Chapter 6
Pricing Multiple Exercise American Options by Linear Programming

Monia Giandomenico and Mustafa Ç. Pınar

Abstract We consider the problem of computing the lower hedging price of American options of the call and put type written on a non-dividend paying stock in a non-recombinant tree model with multiple exercise rights. We prove using a simple argument that an optimal exercise policy for an option with h exercise rights is to delay exercise until the last h periods. The result implies that the mixed-integer programming model for computing the lower hedging price and the optimal exercise and hedging policy has a linear programming relaxation that is exact, i.e., the relaxation admits an optimal solution where all variables required to be integral have integer values.

Keywords American options • Swing options • Multiple exercise rights • Linear programming • Mixed-integer programming • Lower hedging price

6.1 Introduction

Pricing and hedging American options has been an important subject of mathematical finance. Starting with the work of Harrison and Kreps [22], Bensoussan [6] and Karatzas [25], finding a no-arbitrage price for American options has been studied in various settings ranging from discrete-time, discrete probability space to continuous time infinite state space settings in complete and incomplete markets; see e.g., [8, 9, 13, 15, 26, 29, 32, 38]. For a text-book treatment of American options in discrete time the book by Föllmer and Schied [20] is an authoritative source while the monograph by Detemple [16] concentrates on models in continuous time.

For options with early exercise possibility (thus, of American type) but with multiple exercise rights such as the swing options of energy markets [24], one can

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G. Consigli et al. (eds.), *Optimal Financial Decision Making under Uncertainty*,
International Series in Operations Research & Management Science 245,
DOI 10.1007/978-3-319-41613-7_6

consult the following literature [1–5, 11, 12, 18, 21, 23, 27, 31, 36, 37, 39, 40]. Thompson [37] uses lattice based claim evaluation techniques for commodity options with multiple exercise rights. Keppo [27] gives an elementary introduction to electricity swing options. Bardou et al. [1, 2] and Barrera-Esteve et al. [3] consider swing options in complete markets within a stochastic control framework. In particular, in [2] the bang-bang nature of the optimal exercise policy is studied. In [12], Carmona and Touzi study American options with multiple exercise rights in a Black-Scholes [7] framework, whereas in [11] a more general case using linear diffusion models is treated. Bender [5] studies multiple exercise options in continuous time with a finite maturity and proves the existence of the Snell envelope, a reduction principle as nested single stopping problems, and a Doob-Meyer type decomposition for the Snell envelope. He also derives a dual representation that generalizes that of Schoenmakers [36] and gives a primal-dual Monte-Carlo algorithm. In [31] a dual representation in discrete time is given, and its extension to volume constraints is studied in [4]. Haarbrücker and Kuhn [21] use multi-stage stochastic programming to price electricity swing options while Winter and Wilhelm [40] use the finite element method to evaluate swing options. Vayanos et al. [39] consider electricity swing options in incomplete markets as in the present paper using forward contracts for hedging, and compute buyer and seller prices using robust control and constraint sampling techniques. Longstaff and Schwartz [29], Ibáñez [23] and Figueroa [18] use Monte-Carlo simulation techniques to price single and multiple exercise claims. Chalasani and Jha [13], and Pınar and Camcı [34] study American options in the discrete time finite state probability setting as in the present paper, but allow for proportional transaction costs. Camcı and Pınar [10] and Flåm [19] and Pennanen and King [33] treat similar problems from a finite-dimensional optimization point of view.

In the present paper, we concentrate on the problem of finding an optimal exercise and hedging policy, and hence a fair buyer's price for an American option with multiple exercise rights, written on a stock evolving in a non-recombinant tree in the presence of a risk free asset paying no interest, a problem on which little (if anything at all) has been written. We formulate the problem as a mixed-integer programming problem. It is well-known that in discrete-time complete (and arbitrage free) markets the price of a single exercise American call option on a non-dividend paying asset behaves as a sub-martingale, and hence, it is optimal to delay exercise until maturity; see e.g., [20]. The assertion remains true also for an American single exercise put option on a non-dividend paying asset in a zero-interest rate environment [20]. Our main result provides an extension of this well-known fact (delaying exercise until maturity is optimal) for American options with multiple exercise rights. The result not only shows the optimal exercise policy, but also proves the exact nature of the LP relaxation of the mixed-integer model. Therefore, one can obtain the lower hedging price by solving a linear programming problem, a problem that can be solved in polynomial time. To the best of our knowledge, this simple result was not previously available in the mathematical finance literature. We also obtain a min–max expression for the price of an American claim with multiple (two) exercise rights as follows:

