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Orthogonal Polynomials on Generalized Julia Sets

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Abstract We extend results by Barnsley et al. about orthogonal polynomials on Julia sets to the case of generalized Julia sets. The equilibrium measure is considered. In addition, we discuss optimal smoothness of Green's functions and Parreau–Widom criterion for a special family of real generalized Julia sets.

Keywords Julia sets · Parreau-Widom sets · Orthogonal polynomials · Jacobi matrices

Mathematics Subject Classification 37F10 · 42C05 · 30C85

1 Introduction

Let f be a rational function in $\overline{\mathbb{C}}$. Then the set of all points $z \in \overline{\mathbb{C}}$ such that the sequence of iterates $(f^n(z))_{n=1}^{\infty}$ is normal in the sense of Montel is called the *Fatou* set of f. The complement of the Fatou set is called the *Julia* set of f and we denote it by $J_{(f)}$. We use the adjective *autonomous* in order to refer to these usual Julia sets in the text.

Polynomial Julia sets are the most studied objects in one dimensional complex dynamics. Potential theoretical tools for such sets were developed in [8] by Hans

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Brolin. Mañé and Rocha have shown in [23] that Julia sets are uniformly perfect in the sense of Pommerenke and, in particular, they are regular with respect to the Dirichlet problem. For a general exposition we refer to the survey [22] and the monograph [26].

Let (f_n) be a sequence of rational functions. Define $F_0(z) := z$ and $F_n(z) = f_n \circ F_{n-1}(z)$ for all $n \in \mathbb{N}$, recursively. The union of the points z such that the sequence $(F_n(z))_{n=1}^{\infty}$ is normal is called the Fatou set for (f_n) and the complement of the Fatou set is called the Julia set for (f_n). We use the notation $J_{(f_n)}$ to denote it. These sets were introduced in [15]. For a general overview we refer the reader to the paper [10]. For a recent discussion of Chebyshev polynomials on these sets, see [1].

In this paper, we consider orthogonal polynomials with respect to the equilibrium measure of $J_{(f_n)}$ where (f_n) is a sequence of polynomials satisfying some mild conditions, so we extend results from [4–6] where orthogonal polynomials for autonomous Julia sets were studied. We also mention the papers [2] and [27] related to orthogonal polynomials on sets constructed by means of compositions of infinitely many polynomials. While the focus of [27] is quite different from ours, a family of sets considered in [2] presents just a particular case of generalized Julia sets.

The paper is organized as follows. Background information about the properties of $J_{(f_n)}$ regarding potential theory is given in Sect. 2. In Sect. 3 we prove that, for certain degrees, orthogonal polynomials associated with the equilibrium measure of $J_{(f_n)}$ are given explicitly in terms of the compositions F_n. In Sect. 4 we show that the recurrence coefficients can be calculated provided that $J_{(f_n)}$ is real. These two results generalize Theorem 3 in [4] and Theorem 1 in [5] respectively. Techniques that we use here are rather different compared to those of autonomous setting, because of the fact that generalized Julia sets are not completely invariant as opposed to autonomous Julia sets. A weak form of invariance in our case is given by part (e) of Theorem 1. In addition, in Sect. 4 we discuss resolvent functions for generalized Fatou sets. In Sect. 5 we consider a general method to construct real Julia sets. Section 6 is devoted to a quadratic family of polynomials (f_n) such that the set $K_1(\gamma) = J_{(f_n)}$ is a modification of the set $K(\gamma)$ from [19]. In terms of the parameter γ we give a criterion for the corresponding Green function to be optimally smooth. A criterion for $K_1(\gamma)$ to be a Parreau–Widom set is presented in the last section.

For basic notions of logarithmic potential theory we refer the reader to [28], log denotes the natural logarithm. For a compact non-polar set $K \subset \mathbb{C}$ let μ_K denote the equilibrium measure of K, $\mathrm{Cap}(K)$ be the logarithmic capacity of K and $G_{\overline{\mathbb{C}}\setminus K}$ be the Green function with pole at ∞ of the unbounded component Ω of $\mathbb{C}\setminus K$. Recall that Ω is a regular set with respect to the Dirichlet problem if and only if the Green function $G_{\overline{\mathbb{C}}\setminus K}$ is continuous throughout $\mathbb{C}$ (see e.g. [28], Theorem 4.4.9).

For $R > 0$, let $\triangle_R = \{z \in \overline{\mathbb{C}} : |z| > R\}$. Convergence of measures is considered in weak-star topology. In addition, we consider and count multiple roots of a polynomial separately.

2 Preliminaries

Let polynomials $f_n(z) = \sum_{j=0}^{d_n} a_{n,j} \cdot z^j$ be given with $d_n \geq 2$ and $a_{n,d_n} \neq 0$ for all $n \in \mathbb{N}$. Then $F_n = f_n \circ \ldots \circ f_1$ is a polynomial of degree $d_1 \cdots d_n$ with the leading coefficient $(a_{1,d_1})^{d_2 \ldots d_n} (a_{2,d_2})^{d_3 \ldots d_n} \ldots a_{n,d_n}$.

Following [10] (see also [11]), we say that (f_n) is a *regular polynomial sequence* if for some positive real numbers A_1, A_2, A_3, the following properties are satisfied for all $n \in \mathbb{N}$:

- $|a_{n,d_n}| \geq A_1$.
- $|a_{n,j}| \leq A_2 |a_{n,d_n}|$ for $j = 0, 1, \ldots, d_n - 1$.
- $\log |a_{n,d_n}| \leq A_3 \cdot d_n$.

We use the notation $(f_n) \in \mathcal{R}$ if (f_n) is a regular polynomial sequence. We remark that, for a sequence $(f_n) \in \mathcal{R}$, the degrees of polynomials need not to be the same and they do not have to be bounded above either. In the next theorem, which is imported from [10] and [11], the symbol $\xrightarrow{lu}$ denotes locally uniform convergence.

Theorem 2.1 *Let $(f_n) \in \mathcal{R}$. Then the following propositions hold:*

(a) The set $\mathcal{A}_{(f_n)}(\infty) := \{z \in \overline{\mathbb{C}} : F_k(z) \xrightarrow{lu} \infty \text{ as } k \to \infty\}$ is an open connected set containing ∞. Moreover, for every $R > 1$ satisfying the inequality

$$A_1 R \left(1 - \frac{A_2}{R-1}\right) > 2,$$

we have $F_k(z) \xrightarrow{lu} \infty$ whenever $z \in \triangle_R$.

(b) $\mathcal{A}_{(f_n)}(\infty) = \cup_{k=1}^{\infty} F_k{}^{-1}(\triangle_R)$ and $f_n(\overline{\triangle_R}) \subset \triangle_R$ if $R > 1$ satisfies the inequality given in part (a). Furthermore, $J_{(f_n)} = \partial \mathcal{A}_{(f_n)}(\infty)$.

(c) $\mathcal{A}_{(f_n)}(\infty)$ is regular with respect to the Dirichlet problem. Here,

$$G_{\overline{\mathbb{C}} \setminus J_{(f_n)}}(z) = \begin{cases} \lim_{k\to\infty} \frac{1}{d_1 \cdots d_k} \log |F_k(z)| & \text{if } z \in \mathcal{A}_{(f_n)}(\infty), \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

In addition,

$$G_{\overline{\mathbb{C}} \setminus J_{(f_n)}}(z) = \lim_{k\to\infty} \frac{1}{d_1 \cdots d_k} G_{\overline{\mathbb{C}} \setminus J_{(f_n)}}(F_k(z)) \quad \text{whenever} \quad z \in \mathcal{A}_{(f_n)}(\infty). \tag{2.2}$$

In both (2.1) and (2.2) the convergence is locally uniform in $\mathcal{A}_{(f_n)}(\infty)$.

(d)

$$\mathrm{Cap}(J_{(f_n)}) = \exp\left(-\lim_{k\to\infty} \sum_{j=1}^{k} \frac{\log |a_{j,d_j}|}{d_1 \cdots d_j}\right).$$

(e) $F_k^{-1}(F_k(J_{(f_n)})) = J_{(f_n)}$ and $J_{(f_n)} = F_k^{-1}(J_{(f_{k+n})})$ for all $k \in \mathbb{N}$. Here we use the notation $(f_{k+n}) = (f_{k+1}, f_{k+2}, f_{k+3}, \ldots)$.

The statements above follow Sects. 2 and 4 of [10]. In particular, we have (2.1) by the proof of Theorem 4.2 in [10], whereas (2.2) follows from the definitions of the Green function and the set $\mathcal{A}_{(f_n)}(\infty)$.

It should be noted that, for the sequences $(f_n) \in \mathcal{R}$ satisfying the additional condition $d_n = d$ for some $d \geq 2$, there is a qualified theory concerning topological properties of Julia sets. For details, see [13,24].

