Chapter 6
Optical Implementation of Linear Canonical Transforms

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Abstract We consider optical implementation of arbitrary one-dimensional and two-dimensional linear canonical and fractional Fourier transforms using lenses and sections of free space. We discuss canonical decompositions, which are generalizations of common Fourier transforming setups. We also look at the implementation of linear canonical transforms based on phase-space rotators.

6.1 Introduction

In this chapter we consider the problem of designing systems for optically implementing linear canonical transforms (LCTs) and fractional Fourier transforms (FRTs). It is well known that an optical Fourier transformer can be realized by a section of free space followed by a lens followed by another section of free space, and also by a lens followed by a section of free space followed by another lens. Another approach is to use a section of quadratic graded-index media. That these approaches can also be used to implement FRTs has been realized in the nineties. One-dimensional systems have been dealt with in [1, 8, 9, 12, 15, 17, 18, 20, 24, 25, 28, 29] and two-dimensional systems have been dealt with in [9, 11, 15, 19, 23, 25, 29, 34, 36], among others. For an overview of the optical implementation of the FRT, see [27].

LCTs can be interpreted as scaled FRTs with additional phase terms. Thus, in principle, if we have an optical FRT system, we can obtain an LCT system with

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some modifications, although handling the scale and phase may not always be convenient. On the other hand, since FRTs are special cases of LCTs, knowing how to realize a desired LCT means we can also realize any FRT easily.

While the design of one-dimensional systems is relatively straightforward, two-dimensional systems bring additional challenges, mostly arising from the fact that the parameters in the two dimensions can be different and this brings a number of constraints with it. We will deal with these challenges and show how all two-dimensional LCTs can be realized [36].

6.2 FRTs and LCTs

Two-dimensional LCT can be defined as:

$$f_o(\mathbf{r}_o) = \mathscr{L}(\mathbf{T})f_i(\mathbf{r}_i) = \int h(\mathbf{r}_o;\mathbf{r}_i)f_i(\mathbf{r}_i)\,d\mathbf{r}_i,$$

$$h(\mathbf{r}_o;\mathbf{r}_i) = (\det\, i^{-1}\mathbf{L}_{io})^{1/2}\exp\left[i\pi(\mathbf{r}_o^{\mathrm{t}}\mathbf{L}_{oo}\mathbf{r}_o - 2\mathbf{r}_i^{\mathrm{t}}\mathbf{L}_{io}\mathbf{r}_o + \mathbf{r}_i^{\mathrm{t}}\mathbf{L}_{ii}\mathbf{r}_i)\right], \qquad (6.1)$$

where we define the column vector $\mathbf{r}$ as $\mathbf{r} = [x, y]^{\mathrm{t}}$. $\mathbf{L}_{ii}$ and $\mathbf{L}_{oo}$ are symmetric 2×2 matrices and $\mathbf{L}_{io}$ is a non-singular 2×2 matrix given by:

$$\mathbf{L}_{ii} \equiv \begin{bmatrix} \ell_{iix} & 0 \\ 0 & \ell_{iiy} \end{bmatrix}, \quad \mathbf{L}_{io} \equiv \begin{bmatrix} \ell_{iox} & 0 \\ 0 & \ell_{ioy} \end{bmatrix}, \quad \mathbf{L}_{oo} \equiv \begin{bmatrix} \ell_{oox} & 0 \\ 0 & \ell_{ooy} \end{bmatrix}, \qquad (6.2)$$

where $\ell_{oox}, \ell_{iox}, \ell_{iix}$ and $\ell_{ooy}, \ell_{ioy}, \ell_{iiy}$ are real constants. FRTs, Fresnel transforms, chirp multiplication, and scaling operations are widely used in optics to analyze systems composed of sections of free space and thin lenses. These linear integral transforms belong to the class of LCTs. Any LCT is completely specified by its parameters.

Alternatively, LCTs can be specified by using a transformation matrix:

$$f_o(\mathbf{r}_o) = \mathscr{L}(\mathbf{T})f_i(\mathbf{r}_i) = (\det\, i\mathbf{B})^{-1/2} \times \int \exp\left[i\pi(\mathbf{r}_o^{\mathrm{t}}\mathbf{D}\mathbf{B}^{-1}\mathbf{r}_o - 2\mathbf{r}_i^{\mathrm{t}}\mathbf{B}^{-1}\mathbf{r}_o + \mathbf{r}_i^{\mathrm{t}}\mathbf{B}^{-1}\mathbf{A}\mathbf{r}_i)\right] f_i(\mathbf{r}_i)\,d\mathbf{r}_i\,, \qquad (6.3)$$

