The Signature of a Splice

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We study the behavior of the signature of colored links [6, 9] under the splice operation. We extend the construction to colored links in integral homology spheres and show that the signature is almost additive, with a correction term independent of the links. We interpret this correction term as the signature of a generalized Hopf link and give a simple closed formula to compute it.

1 Introduction

The *splice* of two links is an operation defined by Eisenbud and Neumann in [8], which generalizes several other operations on links such as connected sum, cabling, and disjoint union. The precise definition is given in Section 2.1 (see Definition 2.1), but the rough idea is as follows: the splice of two links $K' \cup L' \subset \mathbb{S}'$ and $K'' \cup L'' \subset \mathbb{S}''$ along the distinguished components K' and K'' is the link $L' \cup L''$ in the three-manifold $\mathbb{S}$ obtained by an appropriate gluing of the exteriors of K' and K''. There has been much interest in understanding the behavior of various link invariants under the splice operation. For example, the genus and the fiberability of a link are additive, in a suitable sense, under splicing [8]. The behavior of the Conway polynomial has been studied in [5], and more

recently the relation between the *L*-spaces in Heegaard–Floer homology and splicing has been addressed in [13]. The goal of this paper is to obtain a similar (non-)additivity statement for the multivariate signature of oriented colored links. As a consequence, we show that the conventional univariate Levine–Tristram signature of a splice depends on the *multivariate* signatures of the summands.

In Section 3.2, we define the *signature* of a colored link in an integral homology sphere. This is a natural generalization of the multivariate extension of the Levine–Tristram signature of a link in the three-sphere, considered in [6, 9]. The principal result of the paper is Theorem 2.2, expressing the signature of the splice of two links in terms of the signatures of the summands. We show that the signature is almost additive: there is a defect, but it depends only on some combinatorial data of the links (linking numbers), and not on the links themselves. Geometrically, this defect term appears as the multivariate signature of a certain generalized Hopf link, which is computed in Theorem 2.10. At the end of Section 2, we discuss a few applications of Theorem 2.2 and relate it to some previously known results: namely, we compute the signature of a satellite knot (see Section 2.4 and Theorem 2.12) and that of an iterated torus link (see Section 2.5 and Theorem 2.13). More precisely, we reduce the computation to the signature of cables over the unknot. We also show that the multivariate signature of a link can be computed by means of the conventional Levine–Tristram signature of an auxiliary link (see Section 2.6 and Theorem 2.15).

The paper is organized as follows. Section 2 is devoted to the detailed statement of main results, and the computation of the defect. In Section 3, we introduce the necessary background material on twisted intersection forms and construct the signature of colored links in integral homology spheres. The proofs of the main theorems are carried out in Sections 4 and 5, where the signature of the generalized Hopf links is computed.

2   Principal Results

2.1   The set-up

A μ-*colored link* is an oriented link L in an integral homology sphere $\mathbb{S}$ equipped with a surjective function $\pi_0(L) \twoheadrightarrow \{1, \ldots, \mu\}$, referred to as the *coloring*. The union of the components of L given the same color $i = 1, \ldots, \mu$ is denoted by L_i.

The *signature* of a μ-colored link L is a certain $\mathbb{Z}$-valued function σ_L defined on the *character torus*

$$\mathcal{T}^\mu := \big\{ (\omega_1, \ldots, \omega_\mu) \in (S^1)^\mu \subset \mathbb{C}^\mu \mid \omega_j = \exp(2\pi i \theta_j), \ \theta_j \in \mathbb{Q} \big\}, \tag{2.1}$$

see Definition 3.5 below for details. We let $\mathcal{T}^0 := \{1\} \in \mathbb{C}$. Note that $\mathcal{T}^\mu$ is an abelian group. If $\mu = 1$, the link L is monochrome and σ_L coincides with the restriction (to rational points) of the Levine–Tristram signature [23] (whose definition in terms of Seifert form extends naturally to links in homology spheres). Given a character $\omega \in \mathcal{T}^\mu$ and a vector $\lambda \in \mathbb{Z}^\mu$, we use the common notation $\omega^\lambda := \prod_{i=1}^\mu \omega_i^{\lambda_i}$.

Often, the components of L are split naturally into two groups, $L = L' \cup L''$, on which the coloring takes, respectively, μ' and μ'' values, $\mu' + \mu'' = \mu$. In this case, we regard σ_L as a function of two "vector" arguments $(\omega', \omega'') \in \mathcal{T}^{\mu'} \times \mathcal{T}^{\mu''}$. We use this notation freely, hoping that each time its precise meaning is clear from the context.

Clearly, in the definition of colored link, the precise set of colors is not very important; sometimes, we also admit the color 0. As a special case, we define a $(1, \mu)$-*colored link*

$$K \cup L = K \cup L_1 \cup \ldots \cup L_\mu$$

as a $(1+\mu)$-colored link in which K is the only component given the distinguished color 0. Here, we assume K connected; this component, considered distinguished, plays a special role in a number of operations.

In the following definition, for a $(1, \mu^*)$-colored link $K^* \cup L^* \subset \mathbb{S}^*$, $* = {'}$ or ${''}$, we denote by $T^* \subset \mathbb{S}^*$ a small tubular neighborhood of K^* disjoint from L^* and let $m^*, \ell^* \subset \partial T^*$ be, respectively, its meridian and longitude. (The latter is well defined as $\mathbb{S}^*$ is a homology sphere.)

Definition 2.1.   Given two $(1, \mu^*)$-colored links $K^* \cup L^* \subset \mathbb{S}^*$, $* = {'}$ or ${''}$, their *splice* is the $(\mu' + \mu'')$-colored link $L' \cup L''$ in the integral homology sphere

$$\mathbb{S} := (\mathbb{S}' \smallsetminus \mathrm{int}\, T') \cup_\varphi (\mathbb{S}'' \smallsetminus \mathrm{int}\, T''),$$

where the gluing homeomorphism $\varphi \colon \partial T' \to \partial T''$ takes m' and ℓ' to ℓ'' and m'', respectively.
□

2.2   The signature formula

Given a list (vector, etc.) $a_1, \ldots, a_i, \ldots, a_n$, the notation $a_1, \ldots, \hat{a}_i, \ldots, a_n$ designates that the ith element (component, etc.) has been removed. The complex conjugation is denoted by $\eta \mapsto \bar{\eta}$. The same notation applies to the elements of the character torus $\mathcal{T}^\mu$, where we have $\bar{\omega} = \omega^{-1}$.

The Signature of a Splice

$$\delta_{(1,1)}(\omega) \qquad\qquad \delta_{(1,2)}(\omega) \qquad\qquad \delta_{(2,3)}(\omega)$$
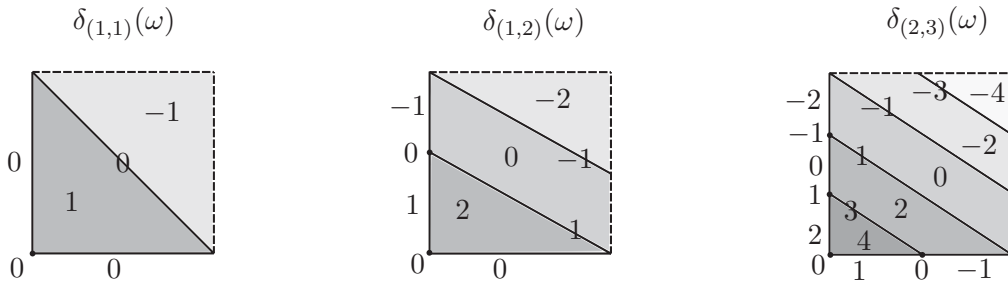

Fig. 1.   The values of three defect functions for $\omega \in \mathcal{T}^2$. The defect is constant on the shaded regions and on the interior of the segments dividing the squares. The values of the defect in the extremal cases, $\omega_1 = 1$ or $\omega_2 = 1$, are given by the numbers on the left and bottom of the squares respectively.

The *linking number* of two disjoint oriented circles K, L in an integral homology sphere $\mathbb{S}$ is denoted by $\ell k_{\mathbb{S}}(K, L)$, with $\mathbb{S}$ omitted whenever understood. For a $(1, \mu)$-colored link $K \cup L$, we also define the *linking vector* $\overline{\ell k}(K, L) = (\lambda_1, \ldots, \lambda_\mu) \in \mathbb{Z}^\mu$, where $\lambda_i :=$ $\ell k(K, L_i)$.

The *index* of a real number x is defined via $\mathrm{ind}(x) := \lfloor x \rfloor - \lfloor -x \rfloor \in \mathbb{Z}$. The Log-*function* $\mathrm{Log} \colon \mathcal{T}^1 \to [0, 1)$ sends $\exp(2\pi i t)$ to $t \in [0, 1)$. This function extends to $\mathrm{Log} \colon \mathcal{T}^\mu \to [0, \mu)$ via $\mathrm{Log}\,\omega = \sum_{i=1}^{\mu} \mathrm{Log}\,\omega_i$; in other words, we specialize each argument to the interval $[0, 1)$ and add the arguments *as real numbers* (rather than elements of $\mathcal{T}^1$) afterwards. For any integral vector $\lambda \in \mathbb{Z}^\mu$, $\mu \geq 0$, we define the *defect function*

$$\delta_\lambda \colon \mathcal{T}^\mu \longrightarrow \mathbb{Z}$$
$$\omega \longmapsto \mathrm{ind}\left(\sum_{i=1}^{\mu} \lambda_i \,\mathrm{Log}\,\omega_i\right) - \sum_{i=1}^{\mu} \lambda_i \,\mathrm{ind}(\mathrm{Log}\,\omega_i).$$

For short, if $\lambda_i = 1$ for all i, we simply denote the defect δ, and omit the subscript. The reader is referred to Figure 1 for a few examples of the defect function on $\mathcal{T}^2$.

The following statement is the principal result of the paper.

Theorem 2.2.   For $* = {}'$ or ${}''$, consider a $(1, \mu^*)$-colored link $K^* \cup L^* \subset \mathbb{S}^*$, and let $L \subset \mathbb{S}$ be the splice of the two links. For characters $\omega^* \in \mathcal{T}^{\mu^*}$, introduce the notation

$$\lambda^* := \overline{\ell k}(K^*, L^*) \in \mathbb{Z}^{\mu^*}, \qquad \upsilon^* := (\omega^*)^{\lambda^*} \in \mathcal{T}^1.$$

Then, assuming that $(v', v'') \neq (1, 1)$, one has

$$\sigma_L(\omega', \omega'') = \sigma_{K' \cup L'}(v'', \omega') + \sigma_{K'' \cup L''}(v', \omega'') + \delta_{\lambda'}(\omega')\delta_{\lambda''}(\omega''). \qquad \square$$

Remark 2.3. Eisenbud and Neumann [8, Theorem 5.2] showed that the Alexander polynomial is multiplicative under the splice. For a μ-colored link L, we denote $\Delta_L(t_1, \ldots, t_\mu)$ the Alexander polynomial of L. Similar to Theorem 2.2, let $t^* = \prod_{i=1}^{\mu^*}(t_i^*)^{\lambda *_i}$. One has

$$\Delta_L(t'_1, \ldots, t'_{\mu'}, t''_1, \ldots, t''_{\mu''}) = \Delta_{K' \cup L'}(t'', t'_1, \ldots, t'_{\mu'}) \cdot \Delta_{K'' \cup L''}(t', t''_1, \ldots, t''_{\mu''}),$$

unless $\mu' = 0$ (i.e. $L' = K'$ is a knot) and $\lambda'' = 0$, in which case

$$\Delta_L(t''_1, \ldots, t''_{\mu''}) = \Delta_{L'' \setminus K''}(t''_1, \ldots, t''_{\mu''}).$$

This formula were refined by Cimasoni [5] for the Conway potential function. Moreover, in relation with the signature of a colored link, one may consider the *nullity*, related to the rank of the twisted first homology of the link complement. This nullity is also additive under the splice operation, in the suitable sense. Detailed statements can be found in [7]. $\qquad \square$

Example 2.4. Consider two copies $K' \cup L'$ and $K'' \cup L''$ of the (1,1)-colored generalized Hopf link $H_{1,2}$, see Section 2.3, where K' and K'' are the single components. Then, $L = L' \cup L'' = H_{2,2}$ is a (1,1)-colored link, and for $\omega \in \mathcal{T}^1 \setminus \{\pm 1\}$, we show by using C-complexes that

$$\sigma_L(\omega, \omega) = \sigma_{K' \cup L'}(\omega^2, \omega) + \sigma_{K'' \cup L''}(\omega^2, \omega) + \delta_{(2)}(\omega)\delta_{(2)}(\omega) = 0 + 0 + \delta_{(2)}(\omega)\delta_{(2)}(\omega).$$

This illustrates trivially that a defect appears. $\qquad \square$

Example 2.5. For the reader convenience we add the following example. Notice the use of the formula in Theorem 2.2 when $\omega_i = 1$ (cf. Remark 3.6). Let $K' \cup L'$ be the (2,4)-torus link and $K'' \cup L''$ be the (4,2)-cable over the unknot with the core retained (cf. Section 2.5). Then, the splice of these two links along the components K' and K'' is the (3,6)-torus link, which we shall denote L.