Before going further, we recall the results from [4] and [5] concerning orthogonal polynomials for the autonomous Julia sets. Let $f(z) = z^n + k_1 z^{n-1} + \cdots + k_n$ be a nonlinear monic polynomial of degree n and let P_j denote the jth monic orthogonal polynomial associated to the equilibrium measure of $J_{(f)}$. Then

(a) $P_1(z) = z + k_1/n$.
(b) $P_{ln}(z) = P_l(f(z))$, for $l = 0, 1, \ldots$
(c) $P_{n^l}(z) = f^l(z) + k_1/n$ for $l = 0, 1, \ldots$, where f^l is the lth iteration of the function f.

Our first aim is to obtain analogous representations for orthogonal polynomials on the generalized Julia sets.

3 Orthogonal Polynomials

We begin with a lemma due to Brolin [8], Lemma 15.5.

Lemma 3.1 *Let K and L be two non-polar compact subsets of $\mathbb{C}$ such that $K \subset L$. Let $(\mu_n)_{n=1}^{\infty}$ be a sequence of probability measures supported on L that converges to a measure μ supported on K. Let U_n denote the logarithmic potential for the measure μ_n and V_K be the Robin constant for K. Suppose that*

(a) $\liminf_{n\to\infty} U_n(z) \geq V_K$ *on* K.
(b) $\operatorname{supp}(\mu_K) = K$.

Then $\mu = \mu_K$.

Let $(f_n) \in \mathcal{R}$. For given $k \in \mathbb{N}$ and $a \in \mathbb{C}$, by the fundamental theorem of algebra (FTA), the equation $F_k(z) - a = 0$ has $d_1 \ldots d_k$ solutions counting multiplicities, let $z_1, \ldots, z_{d_1 \ldots d_k}$. We consider the normalized counting measure $\nu_k^a = \frac{1}{d_1 \ldots d_k} \sum_{j=1}^{d_1 \ldots d_k} \delta_{z_j}$ at these points. In [8] and later on in [9], the convergence $\nu_k^a \to \mu_{J_{(f_n)}}$ was shown for a satisfying a certain condition. In the first article $f_n = f$ with a monic nonlinear polynomial f, whereas in the second one $f_n(z) = z^2 + c_n$. We use the same technique to extend these results to the case of regular polynomial sequences. Due to some minor changes and for the convenience of the reader, we include the proof of the theorem.

Theorem 3.2 *Let $(f_n) \in \mathcal{R}$. Then for $a \in \mathbb{C}\setminus\overline{\mathbb{D}}$ satisfying the condition*

$$|a| A_1 \left(1 - \frac{A_2}{|a| - 1}\right) > 2, \tag{3.1}$$

we have $\nu_k^a \to \mu_{J_{(f_n)}}$ as $k \to \infty$.

Proof Fix $a \in \mathbb{C}\setminus\overline{\mathbb{D}}$ satisfying (3.1). Let $K := J_{(f_n)}$ and $L := \{z \in \mathbb{C} : |z| \leq |a|\}$. Suppose $|z| \geq |a|$. Then, by Theorem 2.1 (b), we have $|F_1(z)| = |f_1(z)| > |a|$ and $z \in \mathcal{A}_{(f_n)}(\infty)$, so $z \notin K$. Therefore, $K \subsetneq L$. Moreover, since K is regular with respect to the Dirichlet problem and K is equal to the boundary of the component of $\overline{\mathbb{C}}\setminus K$ that contains ∞, we have (see e.g. Theorem 4.2.3. of [28]) that $\mathrm{supp}(\mu_K) = K$.

Observe that $F_k^{-1}(a)$ is contained in $L \cap \mathcal{A}_{(f_n)}(\infty)$ for all $k \in \mathbb{N}$. Indeed, let $z \in F_k^{-1}(a)$. If $|z| > |a|$ then applying Theorem 2.1 (b)repeatedly yields $|F_1(z)| > |a|, \ldots, |F_k(z)| > |a|$, contrary to $F_k(z) = a$. Therefore, $F_k^{-1}(a) \subset L$. Similarly, if $F_k(z) = a$ then $|F_{k+1}(z)| > |a|$ and $z \in \mathcal{A}_{(f_n)}(\infty)$.

The measure ν_k^a is supported on the set $F_k^{-1}(a)$. We have a sequence $(\nu_k^a)_{k=1}^{\infty}$ of probability measures on the set L. By Helly's selection principle (see e.g. Theorem 0.1.3. in [30]), there is a subsequence $(\nu_{k_l}^a)_{l=1}^{\infty}$ that converges to some limit μ. The set $\cup_{k=1}^{\infty}\mathrm{supp}(\nu_k^a) = \cup_{k=1}^{\infty} F_k{}^{-1}(a)$, which is a subset of the open set $\mathcal{A}_{(f_n)}(\infty)$, cannot accumulate to a point z in $\mathcal{A}_{(f_n)}(\infty)$, since this would contradict the fact that $F_k(z)$ goes uniformly to infinity in a neighborhood of z. Since $\mathrm{supp}(\mu)$ consists of accumulation points of the set above, we conclude that $\mathrm{supp}(\mu) \subset \partial\mathcal{A}_{(f_n)}(\infty) = K$.

It remains to show that $\liminf_{l\to\infty} U_{k_l}(z) \geq V_K$ for all $z \in K$, where $U_{k_l}(z) = \int \log \frac{1}{|z-\zeta|} d\nu_{k_l}^a(\zeta)$ and $V_K = \lim_{k\to\infty} \sum_{m=1}^{k} \frac{\log |a_{m,d_m}|}{d_1\ldots d_m}$. By Theorem 2.1$(d)$, this limit exists.

For the solutions $(z_{j,k_l})_{j=1}^{d_1\ldots d_{k_l}}$ of the equation $F_{k_l}(z) = a$ and a fixed $z \in K$ we have

$$|F_{k_l}(z) - a| = |(a_{1,d_1})^{d_2\ldots d_{k_l}}||(a_{2,d_2})^{d_3\ldots d_{k_l}}|\ldots|a_{k_l,d_{k_l}}| \prod_{j=1}^{d_1\ldots d_{k_l}} |z - z_{j,k_l}|.$$

Therefore,

$$U_{k_l}(z) = \frac{\sum_{j=1}^{d_1\ldots d_{k_l}} \log|z - z_{j,k_l}|}{-d_1\ldots d_{k_l}} = \sum_{m=1}^{k_l} \frac{\log|a_{m,d_m}|}{d_1\ldots d_m} - \frac{\log|F_{k_l}(z) - a|}{d_1\ldots d_{k_l}}. \tag{3.2}$$

Here, $|F_{k_l}(z)| \leq |a|$, since otherwise, arguing as above, we get $z \in \mathcal{A}_{(f_n)}(\infty)$, in contradiction with $z \in K$. Hence,

$$\liminf_{l\to\infty} U_{k_l}(z) \geq \lim_{l\to\infty} \left(\sum_{m=1}^{k_l} \frac{\log|a_{m,d_m}|}{d_1\ldots d_m} - \frac{\log|2a|}{d_1\ldots d_{k_l}} \right) = V_K.$$

Thus, by Lemma 3.1, we have $\nu_{k_l}^a \to \mu_K$. Since $(\nu_{k_l}^a)$ is an arbitrary convergent subsequence, $\nu_k^a \to \mu_K$ also holds. □

In the next theorem, we use algebraic properties of polynomials as well as analytic properties of the corresponding Julia sets. Let $f(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be a nonlinear polynomial of degree n and let $z_1, z_2, \ldots, z_n$ be the roots of f counting multiplicities. Then, for $k = 1, 2, \ldots, n-1$, we have the following Newton's identities:

$$s_k(f) + \frac{a_{n-1}}{a_n} s_{k-1}(f) + \cdots + \frac{a_{n-k+1}}{a_n} s_1(f) = -k\frac{a_{n-k}}{a_n}, \tag{3.3}$$

where $s_k(f) := \sum_{j=1}^{n} (z_j)^k$.

For the proof of (3.3) see e.g. [25]. Note that none of these equations include the term a_0. This implies that the values $(s_k)_{k=1}^{n-1}$ are invariant under translation, i.e.

$$s_k(f) = s_k(f + c) \tag{3.4}$$

for any $c \in \mathbb{C}$. Let $(P_j)_{j=1}^{\infty}$ denote the sequence of monic orthogonal polynomials associated to $\mu_{J_{(f_n)}}$ where $\deg P_j = j$. Now we are ready to prove our first main result.

Theorem 3.3 *For $(f_n) \in \mathcal{R}$, we have the following identities:*

(a) $P_1(z) = z + \dfrac{1}{d_1}\dfrac{a_{1,d_1-1}}{a_{1,d_1}}$.

