International Journal of Geometric Methods in Modern Physics
Vol. 13, No. 7 (2016) 1650099 (14 pages)

DOI: 10.1142/S0219887816500997

The W_2-curvature tensor on warped product manifolds and applications

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Received 2 August 2015
Accepted 27 April 2016
Published 7 June 2016

The purpose of this paper is to study the W_2-curvature tensor on (singly) warped product manifolds as well as on generalized Robertson–Walker and standard static space-times. Some different expressions of the W_2-curvature tensor on a warped product manifold in terms of its relation with W_2-curvature tensor on the base and fiber manifolds are obtained. Furthermore, we investigate W_2-curvature flat warped product manifolds. Many interesting results describing the geometry of the base and fiber manifolds of a W_2-curvature flat warped product manifold are derived. Finally, we study the W_2-curvature tensor on generalized Robertson–Walker and standard static space-times; we explore the geometry of the fiber of these warped product space-time models that are W_2-curvature flat.

Keywords: W_2-curvature; standard static space-time; generalized Robertson–Walker space-time; warped products.

Mathematics Subject Classification 2010: 53C21, 53C25, 53C50

1. Introduction

In [1], Pokhariyal and Mishra first defined the W_2-curvature tensor and they studied its physical and geometrical properties. Since then the concept of the W_2-curvature tensor has been studied as a research topic by mathematicians and physicists (see [2–5]). Pokhariyal defined many symmetric and skew-symmetric curvature tensors on the same line of the W_2-curvature tensor and studied various geometrical and physical properties of manifolds admitting these tensors in [3]. Among many of his results, we would like to mention that he proved that the vanishing of one of these curvature tensors in an electromagnetic field implies a purely electric field. Another study to establish applications of the W_2-curvature in the theory of general relativity was carried in [6] where the authors particularly prove that a space-time with vanishing W_2-curvature tensor is an Einstein manifold. They also consider the case of vanishing W_2-curvature tensor in relation with a perfect fluid space-time. In [2, 5], the authors study the properties of flat space-time under some conditions regarding the W_2-curvature tensor and W_2-flat space-times. Moreover, there are many studies regarding the geometrical meaning of the W_2-curvature tensor in different types of manifolds (see [7–10] and references therein).

The main aim of this paper is to study and explore the W_2-curvature tensor on warped product manifolds as well as on well-known warped product space-times. The concept of the W_2-curvature tensor has never been studied on warped products before this paper in which we intent to fill this gap in the literature by providing a complete study of the W_2-curvature tensor on such spaces.

This paper is organized as follows. In Sec. 2, we state well-known curvature related formulas of warped product manifolds and the W_2-curvature tensor properties on pseudo-Riemannian manifolds. We also define and study a new curvature tensor, $K(X,Y)Z$, that will be used in the characterization of the W_2-curvature tensor on pseudo-Riemannian manifolds. In Sec. 3, we explore the relation between the W_2-curvature tensor of a warped product manifold and that of the fiber and base manifolds. Section 4 is devoted to the study of the W_2-curvature tensor on generalized Robertson–Walker space-time and standard static space-time.

2. Preliminaries

In this section, we will provide basic definitions and curvature formulas about warped product manifolds.

Suppose that (M_1, g_1, D_1) and (M_2, g_2, D_2) are two $\mathcal{C}^\infty$-pseudo-Riemannian manifolds equipped with pseudo-Riemannian metric tensors g_i where D_i is the Levi-Civita connection of the metric g_i for $i = 1, 2$. Further suppose that $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$ are the natural projection maps of the Cartesian product $M_1 \times M_2$ onto M_1 and M_2, respectively. If $f : M_1 \to (0, \infty)$ is a positive real-valued smooth function, then the warped product manifold $M_1 \times_f M_2$ is the product manifold $M_1 \times M_2$ equipped with the metric tensor $g = g_1 \oplus f^2 g_2$ defined by

$$g = \pi_1^*(g_1) \oplus (f \circ \pi_1)^2 \pi_2^*(g_2),$$

where * denotes the pull-back operator on tensors [11, 12]. The function f is called the warping function of the warped product manifold $M_1 \times_f M_2$. In particular, if $f = 1$, then $M_1 \times_1 M_2 = M_1 \times M_2$ is the usual Cartesian product manifold. It is clear that the submanifold $M_1 \times \{q\}$ is isometric to M_1 for every $q \in M_2$. Moreover, $\{p\} \times M_2$ is homothetic to M_2. Throughout this paper we use the same notation for a vector field and for its lift to the product manifold. Let D, R and Ric be the

Levi–Civita connection, curvature tensor and Ricci curvature of the metric tensor g. Their formulas are well-known (see [11, 12]).

