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Orthogonal Polynomials Associated with Equilibrium Measures on $\mathbb{R}$

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Abstract Let K be a non-polar compact subset of $\mathbb{R}$ and μ_K denote the equilibrium measure of K. Furthermore, let $P_n(\cdot;\mu_K)$ be the n-th monic orthogonal polynomial for μ_K. It is shown that $\|P_n(\cdot;\mu_K)\|_{L^2(\mu_K)}$, the Hilbert norm of $P_n(\cdot;\mu_K)$ in $L^2(\mu_K)$, is bounded below by $\mathrm{Cap}(K)^n$ for each $n \in \mathbb{N}$. A sufficient condition is given for $\left(\|P_n(\cdot;\mu_K)\|_{L^2(\mu_K)}/\mathrm{Cap}(K)^n\right)_{n=1}^{\infty}$ to be unbounded. More detailed results are presented for sets which are union of finitely many intervals.

Keywords Equilibrium measure · Widom factors · Orthogonal polynomials · Jacobi matrices

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1 Introduction and results

Let K be an infinite compact subset of $\mathbb{R}$ and let $\|\cdot\|_{L^\infty(K)}$ denote the sup-norm on K. The polynomial $T_{n,K}(x) = x^n + \cdots$ satisfying

$$\|T_{n,K}\|_{L^\infty(K)} = \min\{\|Q_n\|_{L^\infty(K)} : Q_n \text{ monic real polynomial of degree } n\} \tag{1}$$

is called the n-th Chebyshev polynomial on K. We have (see e.g. Corollary 5.5.5 in [16])

$$\lim_{n\to\infty} \|T_{n,K}\|_{L^\infty(K)}^{1/n} = \mathrm{Cap}(K), \tag{2}$$

where $\mathrm{Cap}(\cdot)$ denotes the logarithmic capacity. For a non-polar compact set $K \subset \mathbb{R}$, let

$$M_{n,K} := \|T_{n,K}\|_{L^\infty(K)}/\mathrm{Cap}(K)^n.$$

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Then $M_{n,K} \geq 2$, see [19]. If $K = \cup_{i=1}^{n}[\alpha_i, \beta_i]$ and $-\infty < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 \cdots < \alpha_n < \beta_n < \infty$, then $(M_{n,K})_{n=1}^{\infty}$ is bounded and many results were obtained (see [26, 28, 29, 32]) regarding the limit points of this sequence. It was recently proved that there are Cantor sets for which $(M_{n,K})_{n=1}^{\infty}$ is bounded, see Theorem 1.4 and Remarks just below the theorem in [9]. In the other direction, for each sequence $(c_n)_{n=1}^{\infty}$ of positive real numbers with subexponential growth, there is a Cantor set $K(\gamma)$ such that $M_{n,K(\gamma)} \geq c_n$ for all $n \in \mathbb{N}$, see Theorem 4.4 [12]. We refer the reader to [22] for a general discussion on Chebyshev polynomials and [16, 18] for basic concepts of potential theory.

Throughout the article, by a measure we mean a unit Borel measure with an infinite compact support on $\mathbb{R}$. For such a measure μ, the polynomial $P_n(x; \mu) = x^n + \cdots$ satisfying

$$\| P_n(\cdot; \mu) \|_{L^2(\mu)} = \min\{\| Q_n \|_{L^2(\mu)} : Q_n \text{ monic real polynomial of degree } n\} \tag{3}$$

is called the n-th monic orthogonal polynomial for μ where $\| \cdot \|_{L^2(\mu)}$ is the Hilbert norm in $L^2(\mu)$. Similarly, the polynomial $p_n(x; \mu) := P_n(x; \mu)/\| P_n(\cdot; \mu)\|_{L^2(\mu)}$ is called n-th orthonormal polynomial for μ. If we assume that $P_{-1}(x; \mu) := 0$ and $P_0(x; \mu) := 1$ then the monic orthogonal polynomials obey a three term recurrence relation, that is

$$P_{n+1}(x; \mu) = (x - b_{n+1}) P_n(x; \mu) - a_n^2 P_{n-1}(x; \mu), \qquad n \in \mathbb{N}_0, \tag{4}$$

where $a_n > 0$, $b_n \in \mathbb{R}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We call $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ as recurrence coefficients for μ. We refer only the a_n's in the text. It is elementary to verify that

$$\| P_n(\cdot; \mu)\|_{L^2(\mu)} = a_1 \cdots a_n \tag{5}$$

for each $n \in \mathbb{N}$.