$$\max_{\tau \in \mathscr{T}^2(T)} \min_{Q \in \tilde{\mathscr{Q}}} \mathbb{E}^Q[F_\tau] = \min_{Q \in \tilde{\mathscr{Q}}} \max_{\tau \in \mathscr{T}^2(T)} \mathbb{E}^Q[F_\tau]$$

where T is the maturity date of the claim, $\mathscr{T}^2(T)$ is the collection of all vectors of stopping times $\tau = (\tau_1, \tau_2) \in [0, 1, \ldots, T] \cup \{+\infty\}$ satisfying some conditions (c.f., end of Sect. 6.4) and $\tilde{\mathscr{Q}}$ represents (the closure of) all equivalent martingale measures. This is reminiscent of the representation

$$\max_{\tau \in \mathscr{T}} \min_{Q \in \tilde{\mathscr{Q}}} \mathbb{E}^Q[F_\tau] = \min_{Q \in \tilde{\mathscr{Q}}} \max_{\tau \in \mathscr{T}} \mathbb{E}^q[F_\tau]$$

for American claims with $\mathscr{T}$ denoting the set of all stopping times. The above representation can easily be generalized to h exercise rights.

A word of caution is in order here. One should bear in mind that for the swing options traded in energy markets, the underlying (e.g., electricity) is not traded in the spot market whereas our analysis in the present paper is based on the assumption that the underlying can be traded.

The rest of the paper is organized as follows. In Sect. 6.2 we review the basics of the stochastic scenario tree and American claims. In Sect. 6.3 we present an optimization model to compute a fair price for an American claim with multiple exercise rights. We prove the main result in Sect. 6.4. We conclude in Sect. 6.5.

6.2 The Stochastic Scenario Tree and American Contingent Claims

An American contingent claim (abbreviated ACC) F is a financial instrument generating a real-valued stochastic (cash-flow) process $(F_t)_{t=0,\ldots,T}$ with $h \geq 1$ exercise rights to the holder. At any stage $t = 0, \ldots, T$, the holder of a single-exercise ACC may decide to take F_t in cash and terminate the process. In the case of $h > 1$ exercise rights, the holder may decide to make up to and including h exercises (at h different time points). The process terminates when the h-th exercise is performed. Of course, the holder may choose to exercise less than h times during the lifetime of the claim. An American call option on a stock S with strike price K has a payoff equal to $F = S - K$. American put is obtained by reversing the sign of F. In our finite probability space setting an American option F with h exercise rights generates payoff opportunities F_n ($F_n = \max\{S_n^1 - K, 0\}$ or $F_n = \max\{K - S_n^1, 0\}$ for some strike price K), ($n \geq 0$) and h exercise possibilities to its holder depending on the states n of the market that we define below.