In the notation of Theorem 2.2, we have $\lambda' = 2$ and $\lambda'' = (1, 1)$. For the C-complexes bounded by these three links one can take those depicted in Figure 2. To simplify the resulting Hermitian matrices H, we re-denote by $t_0, t_1, \ldots$ their arguments (in the order listed) and, for an index set I, introduce the shortcut $\pi_I := 1 + \prod_{i \in I}(-t_i)$. Then

$$H_{K' \cup L'}(\xi', \omega') = -\bar{\pi}_0\bar{\pi}_1\pi_{01},$$

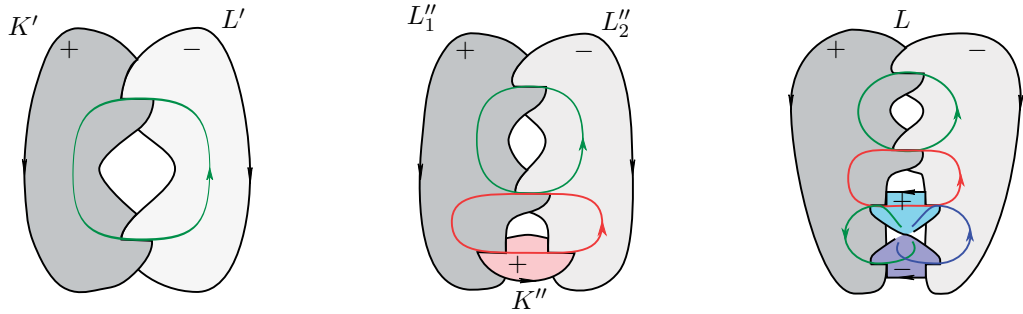ALEX DEGTYAREV, VINCENT FLORENS, AND ANA G. LECUONA

Fig. 2.   The leftmost link is the (2,4)-torus link, depicted as the boundary of a C-complex with rank 1 first homology. In the middle, the (4,2)-cable over the unknot with the core retained, bounding a rank 2 C-complex. The last diagram is the splice of the two preceding ones along K' and K''. It represents the (3,6)-torus link.

$$H_{K'' \cup L''}(\xi'', \omega_1'', \omega_2'') = \bar{\pi}_0 \bar{\pi}_1 \bar{\pi}_2 \begin{pmatrix} -\pi_0 \pi_{12} & t_1 t_2 \pi_0 \\ \pi_0 & -\pi_{012} \end{pmatrix},$$

$$H_{L' \cup L''}(\omega', \omega_1'', \omega_2'') = \bar{\pi}_0 \bar{\pi}_1 \bar{\pi}_2 \begin{pmatrix} -\pi_0 \pi_{12} & t_1 t_2 \pi_0 & 0 & 0 \\ \pi_0 & -\pi_{012} & t_0 t_2 \pi_1 & t_0 \pi_2 \\ 0 & \pi_1 & -\pi_1 \pi_{02} & -t_0 \pi_1 \pi_2 \\ 0 & t_1 \pi_2 & \pi_1 \pi_2 & -\pi_2 \pi_{01} \end{pmatrix},$$

so that, up to units and factors of the form π_i, $i = 0, 1, \ldots$, the Alexander polynomials are

$$\Delta_{K' \cup L'} = \pi_{01}, \qquad \Delta_{K'' \cup L''} = t_0 t_1^2 t_2^2 - 1, \qquad \Delta_{L' \cup L''} = \pi_{012}(t_0 t_1 t_2 + 1)^2.$$

The computation of the signature of these matrices is straightforward: on the respective *open* tori, they are the piecewise constant functions given by the following tables:

$\text{Log}\, \xi' + \text{Log}\, \omega'$		1/2		3/2	
$\sigma_{K' \cup L'}(\xi', \omega')$	1	0	−1	0	1

$\text{Log}\, \xi'' + 2\,\text{Log}\, \omega''$		1		2		3		4	
$\sigma_{K'' \cup L''}(\xi'', \omega'')$	2	1	0	−1	−2	−1	0	1	2

$\text{Log}\, \omega' + \text{Log}\, \omega''$		1/2		1		2		5/2	
$\sigma_{L' \cup L''}(\omega', \omega'')$	4	2	0	−1	−2	−1	0	2	4

Note, however, that L' is the unknot and L'' is homeomorphic to $K' \cup L'$; hence,

$$\sigma_{K' \cup L'}(1, \omega') = 0, \qquad \sigma_{K'' \cup L''}(1, \omega'') = \sigma_{K' \cup L'}(\omega''_1, \omega''_2).$$

Now, it is immediate that the identity

$$\sigma_L(\omega', \omega''_1, \omega''_2) = \sigma_{K' \cup L'}(\omega''_1 \omega''_2, \omega') + \sigma_{K'' \cup L''}(\omega'^2, \omega''_1, \omega''_2) + \delta_{(2)}(\omega')\delta_{(1,1)}(\omega''_1, \omega''_2)$$

given by Theorem 2.2 holds whenever $\omega'^2 \neq 1$ or $\omega''_1 \omega''_2 \neq 1$. (It suffices to compare the values at all triples of eighth roots of unity.) If $\omega'^2 = \omega''_1 \omega''_2 = 1$, we obtain an extra discrepancy of 1; this phenomenon will be explained in [7]. □

As an immediate consequence of Theorem 2.2, we see that the Levine–Tristram signature of a splice cannot be expressed in terms of the Levine–Tristram signature of its summands: in general, the multivariate extension is required.

Corollary 2.6.   Let L be the splice of $(1, 1)$-colored links $K' \cup L'$ and $K'' \cup L''$, and denote $\lambda' = \ell k(K', L')$ and $\lambda'' = \ell k(K'', L'')$. Consider L as a 1-colored link. Then, for a character $\xi \in \mathcal{T}^1$ such that $\xi^{\mathrm{g.c.d.}(\lambda', \lambda'')} \neq 1$, one has

$$\sigma_L(\xi) = \sigma_{K' \cup L'}(\xi^{\lambda''}, \xi) + \sigma_{K'' \cup L''}(\xi^{\lambda'}, \xi) - \lambda' \lambda'' + \delta_{\lambda'}(\xi)\delta_{\lambda''}(\xi),$$

where $\sigma_L(\xi)$ is the Levine–Tristram signature of L. □

Proof.   Consider the two-coloring on L given by the splitting $L' \cup L''$. We have $\sigma_L(\xi, \xi) = \sigma_{K' \cup L'}(\xi^{\lambda''}, \xi) + \sigma_{K'' \cup L''}(\xi^{\lambda'}, \xi) + \delta_{\lambda'}(\xi)\delta_{\lambda''}(\xi)$ by Theorem 2.2. On the other hand, $\sigma_L(\xi) = \sigma_L(\xi, \xi) - \ell k(L', L'')$, see Proposition 3.7. By [8, Proposition 1.2], $\ell k(L', L'') = \lambda' \lambda''$. ■

Theorem 2.2 is proved in Section 4.3. In the special case $L' = \varnothing$, it takes the following stronger form (we do not require that $\upsilon'' \neq 1$); it is proved in Section 4.4.

Addendum 2.7.   Let $L \subset \mathbb{S}$ be the splice of a $(1, 0)$-colored link $K' \subset \mathbb{S}'$ and a $(1, \mu'')$-colored link $K'' \cup L'' \subset \mathbb{S}''$, and let $\lambda'' := \overline{\ell k}(K'', L'')$. Then, for any character $\omega \in \mathcal{T}^{\mu''}$, one has

$$\sigma_L(\omega) = \sigma_{K'}(\omega^{\lambda''}) + \sigma_{L''}(\omega).$$ □

Remark 2.8.   The assumption $(\upsilon', \upsilon'') \neq (1, 1)$ in Theorem 2.2 is essential. If $\upsilon' = \upsilon'' = 1$, the expression for the signature acquires an extra correction term, which can be proved

to take values in $[\![-2, 2]\!]$. In many cases, this term can be computed algorithmically, and simple examples show that typically it does not vanish. Indeed, consider two copies of the Whitehead link $K' \cup L'$ and $K'' \cup L''$. If $\omega = e^{i\pi/3}$, then $\sigma_L(\omega, \omega) = -1$, but $\sigma_{K' \cup L'}(1, \omega) + \sigma_{K'' \cup L''}(1, \omega) + \delta(1) = 0$ and there is a non-zero extra term. (Addendum 2.7 states that the extra term does vanish whenever one of the links L', L'' is empty.) The general computation of this extra term, related to *linkage* invariants (see, e.g., [19]), is addressed in a forthcoming paper [7]. $\qquad\square$

Remark 2.9. We expect that the conclusion of Theorem 2.2 would still hold without the assumption that the characters should be rational. In fact, all ingredients of the proof would work once recast to the language of local systems, and the main difficulty is the very definition of the signature in homology spheres, where the link does not need to bound a surface and the approach of [6] does not apply. (If all links are in S^3, an alternative proof can be given in terms of C-complexes.) This issue will also be addressed in [7]. $\qquad\square$

2.3   The generalized Hopf link

A *generalized Hopf link* is the link $H_{m,n} \subset S^3$ obtained from the ordinary positive Hopf link $H_{1,1} = V \cup U$ by replacing its components V and U with, respectively, m and n parallel copies. This link is naturally $(m + n)$-colored; its signature, which plays a special role in the paper is given by Theorem 2.10 below. Observe the similarity to the correction term in Theorem 2.2; *a posteriori*, Theorem 2.10 can be interpreted as a special case of Theorem 2.2, using the identity $\sigma_{H_{1,n}} \equiv 0$ (which is easily proved independently) and the fact that $H_{m,n}$ is the splice of $H_{1,m}$ and $H_{1,n}$. However, the Hopf links and their signatures are used essentially in the proof of Theorem 2.2.

Theorem 2.10. For any character $(v, u) \in \mathcal{T}^m \times \mathcal{T}^n$, one has $\sigma_{H_{m,n}}(v, u) = \delta(v)\delta(u)$. $\qquad\square$

Certainly, Theorem 2.10 computes as well the signature of a generalized Hopf link equipped with an arbitrary coloring and orientation of components. First, one can recolor the link by assigning a separate color to each component (cf. Proposition 3.7 below). Then, one can reverse the orientation of each negative component L_i; obviously, this operation corresponds to the substitution $\omega_i \mapsto \bar{\omega}_i$. For example, the orientation of the original link can be described in terms of a pair of vectors, viz. the linking vector $\nu \in \{\pm 1\}^m$ of the V-part of $H_{m,n}$ with the U-component of the original Hopf link $H_{1,1}$ and the linking vector $\lambda \in \{\pm 1\}^n$ of the U-part with the V-component. Then, *assuming that*

any two linked components of $H_{m,n}$ are given distinct colors, we have

$$\sigma_{H_{m,n}}(v, u) = \delta_\nu(v)\delta_\lambda(u). \tag{2.2}$$

For future references, we state a few simple properties of the defect function δ and, hence, of the signature $\sigma_{H_{m,n}}$. All proofs are immediate.

Lemma 2.11.   The defect function $\delta \colon \mathcal{T}^\mu \to \mathbb{Z}$ has the following properties:

(1) $\delta(1) = 0$; $\delta \equiv 0$ if $\mu = 0$ or 1;
(2) $\delta(\bar{\omega}) = -\delta(\omega)$ for all $\omega \in \mathcal{T}^\mu$;
(3) δ is preserved by the coordinatewise action of the symmetric group S_μ;
(4) δ commutes with the coordinate embeddings $\mathcal{T}^\mu \hookrightarrow \mathcal{T}^{\mu+1}$, $\omega \mapsto (\omega, 1)$;
(5) δ commutes with the embeddings $\mathcal{T}^\mu \hookrightarrow \mathcal{T}^{\mu+2}$, $\omega \mapsto (\omega, \eta, \bar{\eta})$ for any $\eta \in \mathcal{T}^1$. $\square$

2.4   Satellite knots

As was first observed in [8], the splice operation generalizes many classical link constructions: connected sum, disjoint union, and satellites among others.

Our first application is Litherland's formula for the Levine–Tristram signature of a satellite knot, which is a particular case of Addendum 2.7.

Recall that an embedding of a solid torus in S^3 into another solid torus in another copy of S^3 is called *faithful* if the image of a canonical longitude of the first solid torus is a canonical longitude of the second one. Let V be an unknotted solid torus in S^3, and let k be a knot in the interior of V, with algebraic winding number q, that is, $[k]$ is q times the class of the core in $H_1(V)$. Given any knot $K \subset S^3$, the *satellite knot K^** is defined as the image $f(k)$ under a faithful embedding $f : V \to S^3$ sending the core of V to K.