The W_2-curvature tensor on a pseudo-Riemannian manifold (M, g, D) is defined as follows [1]. Let $X, Y, Z, T \in \mathfrak{X}(M)$, then

$$W_2(X, Y, Z, T) = g(R(X, Y)Z, T) + \frac{1}{n-1}[g(X, Z)\mathrm{Ric}(Y, T) - g(Y, Z)\mathrm{Ric}(X, T)],$$

where $R(X, Y)Z = D_Y D_X Z - D_X D_Y Z + D_{[X,Y]} Z$ is the Riemann curvature tensor. It is clear that $W_2(X, Y, Z, T)$ is skew-symmetric in the first two positions. More explicitly, $W_2(X, Y, Z, T) = -W_2(Y, X, Z, T)$.

Now we redefine W_2-curvature tensor as follows. The W_2-curvature tensor, as shown above, is also given by

$$W_2(X, Y, Z, T) = g(K(X, Y)T, Z),$$

where

$$K(X, Y)T \colon = -R(X, Y)T + \frac{1}{n-1}[\mathrm{Ric}(Y, T)X - \mathrm{Ric}(X, T)Y].$$

The study of the W_2-curvature tensor on warped product manifolds contains large formulas and a huge amount of computations. Thus, this new tool will enable us to minimize computations in our study.

Remark 1. Let M be a pseudo-Riemannian manifold. Then

$$K(X, Y)T + K(T, X)Y + K(Y, T)X = 0$$

for any vector fields $X, Y, T \in \mathfrak{X}(M)$.

The following proposition is a direct consequence of the new definition of the W_2-curvature tensor.

Proposition 2. *Let M be a pseudo-Riemannian manifold. Then the W_2-curvature tensor vanishes if and only if the tensor K vanishes.*

Now, we will note that the tensor K can be simplified if the last position is a concurrent field. First, recall that a vector field ζ is called a concurrent vector field if

$$D_X \zeta = X,$$

for any vector field X. It is clear that a concurrent vector field is a conformal vector field with factor 2. Let ζ be a concurrent vector field, then

$$R(X, Y)\zeta = 0.$$

Now suppose that ζ is a concurrent vector field. Then

$$K(X, Y)\zeta = \frac{1}{n-1}[\mathrm{Ric}(Y, \zeta)X - \mathrm{Ric}(X, \zeta)Y].$$

Finally, a Riemannian metric g on a manifold M is said to be of Hessian type metric if there are two smooth functions k and σ such that $H^\sigma = kg$ where

H^σ is the Hessian of σ. This topic is closely related to the research of Shima on Hessian manifolds (see [13, 14]) and its extension to pseudo-Riemannian manifolds in [15, 16].

3. W_2-Curvature Tensor on Warped Product Manifolds

In this section, we provide an extensive study of W_2-curvature tensor on (singly) warped product manifolds. Throughout the section, (M, g, D) is a (singly) warped product manifold of $(M_i, g_i, D_i), i = 1, 2$ with dimensions $n_i \neq 1$ where $n = n_1 + n_2$. R, R^i denote the curvature tensor and $\mathrm{Ric}, \mathrm{Ric}^i$ denote the Ricci curvature tensor on M, M^i, respectively. Moreover, ∇f denotes the gradient and Δf denotes Laplacian of f on M_1, and also the Hessian of f on M_1 is denoted by H^f. The sharp of f is given by $f^\sharp = f\Delta f + (n_2 - 1)g_1(\nabla f, \nabla f)$. Finally, W_2-curvature tensor and the tensor K on M and M_i are denoted by W_2, K and W_2^i, K^i, respectively for $i = 1, 2$.

The following theorem provides a full description of the W_2-curvature tensor on (singly) warped product manifolds. The proof contains long computations that can be done using previous results on warped product manifolds (see Appendix A).

Theorem 3. *Let $M = M_1 \times_f M_2$ be a singly warped product manifold with the metric tensor $g = g_1 \oplus f^2 g_2$. If $X_i, Y_i, T_i \in \mathfrak{X}(M_i)$ for $i = 1, 2$, then*

$$\begin{aligned} K(X_1, Y_1)T_1 &= K^1(X_1, Y_1)T_1 \\ &\quad - \frac{n_2}{(n-1)(n_1-1)}[\mathrm{Ric}^1(Y_1, T_1)X_1 - \mathrm{Ric}^1(X_1, T_1)Y_1] \\ &\quad - \frac{1}{n-1}\left[\frac{n_2}{f}H^f(Y_1, T_1)X_1 - \frac{n_2}{f}H^f(X_1, T_1)Y_1\right], \end{aligned} \tag{1}$$

$$K(X_1, Y_1)T_2 = K(X_2, Y_2)T_1 = 0, \tag{2}$$

$$K(X_1, Y_2)T_1 = -\left[\frac{1}{n-1}\mathrm{Ric}^1(X_1, T_1) - \frac{n+n_2-1}{(n-1)f}H^f(X_1, T_1)\right]Y_2, \tag{3}$$