(b) $P_{d_1\ldots d_l}(z) = \dfrac{1}{(a_{1,d_1})^{d_2\ldots d_l}(a_{2,d_2})^{d_3\ldots d_l}\cdots a_{l,d_l}}\left(F_l(z) + \dfrac{1}{d_{l+1}}\dfrac{a_{l+1,d_{l+1}-1}}{a_{l+1,d_{l+1}}}\right)$.

Proof Let $(f_n) \in \mathcal{R}$ be given and $a \in \mathbb{C}\backslash\overline{\mathbb{D}}$ satisfy (3.1).

(a) Fix an integer m greater than 1. By FTA, the solutions of the equation $F_m(z) = a$ satisfy an equation of the form

$$\left(F_{m-1}(z) - \beta_{m-1}^{1}\right)\cdots(F_{m-1}(z) - \beta_{m-1}^{d_m}) = 0$$

for certain $\beta_{m-1}^{1}, \ldots, \beta_{m-1}^{d_m} \in \mathbb{C}$. The $d_1 \ldots d_{m-1}$ roots of the equation $F_{m-1}(z) - \beta_{m-1}^{j} = 0$ are the solutions of an equation

$$(F_{m-2}(z) - \beta_{m-2}^{1,j})\cdots(F_{m-2}(z) - \beta_{m-2}^{d_{m-1},j}) = 0$$

with certain $\beta_{m-2}^{1,j}, \ldots, \beta_{m-2}^{d_{m-1},j}$. Continuing in this manner, we see that the points satisfying the equation $F_m(z) = a$ can be grouped into $d_2 \ldots d_m$ parts of size d_1 such that each part consists of the roots of an equation

$$f_1(z) - \beta_1^{j} = 0$$

for $j \in \{1, \ldots, d_2 \ldots d_m\}$ and $\beta_1^{j} \in \mathbb{C}$. For each j we denote by λ_j the normalized counting measure on the roots of $f_1(z) - \beta_1^{j}$. Then

$$\nu_m^{a} = \frac{1}{d_2 \ldots d_m}\sum_{j=1}^{d_2\ldots d_m} \lambda_j.$$

Hence, by (3.3) and (3.4),

$$\int z\,d\nu_m^a = \frac{1}{d_2\ldots d_m}\sum_{j=1}^{d_2\ldots d_m}\int z\,d\lambda_j = \frac{1}{d_2\ldots d_m}\sum_{j=1}^{d_2\ldots d_m}\frac{s_1(f_1-\beta_1^j)}{d_1}$$
$$= \frac{1}{d_1\ldots d_m}\sum_{j=1}^{d_2\ldots d_m} s_1(f_1) = -\frac{1}{d_1}\frac{a_{1,d_1-1}}{a_{1,d_1}}.$$

Since ν_m^a converges to the equilibrium measure of $J_{(f_n)}$ by Theorem 3.2, the result follows.

(b) Let $m, l \in \mathbb{N}$ where $m > l+1$. As above, the roots of the equation $F_m(z) = a$ can be decomposed into $d_{l+2}\ldots d_m$ parts of size $d_1\ldots d_{l+1}$ such that the roots from each part are the solutions of an equation of the form

$$F_{l+1}(z) - \beta_{l+1}^j = 0$$

for $j = 1, 2, \ldots, d_{l+2}\ldots d_m$. Recall that $F_{l+1}(z) = f_{l+1}(t)$ with $t = F_l(z)$.

By FTA, we have $f_{l+1}(t) - \beta_{l+1}^j = (t-\beta_l^{1,j})\cdots(t-\beta_l^{d_{l+1},j})$ for some $\beta_l^{1,j},\ldots,\beta_l^{d_{l+1},j}$. We apply (3.4) for $k \in \{1,\ldots,d_{l+1}-1\}$ and $j, j' \in \{1,\ldots, d_{l+2}\ldots d_m\}$:

$$\sum_{r=1}^{d_{l+1}}(\beta_l^{r,j})^k = s_k(f_{l+1}-\beta_{l+1}^j) = s_k(f_{l+1}-\beta_{l+1}^{j'}) = \sum_{r=1}^{d_{l+1}}(\beta_l^{r,j'})^k.$$

Now we can rewrite $F_{l+1}(z)-\beta_{l+1}^j = 0$ as $(F_l(z)-\beta_l^{1,j})\cdots(F_l(z)-\beta_l^{d_{l+1},j}) = 0$ for j as above. Let us denote by $\lambda_{r,j}$ the normalized counting measures on the roots of $F_l(z)-\beta_l^{r,j} = 0$ for $r = 1,\ldots,d_{l+1}$ and $j = 1,\ldots,d_{l+2}\ldots d_m$. Clearly, this yields

$$\nu_m^a = \frac{1}{d_{l+2}\ldots d_m}\sum_{j=1}^{d_{l+2}\ldots d_m}\frac{1}{d_{l+1}}\sum_{r=1}^{d_{l+1}}\lambda_{r,j} = \frac{1}{d_{l+1}\ldots d_m}\sum_{j=1}^{d_{l+2}\ldots d_m}\sum_{r=1}^{d_{l+1}}\lambda_{r,j}. \tag{3.5}$$

Thus, by (3.3) and (3.4), we deduce that

$$\int F_l(z)\,d\nu_m^a = \frac{1}{d_{l+1}\ldots d_m}\sum_{j=1}^{d_{l+2}\ldots d_m}\sum_{r=1}^{d_{l+1}}\int F_l(z)\,d\lambda_{r,j} = \frac{1}{d_{l+1}\ldots d_m}\sum_{j=1}^{d_{l+2}\ldots d_m}\sum_{r=1}^{d_{l+1}}\beta_l^{r,j}$$
$$= \frac{1}{d_{l+1}\ldots d_m}\sum_{j=1}^{d_{l+2}\ldots d_m} s_1(f_{l+1}-\beta_{l+1}^j) = \frac{1}{d_{l+1}\ldots d_m}\sum_{j=1}^{d_{l+2}\ldots d_m} s_1(f_{l+1})$$
$$= -\frac{1}{d_{l+1}}\frac{a_{l+1,d_{l+1}-1}}{a_{l+1,d_{l+1}}}.$$

To shorten notation, we write c instead of $\frac{1}{d_{l+1}}\frac{a_{l+1,d_{l+1}-1}}{a_{l+1,d_{l+1}}}$. Thus, we have

$$\int (F_l(z)+c)\,d\nu_m^a = 0. \tag{3.6}$$

Let us show that the integrand is orthogonal to z^k with $1 \le k \le d_1 \dots d_l - 1$ as well. Recall that $F_l(z) = \beta_l^{r,j}$ at any point z from the support of $\lambda_{r,j}$. Therefore,

$$\int (F_l(z)+c)\,\overline{z^k}\,d\lambda_{r,j} = \frac{1}{d_1 \dots d_l}\left(\beta_l^{r,j}+c\right)\cdot \overline{s_k\left(F_l - \beta_l^{r,j}\right)}.$$

By (3.4), $\overline{s_k\left(F_l - \beta_l^{r,j}\right)} = \overline{s_k\,(F_l)}$, so it does not depend on r or j. This and the representation (3.5) imply that

$$\int (F_l(z)+c)\,\overline{z^k}\,d\nu_m^a = \frac{1}{d_{l+1}\dots d_m}\sum_{j=1}^{d_{l+2}\dots d_m}\sum_{r=1}^{d_{l+1}}\int (F_l(z)+c)\,\overline{z^k}\,d\lambda_{r,j}$$
$$= \frac{\overline{s_k\,(F_l)}}{d_1\dots d_l}\int (F_l(z)+c)\;d\nu_m^a.$$

The integral in the last line is equal to 0, by (3.6). It follows that if $k \le \deg F_l - 1$ then $(F_l + c) \perp z^k$ in $L^2(\mu_{J_{(f_n)}})$, since ν_m^a converges to the equilibrium measure of $J_{(f_n)}$. This completes the proof of the theorem. $\square$

4 Moments and Resolvent Functions

In this section we consider Julia sets that are subsets of the real line.

Let μ be a probability measure whose support is a compact subset of $\mathbb{R}$ containing infinitely many points. Then the monic polynomials, orthogonal with respect to μ, $(P_n)_{n=1}^{\infty}$ satisfy a recurrence relation

$$P_{n+1}(x) = (x - b_{n+1})P_n(x) - a_n^2 P_{n-1}(x)$$

for $n \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$. We assume here $P_0 = 1$ and $P_{-1} = 0$. The n-th orthonormal polynomial p_n has the following representation in terms of the moments $c_n = \int x^n d\mu$:

$$p_n(x) = \frac{1}{\sqrt{D_n D_{n-1}}}\begin{vmatrix} c_0 & c_1 & \dots & c_n \\ c_1 & c_2 & \dots & c_{n+1} \\ \vdots & \vdots & & \vdots \\ c_{n-1} & c_n & \dots & c_{2n-1} \\ 1 & x & \dots & x^n \end{vmatrix},$$

where D_n is the determinant of the matrix $M_n = (c_{i+j})_{i,j=0}^n$. Thus, by means of the moments of μ, one can calculate recurrence coefficients $(a_n, b_n)_{n=1}^\infty$. See [36] for a theory of general orthogonal polynomials on the real line.