The isotopy class K^* depends of course on the embedding f (and even its concordance class, see [17]). Nevertheless, its Levine–Tristram signature is determined by the signatures of the constituent knots and the winding number:

Theorem 2.12 (cf. [16, Theorem 2]).   In the notation above, the Levine–Tristram signatures of k, K, and K^* are related via

$$\sigma_{K^*}(\omega) = \sigma_K(\omega^q) + \sigma_k(\omega), \quad \omega \in \mathcal{T}^1. \qquad \square$$

Proof. Let C be the core of the solid torus $S^3 \smallsetminus V$. The satellite K^* can be written as the splice of $K \cup \varnothing$ and $C \cup k$. By Addendum 2.7, we have

$$\sigma_{K^*}(\omega) = \sigma_K(\omega^\lambda) + \sigma_k(\omega),$$

where $\lambda := \ell k(C,k)$. By assumption, $\ell k(C,k) = q$, and the statement follows. ∎

2.5 Iterated torus links

Our next application is another special case of Theorem 2.2, which provides an inductive formula for the signatures of iterated torus links. In particular, this class of links contains the algebraic ones, that is, the links of isolated singularities of complex curves in $\mathbb{C}^2$. Partial results on the equivariant signatures of the monodromy were obtained by Neumann [20].

Iterated torus links are obtained from an unknot by a sequence of cabling operations (and maybe, reversing the orientation of some of the components). In order to define the cabling operations (we follow the exposition in [8]), consider two coprime integers p and q (in particular, if one of them is 0, the other is ± 1), a positive integer d, a $(1, \mu')$-colored link $K' \cup L' \subset S^3$, and a small tubular neighborhood T' of K' disjoint from L'. Let m, l be the meridian and longitude of K', and $K'(p,q)$ be the oriented simple closed curve in $\partial T'$ homologous to $pl + qm$. More generally, let $dK'(p,q)$ be the disjoint union of d parallel copies of $K'(p,q)$ in $\partial T'$. We say that the link $L = L' \cup dK'(p,q) - K'$ (resp. $L = L' \cup dK'(p,q)$) is obtained from $K' \cup L'$ by a (dp, dq)-*cabling with the core removed* (resp. *retained*).

Let $H_{1,1} = V \cup U$ be the ordinary Hopf link. The link $V \cup dU(p,q)$ can be regarded as either $(1,d)$-colored or $(1,1)$-colored. We denote the corresponding multivariate and bivariate signature functions by $\tau_{dp,dq}$ and $\tilde{\tau}_{dp,dq}$, respectively. By Proposition 3.7 below,

$$\tilde{\tau}_{dp,dq}(v,u) = \tau_{dp,dq}(v,u,\ldots,u) - \tfrac{1}{2}d(d-1)pq.$$

In the case of core-removing, the link L obtained by the cabling is nothing but the splice of $K' \cup L'$ and $V \cup dU(p,q)$. (Similarly, in the core-retaining case, L is the splice of $K' \cup L'$ and $V \cup U \cup dU(p,q)$.) Hence, the following statement is an immediate consequence of Theorem 2.2.

Theorem 2.13. Let L be obtained from a $(1, \mu')$-colored link $K' \cup L'$ by a (dp, dq)-cabling with the core removed. For a character $\omega := (\omega', \omega'') \in \mathcal{T}^{\mu'} \times \mathcal{T}^d$, let

$$\lambda' := \overline{\ell k}(K', L'), \quad \lambda'' := (p, \ldots, p) \in \mathbb{Z}^d, \quad \text{and} \quad \upsilon^* := (\omega^*)^{\lambda^*}, \; * = \text{' or ''}.$$

Then, assuming that $(\upsilon', \upsilon'') \neq (1, 1)$, one has

$$\sigma_L(\omega) = \sigma_{K' \cup L'}(\upsilon'', \omega') + \tau_{dp,dq}(\upsilon', \omega'') + \delta_{\lambda'}(\omega')\delta_{\lambda''}(\omega''). \qquad \square$$

With the evident modifications, this last corollary can be adapted to give a formula for a (dp, dq)-cabling with the core retained.

The Levine–Tristram signature of the torus link $U(p, q)$ (which coincides with $\tilde{\tau}_{p,q}(1, \zeta)$ in our notation) was computed by Hirzebruch. For the reader's convenience, we cite this result in the next lemma. Unfortunately, we do not know any more general statement.

Lemma 2.14 (see [3]). Let $M = \{1, \ldots, p - 1\} \times \{1, \ldots, q - 1\}$ and let $0 < \theta \leq \frac{1}{2}$. Consider

$$a = \#\{(i,j) \in M \mid \theta < (i/p) + (j/q) < \theta + 1\},$$
$$n = \#\{(i,j) \in M \mid (i/p) + (j/q) = \theta \text{ or } (i/p) + (j/q) = \theta + 1\},$$
$$b = |M| - a - n.$$

Then, one has $\tilde{\tau}_{p,q}(1, \zeta) = b - a$ for $\zeta = \exp(2i\pi\theta)$. $\qquad \square$

2.6 Multivariate versus univariate signature

The last application is the computation of the multivariate signature of a link in terms of the Levine–Tristram signature of an auxiliary link. (One obvious application is the case where the latter auxiliary link is algebraic, so that its Seifert form can be computed in terms of the variation map $H_1(F, \partial F) \to H_1(F)$ in the homology of its Milnor fiber F, see [1].) This result is similar to [9, Theorem 6.22] by the second author and is related to the computation of signature invariants of three-manifolds by Gilmer, see [10, Theorem 3.6].

Let $L = L_1 \cup \ldots \cup L_\mu$ be a μ-colored link. For simplicity, we assume that the coloring is maximal, that is, each component of L is given a separate color. Let $[\lambda_{ij}]$ be the linking matrix of L, that is, $\lambda_{ij} = \ell k(L_i, L_j)$ for $i \neq j$ and $\lambda_{ii} = 0$.

Consider a character $\omega \in \mathcal{T}^\mu$ and assume that $\omega_i = \xi^{n_i}$, where $\xi := \exp(2\pi i/n)$, for some integers $n > 0$ and $0 < n_i < n$. (In particular, all $\omega_i \neq 1$.) For $i = 1, \ldots, \mu$, denote

- $\lambda_i^{\mathrm{w}} := \sum_{j=1}^{\mu} n_j \lambda_{ij}$, the weighted linking number of L_i and $L \smallsetminus L_i$;
- $\upsilon_i := \prod_{j=1}^{\mu} \omega_j^{\lambda_{ij}} = \xi^{\lambda_i^{\mathrm{w}}}$, where λ_i is the ith row of $[\lambda_{ij}]$.

Fix an integral vector $p := (p_1, \ldots, p_\mu) \in \mathbb{Z}^\mu$ and consider the *monochrome* link $\bar{L} := \bar{L}_p(\omega)$ obtained from L by the $(n_i, n_i p_i)$-cabling along the component L_i for each $i = 1, \ldots, \mu$. In other words, each component L_i of L is regarded n_i-fold, and it is replaced with n_i "simple" components, possibly linked (if $p_i \neq 0$).

Theorem 2.15.   In the notation above, one has the identity

$$\sigma_L(\omega) = \sigma_{\bar{L}}(\xi) - \sum_{i=1}^{\mu} \tilde{\tau}_{n_i, n_i p_i}(\upsilon_i, \xi) + \sum_{i=1}^{\mu} (n_i - 1)\operatorname{ind}(\lambda_i^{\mathrm{w}}/n) + \sum_{1 \leq i < j \leq \mu} \lambda_{ij}. \qquad \square$$

Corollary 2.16.   If $p = 0$, the second term in Theorem 2.15 vanishes and one has

$$\sigma_L(\omega) = \sigma_{\bar{L}}(\xi) + \sum_{i=1}^{\mu} (n_i - 1)\operatorname{ind}(\lambda_i^{\mathrm{w}}/n) + \sum_{1 \leq i < j \leq \mu} \lambda_{ij}.$$

For small values of μ, this identity simplifies even further:

(1)   if $\mu = 1$, then $\sigma_L(\omega) = \sigma_{\bar{L}}(\xi)$;

(2)   if $\mu = 2$ and $|\lambda_{12}| \leq 1$, then $\sigma_L(\omega) = \sigma_{\bar{L}}(\xi) + (n_1 + n_2 - 1)\lambda_{12}$. $\square$

Proof of Corollary 2.16.   If $p_i = 0$, then $V \cup U(n_i, 0) = H_{1,n_i}$ is a generalized Hopf link; its signature vanishes due to Theorem 2.10 and Lemma 2.11(1). The only other statement that needs proof is Item (2), where we have $\operatorname{ind}(\lambda_{12} n_i / n) = \lambda_{12}$ whenever $|\lambda_{12}| \leq 1$ and $0 < n_i < n$, $i = 1, 2$.   ∎

Example 2.17.   Let $L = H_{1,1}$ be the ordinary Hopf link, so that $\sigma_L \equiv 0$ by Theorem 2.10 and Lemma 2.11(1). On the other hand, taking $p = 0$, we obtain $\bar{L} = H_{n_1, n_2}$; by Theorem 2.10 and Proposition 3.7, we get $\sigma_{\bar{L}}(\xi) = (1 - n_1)(1 - n_2) - n_1 n_2$, which agrees with Corollary 2.16(2). $\square$

Proof of Theorem 2.15.   Denote $L[0] := L$ and, for $i = 1, \ldots, \mu$, let $L[i]$ be the link obtained from $L[i-1]$ by the $(n_i, n_i p_i)$-cabling along the component L_i. Each link $L[i]$ is naturally μ-colored; we assign to this link the character $\omega[i] := (\xi, \ldots, \xi, \omega_{i+1}, \ldots, \omega_\mu)$. In this notation, $\bar{L}$ is the monochrome version of $L[\mu]$ and, by Proposition 3.7,

$$\sigma_{\bar{L}}(\xi) = \sigma_{L[\mu]}(\omega[\mu]) - \sum_{1 \leq i < j \leq \mu} n_i n_j \lambda_{ij}. \tag{2.3}$$

Introduce the following characters:

- $\tilde{\omega}'[i] := (\xi, \ldots, \xi) \in \mathcal{T}^{n_i}$;
- $\tilde{\omega}''[i]$, obtained from ω by replacing each ω_j with $n_j|\lambda_{ij}|$ copies of $\xi^{\mathrm{sg}\,\lambda_{ij}}$, if $j \leq i$, or $|\lambda_{ij}|$ copies of $\omega_j^{\mathrm{sg}\,\lambda_{ij}}$, if $j > i$;
- $\tilde{\omega}[i]$, obtained from ω by replacing each ω_j with $n_j|\lambda_{ij}|$ copies of $\xi^{\mathrm{sg}\,\lambda_{ij}}$.

By definition, $L[i]$ is the splice of $L[i-1]$ and $V \cup n_i U(1, p_i)$. Then Theorem 2.2 applies and, for each $i = 1, \ldots, \mu$,

$$\sigma_{L[i]}(\omega[i]) = \sigma_{L[i-1]}(\omega[i-1]) + \tilde{\tau}_{n_i, n_i p_i}(\upsilon_i, \xi) + \delta(\tilde{\omega}'[i])\delta(\tilde{\omega}''[i]). \tag{2.4}$$

We have $\mathrm{Log}\,\tilde{\omega}'[i] = n_i/n$; since $0 < n_i < n$, this implies

$$\delta(\tilde{\omega}'[i]) = 1 - n_i. \tag{2.5}$$

One can show that $\delta(\tilde{\omega}''[i]) = \delta(\tilde{\omega}[i]) - \sum_{j=i+1}^{\mu}(1 - n_j)\lambda_{ij}$. Indeed, $\tilde{\omega}[i]$ is obtained from $\tilde{\omega}''[i]$ by $|\lambda_{ij}|$ operations of replacement of a single copy of $\omega_j^{\mathrm{sg}\,\lambda_{ij}}$ with copies of $\xi^{\mathrm{sg}\,\lambda_{ij}}$ for all $j > i$; as in (2.5), one such operation increases the value of δ by $(1 - n_j)\,\mathrm{sg}\,\lambda_{ij}$. The character $\tilde{\omega}[i]$ has all entries equal to ξ or $\bar{\xi}$, with the exponent sum equal to λ_i^{w}. Using Lemma 2.11(5) and (3) to cancel the pairs $\xi, \bar{\xi}$, we get $\delta(\tilde{\omega}[i]) = \mathrm{ind}(\lambda_i^{\mathrm{w}}/n) - \lambda_i^{\mathrm{w}}$; hence,

$$\delta(\tilde{\omega}''[i]) = \mathrm{ind}(\lambda_i^{\mathrm{w}}/n) - \sum_{j=1}^{i-1} n_j\lambda_{ij} - \sum_{j=i+1}^{\mu} \lambda_{ij}. \tag{2.6}$$

Applying (2.4) inductively and taking into account (2.5) and (2.6), we arrive at

$$\sigma_{L[\mu]}(\omega[\mu]) = \sigma_L(\omega) + \sum_{i=1}^{\mu} \tilde{\tau}_{n_i, n_i p_i}(\upsilon_i, \xi) - \sum_{i=1}^{\mu}(n_i - 1)\,\mathrm{ind}(\lambda_i^{\mathrm{w}}/n) + \sum_{1 \leq i < j \leq \mu}(n_i n_j - 1)\lambda_{ij},$$

and the statement of the theorem follows from (2.3). ∎

3   Signature of a Link in a Homology Sphere

In the early 1960s Trotter introduced a numerical knot invariant called the signature [24], which was subsequently extended to links by Murasugi [19]. This invariant was generalized to a function (defined via Seifert forms) on $S^1 \subset \mathbb{C}$ by Levine and Tristram [14, 23]. It was then reinterpreted in terms of coverings and intersection forms of four-manifolds by Viro [25, 26]. Our definition of the signature of a colored link follows Viro's approach and the G-signature theorem, see also [9, 12].