$$\begin{aligned} K(X_1, Y_2)T_2 &= -f g_2(Y_2, T_2)D^1_{X_1}\nabla f + \frac{1}{n-1}\mathrm{Ric}^2(Y_2, T_2)X_1 \\ &\quad - \frac{f^\sharp}{n-1}g_2(Y_2, T_2)X_1, \end{aligned} \tag{4}$$

$$\begin{aligned} K(X_2, Y_2)T_2 &= K^2(X_2, Y_2)T_2 \\ &\quad - \frac{n_1}{(n-1)(n_2-1)}[\mathrm{Ric}^2(Y_2, T_2)X_2 - \mathrm{Ric}^2(X_2, T_2)Y_2] \\ &\quad + \left(\|\nabla f\|_1^2 + \frac{f^\sharp}{n-1}\right)[g_2(X_2, T_2)Y_2 - g_2(Y_2, T_2)X_2]. \end{aligned} \tag{5}$$

In the following part we investigate the geometry of the base factor of the warped product when the product is W_2-curvature flat.

Theorem 4. *Let $M = M_1 \times_f M_2$ be a W_2-curvature flat singly warped product manifold with the metric tensor $g = g_1 \oplus f^2 g_2$. Then*

$$W_2^1(X_1, Y_1, Z_1, T_1) = \frac{2n_2}{(n_1-1)f}[H^f(Y_1, T_1)g_1(X_1, Z_1) - H^f(X_1, T_1)g_1(Y_1, Z_1)] \tag{6}$$

for any vector fields $X_1, Y_1, Z_1, T_1 \in \mathfrak{X}(M_1)$.

Proof. Suppose that M is W_2-curvature flat. Then Eqs. (1) and (3) imply that

$$0 = K^1(X_1, Y_1)T_1 - \frac{n_2}{(n-1)(n_1-1)}[\text{Ric}^1(Y_1, T_1)X_1 - \text{Ric}^1(X_1, T_1)Y_1]$$
$$- \frac{1}{n-1}\left[\frac{n_2}{f}H^f(Y_1, T_1)X_1 - \frac{n_2}{f}H^f(X_1, T_1)Y_1\right],$$
$$0 = \frac{1}{n-1}\text{Ric}^1(X_1, T_1) - \frac{n_1+2n_2-1}{(n-1)f}H^f(X_1, T_1).$$

Now, from the second equation we have

$$\text{Ric}^1(X_1, T_1) = \frac{n_1+2n_2-1}{f}H^f(X_1, T_1). \tag{7}$$

Using this identity in the first equation which eventually turns out to be:

$$K^1(X_1, Y_1)T_1 = \frac{n_2}{(n-1)(n_1-1)}\left[\frac{n_1+2n_2-1}{f}H^f(Y_1, T_1)X_1 - \frac{n_1+2n_2-1}{f}H^f(X_1, T_1)Y_1\right]$$
$$+ \frac{n_2}{n-1}\left[\frac{1}{f}H^f(Y_1, T_1)X_1 - \frac{1}{f}H^f(X_1, T_1)Y_1\right]$$
$$= \frac{2n_2^2}{(n-1)(n_1-1)f}[H^f(Y_1, T_1)X_1 - H^f(X_1, T_1)Y_1].$$

Thus

$$W_2^1(X_1, Y_1, Z_1, T_1) = \frac{2n_2}{(n_1-1)f}[H^f(Y_1, T_1)g_1(X_1, Z_1) - H^f(X_1, T_1)g_1(Y_1, Z_1)].$$

□

Theorem 5. *Let $M = M_1 \times_f M_2$ be a W_2-curvature flat singly warped product manifold with the metric tensor $g = g_1 \oplus f^2 g_2$. Then:*

(1) *M_1 is W_2-curvature flat if and only if $H^f(X_1, Y_1) = 0$ for any vector fields $X_1, Y_1 \in \mathfrak{X}(M_1)$.*
(2) *the scalar curvature S_1 of M_1 is given by*

$$S_1 = \frac{n_1+2n_2-1}{f}\Delta f.$$

(3) *the scalar curvature of M_1 vanishes if M_1 is W_2-curvature flat.*

Proof. The proof just follows from Eqs. (6) and (7). □

Now, we study the geometry of the fiber factor of a warped product admitting flat W_2-curvature.