The transformation matrix of such a system specified by the parameters $\ell_{oox}, \ell_{iox}, \ell_{iix}$ and $\ell_{ooy}, \ell_{ioy}, \ell_{iiy}$ is

$$\mathbf{T} \equiv \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \equiv \begin{bmatrix} A_x & 0 & B_x & 0 \\ 0 & A_y & 0 & B_y \\ C_x & 0 & D_x & 0 \\ 0 & C_y & 0 & D_y \end{bmatrix}$$
$$\equiv \begin{bmatrix} \ell_{iix}/\ell_{iox} & 0 & 1/\ell_{iox} & 0 \\ 0 & \ell_{iiy}/\ell_{ioy} & 0 & 1/\ell_{ioy} \\ -\ell_{iox} + \ell_{oox}\ell_{iix}/\ell_{iox} & 0 & \ell_{oox}/\ell_{iox} & 0 \\ 0 & -\ell_{ioy} + \ell_{ooy}\ell_{iiy}/\ell_{ioy} & 0 & \ell_{ooy}/\ell_{ioy} \end{bmatrix}.$$

with $A_xD_x - B_xC_x = 1$ and $A_yD_y - B_yC_y = 1$ [5, 42].

Propagation in free-space (or a homogeneous medium) and through thin lenses are special forms of LCTs. The transformation matrix for free-space propagation over a distance z and with constant refractive index n can be expressed as

$$\mathbf{T}_{\mathscr{S}}(z) = \begin{bmatrix} 1 & 0 & \frac{\lambda z}{n} & 0 \\ 0 & 1 & 0 & \frac{\lambda z}{n} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{6.4}$$

Similarly, the matrix for a cylindrical lens with focal length f_x along the x direction is

$$\mathbf{T}_{\mathscr{Q}_x}(f_x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-1}{\lambda f_x} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{6.5}$$

and the matrix for a cylindrical lens with focal length f_y along the y direction is

$$\mathbf{T}_{\mathscr{Q}_y}(f_y) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{-1}{\lambda f_y} & 0 & 1 \end{bmatrix}. \tag{6.6}$$

More general anamorphic lenses may be represented by a matrix of the form:

$$\mathbf{T}_{\mathscr{Q}_{xy}}(f_x, f_y, f_{xy}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-1}{\lambda f_x} & \frac{-1}{2\lambda f_{xy}} & 1 & 0 \\ \frac{-1}{2\lambda f_{xy}} & \frac{-1}{\lambda f_y} & 0 & 1 \end{bmatrix}. \tag{6.7}$$

The transformation matrix approach has several advantages. First of all, if several systems are cascaded, the overall system matrix can be found by multiplying the corresponding transformation matrices. Second, the transformation matrix corresponds to the ray-matrix in optics [37]. Third, the effect of the system on the Wigner distribution of the input function can be expressed in terms of this transformation matrix. This topic is extensively discussed in [3–7].

The 2D FRT also belongs to the family of LCTs:

$$\begin{aligned} f_o(\mathbf{r}_o) &= \mathscr{F}(\gamma_x, \gamma_y) f_i(\mathbf{r}_i) \\ &= \int A_{\gamma_\mathbf{r}} \exp[i\pi(\mathbf{r}_o^t \mathbf{C_t} \mathbf{r}_o - 2\mathbf{r}_o^t \mathbf{C_s} \mathbf{r}_i + \mathbf{r}_i^t \mathbf{C_t} \mathbf{r}_i)] f_i(\mathbf{r}_i)\, d\mathbf{r}_i, \end{aligned} \tag{6.8}$$

where

$$\mathbf{C_t} = \begin{bmatrix} \cot\gamma_x & 0 \\ 0 & \cot\gamma_y \end{bmatrix}, \qquad \mathbf{C_s} = \begin{bmatrix} \csc\gamma_x & 0 \\ 0 & \csc\gamma_y \end{bmatrix},$$

$$A_{\gamma_\mathbf{r}} = A_{\gamma_x} A_{\gamma_y}, \quad A_{\gamma_x} = \frac{e^{-i(\pi\hat{\gamma}_x/4 - \gamma_x/2)}}{\sqrt{|\sin\gamma_x|}}, \quad A_{\gamma_y} = \frac{e^{-i(\pi\hat{\gamma}_y/4 - \gamma_y/2)}}{\sqrt{|\sin\gamma_y|}}$$

with $\hat{\gamma}_x = \operatorname{sgn}(\gamma_x)$, $\hat{\gamma}_y = \operatorname{sgn}(\gamma_y)$. γ_x and γ_y are rotational angles of the FRT in the two dimensions, which are related to the fractional orders a_x and a_y through $\gamma_x = a_x\pi/2$ and $\gamma_y = a_y\pi/2$.