3.1   Twisted signature and additivity

We start with recalling the definition and some properties of the twisted signature of a four-manifold.

Let N be a compact smooth oriented four-manifold with boundary and G a finite abelian group. Fix a covering $N^G \to N$, possibly ramified, with G the group of deck transformations. If the covering is ramified, we assume that the ramification locus F is a union of smooth compact surfaces $F_i \subset N$ such that

(1) $\partial F_i = F_i \cap \partial N$;

(2)   each surface F_i is transversal to ∂N, and

(3)   distinct surfaces intersect transversally, at double points, and away from ∂N.

Items (1) and (2) above mean that each component F_i of F is a properly embedded surface. For short, a compact surface $F \subset N$ satisfying all Conditions (1)–(3) will be called *properly immersed*. Under these assumptions, N^G is an oriented rational homology manifold and we have a well-defined Hermitian intersection form

$$\langle \cdot , \cdot \rangle \colon H_2(N^G; \mathbb{C}) \otimes H_2(N^G; \mathbb{C}) \to \mathbb{C}.$$

Regard the homology groups $H_*(N^G; \mathbb{C})$ as $\mathbb{C}[G]$-modules and consider the form

$$\varphi \colon H_2(N^G; \mathbb{C}) \otimes H_2(N^G; \mathbb{C}) \to \mathbb{C}[G], \quad \varphi(x, y) := \sum_{g \in G} \langle x, gy \rangle g.$$

Since G is abelian, this form is sesquilinear, that is, $\varphi(g_1 x, g_2 y) = g_1 g_2^{-1} \varphi(x, y)$ for all $g_1, g_2 \in G$.

Any multiplicative character $\chi \colon G \to \mathbb{C}^*$ induces a homomorphism $\mathbb{C}[G] \to \mathbb{C}$ of algebras with involution ($zg \mapsto \bar{z} g^{-1}$ in $\mathbb{C}[G]$ is mapped to $\eta \mapsto \bar{\eta}$ in $\mathbb{C}$). This makes $\mathbb{C}$ a $\mathbb{C}[G]$-module, and we can consider the *twisted homology*

$$H_*^\chi(N, F) := H_*(N^G; \mathbb{C}) \otimes_{\mathbb{C}[G]} \mathbb{C}.$$

In this notation, the ramification locus F is omitted whenever it is empty or understood. The form φ above induces a $\mathbb{C}$-valued Hermitian form φ^χ on $H_2^\chi(N, F)$; explicitly, the latter is given by

$$\varphi^\chi(x \otimes z_1, y \otimes z_2) = z_1 \bar{z}_2 \sum_{g \in G} \langle x, gy \rangle \chi(g).$$

We will denote by sign(N) the ordinary signature of the four-manifold N, that is, that of the form $\langle \, \cdot \, , \, \cdot \, \rangle$ on $H_2(N)$. The *twisted signature*, denoted by sign$^\chi(N,F)$, is the signature of the above Hermitian form φ^χ.

Remark 3.1.   One can easily see that the twisted homology $H_*^\chi(N,F)$ and twisted signature sign$^\chi(N,F)$ are independent of the group G used in the construction: they only depend on the pair (N,F) and the multiplicative character $\chi \colon H_1(N \smallsetminus F) \to \mathbb{C}^*$, which must be assumed of finite order. In particular, we can always take for G the "smallest" cyclic group, viz. the image of χ. Indeed, there is an obvious canonical isomorphism between $H_*^\chi(N,F)$ and the χ-equitypical summand

$$V_*^\chi(G) := \left\{ x \in H_*(N^G; \mathbb{C}) \mid gx = \chi(g)x \text{ for all } g \in G \right\},$$

and the form φ^χ is $|G|$-times the restriction to $V^\chi(G)$ of the ordinary intersection index form $\langle \, \cdot \, , \, \cdot \, \rangle$. Now, if G is replaced with a larger group $G' \twoheadrightarrow G$, the transfer map induces an isomorphism $V_*^\chi(G) \to V_*^\chi(G')$, multiplying the intersection index form by another positive factor $[G' : G]$; hence, the signature is preserved.   □

Of particular interest is the behavior of the signature under the gluing of manifolds. Recall that, by Novikov's additivity, if N_1 and N_2 are two 4-manifolds such that $\partial N_1 = -\partial N_2$ and $N = N_1 \cup_\partial N_2$, then the ordinary and the twisted signatures of N satisfy

$$\text{sign}(N) = \text{sign}(N_1) + \text{sign}(N_2) \quad \text{and} \quad \text{sign}^\chi(N,F) = \text{sign}^\chi(N_1,F_1) + \text{sign}^\chi(N_2,F_2).$$

Of course, in the twisted version we assume that the ramification loci F_1 and F_2 match along the boundary, $F = F_1 \cup_\partial F_2$, and the characters on N_1, N_2 are the restrictions of a character on N. If N_1 and N_2 are glued along a part of their boundaries only, the above equalities may fail. This situation was completely studied by Wall in [27]. For our purposes we only need a particular case of Wall's theorem, which we state below. The result is given in terms of ordinary signatures, but, as mentioned by Wall at the end of his paper, the same conclusion holds if we consider twisted signatures.

Theorem 3.2 (see [27]).   Suppose that $\partial N_1 \simeq M_1 \cup M_0$ and $\partial N_2 \simeq M_2 \cup -M_0$, where $M_0, M_1,$ and M_2 are three-manifolds glued along their common boundary. Let $N := N_1 \cup_{M_0} N_2$ and $X := \partial M_0 = \partial M_1 = \partial M_2$. Consider the $\mathbb{C}$-vector spaces $A_i := \text{Ker}[H_1(X; \mathbb{C}) \to H_1(M_i; \mathbb{C})]$,

$i = 0, 1, 2$, and let

$$K(A_0, A_1, A_2) := \frac{A_0 \cap (A_1 + A_2)}{(A_0 \cap A_1) + (A_0 \cap A_2)}.$$

If $K(A_0, A_1, A_2)$ is trivial, then we have $\mathrm{sign}(N) = \mathrm{sign}(N_1) + \mathrm{sign}(N_2)$.   □

Remark 3.3.   The additivity in Theorem 3.2 holds if at least two of A_0, A_1, A_2 are equal. Moreover, Wall shows in his article that the vector space $K(A_0, A_1, A_2)$ is independent of the order of the A_i's. When working with twisted signatures, we shall use the notation $A_i^\chi := \mathrm{Ker}[H_1^\chi(X) \to H_1^\chi(M_i)]$, $i = 0, 1, 2$.   □

3.2   The signature of a link

Let L be a μ-colored link in an integral homology sphere $\mathbb{S}$. By Alexander duality, the group $H_1(\mathbb{S} \smallsetminus L)$ is generated by the meridians of the components of L. We shall denote by m_i^k the meridians of the components of the sublink L_i of L of color $i = 1, \dots, \mu$.

Let $\mathbb{Z}^\mu$ be the free multiplicative group generated by $t_1, \dots, t_\mu$. The coloring on L gives rise to a homomorphism $c : H_1(\mathbb{S} \smallsetminus L) \to \mathbb{Z}^\mu, m_i^k \mapsto t_i, i = 1, \dots, \mu$. We consider multiplicative characters $H_1(\mathbb{S} \smallsetminus L) \to \mathbb{C}^*$ that *respect the coloring*, that is, factor through c. They are determined by their values on the generators t_i, and the group of such characters can be identified with $\mathcal{T}^\mu$. Through this identification, the character $\omega \in \mathcal{T}^\mu$ assigns the meridians of the components of the sublink L_i to ω_i. With a certain abuse of the language, we will shortly speak about the character ω on L and say that ω assigns ω_i to (each component of) L_i.

The next proposition asserts that $\omega \colon H_1(\mathbb{S} \smallsetminus L) \to \mathbb{C}^*$ extends to a finite order character $\omega \colon H_1(N \smallsetminus F) \to \mathbb{C}^*$ (also denoted by the same letter ω), where N is a four-manifold bounded by $\mathbb{S}$ and $F \subset N$ is a certain properly immersed surface.

Proposition 3.4.   Let L be a μ-colored link in an integral homology sphere $\mathbb{S}$. Then, there exists a compact smooth oriented four-manifold N and an oriented properly immersed surface $F = F_1 \cup \dots \cup F_\mu$ in N such that

- $\partial N = \mathbb{S}$ and $\partial F_i = L_i$ for $i = 1, \dots, \mu$,
- the group $H_1(N \smallsetminus F) \simeq \mathbb{Z}^\mu$ is freely generated by the meridians $\bar{m}_i$ of F_i, and
- one has $[F_i, \partial F_i] = 0$ in $H_2(N, \partial N)$.

As a consequence, any character $\omega \in \mathcal{T}^\mu$ extends to a unique character

$$\omega \colon H_1(N \smallsetminus F) \to \mathbb{C}^*, \quad \bar{m}_i \mapsto \omega_i. \qquad \square$$

For short, as for characters on links, we will speak about the character ω on F and say that ω assigns ω_i to the component F_i.

We postpone the proof of this statement till Section 3.3.

Now, we are ready to define the main object of study in this paper.

Definition 3.5. The *signature* of a μ-colored link $L \subset \mathbb{S}$ is the map

$$\sigma_L \colon \begin{array}{ccc} \mathcal{T}^\mu & \longrightarrow & \mathbb{Z} \\ \omega & \longmapsto & \mathrm{sign}^\omega(N, F) - \mathrm{sign}(N), \end{array}$$

where N and F are as in Proposition 3.4. $\qquad \square$

The signature of a μ-colored link in $\mathbb{S}$ is related to invariants previously defined by Gilmer [10], Smolinski [22], Levine [15] and the first author [9]. The interested reader can find detailed history in [6]. In the case where $\mathbb{S} = S^3$, the signature considered in this paper coincides with the signature defined by Cimasoni–Florens [6] for $\omega \in \mathcal{T}$ with $\omega_i \neq 1$ for all $i = 1, \ldots, \mu$. In our present work we shall deal also with the case $\omega_i = 1$. The following remark should be clear from the definition of the signature of a colored link.

Remark 3.6. Let L be a μ-colored link in $\mathbb{S}$, and let $\omega \in \mathcal{T}^\mu$ be a vector such that $\omega_i = 1$. Then, the following equality holds:

$$\sigma_L(\ldots, 1, \ldots) = \sigma_{L_1 \cup \ldots \cup \hat{L}_i \cup \ldots \cup L_\mu}(\ldots, \hat{1}, \ldots). \qquad \square$$

Another important observation is the fact that the coloring of the link is essential: it is not enough to merely assign a value of a character to each component of the link. More precisely, we have the following relation (whose proof for S^3 found in [6] extends to integral homology spheres almost literally: the extra term is due to the perturbation of the union $F_\mu \cup F_{\mu+1}$ of two components of the ramification locus into a single surface).

Proposition 3.7 (see [6, Proposition 2.5]). Let $L := L_1 \cup \ldots \cup L_{\mu+1}$ be a $(\mu + 1)$-colored link, and consider the μ-colored link $L' := L_1' \cup \ldots \cup L_\mu'$ defined via $L_i' = L_i$ for $i < \mu$ and $L_\mu' = L_\mu \cup L_{\mu+1}$. Then, for any character $\omega \in \mathcal{T}^\mu$, one has

$$\sigma_{L'}(\omega) = \sigma_L(\omega_1, \ldots, \omega_\mu, \omega_\mu) - \ell k(L_\mu, L_{\mu+1}). \qquad \square$$

Corollary 3.8.   The multivariate signature of a generalized Hopf link $H_{m,n}$ does not depend on the coloring, provided that linked components are given distinct colors.   □

In particular, Proposition 3.7 provides a relation between the restriction of the multivariate signature of a colored link to the diagonal in $\mathcal{T}^\mu$ and the Levine–Tristram signature of the underlying monochrome link.

As asserted in the following proposition, the signature of a colored link is well defined, that is, independent of the pair (N, F) chosen to compute it. This is a consequence of Novikov's additivity and the G-signature theorem.