For a measure μ satisfying $\mathrm{Cap}(\mathrm{supp}(\mu)) > 0$, let

$$W_n(\mu) := \| P_n(\cdot; \mu)\|_{L^2(\mu)} / \mathrm{Cap}(\mathrm{supp}(\mu))^n$$

where $\mathrm{supp}(\cdot)$ stands for the support of the measure. By Eq. 3 and using the assumption that μ is a unit measure, we have

$$\| P_n(\cdot; \mu)\|_{L^2(\mu)} \leq \| T_{n,\mathrm{supp}(\mu)} \|_{L^2(\mu)} \leq \| T_{n,\mathrm{supp}(\mu)} \|_{L^\infty(\mathrm{supp}(\mu))} \tag{6}$$

for each $n \in \mathbb{N}$. Thus, by Eq. 2 it follows that $\limsup_{n\to\infty} \| P_n(\cdot; \mu)\|_{L^2(\mu)}^{1/n} \leq \mathrm{Cap}(\mathrm{supp}(\mu))$. A measure μ satisfying $\lim_{n\to\infty} \| P_n(\cdot; \mu)\|_{L^2(\mu)}^{1/n} = \mathrm{Cap}(\mathrm{supp}(\mu))$ is called regular in the sense of Stahl-Totik and we write $\mu \in \mathbf{Reg}$ if μ is regular.

For a non-polar compact subset K of $\mathbb{R}$, let μ_K denote the equilibrium measure of K. It is due to Widom that $\mu_K \in \mathbf{Reg}$, see [31] and also [20, 23, 30]. Hence, $\lim_{n\to\infty} (W_n(\mu_K))^{1/n} = 1$ holds. But the behavior of $(W_n(\mu_K))_{n=1}^{\infty}$ is unknown for many cases and the main aim of this paper is to study the upper and lower bounds of this sequence for general compact sets on $\mathbb{R}$. We remark that by Lemma 1.2.7 in [23] we have $\mathrm{Cap}(\mathrm{supp}(\mu_K)) = \mathrm{Cap}(K)$, and we use these expressions interchangeably.

A non-polar compact set K on $\mathbb{R}$ which is regular with respect to the Dirichlet problem is called a Parreau-Widom set if $\mathrm{PW}(K) := \sum_j g_K(c_j)$ is finite where g_K denotes the Green function with a pole at infinity for $\overline{\mathbb{C}} \setminus K$ and $\{c_j\}_j$ is the set of critical points of g_K. If $K = \cup_{j=1}^{n}[\alpha_j, \beta_j]$ and $-\infty < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 \cdots < \alpha_n < \beta_n < \infty$ then K is a Parreau-Widom set and each gap (β_j, α_{j+1}) contains exactly one critical point c_j and there are no other critical points of g_K. Some Cantor sets are Parreau-Widom, see e.g. [2, 15].

But a Parreau-Widom set is necessarily of positive Lebesgue measure. We refer the reader to [7, 33] for a discussion on Parreau-Widom sets.

Let K be a Parreau-Widom set and μ be a measure with $\text{supp}(\mu) = K$ which is absolutely continuous with respect to Lebesgue measure, that is $d\mu(t) = \mu'(t)\,dt$ on K where μ' is the Radon-Nikodym derivative of μ with respect to the Lebesgue measure restricted to K. Recall that μ satisfies the Szegő condition on K if $\int \log \mu'(t)\,d\mu_K(t) > -\infty$. In this case we write $\mu \in \text{Sz}(K)$. It is known that $\mu_K \in \text{Sz}(K)$, see Proposition 2 and (4.1) in [7]. By [7], this implies that there is an $M > 0$ such that $1/M < W_n(\mu_K) < M$ holds for all $n \in \mathbb{N}$. In the inverse direction, one can find a Cantor set $K(\gamma)$ such that $W_n\left(\mu_{K(\gamma)}\right) \to \infty$ as $n \to \infty$, see [1].