Theorem 6. *Let $M = M_1 \times_f M_2$ be a singly warped product manifold with the metric tensor $g = g_1 \oplus f^2 g_2$. Assume that f satisfies $H^f = 0$. Then, M is W_2-curvature flat if and only if both M_1 and M_2 are flat and $\nabla f = 0$.*

Proof. Suppose that M is W_2-curvature flat, then M_1 is flat due to Eq. (7) and the first item of Theorem 5. Moreover, from Theorem 3 we have

$$0 = -f g_2(Y_2, T_2) D^1_{X_1} \nabla f + \frac{1}{n-1} \mathrm{Ric}^2(Y_2, T_2) X_1 - \frac{f^\sharp}{n-1} g_2(Y_2, T_2) X_1,$$

$$\begin{aligned} 0 = K^2(X_2, Y_2) T_2 &- \frac{n_1}{(n-1)(n_2-1)} [\mathrm{Ric}^2(Y_2, T_2) X_2 - \mathrm{Ric}^2(X_2, T_2) Y_2] \\ &+ \left(\|\nabla f\|_1^2 + \frac{f^\sharp}{n-1} \right) [g_2(X_2, T_2) Y_2 - g_2(Y_2, T_2) X_2]. \end{aligned}$$

Since $H^f(X_1, Y_1) = 0$, the first equation becomes

$$\mathrm{Ric}^2(Y_2, T_2) = f^\sharp g_2(Y_2, T_2),$$

where $f^\sharp = f \Delta f + (n_2 - 1) g_1(\nabla f, \nabla f) = (n_2 - 1) c^2$ where $c^2 = g_1(\nabla f, \nabla f)$, i.e. M_2 is Einstein with factor $\mu = (n_2 - 1) c^2$ and

$$\mathrm{Ric}^2(Y_2, T_2) = (n_2 - 1) c^2 g_2(Y_2, T_2).$$

The second equation becomes

$$K^2(X_2, Y_2) T_2 = \frac{2(n_2 - 1) c^2}{(n-1)} [g_2(Y_2, T_2) X_2 - g_2(X_2, T_2) Y_2].$$

Thus the W_2-curvature tensor of M_2 is given by

$$W_2^2(X_2, Y_2, Z_2, T_2) = \frac{2(n_2 - 1) c^2}{(n-1)} [g_2(Y_2, T_2) g_2(X_2, Z_2) - g_2(X_2, T_2) g_2(Y_2, Z_2)].$$

But

$$\begin{aligned} W_2^2(X_2, Y_2, Z_2, T_2) &= R^2(X_2, Y_2, Z_2, T_2) \\ &\quad + \frac{1}{n_2 - 1} [g_2(X_2, Z_2) \mathrm{Ric}^2(Y_2, T_2) - g_2(Y_2, Z_2) \mathrm{Ric}^2(X_2, T_2)] \\ &= R^2(X_2, Y_2, Z_2, T_2) \\ &\quad + c^2 [g_2(X_2, Z_2) g_2(Y_2, T_2) - g_2(Y_2, Z_2) g_2(X_2, T_2)]. \end{aligned}$$

Therefore,

$$R^2(X_2, Y_2, Z_2, T_2) = \frac{(n_2 - n_1 - 1) c^2}{(n-1)} [g_2(X_2, Z_2) g_2(Y_2, T_2) - g_2(Y_2, Z_2) g_2(X_2, T_2)],$$

i.e. M_2 has a constant sectional curvature

$$\kappa_2 = \frac{(n_2 - n_1 - 1)c^2}{(n-1)}.$$

But the Einstein factor should be $(n_2 - 1)\kappa_2$ and hence

$$n_1(n_2 - 1)c^2 = 0.$$

Thus M_2 is flat. The converse is straightforward. □

Theorem 7. *Let $M = M_1 \times_f M_2$ be a W_2-curvature flat singly warped product manifold with the metric tensor $g = g_1 \oplus f^2 g_2$. If M_2 is Ricci flat, then the W_2-curvature of M_2 is given by*

$$\begin{aligned} &W_2^2(X_2, Y_2, T_2, Z_2) \\ &\quad = \left(\|\nabla f\|_1^2 + \frac{f^\sharp}{n-1} \right) [g_2(X_2, T_2) g_2(Y_2, Z_2) - g_2(Y_2, T_2) g_2(X_2, Z_2)] \end{aligned}$$

and M_1 is of Hessian type. Moreover, M_2 is flat if $n_2 \geq 3$.

Proof. Suppose that M is W_2-curvature flat, then from Theorem 3 we have

$$\begin{aligned} 0 &= -f g_2(Y_2, T_2) D^1_{X_1} \nabla f + \frac{1}{n-1} \mathrm{Ric}^2(Y_2, T_2) X_1 - \frac{f^\sharp}{n-1} g_2(Y_2, T_2) X_1, \\ 0 &= K^2(X_2, Y_2) T_2 - \frac{n_1}{(n-1)(n_2-1)} [\mathrm{Ric}^2(Y_2, T_2) X_2 - \mathrm{Ric}^2(X_2, T_2) Y_2] \\ &\quad + \left(\|\nabla f\|_1^2 + \frac{f^\sharp}{n-1} \right) [g_2(X_2, T_2) Y_2 - g_2(Y_2, T_2) X_2]. \end{aligned}$$