Proposition 3.9.   For all $\omega \in \mathcal{T}^\mu$, the signature of $(\mathbb{S}, L)$ at ω

$$\sigma_L(\omega) = \mathrm{sign}^\omega(N, F) - \mathrm{sign}(N)$$

does not depend on the pair (N, F).   □

Proof.   Given two pairs (N', F') and (N'', F'') as in Proposition 3.4, consider $W := N' \cup_\partial -N''$ and $F := F' \cup_\partial -F'' \subset W$. By Novikov's additivity, the statement of the proposition would follow if we show that $\mathrm{sign}^\omega(W, F) = \mathrm{sign}\, W$.

To compute the twisted signature, we can use the group $G := C_{q_1} \times \cdots \times C_{q_\mu}$, where q_i is the order of ω_i, $i = 1, \ldots, \mu$, see Remark 3.1. Crucial is the fact that, under the assumptions on (W, F), this group results in a smooth closed manifold W^G.

Consider the equitypical decomposition of the $\mathbb{C}[G]$-module

$$H := H_2(W^G; \mathbb{C}) = \bigoplus_\rho V^\rho, \tag{3.1}$$

where ρ runs over all multiplicative characters $G \to \mathbb{C}^*$. Since the intersection index form $\langle \cdot, \cdot \rangle$ is G-invariant, this decomposition is orthogonal. Denote by $\mathrm{sign}\, V^\rho$ the signature of the restriction of the form to V^ρ. By Remark 3.1, we have $\mathrm{sign}^\omega(W, F) = \mathrm{sign}\, V^\omega$.

The argument below is a slight generalization of [21] (see also [4, Lemma 2.1]).

Each space V^ρ can further be decomposed (not canonically) into the orthogonal sum of two subspaces V^ρ_+ and V^ρ_- with, respectively, positive and negative definite restriction of $\langle \cdot, \cdot \rangle$. Summation over all characters gives us a G-invariant decomposition $H = H_+ \oplus H_-$. Recall that the *G-signature* of an element $g \in G$ is $\mathrm{sign}(g, W) := \mathrm{trace}\, g_*|_{H_+} - \mathrm{trace}\, g_*|_{H_-} \in \mathbb{C}$. It is well defined; in fact, using (3.1), we have

$$\mathrm{sign}(g, W) = \sum_\rho \rho(g)\, \mathrm{sign}\, V^\rho.$$

Multiplying this by $\bar{\omega}(g)$ and summing up over all $g \in G$, we arrive at

$$|G| \operatorname{sign} V^{\omega} = \sum_{g \in G} \bar{\omega}(g) \operatorname{sign}(g, W) = |G| \operatorname{sign} V^1 + \sum_{g \neq 1} (\bar{\omega}(g) - 1) \operatorname{sign}(g, W).$$

(Recall that irreducible characters are orthogonal. For the second equality, we use the identity $\sum_g \operatorname{sign}(g, W) = |G| \operatorname{sign} V^1$, $g \in G$, which is the first equality with $\omega \equiv 1$.) By the usual transfer argument, $\operatorname{sign} V^1 = \operatorname{sign} W$. Summarizing, we conclude that

$$\operatorname{sign}^{\omega}(W, F) - \operatorname{sign}(W) = \frac{1}{|G|} \sum_{g \neq 1} (\bar{\omega}(g) - 1) \operatorname{sign}(g, W) \tag{3.2}$$

is a linear combination of the g-signatures $\operatorname{sign}(g, W)$ with $g \in G$ and $g \neq 1$.

Since W is a smooth manifold, we can use the G-signature theorem [2, 11], which expresses the g-signature $\operatorname{sign}(g, W)$ in terms of the fixed point set of g. We use repeatedly the fact that each surface F_i is connected and the covering is "uniform" along F_i; hence, the extra factor appearing in the G-signature theorem depends on the element $g \in G$ only and does not depend on a particular component of the fixed point set.

If $1 \neq g \in C_{q_i}$ lies in one of the factors of G, its fixed point set is F_i and $\operatorname{sign}(g, W)$ is a multiple of $[F_i]^2$. By Proposition 3.4, $[F_i^*, \partial F_i^*] = 0 \in H_2(N^*, \partial N^*)$ for $* = \prime$ or $\prime\prime$; hence, $[F_i] = 0$ and $\operatorname{sign}(g, W) = 0$.

If $g \in C_{q_i} \times C_{q_j}$ lies in the product of two factors (but not in either of them), the fixed point set is $F_i \cap F_j$ and $\operatorname{sign}(g, W)$ is a multiple of $\langle [F_i], [F_j] \rangle = 0$ (since, as above, $[F_i] = [F_j] = 0$).

In all other cases, the fixed point set is empty (there are no triple intersections); hence, $\operatorname{sign}(g, W) = 0$. Summarizing, $\operatorname{sign}(g, W) = 0$ whenever $g \neq 1$; in view of (3.2), this implies that $\operatorname{sign}^{\omega}(W, F) = \operatorname{sign} W$ and concludes the proof. ∎

3.3   Proof of Proposition 3.4

Consider an integral surgery presentation for $\mathbb{S}$ given by a framed oriented link $T = T_1 \cup \ldots \cup T_k$ in S^3. Since $\mathbb{S}$ is a homology sphere, we may assume that T is algebraically split and that the surgery coefficients of each of its components are ± 1 [18, Theorem A]. The link L can be represented by a collection of curves in $S^3 \smallsetminus T$.

Let N be the four-manifold obtained by attaching two-handles to B^4 along the components of T according to their framings. Since the linking matrix of T is diagonal with ± 1 entries, we may slide the knots in L over the attached handles in order to obtain a presentation of L in $\mathbb{S}$ such that $\ell k_{\mathbb{S}}(L_i, T_j) = 0$ for all $i = 1, \ldots, \mu$ and $j = 1, \ldots, k$. Since

all L_i are disjoint from the attaching tori of the handles, we can consider a surface F in B^4, the 0-handle of N, such that F is a union of compact connected oriented smooth surfaces $F_1, \ldots, F_\mu$, and each F_i is smoothly embedded with $\partial F_i = L_i$.

We have the following commutative diagram:

$$
\begin{array}{ccccccc}
0 = H^1(N) & \longrightarrow & H^1(N \smallsetminus F) & \longrightarrow & H^2(N, N \smallsetminus F) & \longrightarrow & H^2(N) \\
& & & & \downarrow & & \downarrow \\
& & H_2(F, \partial F) & \xrightarrow{\ i_*\ } & H_2(N, \partial N),
\end{array}
$$

where, by Alexander and Lefschetz duality, the two vertical arrows are isomorphisms and the inclusion homomorphism i_* is trivial, as $\ell k_{\mathbb{S}}(L_i, T_j) = 0$ and thus $[F_i, \partial F_i] = 0 \in H_2(N, \partial N)$ for all $i = 1, \ldots, \mu$. It follows that $H^1(N \smallsetminus F)$ is canonically isomorphic to $H_2(F, \partial F) = \mathbb{Z}^\mu$, and the latter group is freely generated by the fundamental classes $[F_i, \partial F_i]$. Repeating the same computation over the finite field $\mathbb{F}_p$, we get $H^1(F \smallsetminus F; \mathbb{F}_p) = H_2(F, \partial F; \mathbb{F}_p)$ and, since the dimension of this vector space does not depend on p, we conclude that the homology group $H_1(F \smallsetminus F) = \operatorname{Hom}(H_1(F \smallsetminus F), \mathbb{Z})$ is freely generated by the elements of the dual basis, that is, the meridians $\bar{m}_i$ of the components $F_i \subset N$.   ∎

4   Proof of Theorem 2.2

4.1   The auxiliary Hopf link

In the proof of Theorem 2.2, it will be useful to have some control over the surface F used to compute the colored signatures; namely, sometimes we want the distinguished component K to bound a disk. The proof of the following lemma is a straightforward adaptation of the proof of Proposition 3.4.

Lemma 4.1.   Let $K \cup L$ be a $(1, \mu)$-colored link in $\mathbb{S}$. Then, the pair (N, F) in Proposition 3.4 can be chosen of the form $(N, D \cup F)$, where D is a disk, $K = \partial D$, and $L_i = \partial F_i$.   □

Proof.   As explained in the proof of Proposition 3.4, we can consider an integral surgery presentation for $\mathbb{S}$ given by a framed oriented link $T = T_1 \cup \ldots \cup T_k$ in S^3, where T is algebraically split and the surgery coefficients of each of its components are ± 1. Moreover, the link $K \cup L$ can be represented by a collection of curves in $S^3 \smallsetminus T$ such that $\ell k(K, T_i) = \ell k(L_j, T_i) = 0$ for all i, j.

Notice that we can obtain $K \cup L \subset \mathbb{S}$ by starting with $U \cup L \subset S^3 \smallsetminus T$, where U is the unknot, and performing surgery on unknotted curves $C_1, \ldots, C_t$ in $S^3 \smallsetminus (T \cup U \cup L)$ with framings $\varepsilon_i = \pm 1$ to do some crossing changes on U to obtain K. It is clear that

we might assume $\ell k(U, T_i) = 0$ for all i and that the curves C_i may be chosen such that $\ell k(C_i, C_j) = 0$ if $i \neq j$ and $\ell k(C_i, T_j) = \ell k(C_i, U) = \ell k(C_i, L_j) = 0$ for all i and j.

The link $U \cup L$ in S^3 bounds a properly immersed surface $D \cup F_1 \cup \ldots$ in B^4. Indeed, one has

(1) $L_i = \partial F_i = F_i \cap \partial B^4$ and $U = \partial D = D \cap \partial B^4$;

(2) D and each surface F_i are transversal to ∂B^4, and

(3)   distinct surfaces intersect transversally, at double points, and away from ∂B^4.

Consider the four-manifold N obtained by attaching two-handles to B^4 along the components of $T \cup C_1 \cup \ldots \cup C_t$ according to their framings. By construction we obtain the link $K \cup L$ sitting in $\mathbb{S} = \partial N$ and bounding F. Moreover, the above conditions on the linking numbers guarantee that the proof of Proposition 3.4 follows word by word with the manifold N and the surface F considered in this proof.   ∎

Let $(N, D \cup F)$ be the pair constructed in Lemma 4.1 and fix a tubular neighborhood $B \cong D \times B^2$ of D in N, see Figure 3. Without loss of generality, by taking B small enough, we may assume that, *up to orientation of the components*, the pair $(B, (D \cup F) \cap B)$ has boundary $(S^3, H_{1,m})$, where m is the number of points in $D \cap F$. The components of $H_{1,m} = V \cup U_1 \cup \ldots \cup U_m$ inherit an orientation from $D \cup F$, and we color them according to the decomposition $D \cup F_1 \cup \ldots \cup F_\mu$.
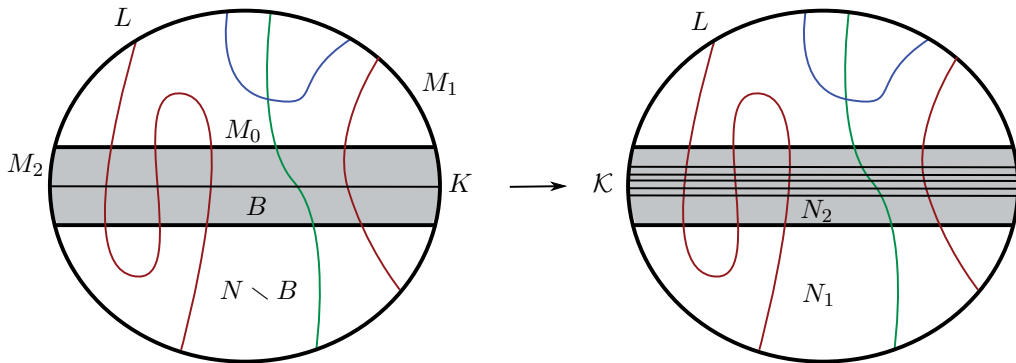
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Fig. 3.   This diagram represents the pairs $(N, D \cup F)$ and $(N, \mathcal{D} \cup F)$. The gray band is the four ball $N_2 = B = D \times B^2$.

Assume that the original link is given a character (v, u). This character extends to $D \cup F$ and restricts to a character, also denoted by (v, u), on $H_{1,m}$. Occasionally, we will replace D with several parallel copies, obtaining a link $H_{n,m}$, and change the character on the V-part of $H_{n,m}$, while keeping u on the U-part. We always assume that linked components are given distinct colors, but we allow a nonstandard orientation of the V part, describing it by a linking vector v, cf. the paragraph prior to (2.2).