First, we restrict our attention to union of several intervals. Let T_N be a real polynomial of degree N with $N \geq 2$ such that it has N real and simple zeros $x_1 < \cdots < x_n$ and $N-1$ critical points $y_1 < \cdots < y_{n-1}$ with $|T_N(y_i)| \geq 1$ for each $i \in \{1, \ldots, N-1\}$. We call such a polynomial admissible. If $K = T_N^{-1}([-1, 1])$ for an admissible polynomial T_N then K is called a T-set. A T-set is of the form $\cup_{i=1}^n [\alpha_i, \beta_i]$ with $n \leq N$ where N is the degree of the associated admissible polynomial. For applications of T-sets to polynomial inequalities and spectral theory of orthogonal polynomials, we refer the reader to [13, 27] and Chapter 5 in [21]. We have the following characterization for T-sets, see Lemma 2.2 in [25]:

Theorem 1 *Let $K = \cup_{j=1}^n [\alpha_j, \beta_j]$ be a disjoint union of n intervals. Then K is a T-set if and only if $\mu_K([\alpha_j, \beta_j]) \in \mathbb{Q}$. If $K = T_N^{-1}[-1, 1]$ for some admissible polynomial T_N then for each $j \in \{1, \ldots, n\}$ there is an $l \in \mathbb{N}$ such that $\mu_K([\alpha_j, \beta_j]) = l/N$.*

If $K = T_N^{-1}[-1, 1]$ for an admissible polynomial T_N then (see Theorem 9 and Lemma 3 in [11]) since $\mu_K \in \text{Sz}(K)$, there is a sequence $(a_n')_{n=1}^\infty$ with $a_k' = a_{k+N}'$ for each $k \in \mathbb{N}$ such that $a_n - a_n' \to 0$ as $n \to \infty$ where $(a_n)_{n=1}^\infty$ is the sequence of recurrence coefficients in Eq. 4 for μ_K. In this case we call $(a_n')_{n=1}^\infty$ the periodic limit for $(a_n)_{n=1}^\infty$ and $(a_n)_{n=1}^\infty$ asymptotically periodic. Our first theorem is about $(W_n(\mu_K))_{n=1}^\infty$ when K is a T-set.

Theorem 2 *Let $K = T_N^{-1}[-1, 1]$ where T_N is an admissible polynomial with leading coefficient c. Furthermore, let $(a_n)_{n=1}^\infty$ be the sequence of recurence coefficients for μ_K and $(a_n')_{n=1}^\infty$ be the periodic limit of it. Then*

(a) $\liminf_{n\to\infty} W_n(\mu_K) = \sqrt{2}$.
(b) $W_n(\mu_K) \geq 1$ *for each* $n \in \mathbb{N}$.
(c) $\inf_l \dfrac{a_1' \cdots a_l'}{\text{Cap}(K)^l} = \dfrac{a_1' \cdots a_N'}{\text{Cap}(K)^N} = 1.$

An arbitrary compact set K on $\mathbb{R}$ can be approximated in an appropriate way by T-sets, see Section 5.8 in [21] and Section 2.4 in [24]. We rely upon these techniques in order to prove our main result:

Theorem 3 *Let K be a non-polar compact subset of $\mathbb{R}$. Then $W_n(\mu_K) \geq 1$ for all $n \in \mathbb{N}$.*

Remark 1 Theorem 3 can be seen as an analogue of Schiefermayr's Theorem (Theorem 2 in [19]). It is unclear whether 1 on the right side of the inequality in Theorem 3 can be improved. This constant can be at most $\sqrt{2}$ by part (a) of Theorem 2. It suffices to find a bigger lower bound for $W_n(\mu_K)$ in part (b) of Theorem 2 to improve the result.

Note that a weaker version of the above theorem was conjectured in [1]. Regularity of μ_K in the sense of Stahl-Totik follows as a corollary of Theorem 3 since the inequality $\liminf_{n\to\infty} (W_n(\mu_K))^{1/n} \geq 1$ directly follows. On the other hand, regularity of a measure μ in the sense of Stahl-Totik does not even imply that $\limsup_{n\to\infty} W_n(\mu) > 0$, see e.g. Example 1.4 in [20]. Hence, the implications of Theorem 3 are profoundly different than those of $\mu_K \in \mathbf{Reg}$. The following result which gives a sufficient condition for unboundedness of $(W_n(\mu_K))_{n=1}^{\infty}$ is also an immediate corollary of Theorem 3:

Corollary 1 *Let K be a non-polar compact subset of $\mathbb{R}$ and $(a_n)_{n=1}^{\infty}$ be the sequence of recurrence coefficients for μ_K. If $\liminf_{n\to\infty} a_n = 0$ then $(W_n(\mu_K))_{n=1}^{\infty}$ and $\left(M_{n,K}\right)_{n=1}^{\infty}$ are unbounded.*