In the next theorem we show that the moments for the equilibrium measure of $J_{(f_n)}$ can be calculated recursively whenever $(f_n) \in \mathcal{R}$. Note that $c_0 = 1$ since the equilibrium measure is of unit mass.

Theorem 4.1 *Let $(f_n) \in \mathcal{R}$ and $F_l = f_l \circ \ldots \circ f_1$ for $l \in \mathbb{N}$. Then $c_k = \int x^k d\mu_{J_{(f_n)}} = \frac{s_k(F_l)}{\deg(F_l)}$ for $k \leq \deg(F_l) - 1$. Here, $s_k(F_l)$ can be calculated recursively by Newton's identities.*

Proof Fix $m > l$ and a as in Theorem 3.2. As in Theorem 3.3, we can divide the roots of the equation $F_m(z) = a$ into $d_{l+1} \ldots d_m$ parts of size $d_1 \cdots d_l$ such that the nodes in each of the groups constitute the roots of an equation of the form

$$F_l(z) - \beta^j = 0$$

for $j = 1, 2, \ldots, d_{l+1} \ldots d_m$. As above, let λ_j be the normalized counting measure on the roots of this equation. Then in view of (3.4)

$$\int x^k \, d\nu_m^a = \frac{1}{d_{l+1} \ldots d_m} \sum_{j=1}^{d_{l+1} \ldots d_m} \int x^k \, d\lambda_j = \frac{1}{d_{l+1} \ldots d_m} \sum_{j=1}^{d_{l+1} \ldots d_m} \frac{s_k(F_l - \beta^j)}{d_1 \ldots d_l}$$
$$= \frac{1}{d_{l+1} \ldots d_m} \sum_{j=1}^{d_{l+1} \ldots d_m} \frac{s_k(F_l)}{d_1 \ldots d_l} = \frac{s_k(F_l)}{d_1 \ldots d_l}$$

for $k < \deg(F_l)$. By Theorem 3.2, the result follows. □

For two bounded sequences $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ with $a_n > 0$ and $b_n \in \mathbb{R}$ for $n \in \mathbb{N}$, the associated (half-line) Jacobi operator $H : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is given by $(Hu)_n = a_n u_{n+1} + b_n u_n + a_{n-1} u_{n-1}$ for $u \in \ell^2(\mathbb{N})$ and $a_0 := 0$. Here, $\ell^2(\mathbb{N})$ denotes the space of square summable sequences in $\mathbb{N}$. The spectral measure of H for the cyclic vector $\delta_1 = (1, 0, 0, \ldots)^T$ is just the one which has a_n, b_n $(n = 1, 2 \ldots)$ as the recurrence coefficients.

Suppose that $(f_n) \in \mathcal{R}$ and $J_{(f_n)} \subset [-M, M]$ for some $M > 0$. Let $H_{(f_n)}$ denote the Jacobi operator associated with $\mu_{J_{(f_n)}}$. Then the *resolvent function* $R_{(f_n)}$ is defined for $z \in \mathbb{C} \backslash J_{(f_n)}$ as

$$R_{(f_n)}(z) := \int \frac{d\,\mu_{J_{(f_n)}}(x)}{x - z} = \langle (H_{(f_n)} - z)^{-1} \delta_1, \delta_1 \rangle.$$

Note that $R_{(f_n)}$ is an analytic function. In the autonomous polynomial case (see e.g [6]), the resolvent function satisfies a functional equation:

$$R_{(f)}(z) = \frac{f'(z)}{\deg f} R_{(f)}(f(z)). \tag{4.1}$$

It is well known that (see e.g. p. 53 in [32]) for $z \in \mathbb{C}\setminus\overline{D_M(0)}$

$$R_{(f_n)}(z) = -\sum_{n=0}^{\infty} c_n z^{-(n+1)} \tag{4.2}$$

where c_n is the nth moment for $\mu_{J_{(f_n)}}$, $D_M(0)$ is the open disc with center at 0 and radius M and the series (4.2) is absolutely convergent in the corresponding domain.

We define the ∂ operator as

$$\partial = \frac{\partial_x - i\partial_y}{2}.$$

If g is a harmonic function on a simply connected domain $D \subset \mathbb{C}$ then (see e.g. Theorem 1.1.2 in [28]) there is an analytic function h on D such that $g = \operatorname{Re} h$ holds. Moreover, we have $h'(z) = 2\partial g(z)$. Furthermore, if $U_{\mu_{J_{(f_n)}}}$ denotes the logarithmic potential for $\mu_{J_{(f_n)}}$, then

$$G_{\overline{\mathbb{C}}\setminus J_{(f_n)}}(z) = \log\left(\operatorname{Cap}(J_{(f_n)})^{-1}\right) - U_{\mu_{J_{(f_n)}}}(z).$$

In addition, for each $z_0 \in \mathbb{C}\setminus J_{(f_n)}$, there exist a $\delta > 0$ and an analytic function h (which may depend on z_0) such that (see e.g. p. 87 in [14]) $h' = R_{(f_n)}$ and $\operatorname{Re} h = U_{\mu_{J_{(f_n)}}}$ on $z \in D_\delta(z_0)$. By harmonicity of $U_{\mu_{J_{(f_n)}}}$ this implies

$$2\partial G_{\overline{\mathbb{C}}\setminus J_{(f_n)}}(z) = -2\partial U_{\mu_{J_{(f_n)}}}(z) = -R_{(f_n)}(z), \quad z \in \mathbb{C}\setminus J_{(f_n)}. \tag{4.3}$$

The next theorem follows from the above discussion.

Theorem 4.2 *Let $(f_n) \in \mathcal{R}$ be such that $J_{(f_n)} \subset \mathbb{R}$. Then*

$$R_{(f_n)} = \lim_{k\to\infty} \frac{F_k' \cdot R_{(f_n)}(F_k)}{d_1 \ldots d_k},$$

where the convergence is locally uniform in $\mathbb{C}\setminus J_{(f_n)}$.

Proof Here the domain $\mathcal{A}_{(f_n)}(\infty)\setminus\infty$ coincides with $\mathbb{C}\setminus J_{(f_n)}$, as $J_{(f_n)} \subset \mathbb{R}$. We apply the operator ∂ in this domain to both sides of (2.2). Since the Green function is harmonic here, we can differentiate the limit on the right side of (2.2) term by term (see e.g. T.1.23 in [3]). Hence, we have

$$\partial G_{\overline{\mathbb{C}}\setminus J_{(f_n)}}(z) = \lim_{k\to\infty} \frac{\partial G_{\overline{\mathbb{C}}\setminus J_{(f_n)}}(F_k(z))F_k'(z)}{d_1 \ldots d_k}, \tag{4.4}$$

where the convergence is locally uniform in $\mathbb{C}\setminus J_{(f_n)}$. Applying (4.3) and (4.4) yields the result. □

5 Construction of Real Julia Sets

Let f be a nonlinear real polynomial with real and simple zeros $x_1 < x_2 < \cdots < x_n$ and distinct extrema $y_1 < \ldots < y_{n-1}$ with $|f(y_i)| > 1$ for $i = 1, 2, \ldots, n-1$. Then we say that f is an *admissible* polynomial. We list useful features of preimages of admissible polynomials.

Theorem 5.1 [16] *Let f be an admissible polynomial of degree n. Then*

$$f^{-1}([-1,1]) = \bigcup_{i=1}^{n} I_i$$

where I_i is a closed non-degenerate interval containing exactly one root x_i of f for each i. These intervals are pairwise disjoint and $\mu_{f^{-1}([-1,1])}(I_i) = 1/n$.

We say that an admissible polynomial f satisfies *the property* (A) if

(a) $f^{-1}([-1,1]) \subset [-1,1]$,
(b) $f(\{-1,1\}) \subset \{-1,1\}$,
(c) $f(a) = 0$ implies $f(-a) = 0$.

Since $f(x) = a_n \prod_{k=1}^{n}(x - x_k)$ with distinct real x_k, the condition (c) implies that f is either even or odd. In addition, $(x_k)_{k=1}^{n} \subset (-1,1)$, by (a) and (b).