Lemma 4.2.   Consider the link $H_{n,m} = V \cup U$ equipped with the coloring, orientation, and character u on the U-part as explained above. Then, for any character v on the V-part and any linking vector v, one has

$$\sigma_{H_{n,m}}(v, u) = -\delta_v(v)\delta_\lambda(u),$$

where $\lambda := \overline{\ell k}(K, L)$. $\square$

Proof.   The U-part of the link can be described as follows: for each $i = 1, \ldots, \mu$, there is a number of components, all carrying the same color and character ω_i, oriented in a random way but so that the entries of the linking vector (with respect to a fixed positive component of the V-part) sum up to λ_i. (These components correspond to the geometric intersection points, and their orientation reflects the sign of the intersection.) Hence, the statement is an immediate consequence of (2.2) and the definition of δ, as the copies of $\pm \operatorname{Log} u_i$ would sum up to $\lambda_i \operatorname{Log} u_i$.   ∎

4.2   A special case

The next lemma is straightforward; it is stated for references. We will use it to apply Wall's Theorem 3.2. Certainly, the statement on $H_0^\chi(X)$ extends to any topological space X, whereas that on $H_1^\chi(X)$ extends to any space with abelian fundamental group.

Lemma 4.3.   Let $X \cong T^2$ be a two-torus and $\chi : H_1(X) \to \mathbb{C}^*$ a multiplicative character. Then $H_1^\chi(X) = H_1(X; \mathbb{C})$, $H_0^\chi(X) = H_0(X; \mathbb{C})$ if $\chi \equiv 1$ and $H_1^\chi(X) = H_0^\chi(X) = 0$ otherwise. $\square$

We start by proving a special case of Theorem 2.2, which will be useful later on and whose proof contains the key ingredients used to establish the general formula. In the following lemma we study the effect on the colored signatures of changing the component K of a $(1, \mu)$-colored link $K \cup L \subset \mathbb{S}$ to a collection of v parallel curves, that is, of performing a $(v, 0)$-cabling. This operation is equivalent to the splice of $K \cup L \subset \mathbb{S}$ and $H_{1,v} \subset S^3$.

Let $\mathcal{K} \cup L$ be the resulting $(\nu + \mu)$-colored link. Denote $\lambda := \overline{\ell k}(K, L)$ and, for a character $\omega \in \mathcal{T}^\mu$, let $\upsilon := \omega^\lambda$. For a character $\zeta \in \mathcal{T}^\nu$, let $\pi := \prod_{i=1}^{\nu} \zeta_i$.

Lemma 4.4.   In the notation above, assuming that $(\upsilon, \pi) \neq (1, 1)$, one has

$$\sigma_{\mathcal{K} \cup L}(\zeta, \omega) = \sigma_{K \cup L}(\pi, \omega) - \delta(\zeta)\delta_\lambda(\omega). \qquad \square$$

Proof.   The diagram in Figure 3 might help one follow the construction. Let $(N, D \cup F)$ be the pair constructed in Lemma 4.1 for the link $K \cup L \subset \mathbb{S} = \partial N$ and fix a tubular neighborhood $B \cong D \times B^2$ of D in N. The pair $(N, D \cup F)$ can be written as the union

$$(N \smallsetminus B, F \cap (N \smallsetminus B)) \cup (B, (D \cup F) \cap B)$$

glued along $(D \times S^1, F \cap \partial B)$. As explained in Section 4.1, the boundary of $(B, (D \cup F) \cap B)$ is $(S^3, H_{1,m})$, where $H_{1,m}$ inherits orientations of the components and coloring with the associated character (π, ω).

We use Wall's Theorem 3.2 to relate the twisted and non-twisted signatures of $(N, D \cup F)$ and $(N \smallsetminus B, F \cap (N \smallsetminus B))$. To this end, define $N_1 = N \smallsetminus B$, $M_1 = \mathbb{S} \smallsetminus \mathring{T}(K)$, $N_2 = B$, $M_2 = T(K)$ and $M_0 = D \times S^1$, where T stands for a small tubular neighborhood and $\mathring{T}$ is its interior. One has $\partial N_1 = M_1 \cup M_0$ and $\partial N_2 = M_2 \cup -M_0$, and in both cases the manifolds are glued along $X := \partial D \times S^1 = K \times S^1$. Let m and ℓ be the meridian and longitude of K, which generate $H_1(X)$. Following the notation of Theorem 3.2, we have $A_0 = A_1 = \langle \ell \rangle$ and $A_2 = \langle m \rangle$, which implies that $K(A_0, A_1, A_2) = 0$ and thus

$$\mathrm{sign}(N) = \mathrm{sign}(N_1 \cup N_2) = \mathrm{sign}(N_1) + \mathrm{sign}(N_2) = \mathrm{sign}(N_1) \qquad (4.1)$$

since N_2 is contractible.

We now make the corresponding computation with twisted coefficients.

Let $\rho := (\pi, \omega)$; we will use the same notation for the extensions of ρ to the other spaces involved. We need to study the relationship between $\mathrm{sign}^\rho(N_1 \cup N_2, D \cup F)$ and the twisted signatures of N_1 and N_2. The group $H_1(X)$ is generated by m, ℓ and, since $\rho(m) = \upsilon$ and $\rho(\ell) = \pi$ are not both trivial, we have $H_1^\rho(X) = 0$, see Lemma 4.3. This trivially implies $K\big(A_0^\rho, A_1^\rho, A_2^\rho\big) = 0$, and Wall's Theorem 3.2 yields

$$\mathrm{sign}^\rho(N_1 \cup N_2, D \cup F) = \mathrm{sign}^\rho(N_1, F \cap N_1) + \mathrm{sign}^\rho(N_2, (D \cup F) \cap N_2). \qquad (4.2)$$

Since the boundary of $(N_2, (D \cup F) \cap N_2)$ is $(S^3, H_{1,m})$, by Lemmas 4.2 and 2.11(1) we have

$$\text{sign}^\rho (\pi, \omega)(N_2, (D \cup F) \cap N_2) - \text{sign}(N_2) = \sigma_{H_{1,m}}(\pi, \omega) = \delta(1)\delta_\lambda(\omega) = 0.$$

Combining equations (4.1) and (4.2) with Definition 3.5 of signature, we get

$$\sigma_{\mathcal{K} \cup L}(\rho) = \text{sign}^\rho(N, D \cup F) - \text{sign}(N) = \text{sign}^\rho(N_1, F \cap N_1) - \text{sign}(N_1). \qquad (4.3)$$

Now, consider the link $\mathcal{K} \cup L$. We can assume that $\mathcal{K}$ lies in the tubular neighborhood $M_2 = \partial D \times B^2$ of K in $\mathbb{S}$. The link $\mathcal{K}$ bounds a collection of ν parallel disks $\mathcal{D} \subset N_2 = B$ and the pair $(N_2, (\mathcal{D} \cup F) \cap N_2)$ has boundary $(S^3, H_{\nu,m})$, with the generalized Hopf link $H_{\nu,m}$ carrying the character (ζ, ω) and corresponding orientations. Similar to (4.2), one has

$$\text{sign}^{(\zeta,\omega)}(N_1 \cup N_2, \mathcal{D} \cup F) = \text{sign}^\rho(N_1, F \cap N_1) + \text{sign}^{(\zeta,\omega)}(N_2, (\mathcal{D} \cup F) \cap N_2). \qquad (4.4)$$

Moreover, we can compute the signature of $H_{\nu,m}$ from the pair $(B, (\mathcal{D} \cup F) \cap B)$; thus, since N_2 is contractible,

$$\text{sign}^{(\zeta,\omega)}(N_2, (\mathcal{D} \cup F) \cap N_2) = -\delta(\zeta)\delta_\lambda(\omega), \qquad (4.5)$$

see Lemma 4.2. Using the pair $(N, \mathcal{D} \cup F)$ to compute the signature of $\mathcal{K} \cup L$, we have

$$
\begin{aligned}
\sigma_{\mathcal{K} \cup L}(\zeta, \omega) \quad = \quad & \text{sign}^{(\zeta,\omega)}(N, \mathcal{D} \cup F) - \text{sign}(N) \\
\underset{(4.1),(4.4)}{=} \quad & \text{sign}^\rho(N_1, F \cap N_1) + \text{sign}^{(\zeta,\omega)}(N_2, (F \cup \mathcal{D}) \cap N_2) - \text{sign}(N_1) \\
\underset{(4.3),(4.5)}{=} \quad & \sigma_{\mathcal{K} \cup L}(\rho) - \delta(\zeta)\delta_\lambda(\omega). \qquad \blacksquare
\end{aligned}
$$

4.3   Proof of Theorem 2.2

The diagram in Figure 4 might be useful to follow the details. Let $(N', D' \cup F')$ be the pair constructed in Lemma 4.1 for the link $K' \cup L' \subset \mathbb{S}'$ and fix a small tubular neighborhood $B' \cong D' \times B^2$ of D' in N'. Since $(\upsilon', \upsilon'') \neq (1, 1)$, we can repeat the arguments in the proof of Lemma 4.4 involving Wall's theorem to obtain

$$\sigma_{K' \cup L'}(\upsilon'', \omega') = \text{sign}^{(\upsilon'',\omega')}(N' \smallsetminus B', F' \cap (N' \smallsetminus B')) - \text{sign}(N' \smallsetminus B'). \qquad (4.6)$$

By construction, the surface $(D' \cup F') \cap B'$ consists of the disk D' and a union of m' parallel disks transversal to D' (those coming from F'). Consider now a pair $(N'', D'' \cup F'')$
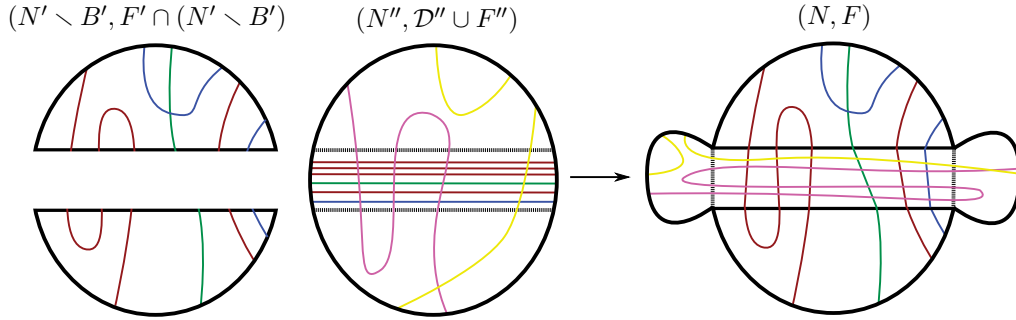
Fig. 4. The third diagram represents the pair (N, F) used to compute the signature of the splice of $K' \cup L'$ and $K'' \cup L''$. This pair is obtained identifying parts of the boundary of $(N' \smallsetminus B', F' \cap (N' \smallsetminus B'))$ and $(N'', \mathcal{D}'' \cup F'')$.

given by Lemma 4.1 for $K'' \cup L'' \subset \mathbb{S}''$. Replace K'' with m' parallel copies (with the orientations coherent with the signs of the intersection points of D' and F') to obtain a $(m' + \mu'')$-colored link $\mathcal{K}'' \cup L'' \subset \mathbb{S}''$, to which we assign the character $(\omega', \omega'') \colon H_1(\mathbb{S} \smallsetminus (\mathcal{K}'' \cup L'')) \to \mathbb{C}^*$. In a similar way, replace the disk D'' with m' parallel copies to obtain a pair $(N'', \mathcal{D}'' \cup F'')$. We may assume that the disks constituting $\mathcal{D}''$ lie in a small neighborhood $B'' \cong D'' \times B^2$, and we color the components of $\mathcal{D}''$ in accordance with the colors of the m' parallel disks coming from the surface F' in $(D' \cup F') \cap B'$.