Corollary 1 cannot be applied to sets having positive measure since in this case we have $\liminf_{n\to\infty} a_n > 0$, see Remark 4.8 in [1]. There are some sets for which the assumptions in Corollary 1 hold, see e.g. [1, 5, 6]. Apart from these particular examples, there is no criterion on an arbitrary set K on $\mathbb{R}$ (except having positive Lebesgue measure) determining if $\liminf_{n\to\infty} a_n = 0$ for μ_K. It would be interesting to calculate $\liminf_{n\to\infty} a_n$ for μ_{K_0} where K_0 is the Cantor ternary set.

To our knowledge, in all known cases when $(W_n(\mu_K))_{n=1}^{\infty}$ is bounded, $\left(M_{n,K}\right)_{n=1}^{\infty}$ is also bounded. Thus, it is plausible to make the following conjecture (see also Conjecture 4.2 in [3]):

Conjecture 1 *Let K be a non-polar compact subset of $\mathbb{R}$. Then $(W_n(\mu_K))_{n=1}^{\infty}$ is bounded if and only if $\left(M_{n,K}\right)_{n=1}^{\infty}$ is bounded.*

In Section 2, we present some aspects of Widom's theory and give proofs for the theorems.

2 Proofs

Let $K = \cup_{j=1}^{p}[\alpha_j, \beta_j]$ be a disjoint union of several intervals, $E_j := [\alpha_j, \beta_j]$ for each $j \in \{1, \ldots, p\}$ and $\{c_j\}_{j=1}^{p-1}$ (for $p = 1$ there are no critical points) be the set of critical points of g_K. Then (see e.g. p. 186 in [14]), we have

$$\mu_K'(t) = \frac{1}{\pi}\frac{|q(t)|}{\sqrt{\prod_{j=1}^{p}|(t-\alpha_j)(t-\beta_j)|}}, \quad t \in K \tag{7}$$

where $q(t) = 1$ if $p = 1$ and $q(t) = \prod_{j=1}^{p-1}(t - c_j)$ if $p > 1$.

Let $\partial g_K/\partial n_+$ and $\partial g_K/\partial n_-$ denote the normal derivatives of g_K in the positive and negative direction respectively. These functions are well defined on K except the end points of the intervals. Moreover by symmetry of K with respect to $\mathbb{R}$, we have $\partial g_K/\partial n_+ = \partial g_K/\partial n_-$, see p. 121 in [18]. Let $\partial g_K/\partial n := \partial g_K/\partial n_+$. Then, $(\partial g_K/\partial n)(t) = \pi\, \mu_K'(t)$, see (5.6.7) in [21]. This is why we can state the functions and theorems in [32] in terms of μ_K instead of $\partial g_K/\partial n$. Similarly, instead of harmonic measure at infinity we use the equilibrium measure, since these two measures are the same, see Theorem 4.3.14 in [16]. The concepts that we describe below can be found in [4, 32] but with somewhat a different terminology.

Let $\mu \in \mathrm{Sz}(K)$ and h be the harmonic function in $\overline{\mathbb{C}} \setminus K$ having boundary values (non-tangential limit exists a.e.) $\log \mu'(t)$. Then following Section 5 and Section 14 of [32], we define the multivalued analytic function R in $\overline{\mathbb{C}} \setminus K$ by $R(z) = \exp(h(z) + i\tilde{h}(z))$ where $\tilde{h}$ is a harmonic conjugate of h and

$$R(\infty) = \exp\left(\int \log \mu'(t) d\mu_K(t)\right).$$

Now, R has no zeros or poles. Moreover, $\log |R(z)|$ is single-valued on $\overline{\mathbb{C}} \setminus K$ and has boundary values $\log \mu'(t)$ on K.

Let F be a multivalued meromorphic function having finitely many zeros and poles in $\overline{\mathbb{C}} \setminus K$ for which $|F(z)|$ is single-valued. Then,

$$\gamma_j(F) := (1/2\pi) \underset{E_j}{\triangle} \arg F,$$

for each $j \in \{1, \dots, p\}$. Here, $\underset{E_j}{\triangle} \arg F$ denotes the increment of the argument of F in going around a positively oriented curve F_j enclosing E_j. The curve is taken so close to E_j that it does not intersect with or enclose any points of E_k with $k \neq j$. A multiple-valued function U in $\overline{\mathbb{C}} \setminus K$ with a single-valued absolute value is of class Γ_γ if $\gamma = (\gamma_1, \dots, \gamma_p) \in [0, 1)^p$ and $\gamma_j(U) = \gamma_j \mod 1$ for each $j \in \{1, \dots, p\}$.