The output of a fairly broad class of optical systems can be expressed as the FRT of the input [27]. This is a generalization of the fact that in certain special planes one observes the ordinary Fourier transform. However, when we are dealing with FRTs, the choice of scale and dimensions must always be noted. To be able to handle the scales explicitly, we will modify the definition of the FRT by introducing input and output scale parameters. Also allowing for additional phase factors that may occur at the output, the kernel can be expressed as

$$\begin{aligned} K_{\gamma_x,\gamma_y}(x, y; x', y') = \ & A_{\gamma_x} \exp[i\pi x^2 p_x] \\ & \times \exp\left[i\pi\left(\frac{x^2}{s_2^2}\cot\gamma_x - \frac{2xx'}{s_1 s_2}\csc\gamma_x + \frac{x'^2}{s_1^2}\cot\gamma_x\right)\right] \\ & \times A_{\gamma_y} \exp[i\pi y^2 p_y] \\ & \times \exp\left[i\pi\left(\frac{y^2}{s_2^2}\cot\gamma_y - \frac{2yy'}{s_1 s_2}\csc\gamma_y + \frac{y'^2}{s_1^2}\cot\gamma_y\right)\right]. \end{aligned} \tag{6.9}$$

In this definition, s_1 stands for the input scale parameter, s_2 stands for the output scale parameter, and p_x and p_y are the parameters of the quadratic phase factors. The transformation matrix corresponding to this kernel can be found as

$$\mathbf{T} \equiv \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}, \tag{6.10}$$

where

$$\mathbf{A} = \begin{bmatrix} \frac{s_2}{s_1}\cos\gamma_x & 0 \\ 0 & \frac{s_2}{s_1}\cos\gamma_y \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \frac{1}{s_1 s_2}[p_x\cos\gamma_x - \sin\gamma_x] & 0 \\ 0 & \frac{1}{s_1 s_2}[p_y\cos\gamma_y - \sin\gamma_y] \end{bmatrix}, \tag{6.11}$$

$$\mathbf{B} = \begin{bmatrix} s_1 s_2\sin\gamma_x & 0 \\ 0 & s_1 s_2\sin\gamma_y \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} \frac{s_1}{s_2}\sin\gamma_x(p_x + \cot\gamma_x) & 0 \\ 0 & \frac{s_1}{s_2}\sin\gamma_y(p_y + \cot\gamma_y) \end{bmatrix}. \tag{6.12}$$

It can be deduced from the above equation that any quadratic-phase system can be implemented by appending lenses at the input and output planes of a fractional Fourier transformer [22, 25, 27].

6.3 Canonical Decompositions, Anamorphic Sections of Free Space, and Optical Implementation of LCTs

One way of designing optical implementations of LCTs is to employ the matrix formulation given in (6.3). The LCT matrix can be decomposed into matrices that corresponds to more elementary operations such as free-space propagation, thin lenses, etc.

6.3.1 One-Dimensional Systems

We first discuss one-dimensional systems, presenting two decompositions that reduce to familiar optical arrangements for the special case of the Fourier transform.

Canonical Decomposition Type-1

The LCT system matrix **T** can be decomposed as

$$\mathbf{T} = \mathbf{T}_{\mathscr{S}}(z_2)\,\mathbf{T}_{\mathscr{Q}_x}(f)\,\mathbf{T}_{\mathscr{S}}(z_1). \tag{6.13}$$

which corresponds to a section of free space of length z_1, followed by a thin lens of focal length f, followed by another section of free space of length z_2, as shown in Fig. 6.1.

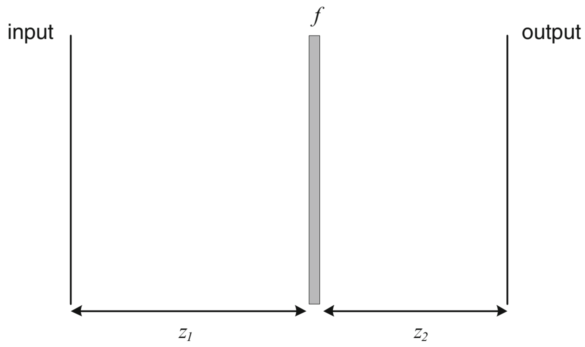

Fig. 6.1 Type-1 system which realizes arbitrary one-dimensional linear canonical transforms [36]

Both the optical system in Fig. 6.1 and the LCT have three parameters. Thus, it is possible to find the system parameters uniquely by solving the above equations. Doing so, the equations for z_1, z_2 and f in terms of ℓ_{oo}, ℓ_{io}, ℓ_{ii} are found as

$$z_1 = \frac{\ell_{io} - \ell_{oo}}{\lambda({\ell_{io}}^2 - \ell_{ii}\ell_{oo})}, \quad z_2 = \frac{\ell_{io} - \ell_{ii}}{\lambda({\ell_{io}}^2 - \ell_{ii}\ell_{oo})}, \quad f = \frac{\ell_{io}}{\lambda({\ell_{io}}^2 - \ell_{ii}\ell_{oo})}. \tag{6.14}$$

Since FRTs are a special case of LCTs, it is possible to implement one-dimensional FRT of any desired order by using this optical setup. The scale parameters s_1 and s_2 may be specified by the designer and the additional phase factors p_x and p_y may be made equal to zero. Letting $\ell_{oo} = \cot\gamma/s_2^2$, $\ell_{ii} = \cot\gamma/s_1^2$ and $\ell_{io} = \csc\gamma/s_1 s_2$, one recovers Lohmann's type-1 fractional Fourier transforming system [15]. In this case, the system parameters are found as

$$z_1 = \frac{(s_1 s_2 - s_1^2\cos\gamma)}{\lambda\sin\gamma}, \qquad z_2 = \frac{(s_1 s_2 - s_2^2\cos\gamma)}{\lambda\sin\gamma}, \qquad f = \frac{s_1 s_2}{\lambda\sin\gamma}. \tag{6.15}$$

Since the additional phase factors are set to zero, they do not appear in the equations. However, if one wishes to set p_x and p_y to a value other than zero, it is again possible by setting $\ell_{oo} = p_x\cot\gamma/s_2^2$ and substituting it in Eq. (6.14).