To lay down a pricing framework based on no-arbitrage arguments for contingent claims, we assume that security prices and other payments are discrete random variables supported on a finite probability space $(\Omega, \mathscr{F}, P)$ whose atoms are sequences of real-valued vectors (asset values) over discrete time periods $t = 0, 1, \ldots, T$. We further assume the market evolves as a discrete, non-recombinant scenario tree. A non-recombinant tree structure is suitable for incomplete markets as discussed in [17] since it allows to work with path-dependent portfolio strategies whereas in recombinant trees one optimizes over path-independent strategies which may be suboptimal. In the scenario tree, the partition of probability atoms $\omega \in \Omega$ generated by matching path histories up to time t corresponds one-to-one with nodes $n \in \mathscr{N}_t$ at level t in the tree. The set $\mathscr{N}_0$ consists of the root node $n = 0$, and the leaf nodes $n \in \mathscr{N}_T$ correspond one-to-one with the probability atoms $\omega \in \Omega$. The σ-algebras are such that, $\mathscr{F}_0 = \{\emptyset, \Omega\}$, $\mathscr{F}_t \subset \mathscr{F}_{t+1}$ for all $0 \leq t \leq T-1$ and $\mathscr{F}_T = \mathscr{F}$. A stochastic process is said to be $(\mathscr{F}_t)_{t=0}^T$-adapted if for each $t = 0, \ldots, T$, the outcome of the process only depends on the element of $\mathscr{F}_t$ that has been realized at stage t. Similarly, a decision process is said to be $(\mathscr{F}_t)_{t=0}^T$-adapted if for each $t = 0, \ldots, T$, the decision depends on the element of $\mathscr{F}_t$ that has been realized at stage t. In the scenario tree, every node $n \in \mathscr{N}_t$ for $t = 1, \ldots, T$ has a unique parent denoted $\pi(n) \in \mathscr{N}_{t-1}$, and every node $n \in \mathscr{N}_t$, $t = 0, 1, \ldots, T-1$ has a non-empty set of child nodes $\mathscr{C}(n) \subset \mathscr{N}_{t+1}$. We denote the set of all nodes in the tree by $\mathscr{N}$. For a given node n, the inverse mapping $t(n)$ gives the time period to which the node n belongs to. The set $\mathscr{A}(n)$ denotes the collection of ascendant nodes or path history of node n including itself. The probability distribution P is obtained by attaching positive weights p_n to each leaf node $n \in \mathscr{N}_T$ so that $\sum_{n \in \mathscr{N}_T} p_n = 1$. For each non-leaf (intermediate level) node in the tree we have, recursively,

$$p_n = \sum_{m \in \mathscr{C}(n)} p_m, \quad \forall\, n \in \mathscr{N}_t,\ t = T-1, \ldots, 0.$$

Hence, each non-leaf node has a probability mass equal to the combined mass of its child nodes.

A random variable X is a real valued function defined on Ω. It can be *lifted* to the nodes of a partition $\mathscr{N}_t$ of Ω if each level set $\{X^{-1}(a) : a \in \mathbb{R}\}$ is either the empty set or is a finite union of elements of the partition. In other words, X can be lifted to $\mathscr{N}_t$ if it can be assigned a value on each node of $\mathscr{N}_t$ that is consistent with its definition on Ω [28]. This kind of random variable is said to be measurable with respect to the information contained in the nodes of $\mathscr{N}_t$. A stochastic process $\{X_t\}$ is a time-indexed collection of random variables such that each X_t is measurable with respect $\mathscr{N}_t$. The expected value of X_t is uniquely defined by the sum

$$\mathbb{E}^P[X_t] := \sum_{n \in \mathscr{N}_t} p_n X_n.$$

The conditional expectation of X_{t+1} on $\mathscr{N}_t$ is given by the expression

$$\mathbb{E}^P[X_{t+1} | \mathscr{N}_t] := \sum_{m \in \mathscr{C}(n)} \frac{p_m}{p_n} X_m.$$

The market consists of two traded securities with prices at node n given by the vector $S_n = (S_n^0, S_n^1)$. We assume that the security indexed by 0 has strictly positive prices at each node of the scenario tree. Our blanket assumption throughout the paper is that $S_n^0 = 1$ for all n i.e., a zero interest rate for the risk-free asset.

This assumption is crucial e.g., for the put option case where non-zero interest rate may lead to strict optimality of exercise earlier than the last h periods. We give a counterexample supporting this claim after the proof of Proposition 1.

The number of shares of security j held by the investor in state (node) $n \in \mathscr{N}_t$ is denoted θ_n^j. Therefore, to each state $n \in \mathscr{N}_t$ is associated a vector $\theta_n \in \mathbb{R}^2$. The value of the portfolio at state n is

$$S_n \cdot \theta_n = \sum_{j=0}^{1} S_n^j \theta_n^j.$$

We need the following definition.

Definition 1. If there exists a probability measure $Q = \{q_n\}_{n \in \mathscr{N}_T}$ such that

$$S_t = \mathbb{E}^Q[S_{t+1} | \mathscr{N}_t] \ (t \leq T-1)$$

then the vector process $\{S_t\}$ is called a vector-valued martingale under Q, and Q is called a martingale probability measure for the process.

It is well-known that a market is free of arbitrage opportunities if and only if the price process S is a martingale; see [28] for a discussion of arbitrage and martingales in finite-state markets. We shall assume this situation to be the case throughout the present paper.

