finally the parameter τ.

Table A.1 displays the accuracy and efficiency of the static hedge portfolio approach for valuing standard American-style call options under the Merton jump-diffusion model for different parameters.

Parameters	Strike	European-style	American-style option			
			Benchmark	SHP 12	SHP 24	SHP 52
	80	20.833	22.257	22.555	22.541	22.527
$r = 0\% \mid q = 5\%$	90	15.503	16.363	16.670	16.683	16.666
$\sigma = 30\%$	100	11.575	12.130	12.458	12.440	12.424
$t = 1$	110	8.783	9.168	9.472	9.454	9.439
	120	6.839	7.130	7.401	7.384	7.369
	80	19.475	21.633	22.035	22.015	21.999
$r = 0\% \mid q = 7\%$	90	14.400	15.665	16.160	16.135	16.115
$\sigma = 30\%$	100	10.712	11.508	12.006	11.979	11.958
$t = 1$	110	8.123	8.669	9.132	9.106	9.085
	120	6.338	6.745	7.161	7.137	7.117
	80	21.323	22.528	22.785	22.772	22.761
$r = 1\% \mid q = 5\%$	90	15.920	16.656	16.939	16.924	16.911
$\sigma = 30\%$	100	11.910	12.391	12.664	12.648	12.635
$t = 1$	110	9.042	9.382	9.631	9.615	9.603
	120	7.035	7.294	7.515	7.501	7.489
	80	17.947	20.166	20.451	20.434	20.419
$r = 0\% \mid q = 5\%$	90	12.067	13.055	13.641	13.609	13.579
$\sigma = 20\%$	100	8.264	8.766	9.397	9.360	9.326
$t = 1$	110	6.035	6.348	6.944	6.906	6.873
	120	4.778	5.018	5.555	5.519	5.489
	80	19.701	20.644	20.751	20.745	20.746
$r = 0\% \mid q = 5\%$	90	13.057	13.514	13.657	13.647	13.648
$\sigma = 30\%$	100	8.398	8.629	8.770	8.758	8.759
$t = 0.5$	110	5.455	5.582	5.713	5.702	5.703
	120	3.732	3.815	3.933	3.922	3.922
CPU (time)			350.14	22.218	36.895	72.908

Table A.1.: Prices of American-style call options under the Merton jump-diffusion model

Table A.2. shows the baseline parameters to pricing the American-style put options: $S = 100$, $r = 7\%$, $q = 5\%$, $\sigma = 30\%$, $\tau = 1$ and $K \in \{80, 90, 100, 110, 120\}$. These parameters were computed through put-call duality and based on the parameters of the call options of the Table 5.3. The jumps were parametrized as,

$$\lambda = 0.043619, \qquad \mu_J = 0.739375, \qquad \sigma_J = 0.425.$$

Afterwards, keeping all other parameters unchanged, single variations are introduced, first with r, then q, after σ and finally the parameter τ.

On Table A.2. it is possible to find put's prices, using the parameters previously established.

Parameters	Strike	European-style	American-style option			
			Benchmark	SHP 12	SHP 24	SHP 52
	80	3.826	3.839	3.839	3.839	3.839
$r = 7\% \mid q = 5\%$	90	7.359	7.398	7.399	7.398	7.398
$\sigma = 30\%$	100	12.215	12.302	12.303	12.303	12.302
$t = 1$	110	18.204	18.410	18.411	18.410	18.410
	120	25.125	25.514	25.516	25.515	25.514
	80	4.060	4.065	4.065	4.065	4.065
$r = 7\% \mid q = 7\%$	90	7.745	7.762	7,762	7.762	7.762
$\sigma = 30\%$	100	12.760	12.805	12.805	12.805	12.805
$t = 1$	110	18.928	19.029	19.030	19.030	19.029
	120	26.010	26.212	26.213	26.213	26.212
	80	4.019	4.027	4.027	4.027	4.027
$r = 6\% \mid q = 5\%$	90	7.668	7.692	7.693	7.693	7.692
$\sigma = 30\%$	100	12.633	12.696	12.697	12.696	12.696
$t = 1$	110	18.739	18.880	18.881	18.880	18.880
	120	25.751	26.025	26.027	26.026	26.025
	80	1.488	1.489	1.489	1.489	1.489
$r = 7\% \mid q = 5\%$	90	4.175	4.181	4.181	4.181	4.181
$\sigma = 20\%$	100	8.743	8.768	8.768	8.768	8.768
$t = 1$	110	15.009	15.092	15.093	15.093	15.093
	120	22.504	22.728	22.729	22.729	22.729
	80	1.579	1.580	1.580	1.580	1.580
$r = 7\% \mid q = 5\%$	90	4.274	4.277	4.277	4.277	4.277
$\sigma = 30\%$	100	8.815	8.829	8.829	8.828	8.828
$t = 0.5$	110	15.082	15.125	15.125	15.125	15.125
	120	22.669	22.779	22.780	22.779	22.779
CPU (time)			350.14	22.218	36.895	72.908

Table A.2.: Prices of American-style put options under the Merton jump-diffusion model

These data reinforce the conclusions drawn in Chapter 5: the static hedge portfolio presents limitations on the pricing for American-style options under Merton jump-diffusion model.

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