Now suppose that M_2 is Ricci flat, then the first equation implies that

$$D^1_{X_1} \nabla f = \frac{-f^\sharp}{(n-1)f} X_1$$

and so

$$H^f = \frac{-f^\sharp}{(n-1)f} g_1,$$

i.e. M_1 is of Hessian type. The second equation implies that

$$K^2(X_2, Y_2) T_2 = \left(\|\nabla f\|_1^2 + \frac{f^\sharp}{n-1} \right) [g_2(X_2, T_2) Y_2 - g_2(Y_2, T_2) X_2]$$

and hence

$$\begin{aligned} &W_2^2(X_2, Y_2, T_2, Z_2) \\ &\quad = \left(\|\nabla f\|_1^2 + \frac{f^\sharp}{n-1} \right) [g_2(X_2, T_2) g_2(Y_2, Z_2) - g_2(Y_2, T_2) g_2(X_2, Z_2)]. \end{aligned}$$

Moreover,

$$R^2(X_2, Y_2, T_2, Z_2) = \left(\|\nabla f\|_1^2 + \frac{f^\sharp}{n-1}\right)[g_2(X_2, T_2)g_2(Y_2, Z_2) - g_2(Y_2, T_2)g_2(X_2, Z_2)].$$

Thus M_2 has a pointwise constant sectional curvature given by

$$\kappa_2 = \|\nabla f\|_1^2 + \frac{f^\sharp}{n-1}.$$

If $n_2 \geq 3$, then by Schur's Lemma, M_2 has a vanishing constant sectional curvature $\kappa_2 = 0$ since M_2 is Ricci flat. □

4. W_2-Curvature on Space-Times

The study of W_2-curvature tensor on space-times is of great interest since this concept provides an access to several geometrical and physical properties of space-times. Among such applications, we want to mention that a W_2-curvature flat 4-dimensional space-time is an Einstein manifold [2, 5]. This section is subsequently devoted to the study of the W_2-curvature tensor on generalized Robertson–Walker space-times and standard static space-times. We will first consider some classical space-times. Obtaining the W_2-curvature tensor for these space-times contains long computations, and hence we omitted them.

- The Minkowski space-time is W_2-curvature flat since it is flat.
- The Friedman–Robertson–Walker with metric
$$ds^2 = -c^2dt^2 + a(t)\left[\frac{d\eta^2}{1-k\eta^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$
is W_2-curvature flat if $\dot{a}(t) = k = 0$.
- The de Sitter space-time metric with cosmological constant $\Lambda > 0$ in conformally flat coordinates reads
$$ds^2 = \frac{\alpha^2}{\tau^2}[-d\tau^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)], \tag{8}$$
where $\alpha^2 = (3/\Lambda)$. This metric is Einstein with factor $\frac{3}{\alpha^2}$ and has a constant sectional curvature $\frac{1}{\alpha^2}$. The non-vanishing components of the W_2-curvature tensor are
$$\begin{aligned} W_2(\partial_i, \partial_j, \partial_i, \partial_j) &= R(\partial_i, \partial_j, \partial_i, \partial_j) + \frac{1}{3}(g(\partial_i, \partial_i)\mathrm{Ric}(\partial_j, \partial_j)) \\ &= R(\partial_i, \partial_j, \partial_i, \partial_j) + \frac{1}{\alpha^2}(g(\partial_i, \partial_i)g(\partial_j, \partial_j)) \\ &= 2R(\partial_i, \partial_j, \partial_i, \partial_j), \\ W_2(\partial_i, \partial_j, \partial_j, \partial_i) &= -W_2(\partial_i, \partial_j, \partial_i, \partial_j), \end{aligned}$$

where $i \neq j$. Direct computations show that the de Sitter space-time with metric (8) is not W_2-curvature flat. Similarly, the anti-de Sitter is not W_2-curvature flat.

- Kasner space-time in (t, x, y, z) coordinates is given by

$$ds^2 = -dt^2 + t^{2\lambda_1} dx^2 + t^{2\lambda_2} dy^2 + t^{2\lambda_3} dz^2,$$

where $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$. This space-time is W_2-curvature flat if $\lambda_1 = 1$.

- The Schwarzschild metric is given by

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) c^2 dt^2 + \left(\frac{1}{1 - \frac{r_s}{r}}\right) dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where r_s is the Schwarzchild radius and c is the speed of light. The Ricci curvatures are all identically zero and so the W_2-curvature tensor is equal to the Riemann tensor.

- A cylindrically symmetric static space-time in (t, r, θ, ϕ) coordinates can be given by

$$ds^2 = -e^v dt^2 + dr^2 + e^v d\theta^2 + e^v d\phi^2,$$

where v is a function of r. A cylindrically symmetric static space-time is W_2-curvature flat if and only if v is constant. If v is a nontrivial function of r, θ, ϕ the situation is more complicated.