6.3 The Formulation

The buyer's problem can be formulated as the following problem that we will refer to as AP1:

$$\begin{array}{lrcl}
\max & V & & \\
\text{s.t.} & S_0 \cdot \theta_0 & = & F_0 e_0 - V \\
& S_n \cdot (\theta_n - \theta_{\pi(n)}) & = & F_n e_n, \ \forall\, n \in \mathscr{N} \setminus \mathscr{N}_0 \\
& S_n \cdot \theta_n & \geq & 0, \ \forall\, n \in \mathscr{N}_T \\
& \sum_{m \in \mathscr{A}(n)} e_m & \leq & h, \ \forall\, n \in \mathscr{N}_T \\
& e_n & \in & \{0, 1\}, \ \forall\, n \in \mathscr{N}
\end{array}$$

where $h \geq 2$ is a fixed integer. In mathematical finance, the theory of incomplete markets involves the price of the seller and the price of the buyer for a contingent claim. These two values can be quite different, leading to an interval in which no arbitrage opportunities for the buyer and seller exist [28, 33]. The fact that these two prices may differ is a matter of active research and discussion in the financial mathematics community (see e.g. [28]) since it brings about the following question: if the maximum the buyer can pay is strictly less than the minimum a seller can settle for, then how are the claims traded in markets? It appears that the present theory—at least in its present form—is not fully capable to explain the prices of contingent claims actually traded in the market. King [28] addresses this problem using existing liabilities of buyers and sellers.

Setting this question aside, for the seller the problem is to form the least costly initial portfolio of traded assets that will cover the potential payments to the holder of the claim (if and when exercised) such that no losses are incurred at the end. By contrast, from the buyer's perspective the problem is to build the most valuable portfolio that can be formed against the ownership rights of the claim. In other words, the buyer initiates a portfolio process (by shorting some instrument(s)), and closes the short positions later by self-financing transactions and the proceeds from the claim in such a way that no losses are incurred at the end of the horizon.

In model AP1, the optimal value of V represents the largest amount that a potential buyer is willing to disburse for acquiring a given American contingent claim F with h exercise rights. The computation of this quantity via the above integer programming problem is performed by construction of the most valuable (today) adapted portfolio process using the proceeds from the exercise of the contingent claim and self-financing transactions using the market-traded securities to avoid any terminal losses. More precisely, the proceeds obtained from the exercise of the claim are used to finance (cover short positions) portfolio transactions initiated by the buyer at time $t = 0$ to acquire the claim. This is expressed in the first and second sets of constraints above in AP1. They represent the requirement that the proceeds from the claim, if exercised, are used in revising the portfolio positions without injection or withdrawal of funds. If there is no exercise at a node, the equation represents self-financing portfolio rebalancing. The third set of constraints makes sure that all terminal portfolio values are non-negative. The integer variables and related constraints represent the h-times exercise of the American contingent claim. The linear programming relaxation of AP1 is the following problem AP2:

$$\begin{array}{lrcl}
\max & V & & \\
\text{s.t.} & S_0 \cdot \theta_0 & = & F_0 e_0 - V \\
& S_n \cdot (\theta_n - \theta_{\pi(n)}) & = & F_n e_n, \ \forall\, n \in \mathscr{N} \setminus \mathscr{N}_0 \\
& S_n \cdot \theta_n & \geq & 0, \ \forall\, n \in \mathscr{N}_T \\
& \sum_{m \in \mathscr{A}(n)} e_m & \leq & h, \ \forall\, n \in \mathscr{N}_T \\
& e_n & \leq & 1, \ \forall\, n \in \mathscr{N} \\
& e_n & \geq & 0, \ \forall\, n \in \mathscr{N}.
\end{array}$$

6.4 The Main Result

The main result of this paper is the following.