Canonical Decomposition Type-2

In this case, instead of one lens and two sections of free space, we have two lenses separated by a single section of free space, as shown in Fig. 6.2. Again, the parameters z, f_1 and f_2 can be solved similar to that for the Type-1 decomposition:

$$z = \frac{1}{\lambda\ell_{io}}, \quad f_1 = \frac{1}{\lambda(\ell_{io} - \ell_{ii})}, \quad f_2 = \frac{1}{\lambda(\ell_{io} - \ell_{oo})}. \tag{6.16}$$

Fig. 6.2 Type-2 system which realizes arbitrary one-dimensional linear canonical transforms [36]

If $\ell_{oo} = \cot\gamma/s_2^2$, $\ell_{ii} = \cot\gamma/s_1^2$ and $\ell_{io} = \csc\gamma/s_1 s_2$ are substituted in these equations, the parameters required to obtain a FRT can be found. The designer can again specify the scale parameters and zero phase factor at the output to find:

$$z = \frac{s_1 s_2 \sin\gamma}{\lambda}, \quad f_1 = \frac{s_1^2 s_2 \sin\gamma}{s_1 - s_2\cos\gamma}, \quad f_2 = \frac{s_1 s_2^2 \sin\gamma}{s_2 - s_1\cos\gamma}. \tag{6.17}$$

Equations (6.14) and (6.16) give the expressions for the system parameters of type-1 and type-2 canonical systems. But for some values of ℓ_{oo}, ℓ_{io}, ℓ_{ii}, the lengths of the free space sections required may turn out to be negative, which is not physically realizable. This constraint will restrict the range of LCTs that can be realized with the suggested setups. However, in Sect. 6.3.3, this constraint is removed by employing an optical setup that simulates anamorphic and negative valued sections of free space. This system is designed in such a way that its effect is equivalent to propagation in free space with different (and possibly negative) distances along the two dimensions.

6.3.2 Two-Dimensional Systems

Now we turn our attention to two-dimensional systems. We first present an elementary result which allows us to analyze two-dimensional systems as two one-dimensional systems, which makes the analysis of two-dimensional systems remarkably easier. We write the output of the system in terms of its input as follows:

$$f_o(\mathbf{r}_o) = \int h(\mathbf{r}_o, \mathbf{r}_i) f_i(\mathbf{r}_i)\, d\mathbf{r}_i.$$

If the kernel $h(\mathbf{r}_o, \mathbf{r}_i)$ is separable, that is, $h(\mathbf{r}_o, \mathbf{r}_i) = h_x(x_o, x_i)\, h_y(y_o, y_i)$, then the response in the x direction is the result of the one-dimensional transform

$$f_x(x_o, y_i) = \int h_x(x_o, x_i) f(x_i, y_i)\, dx_i, \tag{6.18}$$

and similar in the y direction. Moreover if the function is also separable, that is, if $f(\mathbf{r}) = f_x(x) f_y(y)$, the overall response of the system is

$$f_o(\mathbf{r}) = f_{o_x}(x) f_{o_y}(y),$$

where

$$f_{o_x}(x) = \int h_x(x, x_i) f_{i_x}(x_i)\, dx_i,$$

$$f_{o_y}(y) = \int h_y(y, y_i) f_{i_y}(y_i)\, dy_i.$$

This result has a nice interpretation in optics which makes the analysis of two-dimensional systems easier. For example, in order to design an optical setup that realizes imaging in the x direction and Fourier transforming in the y direction, one can design two one-dimensional systems that realize the given transformations. When these two systems are merged, the overall effect of the system is imaging in the x direction and Fourier transforming in the y direction. Similarly, if we have a system that realizes a FRT with rotational angle γ_x in the x direction and another system which realizes a FRT with rotational angle γ_y in the y direction, then these two optical setups will together implement a two-dimensional FRT with the rotational angles γ_x and γ_y. So the problem of designing a two-dimensional fractional Fourier transformer reduces to the problem of designing two one-dimensional fractional Fourier transformers.