4.1. *W_2-curvature on generalized Robertson–Walker space-times*

We first define generalized Robertson–Walker space-times. Let (M, g) be an n-dimensional Riemannian manifold and $f : I \to (0, \infty)$ be a smooth function. Then $(n+1)$-dimensional product manifold $I \times M$ furnished with the metric tensor

$$\bar{g} = -dt^2 \oplus f^2 g$$

is called a generalized Robertson–Walker space-time and is denoted by $\bar{M} = I \times_f M$ where I is an open, connected subinterval of $\mathbb{R}$ and dt^2 is the Euclidean metric tensor on I. This structure was introduced to the literature to extend Robertson–Walker space-times [17–20].

From now on, we will denote $\frac{\partial}{\partial t} \in \mathfrak{X}(I)$ by ∂_t to state our results in simpler forms.

Theorem 8. *Let $\bar{M} = I \times_f M$ be a generalized Robertson–Walker space-time equipped with the metric tensor $\bar{g} = -dt^2 \oplus f^2 g$. Then the curvature tensor $\bar{K}$ on $\bar{M}$ is given by*

(1) $\bar{K}(\partial_t, \partial_t)\partial_t = \bar{K}(\partial_t, \partial_t)X = \bar{K}(X, Y)\partial_t = 0$,

(2) $\bar{K}(\partial_t, X)\partial_t = -\frac{\ddot{f}}{f} X$,

(3) $\bar{K}(X, \partial_t)Y = [\frac{n-1}{n} g(X, Y)(f\ddot{f} - \dot{f}^2) - \frac{1}{n}\mathrm{Ric}(X, Y)]\partial_t$,

(4) $\bar{K}(X,Y)Z = -R(X,Y)Z + \dot{f}^2[g(Y,Z)X - g(X,Z)Y] + \frac{1}{n}[\mathrm{Ric}(Y,Z)X - \mathrm{Ric}(X,Z)Y] + \frac{1}{n}[g(Y,Z)X - g(X,Z)Y](f\ddot{f} + (n-1)\dot{f}^2)$

for any $X,Y,Z \in \mathfrak{X}(M)$.

Now we investigate the implications of a W_2-curvature flat generalized Robertson–Walker space-time to its fiber.

Theorem 9. *Let* $\bar{M} = I \times_f M$ *be a generalized Robertson–Walker space-time equipped with the metric tensor* $\bar{g} = -dt^2 \oplus f^2 g$. *Then,* $\bar{M}$ *is* W_2*-curvature flat if and only if* M *has a constant sectional curvature* $\kappa = -\dot{f}^2$.

Proof. Assume that $\bar{M} = I \times_f M$ is W_2-curvature flat, then

$$\begin{aligned}
0 &= -f\ddot{f}g(X,Y),\\
0 &= \frac{1}{n}\mathrm{Ric}(X,Y) - \frac{n-1}{n}g(X,Y)(f\ddot{f} - \dot{f}^2),\\
0 &= -f^2R(X,Y,Z,T) + f^2\dot{f}^2[g(Y,Z)g(X,T) - g(X,Z)g(Y,T)]\\
&\quad + \frac{f^2}{n}[\mathrm{Ric}(Y,Z)g(X,T) - \mathrm{Ric}(X,Z)g(Y,T)]\\
&\quad + \frac{f^2}{n}[g(Y,Z)g(X,T) - g(X,Z)g(Y,T)](f\ddot{f} + (n-1)\dot{f}^2).
\end{aligned}$$

The first equation implies that $\ddot{f} = 0$, i.e. $f = \mu t + \lambda$ and so the second equation yields

$$\mathrm{Ric}(X,Y) = -\mu^2(n-1)g(X,Y).$$

The third equation implies that

$$R(X,Y,Z,T) = \mu^2[g(Y,Z)g(X,T) - g(X,Z)g(Y,T)].$$

Thus the sectional curvature of M is

$$\kappa = -\mu^2.$$

The converse is direct by using the fact that $\bar{M}$ is Einstein with factor $(n-1)\kappa$. □

A 4-dimensional space-time is called Petrov type O if the Weyl conformal tensor vanishes. There are many generalizations of Petrov classification for higher dimensions (see for instance [21]) but type O still has the same definition. From the above theorem, we conclude that $\bar{M}$ is flat and hence the Weyl conformal tensor vanishes.

4.2. *W_2-curvature tensor on standard static space-times*

We begin by defining standard static space-times. Let (M,g) be an n-dimensional Riemannian manifold and $f : M \to (0,\infty)$ be a smooth function. Then

$(n+1)$-dimensional product manifold $I \times M$ furnished with the metric tensor

$$\bar{g} = -f^2 dt^2 \oplus g$$

is called a standard static space-time and is denoted by $\bar{M} = I_f \times M$ where I is an open, connected subinterval of $\mathbb{R}$ and dt^2 is the Euclidean metric tensor on I.