Proposition 1. *Assuming that the underlying is a traded instrument, in a financial market described as a non-recombinant tree with two traded instruments (one risky*

Table 6.1 Cash flows of the two strategies 1 and 2 for a call option

Strategy	t_i	t_j	$T-1$	T
Strategy 1	$S_{t_i}-K$	$S_{t_j}-K$	0	0
Strategy 2	$S_{t_i}-K$	$S_{t_j}-K$	$K-S_{T-1}+(S_{T-1}-K)_+$	$K-S_T+(S_T-K)_+$

Table 6.2 Cash flows of the two strategies 1 and 2 for a put option

Strategy	t_i	t_j	$T-1$	T
Strategy 1	$K-S_{t_i}$	$K-S_{t_j}$	0	0
Strategy 2	$K-S_{t_i}$	$K-S_{t_j}$	$S_{T-1}-K+(K-S_{T-1})_+$	$S_T-K+(K-S_T)_+$

asset which is the underlying, and one riskless asset), T time periods to maturity, and zero interest rate, the following holds for an American contingent claim with $h \geq 2$ exercise rights :

1. *It is optimal to delay exercise until the periods $T-h+1, T-h+2, \ldots, T-1$ and T,*
2. *AP2 has an optimal solution with all e variables binary.*

Proof. [1] For the sake of simplicity we shall give the proof of part 1 for the case of $h = 2$. The proof is based on a simple argument of no-arbitrage adapted from the book by Cox and Rubinstein [14], pp. 139–140 for the case $h = 1$.

Assume an exercise strategy that exercises the two rights of a call at times t_i, t_j with $t_i < t_j \leq T, S_{t_i} \geq K, S_{t_j} \geq K$. Now, we can see that exercising at times $T-1$ and T does no worse, in a path-wise sense, than exercising at times t_i and t_j. To see this, compare the cash flows generated by the exercise strategy of times t_i, t_j (referred to as strategy 1), and strategy that exercises the option at times $T-1$ and T, together with shorting a unit of the stock and lending K dollars at times t_i and t_j while closing the positions at times $T-1$ and T (referred to as strategy 2) for a call option. In the case of a put, simply reverse strategy 2 in the following sense: borrow K dollars and go long one unit of stock to close positions at times $T-1$ and T. The following two tables show the cash flows of the two respective strategies in the case of call and put options (Tables 6.1 and 6.2).

It is immediate to see from the cash flows of the two strategies that either strategy 2 has a cash flow identical to strategy 1 or it dominates strategy 1. To see this, note that if $K - S_{T-1} < 0$ then $(S_{T-1} - K)_+ = -(K - S_{T-1})$. On the other hand if $K - S_{T-1} > 0$ then $(S_{T-1} - K)_+ = 0 < (K - S_{T-1})$. A similar observation holds for period T. Therefore, using strategy 2, the holder has a non-negative surplus which is immediately translated into a portfolio process with an objective function value at least as large as that of strategy 1. The reason is that the potential surplus at the

[1] An earlier version of the paper had quite a long proof for the case $h = 2$ and restricted to binomial and trinomial trees. It was based on an elaborate primal-dual construction. The present proof was offered by an anonymous reviewer of the earlier version, to whom we are thankful.

last two periods can be placed in the riskless asset, which (carried backward at no interest) corresponds to a larger initial short position (borrowing) in one of the two assets at period 0, thus a larger value for V. Hence, exercising at the last two periods is at least as good a strategy as any other exercise strategy.

Based on part 1, we can fix the binary variables e to one in the nodes of the last two periods where the payoff is positive, and solve the resulting linear program. The result is an optimal hedging strategy. Therefore, AP1 is equivalent to a linear programming problem.

For the general case of $h > 2$ it suffices to extend the above construction using h exercise rights. □

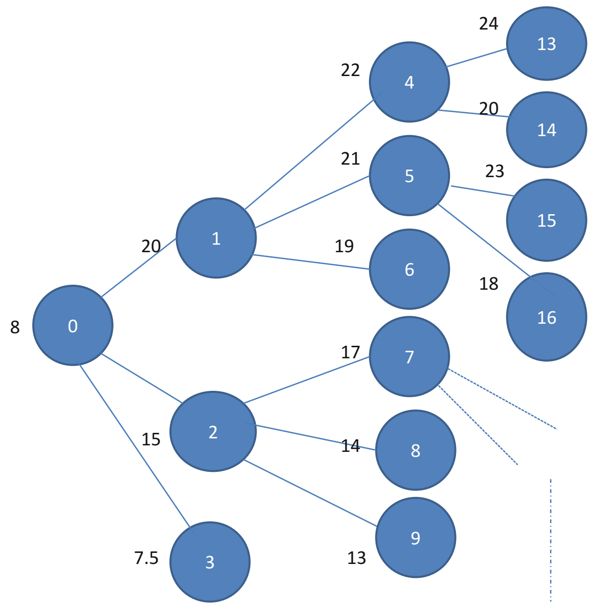
Note that the above result would be valid also in recombining trees. However, we stated the result for the more general non-recombinant tree structure where one can optimize over path-dependent policies.