Let $H^2(\overline{\mathbb{C}} \setminus K, \mu', \Gamma_\gamma)$ denote the space of multi-valued analytic functions F from Γ_γ in $\overline{\mathbb{C}} \setminus K$ such that $|F(z)^2 R(z)|$ has a harmonic majorant. Then

$$\nu(\mu', \Gamma_\gamma) := \inf_F \int_E |F(t)|^2 \mu'(t) dt.$$

where $F \in H^2(\overline{\mathbb{C}} \setminus K, \mu', \Gamma_\gamma)$ and $|F(\infty)| = 1$.

For the class associated with $(-n\mu_E(E_1) \mod 1, \dots, -n\mu_E(E_p) \mod 1)$ we use Γ_n.

Before giving the proofs, we state some results from [32] in a unified way. The part (a) is Theorem 12.3, part (c) is Theorem 9.2 (see p. 223 for the explanation of why it is applicable) and part (b) is given in p. 216 in [32].

Theorem 4 *Let $K = \cup_{j=1}^p [\alpha_j, \beta_j]$ be a disjoint union intervals and let $\mu \in \mathrm{Sz}(K)$. Then*

(a) $(W_n(\mu))^2 \sim \nu(\mu', \Gamma_n)$ *where $a_n \sim b_n$ means that $\frac{a_n}{b_n} \to 1$ as $n \to \infty$.*
(b) $(W_n(\mu))^2 \geq \frac{\nu(\mu', \Gamma_n)}{2}$ *for all $n \in \mathbb{N}$.*
(c) *The limit points of $\left((W_n(\mu))^2\right)_{n=1}^\infty$ are bounded below by*

$$2\pi R(\infty) \mathrm{Cap}(K) \exp(-\mathrm{PW}(K)).$$

Proof of Theorem 2 Let $\{\alpha_j\}_j$ and $\{\beta_j\}_j$ be the set of left and right endpoints of the connected components of K respectively so that $\alpha_1 < \beta_1 < \cdots < \alpha_p < \beta_p$. Moreover let $E_j := [\alpha_j, \beta_j]$ for each $j \in \{1, \dots, p\}$ and $\{c_j\}_j$ be the set of critical points of g_K.

(a) First, let us show that $\liminf_{n\to\infty} (W_n(\mu_K))^2 \geq 2$. Since $\mu_K \in \mathrm{Sz}(K)$, Theorem 4 is applicable. We need to compute

$$\log R(\infty) = \int \log \mu_K'(t)\, d\mu_K(t).$$

Using Eq. 7, we can write

$$\log R(\infty) = -\log \pi + D_1 + D_2 + D_3$$

where

$$D_1 = -\frac{1}{2}\sum_{j=1}^{p}\int \log|t-\alpha_j|\,d\mu_K(t),$$

$$D_2 = -\frac{1}{2}\sum_{j=1}^{p}\int \log|t-\beta_j|\,d\mu_K(t),$$

$$D_3 = \sum_{j=1}^{p-1}\int \log|t-c_j|\,d\mu_K(t), \text{ if } p \geq 2$$

and $D_3 = 0$ if $p = 1$.

Since K is regular with respect to the Dirichlet problem, g_K can be extended to $\overline{\mathbb{C}}$ by taking $g_K(z) = 0$ for $z \in K$ so that g_K is continuous everywhere in $\mathbb{C}$. Besides,

$$g_K(z) = -U^{\mu_K}(z) - \log \mathrm{Cap}(K) \tag{8}$$

holds in $\mathbb{C}$ where $U^{\mu_K}(z) = -\int \log|z-t|\,d\mu_K(t)$. See p. 53-54 in [18].

By Eq. 8, for any $z \in K$ we have $\int \log|z-t|\,d\mu_K(t) = \log \mathrm{Cap}(K)$. Hence, $D_1 + D_2 = 2p(-1/2)\log \mathrm{Cap}(K) = -\log(\mathrm{Cap}(K)^p)$.