Note that standard static space-times can be considered as a generalization of the Einstein static universe[22–25].

Now, we are ready to study both K and W_2 tensors on $\bar{M} =_f I \times M$. The following two theorems describe both tensors on $\bar{M} =_f I \times M$.

Theorem 10. *Let $\bar{M} =_f I \times M$ be a standard static space-time with the metric tensor $\bar{g} = -f^2 dt^2 \oplus g$. If $\partial_t \in \mathfrak{X}(I)$ and $X, Y, Z \in \mathfrak{X}(M)$, then*

(1) $\bar{K}(\partial_t, \partial_t)\partial_t = \bar{K}(\partial_t, \partial_t)X = \bar{K}(X, Y)\partial_t = 0$,
(2) $\bar{K}(\partial_t, X)\partial_t = -f(D_X \nabla f + \frac{\Delta f}{n} X)$,
(3) $\bar{K}(\partial_t, X)Y = \frac{1}{n}(\mathrm{Ric}(X, Y) - (n+1)\frac{H^f(X,Y)}{f})\partial_t$,
(4) $\bar{K}(X, Y)Z = -R(X, Y)Z + \frac{1}{n}[\mathrm{Ric}(Y, Z)X - \mathrm{Ric}(X, Z)Y] + \frac{1}{nf}[-H^f(Y, Z)X + H^f(X, Z)Y]$.

Theorem 11. *Let $\bar{M} =_f I \times M$ be a standard static space-time with the metric tensor $\bar{g} = -f^2 dt^2 \oplus g$. Then, $\bar{M}$ is W_2-curvature flat if and only if M is flat and $H^f = -\frac{\Delta f}{n} g$.*

Proof. Suppose that $\bar{M} =_f I \times M$ is W_2-curvature flat, then the second item of Theorem 10 implies that

$$D_X \nabla f = -\frac{\Delta f}{n} X, \quad H^f = -\frac{\Delta f}{n} g.$$

Taking the trace of both sides implies $\Delta f = 0$ and consequently $H^f = 0$. The third item implies that

$$\mathrm{Ric}(X, Y) = 0$$

and so M is Ricci flat. The last item of Theorem 10 implies that

$$R(X, Y)Z = \frac{1}{n}[\mathrm{Ric}(Y, Z)X - \mathrm{Ric}(X, Z)Y] + \frac{1}{nf}[-H^f(Y, Z)X + H^f(X, Z)Y],$$
$$R(X, Y)Z = 0.$$

Thus M is flat. The converse is straightforward. □

Appendix A. A Proof of Theorem 3

Let $M = M_1 \times_f M_2$ be a warped product manifold equipped with the metric tensor $g = g_1 \oplus f^2 g_2$ where $\dim(M_i) = n_i, i = 1, 2$ and $n = n_1 + n_2$. Let $X_i, Y_i, Z_i, T_i \in$

$\mathfrak{X}(M_i)$ for $i = 1, 2$. Then

$$\begin{aligned} K(X_1, Y_1)T_1 &= -R(X_1, Y_1)T_1 + \frac{1}{n-1}[\mathrm{Ric}(Y_1, T_1)X_1 - \mathrm{Ric}(X_1, T_1)Y_1] \\ &= -R^1(X_1, Y_1)T_1 + \frac{1}{n-1}\left(\mathrm{Ric}^1(Y_1, T_1) - \frac{n_2}{f}H^f(Y_1, T_1)\right)X_1 \\ &\quad - \frac{1}{n-1}\left(\mathrm{Ric}^1(X_1, T_1) - \frac{n_2}{f}H^f(X_1, T_1)\right)Y_1 \\ &= -R^1(X_1, Y_1)T_1 + \frac{1}{n-1}[\mathrm{Ric}^1(Y_1, T_1)X_1 - \mathrm{Ric}^1(X_1, T_1)Y_1] \\ &\quad - \frac{1}{n-1}\left[\frac{n_2}{f}H^f(Y_1, T_1)X_1 - \frac{n_2}{f}H^f(X_1, T_1)Y_1\right] \\ &= K^1(X_1, Y_1)T_1 \\ &\quad - \frac{n_2}{(n-1)(n_1-1)}[\mathrm{Ric}^1(Y_1, T_1)X_1 - \mathrm{Ric}^1(X_1, T_1)Y_1] \\ &\quad - \frac{1}{n-1}\left[\frac{n_2}{f}H^f(Y_1, T_1)X_1 - \frac{n_2}{f}H^f(X_1, T_1)Y_1\right]. \end{aligned}$$

The second case is

$$\begin{aligned} K(X_1, Y_1)T_2 &= -R(X_1, Y_1)T_2 + \frac{1}{n-1}[\mathrm{Ric}(Y_1, T_2)X_1 - \mathrm{Ric}(X_1, T_2)Y_1] \\ &= 0. \end{aligned}$$