A result similar in spirit to Proposition 1 is given in Bardou et al. [2] where the bang-bang nature of the optimal exercise quantities for swing options is proved in complete markets using a stochastic control framework. The bang-bang property corresponds in our case to the fact that the LP relaxation allows an optimal solution with 0–1 valued exercise variables, i.e., either no exercise or full exercise at each time point.

We proved that it is always optimal to use the exercise rights at the final h periods. This statement does not mean that earlier exercise is sub-optimal, though. There exist examples where exercise at node 0 may also be part of another optimal exercise policy as the following example demonstrates.

Example (Put Option in a Four-Period Market). Consider a financial market with four trading points, i.e., $T = 3$ evolving as a trinomial tree up to $t = 2$, and from each node of the tree at $t = 2$ two nodes emerge, i.e., the tree behaves binomially at the last period. Hence, the tree has 31 nodes. The risky asset price evolves as follows: at time $t = 0$, we have $S_0 = 8$. At time $t = 1$ the price evolves to either $S_1 = 20$, or $S_2 = 15$ or $S_3 = 7.5$ with equal probability. At time $t = 2$, if the price were equal to 20 at $t = 1$, it becomes either $S_4 = 22$ or $S_5 = 21$ or $S_6 = 19$ with equal probability. If the price were equal to 15 at $t = 1$, it becomes either $S_7 = 17$ or $S_8 = 14$ or $S_9 = 13$ with equal probability. Finally, given that the price were equal to 15 at $t = 1$, it evolves into either $S_{10} = 9$ or $S_{11} = 8$ or $S_{12} = 7$ with equal probability. The remaining nodes, numbered 13–30, have the following price values respectively, $(24, 20, 23, 18, 21, 16, 19, 16.5, 17, 12, 15, 11, 10, 8, 9.5, 7.5, 8.5, 6)$. A partial representation of the tree is given in Fig. 6.1 for the convenience of the reader. The number next to each node is the stock price at that node. An option of the put type with two exercise rights and strike $K = 15$ is introduced into this financial market. Solving the optimization problem (AP1) we observe that it is equally optimal to use one exercise right at the node 3 or suppressing exercise at node 3 and delay exercise to periods $t = 2$ and $t = 3$. Both strategies lead to equal objective function value, hence there exist two different optimal hedging strategies resulting in identical price for the option.

Fig. 6.1 The non-recombinant tree of example for put option in four periods with 31 nodes (partially depicted)



6.4.1 The Case of Non-zero Interest Rate

Corollary 1. *The statement of Proposition 1 is valid for a call option in a market where the risk-less asset has positive per period growth equal to* $R > 1$.

Proof. The proof is similar to the proof of Proposition 1 with a slight modification. For the sake of simplicity, let us consider again the case $h = 2$. Due to non-zero interest rate, the cash flows at the last two periods change as shown in the table below (Table 6.3). It is immediate to see that the cash flows of strategy 2 are at least as good as those of strategy 1. □

However, a similar statement cannot be made in the case of an American put in the presence of a non-zero interest rate even in complete markets and single exercise. According to Luenberger [30] which has an elementary discussion and numerical example for American (single exercise) put options in complete markets, "intuitively, early exercise of a put may be optimal because the upside profit is bounded (unlike the case of call options). Clearly, for example, if the stock price

Table 6.3 Cash flows of the two strategies 1 and 2 for a call option under non-zero interest rate

Strategy	t_i	t_j	$T-1$	T
Strategy 1	$S_{t_i} - K$	$S_{t_j} - K$	0	0
Strategy 2	$S_{t_i} - K$	$S_{t_j} - K$	$KR^{T-1-t_i} - S_{T-1} + (S_{T-1} - K)_+$	$KR^{T-t_j} - S_T + (S_T - K)_+$

tails to zero, one should exercise there, since no greater profit can be achieved." (The reader is referred to pp. 334–335 of [30].) The following example shows that the removal of zero-interest rate assumption may lead to a change in the optimal exercise policy in the case of multiple exercise and incomplete markets as well.