For $p \geq 2$, $\int \log|t-c_j|\,d\mu_K(t) = g_K(c_j) + \log \mathrm{Cap}(K)$ by Eq. 8. Thus,

$$D_3 = \mathrm{PW}(K) + \log\left(\mathrm{Cap}(K)^{p-1}\right). \tag{9}$$

But since $\mathrm{PW}(K) + \log(\mathrm{Cap}(K)^{p-1}) = 0$ for $p = 1$, Eq. 9 is valid for $p \geq 1$. Therefore,

$$\log R(\infty) = -\log\pi + \mathrm{PW}(K) - \log \mathrm{Cap}(K).$$

Using part (c) of Theorem 4, we have

$$\liminf_{n\to\infty}(W_n(\mu_K))^2 \geq \frac{2\pi\exp(\mathrm{PW}(K))\mathrm{Cap}(K)}{\pi\exp(\mathrm{PW}(K))\mathrm{Cap}(K)} \geq 2.$$

In order to complete the proof, it is enough to show that

$$\liminf_{n\to\infty}(W_n(\mu_K))^2 \leq 2. \tag{10}$$

On $[-1,1]$, we have the formula $p_l(x;\mu_{[-1,1]}) = \sqrt{2}S_l(x)$ where S_l is the l-th Chebyshev polynomial on $[-1,1]$ of the first kind, see (1.89b) in [17]. By Theorem 1 and Theorem 11 in [11] this gives,

$$p_{lN}(x;\mu_K) = p_l\left(T_N(x);\mu_{[-1,1]}\right) = \sqrt{2}S_l(T_N(x)),$$

for each $l \in \mathbb{N}$. The leading coefficient of $p_{lN}(x;\mu_K)$ is $\sqrt{2}\cdot 2^{l-1}\cdot c^l$ or in other words $\|P_{lN}(\cdot;\mu_K)\|_{L^2(\mu_K)} = (\sqrt{2}\cdot 2^{l-1}\cdot c^l)^{-1}$. By (5.2) in [11], $\mathrm{Cap}(K)^{lN} = (2c)^{-l}$ since (see e.g. p. 135 in [16]) $\mathrm{Cap}[-1,1] = 1/2$. Therefore, $W_{lN}(\mu_K) = \sqrt{2}$ for each $l \in \mathbb{N}$ and Eq. 10 holds. This completes the proof of part (a).

(b) By Theorem 1, $(lN+s)\mu_K(E_j) = s\cdot\mu_K(E_j) \mod 1$ for all $l \in \mathbb{N}$, $s \in \{0,\ldots,N-1\}$ and $j \in \{1,\ldots,N\}$. Hence $\Gamma_{lN+s} = \Gamma_s$ where l and s are as above. Therefore, $\left(\nu\left(\mu_K',\Gamma_n\right)\right)_{n=1}^{\infty}$ is a periodic sequence of period N. This implies that

$\inf_{n\in\mathbb{N}} \nu\left(\mu'_K, \Gamma_n\right) = \liminf_{n\to\infty} \nu\left(\mu'_K, \Gamma_n\right)$. By part (a) of Theorem 4 and part (a) of this theorem, we have

$$\liminf_{n\to\infty} \nu\left(\mu'_K, \Gamma_n\right) = \liminf_{n\to\infty} (W_n(\mu_K))^2 = 2. \tag{11}$$

From Eq. 11, it follows that, $\inf_{n\in\mathbb{N}} \nu\left(\mu'_K, \Gamma_n\right) = 2$. By part (b) of Theorem 4, we get $(W_n(\mu_K))^2 \geq 1$ for each $n \in \mathbb{N}$ which gives the desired result.

(c) Equality on the right can be found in the literature, see e.g. (2.23) in [10]. As we see, in the proof of part (b), $(W_n(\mu_K))_{n=1}^{\infty}$ is asymptotically periodic with the periodic limit $\left(\sqrt{\nu\left(\mu'_K, \Gamma_n\right)}\right)_{n=1}^{\infty}$. The periodic limit can be written in the form

$$\left(d\frac{a'_1\cdots a'_n}{\mathrm{Cap}(K)^n}\right)_{n=1}^{\infty},$$

by Corollary 6.7 of [8] where $d \in \mathbb{R}^+$. Since $W_{lN}(\mu_K) = \sqrt{2}$ by the proof of part (a) and $\frac{a'_1\cdots a'_{lN}}{\mathrm{Cap}(K)^{lN}} = 1$ holds for all $l \in \mathbb{N}$, we obtain $d = \sqrt{2}$. Besides,

$$\liminf_{l\to\infty} \sqrt{2}\frac{a'_1\cdots a'_l}{\mathrm{Cap}(K)^l} = \liminf_{l\to\infty} W_l(\mu_K) = \sqrt{2} \tag{12}$$

holds by part (a). Using periodicity and Eq. 12, we have

$$\inf_{l\in\mathbb{N}} \frac{a'_1\cdots a'_l}{\mathrm{Cap}(K)^l} = \liminf_{l\to\infty} \frac{a'_1\cdots a'_l}{\mathrm{Cap}(K)^l} = 1.$$

This concludes the proof.