The third case is

$$\begin{aligned} K(X_1, Y_2)T_1 &= -R(X_1, Y_2)T_1 + \frac{1}{n-1}[\mathrm{Ric}(Y_2, T_1)X_1 - \mathrm{Ric}(X_1, T_1)Y_2] \\ &= \frac{1}{f}H^f(X_1, T_1)Y_2 - \frac{1}{n-1}\mathrm{Ric}^1(X_1, T_1)Y_2 + \frac{n_2}{(n-1)f}H^f(X_1, T_1)Y_2 \\ &= -\left[\frac{1}{n-1}\mathrm{Ric}^1(X_1, T_1) - \frac{n+n_2-1}{(n-1)f}H^f(X_1, T_1)\right]Y_2. \end{aligned}$$

The next case is

$$\begin{aligned} K(X_1, Y_2)T_2 &= -R(X_1, Y_2)T_2 + \frac{1}{n-1}[\mathrm{Ric}(Y_2, T_2)X_1 - \mathrm{Ric}(X_1, T_2)Y_2] \\ &= -fg_2(Y_2, T_2)D^1_{X_1}\nabla f + \frac{1}{n-1}\mathrm{Ric}^2(Y_2, T_2)X_1 \\ &\quad - \frac{f^\sharp}{n-1}g_2(Y_2, T_2)X_1. \end{aligned}$$

Also,

$$\begin{aligned} K(X_2, Y_2)T_1 &= -R(X_2, Y_2)T_1 + \frac{1}{n-1}[\mathrm{Ric}(Y_2, T_1)X_2 - \mathrm{Ric}(X_2, T_1)Y_2] \\ &= 0. \end{aligned}$$

Finally,

$$\begin{aligned}K(X_2,Y_2)T_2 &= -R(X_2,Y_2)T_2 + \frac{1}{n-1}[\mathrm{Ric}(Y_2,T_2)X_2 - \mathrm{Ric}(X_2,T_2)Y_2]\\ &= -R^2(X_2,Y_2)T_2 + \|\nabla f\|_1^2[g_2(X_2,T_2)Y_2 - g_2(Y_2,T_2)X_2]\\ &\quad + \frac{1}{n-1}[\mathrm{Ric}^2(Y_2,T_2) - f^\sharp g_2(Y_2,T_2)]X_2\\ &\quad - \frac{1}{n-1}[\mathrm{Ric}^2(X_2,T_2) - f^\sharp g_2(X_2,T_2)]Y_2.\end{aligned}$$

Then

$$\begin{aligned}K(X_2,Y_2)T_2 &= -R^2(X_2,Y_2)T_2 + \frac{1}{n-1}\mathrm{Ric}^2(Y_2,T_2)X_2 - \frac{1}{n-1}\mathrm{Ric}^2(X_2,T_2)Y_2\\ &\quad + \|\nabla f\|_1^2[g_2(X_2,T_2)Y_2 - g_2(Y_2,T_2)X_2]\\ &\quad - \frac{f^\sharp}{n-1}(g_2(Y_2,T_2)X_2 - g_2(X_2,T_2)Y_2)\end{aligned}$$

and so

$$\begin{aligned}K(X_2,Y_2)T_2 &= -R^2(X_2,Y_2)T_2 + \frac{1}{n-1}[\mathrm{Ric}^2(Y_2,T_2)X_2 - \mathrm{Ric}^2(X_2,T_2)Y_2]\\ &\quad - \left(\|\nabla f\|_1^2 + \frac{f^\sharp}{n-1}\right)[g_2(Y_2,T_2)X_2 - g_2(X_2,T_2)Y_2]\\ &= -R^2(X_2,Y_2)T_2 + \frac{1}{n-1}[\mathrm{Ric}^2(Y_2,T_2)X_2 - \mathrm{Ric}^2(X_2,T_2)Y_2]\\ &\quad + \left(\|\nabla f\|_1^2 + \frac{f^\sharp}{n-1}\right)[g_2(X_2,T_2)Y_2 - g_2(Y_2,T_2)X_2].\end{aligned}$$

Thus

$$\begin{aligned}&K(X_2,Y_2)T_2\\ &\quad = K^2(X_2,Y_2)T_2 - \frac{n_1}{(n-1)(n_2-1)}[\mathrm{Ric}^2(Y_2,T_2)X_2 - \mathrm{Ric}^2(X_2,T_2)Y_2]\\ &\qquad + \left(\|\nabla f\|_1^2 + \frac{f^\sharp}{n-1}\right)[g_2(X_2,T_2)Y_2 - g_2(Y_2,T_2)X_2]\end{aligned}$$

and the proof is now complete.

Acknowledgment

We would like to thank the referee for the careful review and the valuable comments, which provided insights that helped us to improve the quality of the paper.

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