In the boundary of B'', we obtain a generalized Hopf link $H_{m',m''}$ (up to orientation of the components, cf. Section 4.1) carrying the character (ω', ω''). Lemma 4.4 applied to the $(m', 0)$-cabling of $K'' \cup L''$ along K'' yields

$$\sigma_{\mathcal{K}'' \cup L''}(\omega', \omega'') = \text{sign}^{(\omega', \omega'')}(N'', \mathcal{D}'' \cup F'') - \text{sign}(N'')$$

$$= \sigma_{K'' \cup L''}(\upsilon', \omega'') - \delta_{\lambda'}(\omega')\delta_{\lambda''}(\omega''). \tag{4.7}$$

(For the last term, Lemma 4.2 is applied twice, first to ω', then to ω''.) Now, let us look at the pair (N, F) obtained as the gluing

$$(N' \smallsetminus B', F' \cap (N' \smallsetminus B')) \cup (N'', \mathcal{D}'' \cup F''), \tag{4.8}$$

with the solid torus $T' = D' \times \partial B^2$ in the boundary of B' identified with the solid torus $T'' = \partial D'' \times B^2$, which is a tubular neighborhood $T(K'')$ of K'' in $\mathbb{S}''$. The identification is made in such a way that the disk $D' \subset T'$ is glued to $B^2 \subset T''$. Moreover, the m' disks removed from the surfaces F' in the intersection $F' \cap (N' \smallsetminus B')$ are filled with the

corresponding m' disks constituting $\mathcal{D}''$. Notice that the boundary of (N, F) is nothing but $(\mathbb{S}, L)$, that is, the splice in question. Furthermore, by the construction, the pair (N, F) can be used to compute the colored signature of $(\mathbb{S}, L)$, that is,

$$\sigma_L(\omega', \omega'') = \text{sign}^{(\omega', \omega'')}(N, F) - \text{sign}(N).$$

To complete the proof, we shall study the behavior of the twisted and classical signatures of N with respect to the decomposition (4.8). By Theorem 3.2, the signatures will be additive with respect to this decomposition if at least two of the kernels A_0, A_1, A_2 coincide in the classical and in the twisted version. In the classical version, we are dealing with the group $H_1(\partial T')$, generated by $m_{K'} = \ell_{K''}$ and $\ell_{K'} = m_{K''}$, the meridian and longitude of K' and K'' which are identified in (4.8). It is clear that the kernels of the inclusion of $H_1(\partial T')$ into both $H_1(\mathbb{S}' \smallsetminus \text{int } T(K'))$ and $H_1(T(K''))$ are generated by $\ell_{K'} = m_{K''}$, and thus, by Wall's theorem we have

$$\text{sign}(N) = \text{sign}(N' \smallsetminus B') + \text{sign}(N''). \tag{4.9}$$

In the twisted version, the space $H_1^{(\omega', \omega'')}(\partial T') = H_1^{(\upsilon', \upsilon'')}(\partial T')$ vanishes due to Lemma 4.3 and the assumption $(\upsilon', \upsilon'') \neq (1, 1)$. Hence, Theorem 3.2 yields

$$\text{sign}^{(\omega', \omega'')}(N, F) = \text{sign}^{(\upsilon'', \omega')}(N' \smallsetminus B', F' \cap (N' \smallsetminus B')) + \text{sign}^{(\omega', \omega'')}(N'', \mathcal{D}'' \cup F''). \tag{4.10}$$

Putting these equations together, we obtain

$$\sigma_L(\omega', \omega'') \underset{(4.9),(4.10)}{=} \text{sign}^{(\upsilon'', \omega')}(N' \smallsetminus B', F' \cap (N' \smallsetminus B')) + \text{sign}^{(\omega', \omega'')}(N'', \mathcal{D}'' \cup F'')$$
$$- \text{sign}(N' \smallsetminus B') - \text{sign}(N'')$$
$$\underset{(4.6),(4.7)}{=} \sigma_{K' \cup L'}(\upsilon'', \omega') + \sigma_{K'' \cup L''}(\upsilon', \omega'') - \delta_{\lambda'}(\omega')\delta_{\lambda''}(\omega''). \qquad \blacksquare$$

4.4   Proof of Addendum 2.7

Applying Theorem 2.2 to the splice of K' and $K'' \cup L''$, we obtain

$$\sigma_L(\omega) = \sigma_{K'}(\upsilon'') + \sigma_{K'' \cup L''}(1, \omega'') - \delta(1)\delta_{\lambda''}(\omega'') = \sigma_{K'}(\upsilon'') + \sigma_{L''}(\omega'').$$

Thus, it suffices to justify that, in this particular case, Theorem 2.2 holds even if $\upsilon'' = 1$.

Let $\rho := (\upsilon', \upsilon'')$. In the proof of Theorem 2.2, the assumption $\rho \neq (1,1)$ was only used to establish that the twisted homology group $H_1^\rho(\partial T')$ is trivial, yielding (4.10). If $\rho = (1,1)$, this group is no longer trivial, but we shall see that (4.10) still holds if L' is empty.

By Remark 3.3, we only need to show that two among the three groups A_0^ρ, A_1^ρ, and A_2^ρ are equal. We are dealing with the kernels of the inclusions $H_1^\rho(\partial T') \to H_1^\omega(M_i)$, where $M_0 = \mathbb{S}' \smallsetminus \operatorname{int} T(K')$, $M_1 = \mathbb{S}'' \smallsetminus \operatorname{int} T(K'')$, and $M_2 = T(K'')$. Since $\rho = (1,1)$, the restriction of the covering to $\partial T'$ is trivial and the group $H_1^\rho(\partial T')$ is generated by the lifts $\widetilde{m}_{K'}$ and $\widetilde{\ell}_{K'}$ of the meridian and longitude of K', which are identified, respectively, with the longitude and meridian of K''. While the generators of A_1^ρ are not evident, the groups A_0^ρ and A_2^ρ are easily seen to be equal. Indeed, since L' is empty and $\upsilon'' = 1$, the group $H_1^{\upsilon''}(\mathbb{S}' \smallsetminus \operatorname{int} T(K'))$ is the homology group of the trivial covering of M_0; therefore, $\widetilde{\ell}_{K'} = \widetilde{m}_{K''}$ generates A_0^ρ. On the other hand, since L' is empty, $|\lambda'| = 0$ and we do not need to work with parallel copies of K''. It follows that the group $H_1^{\upsilon''}(T(K''))$ is the homology group of the trivial covering of M_2 and $\widetilde{m}_{K''} = \widetilde{\ell}_{K'}$ generates A_2^ρ. We conclude that $A_0^\rho = A_2^\rho$, completing the proof. ∎

5   The Generalized Hopf Link

In this section, we compute the signature of a generalized Hopf link using the C-complex approach of [6]. This approach works only for characters with all components distinct from one. Thus, we define the *open character torus* $\mathring{T}^\mu$, obtained from $\mathcal{T}^\mu$ by removing all "coordinate planes" of the form $\omega_i = 1$, $i = 1, \ldots, \mu$.

5.1 C-complexes and Seifert forms

We recall briefly the notion of C-complex of a μ-colored link and its application to the computation of the signature. To avoid excessive indexation, we consider the special case of the bivariate signature of a bicolored link; for the general case and further details, see [6].

Thus, let $K \cup L$ be a bicolored link, with the coloring $K \mapsto 1$, $L \mapsto 2$. (We do not assume K or L connected.) A *C-complex* is a pair of Seifert surfaces E for K and F for L, possibly disconnected, which intersect each other transversally (in the stratified sense) and only at *clasps*, that is, smooth simple arcs, each connecting a point of K to a point of L. (In the general case of more than two colors, the only additional requirement is that all triple intersections of Seifert surfaces involved must be empty.) Let $S := E \cup F$.

Then, for each pair $\varepsilon, \delta = \pm 1$, one can consider the *Seifert form*

$$\theta^{\varepsilon\delta} \colon H_1(S) \otimes H_1(S) \to \mathbb{Z},$$

defined as follows. Pick a class $\alpha \in H_1(S)$ and represent it by a simple closed curve $\mathbf{a} \subset S$ satisfying the following condition: each clasp $\mathbf{c} \subset E \cap F$ is either disjoint from $\mathbf{a}$ or entirely contained in $\mathbf{a}$. It is immediate that such a curve $\mathbf{a}$ can be pushed off E in the direction ε (with respect to the coorientation of E, which is part of the structure) and off F in the direction δ, so that the resulting curve $\mathbf{a}'$ is disjoint from S. Then, for another class $\beta \in H_1(S)$, the value $\theta^{\varepsilon\delta}(\alpha \otimes \beta)$ is the linking coefficient of the shift $\mathbf{a}'$ and a cycle representing β.

Now, given a pair of complex units $(\eta, \zeta) \in \mathring{T}^2$, consider the form

$$H(\eta, \zeta) := (1 - \bar{\eta})(1 - \bar{\zeta})\left(\theta^{1,1} - \zeta\theta^{1,-1} - \eta\theta^{-1,1} + \eta\zeta\theta^{-1,-1}\right). \tag{5.1}$$

The extensions of $\theta^{\varepsilon\delta}$ to $H_1(S) \otimes \mathbb{C}$ are chosen *sesquilinear*; hence, this form is Hermitian and it has a well-defined signature. It computes the signature of $K \cup L$.

Theorem 5.1 (see [6]).   The restriction to the open torus $\mathring{T}^2$ of the bivariate signature of a bicolored link $K \cup L$ is given by

$$\sigma_{K \cup L} \colon (\eta, \zeta) \mapsto \operatorname{sign} H(\eta, \zeta). \qquad \square$$

Remark 5.2.   Strictly speaking, the statement of Theorem 5.1 is the *definition* of signature in [6]. This definition is equivalent to the conventional one, see [6, Section 6.2]. $\qquad \square$

In general, for a μ-colored link L, one should consider a μ-component C-complex S and all 2^μ possible shift directions, arriving at a Hermitian form $H(\omega)$, $\omega \in \mathring{T}^\mu$, computing the signature $\sigma_L(\omega)$. The *nullity* $\operatorname{null}_L(\omega) := \operatorname{null} H(\omega)$ is also an invariant of L; it is given by the following theorem.

Theorem 5.3 (see [6, Theorem 6.1]).   For any character $\omega \in \mathring{T}^\mu$ in the open character torus, one has $\operatorname{null}_L(\omega) = \dim H_1^\omega(S^3 \smallsetminus L)$. $\qquad \square$

Proposition 5.4. Let $H := H_{m,n}$ be a generalized Hopf link. Then, for any $(\eta, \zeta) \in \mathring{T}^m \times \mathring{T}^n$, one has

$$\mathrm{null}_H(\eta, \zeta) = \begin{cases} m + n - 3, & \text{if } \mathrm{Log}\,\eta \in \mathbb{Z} \text{ and } \mathrm{Log}\,\zeta \in \mathbb{Z}, \\ m - 1, & \text{if } \mathrm{Log}\,\eta \notin \mathbb{Z}, \mathrm{Log}\,\zeta \in \mathbb{Z}, \\ n - 1, & \text{if } \mathrm{Log}\,\eta \in \mathbb{Z}, \mathrm{Log}\,\zeta \notin \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases} \qquad \square$$

Proof. The generalized Hopf link $H_{m,n}$ can be thought of as the splice of the links $H_{1,m}$ and $H_{1,n}$. Since obviously $S^3 \smallsetminus H_{1,m} \cong S^1 \times D_m$, where D_m is an m-punctured disk, for any pair $(\upsilon, \eta) \in \mathcal{T}^1 \times \mathring{T}^m$ we have

$$\dim H_1^{(\upsilon, \eta)}(S^3 \smallsetminus H_{1,m}) = \begin{cases} m - 1, & \text{if } \upsilon = 1, \\ 0, & \text{if } \upsilon \neq 1. \end{cases}$$

A similar relation holds for $H_{1,n}$; in view of Theorem 5.3, the statement of the proposition follows from the Mayer–Vietoris exact sequence, with Lemma 4.3 taken into account. $\blacksquare$

5.2   Proof of Theorem 2.10

Due to Remark 3.6, we have

$$\sigma_{H_{m,n}}(\ldots, 1, \ldots, \zeta) = \sigma_{H_{m-1,n}}(\ldots, \hat{1}, \ldots, \zeta), \qquad \sigma_{H_{m,n}}(\eta, \ldots, 1, \ldots) = \sigma_{H_{m,n-1}}(\eta, \ldots, \hat{1}, \ldots).$$

These formulas agree with the statement of the theorem, see Lemma 2.11(4), and it suffices to compute the restriction of $\sigma_{H_{m,n}}$ to the open character torus $\mathring{T}^{m+n}$.

Consider the group $G := \mathbb{Z}/m \times \mathbb{Z}/n$. We will use the cyclic indexing for the components of the link and other related objects. Let K_i, $i \in \mathbb{Z}/m$ be the first m parallel components and $L_j, j \in \mathbb{Z}/n$, the last n parallel components.