Lemma 5.2 *Let g_1 and g_2 be admissible polynomials satisfying (A). Then $g_3 := g_2 \circ g_1$ is also an admissible polynomial that satisfies (A).*

Proof Let $\deg g_k = n_k$. Moreover, let $(x_{j,1})_{j=1}^{n_1}, (x_{j,2})_{j=1}^{n_2}$ be the zeros and $(y_{j,1})_{j=1}^{n_1-1}$ and $(y_{j,2})_{j=1}^{n_2-1}$ be the critical points of $g_1,\ g_2$ respectively. Then the equation $g_3(z) = 0$ implies that $g_1(z) = x_{j,2}$ for some $j \in \{1, \ldots, n_2\}$. By (a) and (b), the equation $g_1(z) = \beta$ has n_1 distinct roots for $|\beta| \le 1$ and the sets of roots of $g_1(z) = \beta_1$ and $g_1(z) = \beta_2$ are disjoint for different $\beta_1, \beta_2 \in [-1,1]$. Therefore, g_3 has $n_1 n_2$ distinct zeros. Similarly, $(g_3)'(z) = g_2'(g_1(z)) g_1'(z) = 0$ implies $g_1'(z) = 0$ or $g_1(z) = y_{j,2}$ for some $j \in \{1, \ldots, n_2 - 1\}$. The equation $g_1'(z) = 0$ has $n_1 - 1$ distinct solutions in $(-1,1)$. For each of them $|g_1(z)| > 1$ and $g_2{}'(g_1(z)) \neq 0$. On the other hand, for each $j \le n_2 - 1$, the equation $g_1(z) = y_{j,2}$ has n_1 distinct solutions with $g_1{}'(y_{j,2}) \neq 0$. Thus, the total number of solutions for the equation $g_3{}'(z) = 0$ is $n_1 - 1 + n_1(n_2 - 1) = n_1 n_2 - 1$ which is required. Hence, g_3 is admissible. It is straightforward that for the function g_3 parts (a) and (b) are satisfied. The part (c) is also satisfied for g_3, since arbitrary compositions of even and odd functions are either even or odd. $\square$

Lemma 5.3 *Let $(f_n) \in \mathcal{R}$ be a sequence of admissible polynomials satisfying (A). Then F_n is an admissible polynomial with the property (A). Besides, $F_{n+1}^{-1}([-1,1]) \subset F_n^{-1}([-1,1]) \subset [-1,1]$ and $K = \cap_{n=1}^{\infty} F_n^{-1}([-1,1])$ is a Cantor set in $[-1,1]$.*

Proof All statements except the last one follow directly from Lemma 5.2 and the representation $F_n(z) = f_n \circ F_{n-1}(z)$. Let us show that K is totally disconnected.

By the construction, the set K is uncountable. If, contrary to our claim, K is not totally disconnected, then K contains an interval I of positive length such that $I \subset$

$F_n^{-1}([-1,1])$ for all n. By Theorem A.16. of [31], we have $\mu_{F_n^{-1}([-1,1])} \to \mu_K$. In addition, by Theorem 5.1, $\mu_{F_n^{-1}([-1,1])}(I) \le 1/(d_1 \ldots d_n)$. Therefore, $\mu_K(I) = 0$. Thus all interior points of I in $\mathbb{R}$ are outside of the support of μ_K. This is impossible since $K = \partial(\overline{\mathbb{C}} \setminus K)$ and $\mathrm{Cap}(I) > 0$ ([28], Theorem 4.2.3). □

Corollary 5.4 *Let (f_n) be as in Lemma 5.3. Then $F_n^{-1}([-1,1]) = \bigcup_{j=1}^{d_1 \ldots d_n} I_{j,n}$ where $I_{j,n}$ are closed disjoint intervals of positive length. Moreover,* $\max_{1 \le j \le d_1 \ldots d_n} |I_{j,n}| \to 0$ *as $n \to \infty$.*

Indeed, the desired representation is a subject of Theorem 5.1, whereas the second statement follows from the fact that the interior of K is empty.

Lemma 5.5 *Let f be an admissible polynomial satisfying (A). Then $|f(z)| > 1 + 2\epsilon$ provided $|z| > 1 + \epsilon$ for $\epsilon > 0$. If $|z| = 1$ then $|f(z)| > 1$ unless $z = \pm 1$.*

Proof Let $\deg f = n$ and $x_1 < x_2 < \cdots < x_n$ be the zeros of f. It follows from (c) that $x_k = -x_{n+1-k}$ for $k \le n$. In particular, if n is odd, then $x_{(n+1)/2} = 0$.

Let $x_i \ne 0$ and $\epsilon > 0$. Then, by the law of cosines, the polynomial $P_{x_i}(z) := z^2 - x_i^2$ attains minimum of its modulus on the set $\{z : |z| = 1 + \epsilon\}$ at the point $z = 1 + \epsilon$. Therefore $|P_{x_i}(z)|/|P_{x_i}(\pm 1)| > 1 + 2\epsilon$ for any z with $|z| = 1 + \epsilon$. Using the symmetry of the roots of f about $x = 0$, we see that $|f(z)| = |f(z)|/|f(\pm 1)| > 1 + 2\epsilon$ for such z.

Similarly, $|P_{x_i}(z)| \ge |P_{x_i}(\pm 1)|$ for $|z| = 1$ and the inequality is strict for $z \ne \pm 1$. Hence, $|f(z)| > 1$ in this case as well. □

In the next theorem we use the argument of Theorem 1 in [19].

Theorem 5.6 *Let $(f_n) \in \mathcal{R}$ be a sequence of admissible polynomials satisfying (A). Then $K = \cap_{n=1}^{\infty} F_n^{-1}([-1,1]) = J_{(f_n)}$.*

Proof Let us first prove the inclusion $J_{(f_n)} \subset K$.

Recall that, by Theorem 2.1(b), $\mathcal{A}_{(f_n)}(\infty) = \cup_{k=1}^{\infty} F_k{}^{-1}(\triangle_R)$ and $f_n(\overline{\triangle}_R) \subset \triangle_R$ for all n provided R with $R > 1$ satisfies the condition $A_1 R(1 - (A_2/(R-1))) > 2$. Fix such R.

Fix $z \notin K$, so $F_m(z) \notin [-1,1]$ for some m. We aim to show that $|F_n(z)| > 1 + \epsilon$ for some $n \in \mathbb{N}$ and $\epsilon > 0$. Then we repeatedly apply Lemma 5.5 and show that $F_{n+k}(z) \in \triangle_R$ for given R and some k. This will imply $z \notin J_{(f_n)}$.

Let us consider different positions of z.

If $|z| = 1 + \epsilon$ with $\epsilon > 0$ then using Lemma 5.5 gives $|F_1(z)| > 1 + 2\epsilon$.

Similarly, $|F_1(z)| > 1$ for $|z| = 1$ with $z \ne \pm 1$.

If $z \in [-1,1] \setminus K$ then, by the construction of the set K, there exists N such that $|F_N(z)| > 1$, which is the desired result.

Suppose that $z = x + iy \in \mathbb{D}$ with $|y| > 0$. Assume first that $x \notin K$. Then there exists N such that $|F_N(x)| > 1$. Since, by Lemma 5.3, all zeros of F_N are real, we have $|F_N(z)| > |F_N(x)| > 1$.

Now, let z be as above, yet with $x \in K$. By Corollary 5.4, there exists n such that the length of each component of $F_n^{-1}([-1,1])$ is less than $y^2/8$. We fix this n. Let $x_1 < x_2 < \cdots < x_{d_1 \cdots d_n}$ be the roots of the polynomial F_n and $I_{j,n}$ denote the interval from $F_n^{-1}([-1,1])$ that contains x_j for $j = 1, 2, \ldots, d_1 \ldots d_n$. Furthermore, let $I = [a, b]$ be the component containing the point x. Observe that $|F_n(a)| = |F_n(b)| = 1$. So, in order to get $z \notin J_{(f_n)}$, it is enough to show that $|F_n(z)| > |F_n(a)|$ or $\prod_j |z - x_j| > \prod_j |a - x_j|$.

If $j < s$ then $|a - x_j| \leq |x - x_j| < |z - x_j|$.

If $j = s$ then $|a - x_j| < y^2/8 < |y| \leq |z - x_j|$.

If $j > s$ then $|a - x_j| = \sqrt{|x_j - a|^2} \leq \sqrt{|x_j - x|^2 + |x - a|^2 + 2|x_j - x||x - a|}$ $< \sqrt{|x_j - x|^2 + \frac{y^4}{64} + \frac{y^2}{2}} < \sqrt{|x_j - x|^2 + y^2} = |z - x_j|$.

Therefore, $|F_n(z)| > 1$ and $J_{(f_n)} \subset K$.

For the inverse inclusion, fix $z \notin K$. We aim to show that $z \notin J_{(f_n)}$. Since $J_{(f_n)} \subset K \subset [-1, 1]$, it's enough to consider only $z = x \in [-1, 1]$. The condition $x \notin K$ means that there exists n such that $|F_n(x)| > 1$ in some neighborhood of x. By Lemma 5.5, we have $F_{n+k}(x) \overset{lu}{\to} \infty$ as $k \to \infty$. Thus, $x \in \mathcal{A}_{(f_n)}(\infty)$ and the result follows. □

Remark Orthogonal polynomials associated to the equilibrium measure of K and the corresponding recurrence coefficients can be calculated by Theorem 3.3 and Theorem 4.1.