Canonical Decomposition Type-1

According to the above result, the x and y directions can be considered independent of each other, since the kernel given in Eq. (6.1) or Eq. (6.3) is separable. Hence if two optical setups realizing one-dimensional LCTs are put together, one can implement the desired two-dimensional FRT. The suggested optical system is shown in Fig. 6.3 and employs the following parameters:

$$z_{1x} = \frac{\ell_{iox} - \ell_{oox}}{\lambda(\ell_{io_x}^2 - \ell_{iix}\ell_{oox})}, \qquad z_{2x} = \frac{\ell_{iox} - \ell_{iix}}{\lambda(\ell_{io_x}^2 - \ell_{iix}\ell_{oox})}, \qquad f_x = \frac{\ell_{iox}}{\lambda(\ell_{io_x}^2 - \ell_{iix}\ell_{oox})}, \tag{6.19}$$

$$z_{1y} = \frac{\ell_{ioy} - \ell_{ooy}}{\lambda(\ell_{io_y}^2 - \ell_{iiy}\ell_{ooy})}, \qquad z_{2y} = \frac{\ell_{ioy} - \ell_{iiy}}{\lambda(\ell_{io_y}^2 - \ell_{iiy}\ell_{ooy})}, \qquad f_y = \frac{\ell_{ioy}}{\lambda(\ell_{io_y}^2 - \ell_{iiy}\ell_{ooy})}. \tag{6.20}$$

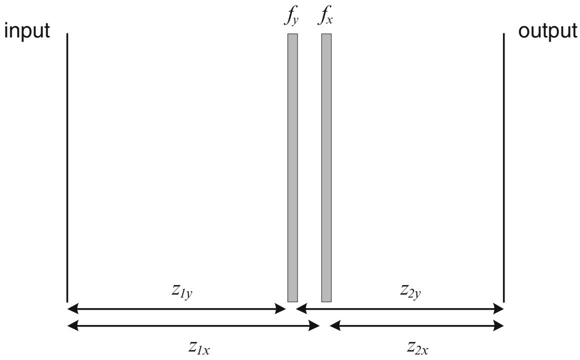

Fig. 6.3 Type-1 system that realizes arbitrary two-dimensional linear canonical transforms [36]

Arbitrary two-dimensional fractional Fourier transforming systems can be obtained as a special case by using:

$$\ell_{oox} = \cot\gamma_x/s_2^2, \qquad \ell_{iix} = \cot\gamma_x/s_1^2, \qquad \ell_{iox} = \csc\gamma_x/s_1s_2, \tag{6.21}$$

$$\ell_{ooy} = \cot\gamma_y/s_2^2, \qquad \ell_{iiy} = \cot\gamma_y/s_1^2, \qquad \ell_{ioy} = \csc\gamma_y/s_1s_2. \tag{6.22}$$

When these equations are substituted into (6.19) and (6.20), the parameters of the fractional Fourier transforming optical system can be found.

We saw that the derivations of the required system parameters can be carried out by treating x and y independently. However, $z_{1x} + z_{2x} = z_x = z_{1y} + z_{2y} = z_y$ should always be satisfied so that the actions in the x and y dimensions meet at a single output plane. Another constraint that needs to be satisfied is the positivity of the lengths of the free space sections. z_{1x},z_{1y},z_{2x},z_{2y} should always be positive. These two constraints restrict the set of LCTs that can be implemented. As before, this restriction can be dealt with by simulating anamorphic sections of free space which provides us a propagation distance of z_x in the x direction and a distance of z_y in the y direction where z_x and z_y may take negative values. By removing the restriction that the propagation distance in the two dimensions has to be equal and positive, all LCTs can be realized. This problem is solved in Sect. 6.3.3.

Canonical Decomposition Type-2

Two type-2 systems can also realize arbitrary two-dimensional LCTs, by using the parameters

$$z_x = \frac{1}{\lambda\ell_{iox}}, \qquad f_{1x} = \frac{1}{\lambda(\ell_{iox} - \ell_{iix})}, \qquad f_{2x} = \frac{1}{\lambda(\ell_{iox} - \ell_{oox})}, \tag{6.23}$$

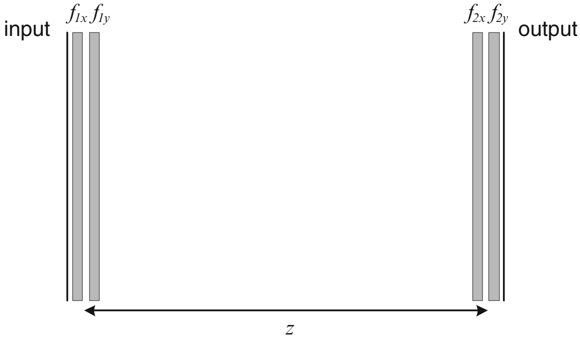

Fig. 6.4 Type-2 system that realizes arbitrary two-dimensional linear canonical transforms [36]

$$z_y = \frac{1}{\lambda \ell_{ioy}}, \quad f_{1y} = \frac{1}{\lambda(\ell_{ioy} - \ell_{iiy})}, \quad f_{2y} = \frac{1}{\lambda(\ell_{ioy} - \ell_{ooy})}. \tag{6.24}$$

As before, if Eqs. (6.21) and (6.22) are substituted in (6.23) and (6.24), the design parameters for the FRT can be obtained.