□

Proof of Theorem 3 By Theorem 5.8.4 in [21], there is a sequence $(F_s)_{s=1}^{\infty}$ of T-sets such that

$$K \subset \cdots \subset F_{s+1} \subset F_s \subset \cdots \subset \mathbb{R} \tag{13}$$

and

$$\cap_{s=1}^{\infty} F_s = K \tag{14}$$

hold. Moreover, Eqs. 13 and 14 imply that

$$\mu_{F_s} \to \mu_K \tag{15}$$

in weak star sense, and

$$\mathrm{Cap}(F_s) \to \mathrm{Cap}(K)$$

as $s \to \infty$.

Let $n \in \mathbb{N}$. Then for each $s \in \mathbb{N}$, we have

$$\|P_n(\cdot;\mu_{F_s})\|_{L^2(\mu_{F_s})} \leq \|P_n(\cdot;\mu_K)\|_{L^2(\mu_{F_s})} \tag{16}$$

by minimality of $P_n(x;\mu_{F_s})$ in $L^2\left(\mu_{F_s}\right)$. It follows from monotonicity (see e.g. Theorem 5.1.2 in [16]) of capacity that

$$\mathrm{Cap}(K) \leq \mathrm{Cap}(F_s) \text{ for each } s \in \mathbb{N}. \tag{17}$$

Hence,

$$(W_n(\mu_K))^2 = \frac{\int P_n^2(t;\mu_K)\,d\mu_K(t)}{\mathrm{Cap}(K)^{2n}} \tag{18}$$

$$= \frac{\lim_{s\to\infty}\int P_n^2(t;\mu_K)\,d\mu_{F_s}(t)}{\mathrm{Cap}(K)^{2n}} \tag{19}$$

$$\geq \liminf_{s\to\infty} \frac{\int P_n^2(t;\mu_{F_s})\,d\mu_{F_s}(t)}{\mathrm{Cap}(F_s)^{2n}} \tag{20}$$

$$= \liminf_{s\to\infty} \left(W_n\left(\mu_{F_s}\right)\right)^2 \tag{21}$$

$$\geq 1. \tag{22}$$

In order to obtain Eq. 19, we use Eq. 15. The inequality (20) follows from Eqs. 16, 17 and 22 is obtained by using part (b) of Theorem 2. Thus, the proof is complete. □

Proof of Corollary 1 Let $\left(a_{n_j}\right)_{j=1}^{\infty}$ be a subsequence of $(a_n)_{n=1}^{\infty}$ such that $a_{n_j} \to 0$ as $j \to \infty$. By Eq. 5 and Theorem 3, for each $j > 1$, we have

$$W_{n_j-1}(\mu_K) = W_{n_j}(\mu_K)\frac{\mathrm{Cap}(K)}{a_{n_j}} \geq \frac{\mathrm{Cap}(K)}{a_{n_j}} \tag{23}$$

Since $a_{n_j} \to 0$ as $j \to \infty$, the right hand side of Eq. 23 goes to infinity as $j \to \infty$. Hence $\lim_{j\to\infty} W_{n_j-1}(\mu_K) = \infty$ and in particular $(W_n(\mu_K))_{n=1}^{\infty}$ is unbounded. Since $\mathrm{supp}(\mu_K) \subset K$, $\|T_{n,\mathrm{supp}(\mu_K)}\|_{L^\infty(\mathrm{supp}(\mu_K))} \leq \|T_{n,K}\|_{L^\infty(K)}$ holds for all $n \in \mathbb{N}$. Thus, by Eq. 6, we have $W_n(\mu_K) \leq M_{n,K}$ for each $n \in \mathbb{N}$. This implies that $\left(M_{n,K}\right)_{n=1}^{\infty}$ is also unbounded. □

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