Example (Put Option with Non-zero Interest Rate). Consider a trinomial incomplete financial market with three trading points, i.e., $T = 2$. The risky asset price evolves as follows: at time $t = 0$, we have $S_0 = 8$. At time $t = 1$ the price evolves to either $S_1 = 20$, or $S_2 = 15$ or $S_3 = 7.5$ with equal probability while 1 unit of risk-less asset at time $t = 0$ has a value of 1.01 at time $t = 1$. At time $t = 2$, if the price were equal to 20 at $t = 1$, it becomes either $S_4 = 22$ or $S_5 = 21$ or $S_6 = 19$ with equal probability. If the price were equal to 15 at $t = 1$, it becomes either $S_7 = 17$ or $S_8 = 14$ or $S_9 = 13$ with equal probability. Finally, given that the price were equal to 15 at $t = 1$, it evolves into either $S_{10} = 9$ or $S_{11} = 8$ or $S_{12} = 7$ with equal probability. The risk-less account again appreciates by a factor of 1.01, i.e., it has a value equal to 1.0201 at time $t = 2$. An option of the put type with two exercise rights and strike $K = 15$ is introduced into this financial market. Solving the optimization problem (AP1) we observe that it is strictly optimal to use one exercise right at the root node, node 0, i.e., suppressing exercise at node 0 leads to a strictly smaller objective function value, hence a sub-optimal price for the option.

In our computational experience the exactness property of the LP relaxation appears to continue to hold also in that case.

Conjecture 1. The LP relaxation AP2 is tight in the case of a put option with h exercise rights in a market where the risk-less asset has positive per period growth equal to $R > 1$.

If the conjecture is true, then one can obtain the buyer's price for an American put with multiple exercise rights in a non-zero interest rate market by solving a linear programming problem.

6.4.2 A Min–Max Representation

The usual method to describe exercise strategies of American contingent claims involves stopping times. These are functions $\tau : \Omega \to \{0, \dots, T\} \cup \{+\infty\}$ such that $\{\omega \in \Omega \mid \tau(\omega) = t\} \in \mathscr{F}_t$, for each $t = 0, \dots, T$. The relation $e_t = 1 \Leftrightarrow \tau = t$ defines a one-to-one correspondence between stopping times and decision processes $e \in E$ where

$$E = \{e \mid e \text{ is } (\mathscr{F}_t)_{t=0}^{T}\text{-adapted}, \sum_{t=0}^{T} e_t \leq 1 \text{ and } e_t \in \{0, 1\} \text{ } P\text{-a.s.}\}.$$

The set of stopping times will be denoted by $\mathscr{T}$. Let $\tilde{\mathscr{Q}}$ denote the closure of the set of all martingale measures equivalent to P, i.e., the set

$$\tilde{\mathscr{Q}} = \{q \mid q_0 = 1, q_n S_n = \sum_{m \in \mathscr{C}(n)} q_m S_m, \forall n \in \mathscr{N} \backslash \mathscr{N}_T; 0 \le q_n, \forall n \in \mathscr{N}_T\}.$$

The following expression for American contingent claims is well-known:

$$\max_{\tau \in \mathscr{T}} \min_{Q \in \tilde{\mathscr{Q}}} \mathbb{E}^Q[F_\tau] = \min_{Q \in \tilde{\mathscr{Q}}} \max_{\tau \in \mathscr{T}} \mathbb{E}^q[F_\tau].$$

In the case of multiple rights we can also obtain a similar expression as a result of Proposition 1. For $h = 2$ we shall denote by $\mathscr{T}^2(T)$ the collection of all vectors of stopping times $\tau = (\tau_1, \tau_2)$ such that

$$\tau_1 \le T \text{ and } \tau_2 - \tau_1 \ge 1 \text{ on } \{\tau_2 \le T\} \text{ a.s.,}$$

where we implicitly assumed that the minimum allowed elapsed time (a.k.a. latency) between two consecutive exercise dates is smaller than (or equal to) the discrete time step used in constructing the scenario tree (e.g., using an appropriate discretization of a continuous stochastic process). If this is not the case, then the constraint $\tau_2 - \tau_1 \ge 1$ should be modified accordingly.