6 Smoothness of Green's Functions

For some generalized Julia sets a deeper analysis can be done. In this section we consider a modification $K_1(\gamma)$ of the set $K(\gamma)$ from [19] that corresponds to Theorem 5.6. We give a necessary and sufficient condition on the parameters that makes the Green function $G_{\overline{\mathbb{C}} \setminus K_1(\gamma)}$ optimally smooth. Although smoothness properties of Green functions are interesting in their own rights, in our case the optimal smoothness of $G_{\overline{\mathbb{C}} \setminus K_1(\gamma)}$ is necessary for $K_1(\gamma)$ to be a Parreau-Widom set.

Let $K \subset \mathbb{C}$ be a non-polar compact set. Then $G_{\overline{\mathbb{C}} \setminus K}$ is said to be *Hölder continuous* with exponent β if there exists a number $A > 0$ such that

$$G_{\overline{\mathbb{C}} \setminus K}(z) \leq A(\mathrm{dist}(z, K))^{\beta},$$

holds for all z satisfying $\mathrm{dist}(z, K) \leq 1$, where $\mathrm{dist}(\cdot)$ stands for the distance function. For applications of smoothness of Green functions, we refer the reader to [7].

Smoothness properties of Green functions are examined for a variety of sets. For the complement of autonomous Julia sets, see [21] and for the complement of $J_{(f_n)}$ see [9,10]. In Corollary 2 from [20], a quadratic generalized Julia set was investigated, for which the associated Green function is continuous but is not Hölder continuous. When K is a symmetric Cantor-type set in $[0, 1]$, it is possible to give a sufficient and necessary condition for the corresponding Green function to be Hölder continuous with the exponent $1/2$, i.e. optimally smooth. See Chapter 5 in [35] for details.

It is possible to associate the density properties of equilibrium measures with the smoothness properties of Green's functions.

Theorem 6.1 [34] *Let $K \subset \mathbb{C}$ be a compact set such that the unbounded component Ω of $\overline{\mathbb{C}}\setminus K$ is regular. Then for each $z_0 \in \partial\Omega$ and $0 < r < 1$ we have*

$$\int_0^r \frac{\mu_K(D_t(z_0))}{t}dt \leq \sup_{|z-z_0|=r} G_{\overline{\mathbb{C}}\setminus K}(z) \leq 3\int_0^{4r} \frac{\mu_K(D_t(z_0))}{t}dt.$$

The rest of the paper is devoted to a special family of quadratic generalized Julia set.

Let $\gamma := (\gamma_n)_{n=1}^{\infty}$ be given such that $0 < \gamma_n < 1/4$ for all n, $\epsilon_n := 1/4 - \gamma_n$. Take $f_n(z) = \frac{1}{2\gamma_n}(z^2 - 1) + 1$ for $n \in \mathbb{N}$. Thus, $F_1(z) = \frac{1}{2\gamma_1}(z^2 - 1) + 1$ and similarly $F_n(z) = \frac{1}{2\gamma_n}(F_{n-1}^2(z) - 1) + 1$ for $n \geq 2$. It is easy to see that, as a polynomial of real variable, F_n is admissible, it satisfies (A) and, in addition, all minimums of F_n are the same and equal to $1 - \frac{1}{2\gamma_n}$. Then $K_1(\gamma) = \cap_{n=1}^{\infty} F_n^{-1}([-1, 1])$ is a stretched version of the set $K(\gamma)$ from [19]. Here,

$$G_{\overline{\mathbb{C}}\setminus K_1(\gamma)}(z) = \lim_{n\to\infty} 2^{-n} \log|F_n(z)|.$$

Since the leading coefficient of F_n is $2^{1-2^n}\gamma_n\,\gamma_{n-1}^2 \cdots \gamma_1^{2^{n-1}}$, the logarithmic capacity of $K_1(\gamma)$ is $2\exp(\sum_{n=1}^{\infty} 2^{-n}\log\gamma_n)$.

In addition, as is easy to check, $(f_n) \in \mathcal{R}$ if and only if $\inf_n \gamma_n > 0$. Thus, provided this condition, Theorem 5.6 implies $K_1(\gamma) = J_{(f_n)}$.

In the limit case, when all $\gamma_n = 1/4$, F_n is the Chebyshev polynomial (of the first kind) T_{2^n} and $K_1(\gamma) = [-1, 1]$. This does not contradict to Lemma 5.3, because the Chebyshev polynomials are not admissible according to our definition. Note that in the literature, in the definition of admissible polynomials, the condition $|f(y_i)| > 1$ for values at extrema points is often given in non-strict sense.

Let $I_{1,0} := [-1, 1]$. The set $F_n^{-1}([-1, 1])$ is a disjoint union of 2^n non-degenerate closed intervals $I_{j,n} = [a_{j,n}, b_{j,n}]$ with length $l_{j,n}$ for $1 \leq j \leq 2^n$. We call them *basic intervals of n−th level*. The inclusion $F_{n+1}^{-1}([-1, 1]) \subset F_n^{-1}([-1, 1])$ implies that $I_{2j-1,n+1} \cup I_{2j,n+1} \subset I_{j,n}$ where $a_{2j-1,n+1} = a_{j,n}$ and $b_{2j,n+1} = b_{j,n}$. We denote the gap $(b_{2j-1,n+1}, a_{2j,n+1})$ by $H_{j,n}$ and the length of the gap by $h_{j,n}$. Thus,

$$K_1(\gamma) = [-1, 1]\setminus\left(\bigcup_{n=0}^{\infty}\bigcup_{1\leq j\leq 2^n} H_{j,n}\right).$$

Let us consider the parameter function $v_\gamma(t) = \sqrt{1 - 2\gamma(1-t)}$ for $|t| \leq 1$ with $0 < \gamma \leq 1/4$. This increasing and concave function is an analog of u from [19]. By means of v_γ we can write the endpoints of the basic intervals of n−th level. The set of these endpoints consists of the solutions of the equations $F_k(x) = -1$ for $1 \leq k \leq n$ and the points ± 1. Namely, $F_n(x) = -1$ gives $F_{n-1}(x) = \pm v_{\gamma_n}(-1)$, then $F_{n-2}(x) = \pm v_{\gamma_{n-1}}(\pm v_{\gamma_n}(-1))$, etc. The iterates eventually give 2^n values

$$x = \pm v_{\gamma_1} \circ (\pm v_{\gamma_2} \circ (\cdots \pm v_{\gamma_{n-1}} \circ (\pm v_{\gamma_n}(-1) \cdots), \tag{6.1}$$

which are the endpoints $\{b_{2j-1,n}, a_{2j,n}\}_{j=1}^{2^{n-1}}$. The remaining 2^n points can be found similarly, as the solutions of $F_k(x) = -1$ for $1 \le k < n$ and ± 1.

As in Lemma 2 in [19], $\min_{1\le j\le 2^n} l_{j,n}$ is realized on the first and the last intervals. Since the rightmost solution of $F_n(x) = -1$, namely $a_{2^n,n}$, is given by (6.1) with all signs positive, we have

$$l_{1,n} = l_{2^n,n} = 1 - v_{\gamma_1}(v_{\gamma_2}(\cdots v_{\gamma_{n-1}}(v_{\gamma_n}(-1)\cdots). \tag{6.2}$$

The next lemma shows that $l_{1,n}$ can be evaluated in terms of $\delta_n := \gamma_1\gamma_2\cdots\gamma_n$.

Lemma 6.2 *For each γ with $0 < \gamma_k \le 1/4$ and for all $n \in \mathbb{N}$ we have*

$$2\,\delta_n \le l_{1,n} \le (\pi^2/2)\,\delta_n.$$

Proof Clearly, $1 - v_\gamma(t) = \frac{2}{1+v_\gamma(t)}\,\gamma(1-t)$. Repeated application of this to (6.2) gives the representation $l_{1,n} = 2\,\varkappa_n(\gamma)\,\delta_n$, where $\varkappa_n(\gamma)$ is equal to

$$\frac{2}{1+v_{\gamma_1}(v_{\gamma_2}(\cdots v_{\gamma_n}(-1)\cdots))}\cdot\frac{2}{1+v_{\gamma_2}(\cdots v_{\gamma_n}(-1)\cdots)}\cdots\frac{2}{1+v_{\gamma_n}(-1)}.$$

Since $v_{1/4}(t) \le v_\gamma(t) \le 1$, we have $1 \le \varkappa_n(\gamma) \le \varkappa_n(1/4)$, where the last denotes the value of $\varkappa_n$ in the case when all $\gamma_k = 1/4$. This gives the left part of the inequality. Let C_{2^n} be the distance between 1 and the rightmost extremum of T_{2^n}. Hence, see e.g. p.7. of [29], $C_{2^n} = 1-\cos(\pi/2^n) < \pi^2/(2\cdot 4^n)$. On the other hand, $C_{2^n} = 2\,\varkappa_n(1/4)\,4^{-n}$. Therefore, $\varkappa_n(1/4) < \pi^2/4$, and the lemma follows. □

In the case of $\gamma_n \le 1/32$ for all n, smoothness of the Green's function of $\overline{\mathbb{C}}\setminus K(\gamma)$ and related properties are examined in [18], [19]. The next theorem is complementary to Theorem 1 of [18] and examines the smoothness of the Green function as $\gamma_n \to 1/4$.