In the optical setup in Fig. 6.4, we have the constraint $z_x = z_y = z$, which is even more restrictive than with type-1 systems. Again z_x and z_y cannot be negative. In order to overcome these difficulties, in the following section, we show how to simulate anamorphic sections of free space with physically realizable components.

6.3.3 Simulation of Anamorphic Sections of Free Space

While designing optical setups that implement one-dimensional LCTs, we treated the lengths of the sections of free space as free parameters. But some LCTs specified by the parameters $\ell_{oo}, \ell_{ii}, \ell_{io}$, turned out to require the use of free space sections with negative length. This problem is again encountered in the optical setups realizing two-dimensional LCTs. Besides, two-dimensional optical systems may require different propagation distances in the x and y directions. In order to implement all possible one-dimensional and two-dimensional LCTs, we will design a physically realizable optical system simulating the required, but physically unrealizable free space sections.

The optical system in Fig. 6.5 is composed of a Fourier block, an anamorphic lens and an inverse Fourier block. It can simulate two-dimensional anamorphic sections of free space with propagation distance z_x in the x direction and z_y in the y direction. When the analysis of the system in Fig. 6.5 is carried out, the relation between the input light distribution $f_i(x, y)$ and the output light distribution $f_o(x, y)$ is found as

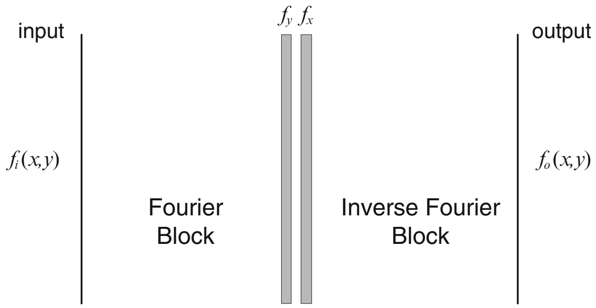

Fig. 6.5 Optical system that simulates anamorphic free space propagation [36]

$$f_o(x,y) = C \iint \exp[i\pi(x-x_i)^2/\lambda z_x + (y-y_i)^2/\lambda z_y] f_i(x_i,y_i)\,dx_i\,dy_i, \tag{6.25}$$

where

$$z_x = \frac{s^4}{\lambda^2 f_x}, \qquad z_y = \frac{s^4}{\lambda^2 f_y}. \tag{6.26}$$

and where s is the scale of the Fourier and inverse Fourier blocks. f_x and f_y can take any real value including negative ones. Thus it is possible to obtain any combination of z_x and z_y by using the optical setup in Fig. 6.5. The anamorphic lens which is used to control z_x and z_y may be composed of two orthogonally situated cylindrical thin lenses with different focal lengths. The Fourier block and inverse Fourier block are 2-f systems with a spherical lens between two sections of free space. Thus, simulating an anamorphic section of free space requires 2 cylindrical and two spherical lenses.

The system in Fig. 6.5 can also be adapted for the one-dimensional case, allowing us to simulate propagation with negative distances. When the required free space sections in the type-1 and type-2 implementations are realized by the optical setup in Fig. 6.5, the optical implementation of all separable LCTs can be realized.

Specializing to the FRT, it is possible to implement all combinations of orders if we can replace the free space sections with sections of anamorphic free space, if need be. All combinations of orders a_x and a_y can be implemented with full control on the scale parameters s_1, s_2 and the phase factors p_x, p_y, the latter which we can set to zero if desired.

6.4 Iwasawa Decomposition, Phase-Space Rotators, and Optical Implementation of LCTs

The modified Iwasawa decomposition [38, 41, 43] states that any ray transformation matrix $\mathbf{T}$ can be written as the product

$$\mathbf{T} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{G} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ -\mathbf{Y} & \mathbf{X} \end{bmatrix} = \mathbf{T}_L \mathbf{T}_S \mathbf{T}_O, \tag{6.27}$$

where

$$\begin{aligned} \mathbf{G} &= -\left(\mathbf{C}\mathbf{A}^t + \mathbf{D}\mathbf{B}^t\right)\left(\mathbf{A}\mathbf{A}^t + \mathbf{B}\mathbf{B}^t\right)^{-1} &&= \mathbf{G}^t, \\ \mathbf{S} &= \left(\mathbf{A}\mathbf{A}^t + \mathbf{B}\mathbf{B}^t\right)^{1/2} &&= \mathbf{S}^t, \\ \mathbf{X} + i\mathbf{Y} &= \left(\mathbf{A}\mathbf{A}^t + \mathbf{B}\mathbf{B}^t\right)^{-1/2}\left(\mathbf{A} + i\mathbf{B}\right) &&= \left(\mathbf{X}^t - i\mathbf{Y}^t\right)^{-1}. \end{aligned} \tag{6.28}$$

The first matrix $\mathbf{T}_L$ corresponds to an anamorphic quadratic-phase modulation, which can be realized with a generalized lens. $\mathbf{T}_S$ is a scaling operation, which corresponds to optical magnification or demagnification. The last one, $\mathbf{T}_O$, is an ortho-symplectic matrix (both orthogonal $\left(\mathbf{T}_O^t = \mathbf{T}_O^{-1}\right)$ and symplectic) [39, 40, 43]. The key to implementing an arbitrary LCT by using the Iwasawa decomposition above is the ortho-symplectic matrix, which corresponds to an optical phase-space rotator. If we know how to realize optical phase-space rotators, we can implement any desired LCT.