Define the sets

$$E_2 = \{e \mid e \text{ is } (\mathscr{F}_t)_{t=0}^T\text{-adapted}, \sum_{t=0}^T e_t \le 2 \text{ and } e_t \in \{0, 1\} \ P\text{-a.s.}\},$$

$$\tilde{E}_2 = \{e \mid e \text{ is } (\mathscr{F}_t)_{t=0}^T\text{-adapted}, \sum_{t=0}^T e_t \le 2 \text{ and } 0 \le e_t \le 1 \ P\text{-a.s.}\}.$$

The following result follows the ideas of Theorem 4 in [33].

Proposition 2. *If there is no arbitrage in a financial market represented by a non-recombinant tree with two traded instruments (one risky asset which is the underlying, and one riskless asset), T time periods to maturity, the buyer's price for American contingent claim F (call option under zero or positive interest rate, put option with zero interest rate) with two exercise rights can be expressed as*

$$\max_{\tau \in \mathscr{T}^2(T)} \min_{Q \in \tilde{\mathscr{Q}}} \mathbb{E}^Q[F_\tau] = \min_{Q \in \tilde{\mathscr{Q}}} \max_{\tau \in \mathscr{T}^2(T)} \mathbb{E}^q[F_\tau]. \tag{6.1}$$

Proof. If we set e fixed in AP1 and maximize with respect to θ, we have a contingent claim with payoffs $F_t e_t$ for $t = 0, 1, \ldots, T$. Then, for the buyer's price of this claim, we have

$$\min_{Q\in\tilde{\mathscr{Q}}} \mathbb{E}^Q[\sum_{t=0}^{T} F_t e_t].$$

Then, maximizing with respect to e, for the buyer's price of the American claim with two exercise rights we have

$$\max_{e\in E_2} \min_{Q\in\tilde{\mathscr{Q}}} \mathbb{E}^Q[\sum_{t=0}^{T} F_t e_t].$$

The correspondence between multiple stopping times in $\mathscr{T}^2(T)$ and the vectors $e \in E_2$ implies that the buyer's price for the American claim with two exercise rights can be expressed as the left hand side of Eq. (6.1) since maximization over $\mathscr{T}^2(T)$ is equivalent to maximization over E_2 after making the appropriate change in the objective function. By Proposition 1, instead of the last expression we can use

$$\max_{e\in\tilde{E}_2} \min_{Q\in\tilde{\mathscr{Q}}} \mathbb{E}^Q[\sum_{t=0}^{T} F_t e_t]. \tag{6.2}$$

Since $\tilde{E}_2$ and $\tilde{\mathscr{Q}}$ are bounded convex sets, by Corollary 37.6.1 of [35] we can change the order of max and min without changing the value. Then, for each fixed $Q \in \tilde{\mathscr{Q}}$, the objective in (6.2) is linear in e. So the maximum over $\tilde{E}_2$ is attained at an extreme point of $\tilde{E}_2$. We know that the extreme points of $\tilde{E}_2$ are the elements of the set E_2 since $\tilde{E}_2$ is an integral polytope. Thus, we reach the expression on the right hand side in Eq. (6.1). □

6.5 Conclusions

In this paper we have dealt with the pricing of American options with multiple exercise rights in a financial market composed of a risky stock following a non-recombinant tree process and a risk free asset. We established that it is optimal to delay exercise until the last h periods. The result also implies that the LP relaxation of the associated mixed-integer programming formulation to find a no-arbitrage price and hedging policy has an integral solution. Hence, the lower hedging price can be obtained by solving a linear programming problem.

An open problem remains to confirm or refute the claim (made after numerical experimentation) that the LP relaxation continues to be exact in the case of a put option in the presence of a non-zero interest rate.

Acknowledgements Supported in part by a grant from the University of L'Aquila, as part of Research Grant *Reti per la conoscenza e l'orientamento tecnico-scientifico per lo sviluppo della competitività* (RE.C.O.TE.S.S.C.) POR 2007–2013-Action 4, funded by Regione Abruzzo and the European Social Fund 2007–2013. Revised twice, November and December 2013. The paper benefited from the comments of two anonymous reviewers.

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