Theorem 6.3 *The function $G_{\overline{\mathbb{C}}\setminus K_1(\gamma)}$ is Hölder continuous with the exponent $1/2$ if and only if $\sum_{k=1}^\infty \epsilon_k < \infty$.*

Proof Let us assume that $\sum_{k=1}^\infty \epsilon_k < \infty$. Then $\prod_{k=1}^\infty(1-4\epsilon_k) = a$ for some $0 < a < 1$, $\delta_n = 4^{-n}\prod_{k=1}^n(1-4\epsilon_k) > a\,4^{-n}$ and, by Lemma 6.2, $2a\cdot 4^{-n} \le l_{1,n}$ for all $n \in \mathbb{N}$.

Let z_0 be an arbitrary point of $K_1(\gamma)$. We claim that $\mu_{K_1(\gamma)}(D_t(z_0)) \le \frac{4\sqrt{2}}{\sqrt{a}}\sqrt{t}$ for all $t > 0$. It is evident for $t \ge 1/32$, as $\mu_{K_1(\gamma)}$ is a probability measure. Let $0 < t < 1/32$. Fix n with $l_{1,n} < t \le l_{1,n-1}$. We have $t > 2a\cdot 4^{-n}$.

On the other hand, $D_t(z_0)$ can contain points from at most 4 basic intervals of level $n-1$. Since $\mu_{F_n^{-1}([-1,1])} \to \mu_{K_1(\gamma)}$, by Theorem A.16 from [31], we have $\mu_{K_1(\gamma)}(I_{j,k}) = 1/2^k$ for all $k \in \mathbb{N}$ and $1 \le j \le 2^k$. Therefore, $\mu_{K_1(\gamma)}(D_t(z_0)) \le 2^{3-n} < 8\sqrt{t/2a}$, which is our claim. The optimal smoothness of $G_{\overline{\mathbb{C}}\setminus K_1(\gamma)}$ follows from Theorem 6.1.

Conversely, suppose that, on the contrary, $\sum_{k=1}^\infty \epsilon_k = \infty$. This is equivalent to the condition $4^n\delta_n \to 0$ as $n \to \infty$. Thus, for any $\sigma > 0$, there is a number N such that

$n > N$ implies that $4^n \delta_n < \sigma$. For any $t \le l_{1,N+1}$, there exists $m \ge N+1$ such that $l_{1,m+1} < t \le l_{1,m}$. Then, $\mu_{K_1(\gamma)}(D_t(0)) \ge \mu_{K_1(\gamma)}(I_{1,m+1}) = 2^{-m-1}$. On the other hand, by Lemma 6.2, $t \le 2\pi^2\sigma\, 4^{-m-1}$. Therefore, for any $t \le l_{1,N+1}$ we have $\frac{\sqrt{t}}{\pi\sqrt{2\sigma}} \le \mu_{K_1(\gamma)}(D_t(0))$. Hence, the inequality

$$\frac{\sqrt{2}}{\pi\sqrt{\sigma}}\sqrt{r} \le \int_0^r \frac{\mu_{K_1(\gamma)}(D_t(0))}{t}dt,$$

holds for $r \le l_{1,N+1}$. By Theorem 6.1, $G_{\overline{\mathbb{C}}\setminus K_1(\gamma)}(-r) \ge \frac{\sqrt{2}}{\pi\sqrt{\sigma}}\sqrt{r}$. Since σ is as small as we wish here, the Green function is not optimally smooth. □

7 Parreau–Widom Sets

Parreau–Widom sets are of special interest in the recent spectral theory of orthogonal polynomials. For different aspects of the theory, we refer the reader to the articles [12,17,33,37].

A compact set $K \subset \mathbb{R}$ which is regular with respect to the Dirichlet problem is called a *Parreau–Widom* set if $PW(K) := \sum_j G_{\overline{\mathbb{C}}\setminus K}(c_j) < \infty$ where $\{c_j\}$ is the set of critical points of $G_{\overline{\mathbb{C}}\setminus K}$, which, clearly, is at most countable. A Parreau–Widom set has always positive Lebesgue measure, see [12].

Our aim is to give a criterion for $K_1(\gamma)$ to be a Parreau–Widom set. Note that, since autonomous Julia-Cantor sets in $\mathbb{R}$ have zero Lebesgue measure (see e.g. Section 1.19. in [22]), such sets cannot be Parreau–Widom.

We begin with a technical lemma.

Lemma 7.1 *Given $p \in \mathbb{N}$, let $b_0 = 1$ and $b_{k+1} = b_k(1+4^{-p+k}\,b_k)$ for $0 \le k \le p-1$. Then $b_p < 2$.*

Proof We have $b_1 = 1+4^{-p}$, $b_2 = 1+(1+4)\,4^{-p}+2\cdot 4\cdot 4^{-2p}+4\cdot 4^{-3p}, \ldots$, so $b_k = \sum_{n=0}^{N_k} a_{n,k}4^{-np}$ with $N_k = 2^k - 1$ and $a_{0,k} = 1$. Let $a_{n,k} := 0$ if $n > N_k$. The definition of b_{k+1} gives the recurrence relation

$$a_{n,k+1} = a_{n,k} + 4^k \sum_{j=1}^{n} a_{n-j,k}\, a_{j-1,k} \quad \text{for} \quad 1 \le n \le N_{k+1}. \tag{7.1}$$

If $N_k < n \le N_{k+1}$, that is $n = N_k+m$ with $1 \le m \le N_k+1$, then the formula takes the form $a_{n,k+1} = 4^k \sum_{j=m}^{n-m+1} a_{n-j,k}\, a_{j-1,k}$, since $a_{n-j,k} = 0$ for $j < m$ and $a_{j-1,k} = 0$ for $j > n-m+1$. In particular, $a_{N_{k+1},k+1} = 4^k\, a^2_{N_k,k}$ and $a_{1,k+1} = a_{1,k} + 4^k$. Therefore, $a_{1,k} = 1+4+\cdots+4^{k-1} < 4^k/3$. Let us show that $a_{n,k} < C_n\, 4^{nk}$ with $C_n = 4^{1-n}/3$ for $n \ge 2$. This gives the desired result, as $b_p = \sum_{n=0}^{N_p} a_{n,p}4^{-np} < 1 + 1/3 \cdot \sum_{n=1}^{N_p} 4^{1-n} < 2$.

By induction, suppose the inequality $a_{j,k} < C_j\, 4^{jk}$ is valid for $1 \le j \le n-1$ and for all $k > 0$. We consider $j = n$. The bound $a_{n,i} < C_n\, 4^{ni}$ is valid for $i = 1$, as $a_{n,1} = 0$ for $n \ge 2$. Suppose it is valid as well for $i \le k$.

We use (7.1) repeatedly, in order to reduce the second index, and, after this, the induction hypothesis:

$$a_{n,k+1} = \sum_{q=1}^{k} 4^q \sum_{j=1}^{n} a_{n-j,q}\, a_{j-1,q} < \sum_{q=1}^{k} 4^{nq} \sum_{j=1}^{n} C_{n-j}\, C_{j-1} < \sum_{q=1}^{k} 4^{nq} < C_n\, 4^{n(k+1)},$$

where $C_0 := 1$. Therefore the desired bound is valid for all positive n and k. □

Theorem 7.2 *$K_1(\gamma)$ is a Parreau–Widom set if and only if $\sum_{n=1}^{\infty} \sqrt{\epsilon_n} < \infty$.*