By an obvious semicontinuity argument, for any μ-colored link L, the multivariate signature $\sigma_L(\omega)$ is constant on each connected component of each stratum $\{\omega \in \mathring{T}^\mu \mid \mathrm{null}_L(\omega) = \mathrm{const}\}$. If $L = H_{m,n}$, the strata are given by Proposition 5.4: they are the hyperplanes $P_p \times \mathring{T}^n$ and $\mathring{T}^m \times Q_q$, where

$$P_p := \{\eta \in \mathring{T}^m \mid \mathrm{Log}\,\eta = p\}, \quad Q_q := \{\zeta \in \mathring{T}^n \mid \mathrm{Log}\,\zeta = q\}, \quad p, q \in \mathbb{Z},$$

and all pairwise intersections thereof. It is immediate that the bi-diagonal $\eta_1 = \ldots = \eta_m$, $\zeta_1 = \ldots = \zeta_n$ meets each component of each stratum; hence, it suffices to compute the

restriction of the signature function to this bi-diagonal. Due to Corollary 3.8, this is equivalent to computing the *bivariate* signature $\tilde{\sigma} \colon \mathring{T}^2 \to \mathbb{Z}$, of the *bicolored* generalized Hopf link (with the coloring $K_i \mapsto 1, L_j \mapsto 2, (i,j) \in G$), and the formula to be established takes the form

$$\tilde{\sigma}(\eta,\zeta) = \delta_{[m]}(\eta)\delta_{[n]}(\zeta) = \big(\mathrm{ind}(m \operatorname{Log} \eta) - m\big)\big(\mathrm{ind}(n \operatorname{Log} \zeta) - n\big), \qquad (\eta,\zeta) \in \mathring{T}^2.$$

Consider m disjoint parallel disks E_i and n disjoint parallel disks F_j, so that $\partial E_i = K_i, i \in \mathbb{Z}/m$, and $\partial F_j = L_j, j \in \mathbb{Z}/n$. We can assume that each component L_j intersects each disk E_i at a single point e_{ij}, so that these points appear in L_j in the cyclic order given by the orientation. These points cut L_j into segments $\mathbf{l}_{ij} := [e_{ij}, e_{i+1\,j}], i \in \mathbb{Z}/m$. Likewise, each component K_i intersects each disk F_j at a single point f_{ij}, the points appearing in K_i in the cyclic order given by the orientation, and we will speak about the segments $\mathbf{k}_{ij} := [f_{ij}, f_{i,j+1}] \subset K_i, j \in \mathbb{Z}/n$. Finally, assume that the intersection $E_i \cap F_j$ is a segment $\mathbf{c}_{ij} := [e_{ij}, f_{ij}]$ (a clasp). Then, letting $E := \bigcup_i E_i$ and $F := \bigcup_j F_j$, the union $S := E \cup F$ is a bicolored C-complex for $H_{m,n}$, and we can apply Theorem 5.1.

Remark 5.5. If $m \leq 1$ or $n \leq 1$, then $H_1(S) = 0$ and the signature is trivially zero. Hence, from now on we can assume that $m, n \geq 2$. Note though that formally this case does agree with the statement of the theorem, as $\delta \equiv 0$ on $\mathcal{T}^0$ and $\mathcal{T}^1$. $\qquad \square$

In each disk E_i, consider a collection of segments (simple arcs) $\mathbf{e}_{ij} := [e_{ij}, e_{i,j+1}]$, $j \in \mathbb{Z}/n$, disjoint except the common boundary points and such that their union is a circle C_i parallel to $\partial E_i = K_i$ (and the points appear in this circle in accordance with their cyclic order). Consider similar segments $\mathbf{f}_{ij} := [f_{ij}, f_{i+1\,j}] \subset F_j, i \in \mathbb{Z}/m$, forming circles $D_j \subset F_j$ parallel to $\partial F_j = L_j$. Then, the group $H_1(S)$ is generated by the classes α_{ij} of the loops

$$\mathbf{a}_{ij} := \mathbf{c}_{ij} \cdot \mathbf{f}_{ij} \cdot \mathbf{c}_{i+1\,j}^{-1} \cdot \mathbf{e}_{i+1\,j} \cdot \mathbf{c}_{i+1\,j+1} \cdot \mathbf{f}_{i,j+1}^{-1} \cdot \mathbf{c}_{i,j+1}^{-1} \cdot \mathbf{e}_{ij}^{-1},$$

$(i,j) \in G$, connecting the points

$$e_{ij} \to f_{ij} \to f_{i+1\,j} \to e_{i+1\,j} \to e_{i+1\,j+1} \to f_{i+1\,j+1} \to f_{i,j+1} \to e_{i,j+1} \to e_{ij}$$

(in the order of appearance). The construction is illustrated in Figure 5. We do not assert that these elements form a basis: they are linearly dependent. However, we will do the computations in the free abelian group $\mathcal{H} := \bigoplus_{ij} \mathbb{Z}\alpha_{ij}, (i,j) \in G$; this change will increase the kernel of the form, but it will not affect the signature.
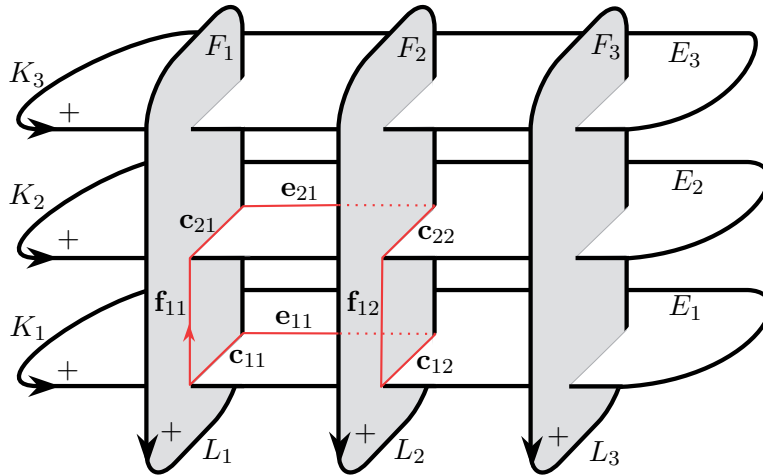
THE SIGNATURE OF A SPLICE

Fig. 5. This diagram represents a generalized Hopf link of type $H_{3,3}$. The link is depicted bounding a bicolored oriented C-complex S, which is the union of the disks E_i and F_j. The red loop is $\mathbf{a}_{11}$, whose homology class α_{11} is an element of $H_1(S)$.

The proof of the following lemma is postponed till Section 5.3.

Lemma 5.6.   Given $\varepsilon, \delta = \pm 1$, the only nontrivial values taken by the Seifert form $\theta^{\varepsilon\delta}$ on the pairs of generators α_{ij} are as follows:

$$\alpha_{ij} \otimes \alpha_{ij} \mapsto -\varepsilon\delta, \quad \alpha_{ij} \otimes \alpha_{i-\varepsilon j} \mapsto \varepsilon\delta, \quad \alpha_{ij} \otimes \alpha_{i,j+\delta} \mapsto \varepsilon\delta, \quad \alpha_{ij} \otimes \alpha_{i-\varepsilon j+\delta} \mapsto -\varepsilon\delta,$$

where $(i,j) \in G$.   □

Consider the Hermitian inner product $\langle \cdot , \cdot \rangle$ on $\mathcal{H} \otimes \mathbb{C}$ with respect to which α_{ij} is an orthonormal basis, and use this inner product to identify operators $A \colon \mathcal{H} \otimes \mathbb{C} \to \mathcal{H} \otimes \mathbb{C}$ and sesquilinear forms $\alpha \otimes \beta \mapsto \langle \alpha A, \beta \rangle$. (In accordance with the contemporary right group action conventions, our matrices act on *row* vectors by the *right* multiplication.) Then, in order to complete the proof, we need to find the eigenvalues of the self-adjoint operator $H(\eta, \zeta)$ as in (5.1). To this end, consider the unitary representation $\rho \colon G \to U(\mathcal{H} \otimes \mathbb{C})$ given by the index shifts of the basis elements, viz.

$$\rho(p,q) \colon \alpha_{ij} \mapsto \alpha_{i+p, j+q}, \qquad (p,q), (i,j) \in G.$$

This is the regular representation of G, and its equitypical summands are all of dimension 1; letting $\xi_k := \exp(2\pi i/k)$, the summands are generated by the bi-eigenvectors

$$v_{ij} := \frac{1}{mn} \sum_{(r,s)\in G} \xi_m^{-r}\xi_n^{-s}\alpha_{i+r,i+s}, \qquad (i,j) \in G,$$

so that v_{ij} is an eigenvector of $\rho(p,q)$ with the eigenvalue $\xi_m^{pi}\xi_n^{qj}$, $(p,q) \in G$. It is immediate from Lemma 5.6 that all forms $\theta^{\varepsilon\delta}$ are G-invariant; hence, they all have the same eigenvectors v_{ij}. In fact, we have more: using Lemma 5.6, one easily concludes that

$$\theta^{\varepsilon\delta} = -\varepsilon\delta\big(\rho(0,0) - \rho(-\varepsilon,0) - \rho(0,\delta) + \rho(-\varepsilon,\delta)\big).$$

Combining this with (5.1) and simplifying, we see that the spectrum of $H(\eta,\zeta)$ consists of the mn real numbers $\lambda(\eta,\xi_m^i)\lambda(\zeta,\bar{\xi}_n^j)$, $(i,j) \in G$, where

$$\lambda(x,y) := i(1 - \bar{x})(1 - \bar{y})(1 - xy).$$

(Note that $\lambda(x,y) \in \mathbb{R}$ whenever $|x| = |y| = 1$, which we always assume.) Thus, the signature of $H(\eta,\zeta)$ equals $\sigma_m(\eta)\sigma_n(\zeta)$, where $\sigma_k(x)$ stands for the "signature" of the sequence of real numbers $\lambda(x,\xi_k^i)$, $i \in \mathbb{Z}/k$.

To compute $\sigma_k(x)$, we make the following simple observations (where $x,y \in \mathbb{C}$ are complex units, $|x| = |y| = 1$):

(1) $\lambda(x,1) = \lambda(1,x) = \lambda(x,\bar{x}) = 0$;
(2) $\lambda(x,-1) = \lambda(-1,x) = -4\,\mathrm{Im}\,x$; and
(3)   with $y \neq 1$ fixed, the function $\lambda(x,y)$, $x \neq 1$, changes sign only at $x = \bar{y}$.

Combining Items (2) and (3), we see that $\mathrm{sg}\,\lambda(x,y) = \mathrm{sg}(\mathrm{Log}\,x + \mathrm{Log}\,y - 1)$ for all $x,y \neq 1$. Consider the function $\phi\colon (0,1) \to \mathbb{Z}$, $t \mapsto \sigma_k(\exp(2\pi it))$. It follows that ϕ is locally constant at each point $t \in (0,1)$ such that $kt \notin \mathbb{Z}$, whereas $\phi(t+0)-\phi(t) = \phi(t)-\phi(t-0) = 1$ for $kt \in \mathbb{Z}$. Together with the normalization $\phi(\frac{1}{2}) = \sigma_k(-1) = 0$ given by Items (1) and (2) above, we have $\phi(t) = \mathrm{ind}(kt) - k$. In other words, $\sigma_k(x) = \mathrm{ind}(k\,\mathrm{Log}\,x) - k$, which concludes the proof of the theorem. ∎

5.3   Proof of Lemma 5.6

We keep the notation introduced in Section 5.2.

The cycles $\mathbf{a}_{ij}$ do satisfy the conditions imposed in the definition of $\theta^{\varepsilon,\delta}$, see Section 5.1. To compute the linking coefficients, we consider the Π-shaped Seifert surface Π_{ij} for $\mathbf{a}_{ij}$ composed of three "squares":

- one in E_i, bounded by the loop $\mathbf{c}_{ij} \cdot \mathbf{k}_{ij} \cdot \mathbf{c}_{i,j+1}^{-1} \cdot \mathbf{e}_{ij}^{-1}$,
- one in E_{i+1}, bounded by the loop $\mathbf{e}_{i+1,j} \cdot \mathbf{c}_{i+1,j+1} \cdot \mathbf{k}_{i+1,j}^{-1} \cdot \mathbf{c}_{i+1,j}^{-1}$, and
- one bounded by $\mathbf{f}_{ij} \cdot \mathbf{k}_{i+1,j} \cdot \mathbf{f}_{i,j+1}^{-1} \cdot \mathbf{k}_{ij}^{-1}$, disjoint from S except the boundary.

Shift this surface together with the cycle, first off S in the direction (ε, δ), and then "toward the reader," in the direction of the clasp $\mathbf{c}_{ij}$, that is, from e_{ij} to f_{ij}. Then, the intersection index of the shift Π'_{ij} and another cycle $\mathbf{a}_{pq}$, $(p,q) \in G$ is easily seen geometrically; below, we give a simple visual description of the result.

Observe that all cycles $\mathbf{a}_{ij}$ lie in the graph $S' := \bigcup_i C_i \cup \bigcup_j D_j \cup \bigcup_{i,j} \mathbf{c}_{ij}$. Contracting each clasp to a point, we project S' to an $(m \times n)$-grid in the torus T^2, identified with $D_0 \times C_0$. The cycle $\mathbf{a}_{ij}$ projects to the boundary ∂s_{ij} of the (i,j)-th cell s_{ij} of the grid. (This cell can be visualized as the projection of the third square, the one not contained in E, in the description of Π_{ij}; the two other squares collapse to the two horizontal edges of s_{ij}.) Let $s'_{ij} \subset T^2$ be a small shift of s_{ij} off the grid in the direction (ε, δ). Then one has

$$\Pi'_{ij} \circ_{S^3} \mathbf{a}_{pq} = -\operatorname{vert} s_{pq} \circ_{T^2} \operatorname{hor} s'_{ij},$$

where $\operatorname{vert} s$ stands for the sum of the two "vertical" edges in ∂s, $\operatorname{hor} s$ stands for the sum of the two "horizontal" edges in the boundary of a (shifted) cell s (with their boundary orientation), and the second intersection index is in the torus T^2, oriented so that the projection of each cycle $\mathbf{a}_{ij}$ is the *positive* boundary of the respective cell s_{ij}. From here, the statement of the lemma is immediate. ∎

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