The design of an arbitrary phase-space rotator is significantly simplified by using the FRT. Indeed, any phase-space rotator can be written as an FRT, $\mathscr{F}(\gamma_x, \gamma_y)$, embedded between two ordinary image rotators: $\mathscr{R}(\beta)\,\mathscr{F}(\gamma_x, \gamma_y)\,\mathscr{R}(\alpha)$ [30]. Thus, ultimately, the design of arbitrary LCTs boils down to our ability to design arbitrary FRTs.

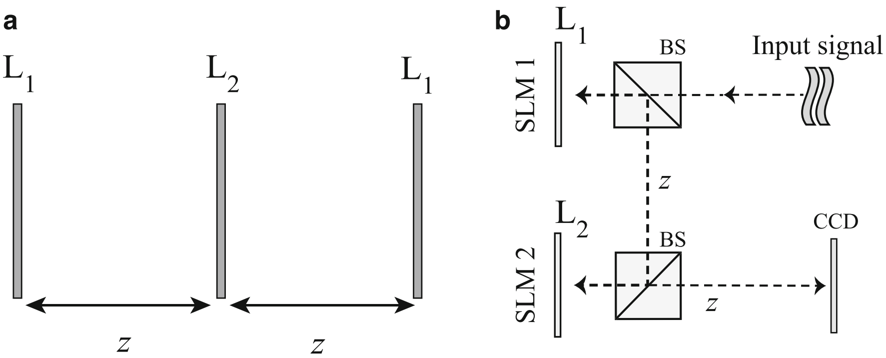

Fig. 6.6 (**a**) Optical system for the FRT using three generalized lenses separated by distance z. (**b**) Experimental implementation of a programmable optical FRT setup: two reflective phase-only SLMs are used to realize the generalized lenses $\mathbf{L}_1$ and $\mathbf{L}_2$. The output signal is registered by a CCD camera in real time [33]

Here we consider a flexible optical setup for the FRT that is suitable for use in many applications. In this setup, a change of the fractional angle γ_x or γ_y does not lead to an additional scaling and/or phase factor, that occurs in other proposed systems [16, 21, 35]. Specifically, this FRT system consists of three generalized lenses with a fixed distance z between them, as shown in Fig. 6.6a. The first and the last lens are identical ($\mathbf{L}_3 = \mathbf{L}_1$). Each generalized lens $\mathbf{L}_j$ ($j = 1, 2$) is an assembled set of two crossed cylindrical lenses, active in the two orthogonal directions x and y, with phase modulation functions $\exp[-\mathrm{i}\pi g_x^{(j)} x^2/\lambda]$ and $\exp[-\mathrm{i}\pi g_y^{(j)} y^2/\lambda]$, respectively, where we still have the possibility to choose a proper normalization parameter s. The lens powers $g_x^{(j)}$ and $g_y^{(j)}$ are given by [31]

$$\begin{aligned} g_x^{(1)}\, z &= 1 - (\lambda z/s)\ \cot(\gamma_x/2), \\ g_y^{(1)}\, z &= 1 - (\lambda z/s)\ \cot(\gamma_y/2), \\ g_x^{(2)}\, z &= 2 - (s/\lambda z)\ \sin\gamma_x, \\ g_y^{(2)}\, z &= 2 - (s/\lambda z)\ \sin\gamma_y. \end{aligned} \tag{6.29}$$

The multiplication of the matrices corresponding to the constituent optical elements yields the FRT transformation matrix [31]. The cylindrical lenses are oriented such that $\phi_1^{(1,2)} = 0$ and $\phi_2^{(1,2)} = \pi/2$, where the angles are measured in the counterclockwise direction and $\phi = 0$ corresponds to the y axis. Using the matrix formalism it is easy to prove that the matrix of the composite system corresponds to the separable phase space rotator and therefore the relation of the complex field amplitudes at the input $f_i(\mathbf{r}_i)$ and output $f_o(\mathbf{r}_o) = \mathscr{F}(\gamma_x, \gamma_y) f_i(\mathbf{r}_i)$ planes are given by the separable FRT.