Proof Let $E_n = \{z \in \mathbb{C} : |F_n(z)| \le 1\}$. Then $G_{\overline{\mathbb{C}}\setminus E_n}(z) = 2^{-n} \log |F_n(z)|$ for $z \notin E_n$. Clearly, the critical points of $G_{\overline{\mathbb{C}}\setminus E_n}$ coincide with the critical points of F_n and thus they are real. Let $Y_n = \{x : F_n^{'}(x) = 0\}$, $Z_n = \{x : F_n(x) = 0\}$. We see at once that $Y_n \cap Z_n = \emptyset$ and $Z_k \cap Z_n = \emptyset$ for $n \ne k$. Since $F_n^{'} = F_{n-1} F_{n-1}^{'}/\gamma_n$, we have $Y_n = Y_{n-1} \cup Z_{n-1}$, so $Y_n = Z_{n-1} \cup Z_{n-2} \cup \cdots \cup Z_0$, where $Z_0 = \{0\}$. Therefore, the critical points of $G_{\overline{\mathbb{C}}\setminus E_n}$ are also critical for $G_{\overline{\mathbb{C}}\setminus E_{n+1}}$. Of course, $G_{\overline{\mathbb{C}}\setminus E_n} \nearrow G_{\overline{\mathbb{C}}\setminus K_1(\gamma)}$, so the set of critical points for $G_{\overline{\mathbb{C}}\setminus K_1(\gamma)}$ is $\cup_{n=0}^{\infty} Z_n$. It follows that $PW(K_1(\gamma)) = \sum_{n=1}^{\infty} \sum_{z \in Z_{n-1}} G_{\overline{\mathbb{C}}\setminus K_1(\gamma)}(z)$. In addition, for each $k \ge n$ the function F_k is constant on the set Z_{n-1} which contains 2^{n-1} points. By Theorem 2.1(*c*), the Green function is also constant on this set. Let $s_n = 2^{n-1} G_{\overline{\mathbb{C}}\setminus K_1(\gamma)}(z)$, where z is any point from Z_{n-1}. Then

$$PW(K_1(\gamma)) = \sum_{n=1}^{\infty} s_n.$$

We can certainly assume that $\sum_{n=1}^{\infty} \epsilon_n < \infty$. Indeed, it is immediate if $\sum_{n=1}^{\infty} \sqrt{\epsilon_n} < \infty$. On the other hand, if $z \in Z_{n-1}$, that is $F_{n-1}(z) = 0$, then $F_n(z) = 1 - \frac{1}{2\gamma_n} = -1 - \frac{8\epsilon_n}{1-4\epsilon_n}$. Since $G_{\overline{\mathbb{C}}\setminus E_n} \nearrow G_{\overline{\mathbb{C}}\setminus K_1(\gamma)}$, we have $s_n > 1/2 \log |F_n(z)| > 1/2 \log(1 + 8\epsilon_n) > 2\epsilon_n$, as $\log(1 + t) > t/2$ for $0 < t < 2$. Therefore the supposition $PW(K_1(\gamma)) < \infty$ implies that $\sum_{n=1}^{\infty} \epsilon_n < \infty$.

Let $a = \prod_{n=1}^{\infty}(1 - 4\epsilon_n)$. Since $\epsilon_n \in (0, \frac{1}{4})$ is a term of convergent series, we have $0 < a < 1$.

Our aim is to evaluate s_n from both sides for large enough n. From now on we consider only n such that $\epsilon_n \le a/36$. Then $1 - 4\epsilon_n > 8/9$ and for $\sigma_n := \frac{8\epsilon_n}{1-4\epsilon_n}$ we have $0 < \sigma_n < 1/4$. Given n, we fix $p = p(n) \in \mathbb{N}$ with

$$a \cdot 4^{-1-p} < \sigma_n \le a \cdot 4^{-p}. \tag{7.2}$$

This gives

$$\frac{1}{\sqrt{a}}\sqrt{\sigma_n} \le 2^{-p} < \frac{2}{\sqrt{a}}\sqrt{\sigma_n}. \tag{7.3}$$

Clearly, $\sum_{n=1}^{\infty} \sqrt{\epsilon_n} < \infty$ if and only if $\sum_{n=1}^{\infty} \sqrt{\sigma_n} < \infty$.

Consider the function $f(t) = \frac{1}{2\beta}(t^2 - 1) + 1$ for $t > 1$, where $\beta = 1/4 - \epsilon$ with $\epsilon < 1/36$. If $t = 1 + \sigma$ with small σ, then we use the representation $f(t) = 1 + \sigma_1$

where $4\sigma < \sigma_1 = 4\sigma\ \frac{1+\sigma/2}{1-4\epsilon}$. On the other hand, for large t we have $t^2 \le f(t) < \frac{1}{2\beta}\,t^2 < \frac{9}{4}\,t^2$.

Let us fix $z \in Z_{n-1}$. Then, as above, $|F_n(z)| = 1+\sigma_n$. Clearly, $F_{n+1}(z) = f(F_n(z))$ with $\beta = \gamma_{n+1}$. Hence, $F_{n+1}(z) = 1 + \sigma_{n+1}$ with $4\sigma_n < \sigma_{n+1} = 4\sigma_n\ \frac{1+\sigma_n/2}{1-4\epsilon_{n+1}}$. We continue in this fashion to obtain $F_{n+p}(z) = 1 + \sigma_{n+p}$ with

$$4^p\,\sigma_n < \sigma_{n+p} = 4^p\,\sigma_n \cdot \prod_{k=n}^{n+p-1} \frac{1+\sigma_k/2}{1-4\epsilon_{k+1}} < a^{-1}\,4^p\,\sigma_n \cdot \prod_{k=n}^{n+p-1} (1+\sigma_k/2). \quad (7.4)$$

After that we use the second estimation for f. This gives $F^2_{n+p}(z) \le F_{n+p+1}(z) < \frac{9}{4}\,F^2_{n+p}(z)$ and, for each $k \in \mathbb{N}$,

$$F^{2^k}_{n+p}(z) \le F_{n+p+k}(z) < (9/4)^{2^k-1}\,F^{2^k}_{n+p}(z).$$

From this, we have

$$2^{-n-p}\,\log F_{n+p}(z) \le G_{\overline{\mathbb{C}}\setminus E_{n+p+k}}(z) \le 2^{-n-p}[\log(9/4) + \log F_{n+p}(z)].$$

Recall that $G_{\overline{\mathbb{C}}\setminus E_{n+p+k}}(z) \nearrow G_{\overline{\mathbb{C}}\setminus K_1(\gamma)}(z)$ as $k \to \infty$. Also $s_n = 2^{n-1}G_{\overline{\mathbb{C}}\setminus K_1(\gamma)}(z)$ and $F_{n+p}(z) = 1 + \sigma_{n+p}$. Hence,

$$2^{-p-1}\,\log(1+\sigma_{n+p}) \le s_n \le 2^{-p-1}[\log(9/4) + \log(1+\sigma_{n+p})]. \quad (7.5)$$

We are in a position to prove the statement of the theorem. Suppose that $K_1(\gamma)$ is a Parreau–Widom set, so the series $\sum_{n=1}^{\infty} s_n$ converges. By (7.5) and (7.4), we have $s_n \ge 2^{-p-1}\,\log(1+4^p\,\sigma_n)$. By (7.2), $4^p\,\sigma_n < 1$ and $\log(1+4^p\,\sigma_n) > 4^p\,\sigma_n/2$. Therefore, $s_n \ge 2^p\,\sigma_n/4$. Finally, we use (7.3) to obtain $s_n \ge \sqrt{a\,\sigma_n}/8$, which implies the convergence of $\sum_{n=1}^{\infty}\sqrt{\sigma_n}$ and $\sum_{n=1}^{\infty}\sqrt{\epsilon_n}$.

Conversely, suppose that $\sum_{n=1}^{\infty}\sqrt{\sigma_n} < \infty$. By (7.5), $s_n \le 2^{-p}\log(3/2) + 2^{-p-1}\sigma_{n+p}$. Here, by (7.3), the series $\sum_{n=1}^{\infty} 2^{-p(n)}$ converges. For the addend, (7.4) implies

$$2^{-p-1}\sigma_{n+p} < (2a)^{-1} \cdot 2^p\,\sigma_n \prod_{k=n}^{n+p-1} (1+\sigma_k/2).$$

From (7.3) it follows that $2^p\,\sigma_n \le \sqrt{a\sigma_n}$, a term of convergent series. Let us show that

$$\prod_{k=n}^{n+p-1} (1+\sigma_k/2) < 2. \quad (7.6)$$

This will give the convergence of $\sum_{n=1}^{\infty} s_n$, which is the desired conclusion.

We use notations of Lemma 7.1. By (7.2), we have $1+\sigma_n/2 \le 1+a\,4^{-p}/2 < b_1$. Then

$$1+\sigma_{n+1}/2 < 1+\frac{a}{1-4\epsilon_{n+1}}\,4^{-p+1}(1+\sigma_n/2) < 1+4^{-p+1}\,b_1 = b_2/b_1$$

and $(1+\sigma_n/2)(1+\sigma_{n+1}/2) < b_2$. Similarly, by (7.4) and (7.2),

$$1+\sigma_{n+k+1}/2 < 1+\frac{a}{(1-4\epsilon_{n+1})\cdots(1-4\epsilon_{n+k})}\,4^{-p+k}\,b_k < b_{k+1}/b_k$$

for $k \le p-2$. Lemma 7.1 now yields (7.6). □

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