If we choose the normalization parameter as $s = 2\lambda z$, the lens powers are given by $g_x^{(1)} z = 1 - \cot(\gamma_x/2)/2$, $g_y^{(1)} z = 1 - \cot(\gamma_y/2)/2$, $g_x^{(2)} z = 2 - 2\sin\gamma_x$ and $g_y^{(2)} z = 2 - 2\sin\gamma_y$. Although γ_x or γ_y may take any value in the interval $(0, 2\pi)$, we use the interval $[\pi/2, 3\pi/2]$ because it corresponds to convergent lenses. This interval will be sufficient in most applications. Nevertheless, the entire interval $(0, 2\pi)$ can be covered, if necessary, thanks to the relation $F^{\gamma_x+\pi,\gamma_y+\pi}(\mathbf{r}) = F^{\gamma_x,\gamma_y}(-\mathbf{r})$.

The phase-space rotator $\mathscr{R}(-\alpha)\,\mathscr{F}(\gamma_x, \gamma_y)\,\mathscr{R}(\alpha)$ can be easily realized by rotating the above FRT system by an angle α around the optical axis [30]. In other words, the cylindrical lenses are now oriented according to the angles $\phi_1^{(1,2)} = \alpha$ and $\phi_2^{(1,2)} = \alpha + \pi/2$. Thus, the phase modulation function associated with each generalized lens $\mathbf{L}_j$ ($j = 1, 2$) takes the form

$$\begin{aligned} \Psi^{(j)}(x, y) = \exp&\left[-\mathrm{i}\pi \frac{g_x^{(j)}}{\lambda}\ (x\cos\alpha - y\sin\alpha)^2\right] \\ &\times \exp\left[-\mathrm{i}\pi \frac{g_y^{(j)}}{\lambda}\ (y\cos\alpha + x\sin\alpha)^2\right]. \end{aligned} \tag{6.30}$$

This optical configuration permits us to perform various attractive operations. For example, for $\alpha = 0$ we recover the basic FRT setup, whereas for $\alpha = \pi/4$ and $\gamma_x = -\gamma_y = \gamma$, the gyrator operation $\mathscr{R}(-\pi/4)\,\mathscr{F}(\gamma,-\gamma)\,\mathscr{R}(\pi/4)$ is obtained.

One way of implementing a generalized lens is to use a programmable SLM. This type of digital lens implementation allows one to modify the transformation angles α, γ_x, γ_y in real time. The corresponding optical setup is shown in Fig. 6.6b, where two reflective phase-only SLMs are used for the generalized lens implementation. Note that the third generalized lens is not required here because it only modulates the phase of the output beam, which will be recorded as an intensity image by a CCD camera. The feasibility of such a programmable setup has been demonstrated experimentally [33].

We note that for the special case $\gamma_x = -\gamma_y = \gamma$, the corresponding setup can also be built using glass cylindrical lenses (of fixed power) instead of digital lenses. This subclass of phase-space rotators include the gyrator and the antisymmetric FRT. In such a case, the generalized lens is an assembled set of two identical convergent cylindrical lenses, which are in contact with each other. The distance z between the generalized lenses $\mathbf{L}_j$ is fixed and the lens powers are set according to $g_x^{(j)} = j/z$ and $g_y^{(j)} = j/z$. Note that the first and last generalized lens are identical. While the transverse axes of the cylindrical lenses form angles $\phi_1^{(j)} = \varphi^{(j)} + \alpha + \pi/4$ and $\phi_2^{(j)} = -\varphi^{(j)} + \alpha - \pi/4$ with the y axis, note that the two cylindrical lenses cross at an angle $\phi_1^{(j)} - \phi_2^{(j)} = 2\varphi^{(j)} + \pi/2$. The angles $\varphi^{(1,2)}$ follow from $\sin 2\varphi^{(1)} = (\lambda z/s)\cot(\gamma/2)$ and $2\sin 2\varphi^{(2)} = (s/\lambda z)\sin\gamma$, where s is the normalization parameter. Because of the requirement $|\cot(\gamma/2)| \leq 1$, we conclude that the angle interval $\gamma \in [\pi/2, 3\pi/2]$ is covered if $\lambda z/s = 1$. This scheme (with normalization parameter $s = \lambda z$) has been used for the experimental realization of the gyrator (when $\alpha = 0$) reported in [32] and the antisymmetric fractional FT (when $\alpha = \pi/4$) reported in [10].

6.5 Conclusion

We reviewed some methods for optical implementation of one-dimensional and two-dimensional fractional Fourier transforms (FRTs) and linear canonical transforms (LCTs).

The systems we discussed are good for realizing arbitrary LCTs, which are a more general class of transforms than FRT. Thus, they can be specialized to obtain FRTs with desired orders and parameters as well.

We considered two main groups of approaches. The first is based on canonical decompositions and involves anamorphic sections of free space. The second is based on the modified Iwasawa decomposition and involves phase-space rotators.

LCTs represent a fairly general and important class of optical systems. Thus, their optical implementation is of interest for a variety of optical signal and image processing systems. In particular, these systems can be used for optical implementations of filtering in fractional Fourier or LCT domains [2, 13, 14, 26] and for optical mode converters [32, 33].

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