

## LUMPABILITY OF LINEAR EVOLUTION EQUATIONS IN BANACH SPACES

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**ABSTRACT.** We analyze the lumpability of linear systems on Banach spaces, namely, the possibility of projecting the dynamics by a linear reduction operator onto a smaller state space in which a self-contained dynamical description exists. We obtain conditions for lumpability of dynamics defined by unbounded operators using the theory of strongly continuous semigroups. We also derive results from the dual space point of view using sun dual theory. Furthermore, we connect the theory of lumping to several results from operator factorization. We indicate several applications to particular systems, including delay differential equations.

**1. Introduction.** Consider a linear dynamical system defined on a Banach space  $X$ :

$$\begin{cases} \dot{x}(t) = Ax(t), \\ x(0) = x_0, \quad x_0 \in X, \end{cases} \quad (1)$$

with  $A : \mathcal{D}(A) \subseteq X \rightarrow X$ . We assume that the dynamics (1) is well defined, in the sense that for every  $x_0 \in \mathcal{D}(A)$  there exists a unique classical solution  $x \in C^1([0, +\infty), \mathcal{D}(A))$  that depends continuously on the initial condition  $x_0$ . In addition, consider a linear bounded map  $M : X \rightarrow Y$  where  $Y$  is another Banach space. We view the operator  $M$  as representing a *reduction* of the state space: it is surjective but not an isomorphism. The question we are interested in is whether the variable  $y = Mx$  also satisfies a well-posed and self-contained linear dynamics on  $Y$ , say

$$\dot{y}(t) = \widehat{A}y(t), \quad y = Mx.$$

If this is the case, then we refer to  $M$  as a *reduction* or *lumping* operator.

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Diagrammatically, the system (1) is said to be *lumpable* by the operator  $M$  if there exists a linear operator  $\widehat{A} : Y \rightarrow Y$  such that the following diagram commutes

$$\begin{array}{ccc}
 Y & \xrightarrow{\widehat{A}} & Y \\
 M \uparrow & & \uparrow M \\
 X & \xrightarrow{A} & X
 \end{array} \tag{2}$$

that is,

$$MA = \widehat{A}M. \tag{3}$$

Typically, the operator  $\widehat{A}$  here is required to have analogous properties as  $A$ ; for example,  $\widehat{A}$  should be bounded if  $A$  is bounded, or they should both be generators of strongly continuous semigroups on  $X$  and  $Y$ , respectively, in case they are unbounded (and defined on proper subsets of their respective spaces; see Diagram 8).

The term *lumping* originates from chemical reaction systems, where the aim is to aggregate the species involved in the reaction into a few groups, called *lumps* of chemical reagents, and describe the reaction with a reduced number of equations. [29]. A similar concept of *aggregation* of states has been used in the theory of Markov chains, where the question is whether the newly-formed aggregates also admit a Markovian description for the state transitions [10, 19, 21], as well as in population dynamics [3]. Diagram 2, however, is more general, as the operator  $M$  can also represent other types of reduction, for example projections or averages. It can also be interpreted in the context of *multi-level* systems, where  $X$  and  $Y$  are sometimes referred to as micro (lower) and macro (upper) levels, respectively. Here the question is, given some dynamics  $A$  on the micro states  $X$ , finding the conditions on  $M$  such that  $Y$  represents a new *level* with its own autonomous dynamics. We also mention the connection to the notion of *semi-conjugacy* in nonlinear dynamical systems, where two flows  $\phi_t$  and  $\psi_t$ , generated by the nonlinear operators  $A$  and  $\widehat{A}$  defined on topological spaces  $X$  and  $Y$ , respectively, are called semi-conjugate if there exists a surjection  $M : X \rightarrow Y$  such that  $M(\phi_t) = \psi_t(M)$  [8, 9]. However, the interpretation of Diagram 2 is then rather different, because in semi-conjugacy the flow  $\psi_t$  is already given and the question is the existence of a surjection  $M$ , while the lumping problem starts from a given surjection  $M$  and asks whether there exists a reduced flow on the space  $Y$ . In fact, a more commonly used property in dynamical systems is *conjugacy*, which is obtained when  $M$  is invertible, in which case there is no reduction at all.

An interesting connection exists between lumpability and *factorization* of operators: given two linear operators  $E$  and  $D$  on a Banach space,  $D$  is said to be a left multiple of  $E$  if there exists another linear operator  $C$  such that

$$D = CE. \tag{4}$$

With  $D = MA$ ,  $E = M$ , and  $C = \widehat{A}$ , (4) corresponds to the lumping relation (3). Factorization has been studied for bounded operators on Hilbert [13] and Banach spaces [5, 14]. Some generalizations to unbounded operators can be found in [17], under the assumption of a *pseudoinverse operator* for  $E$ . The operator  $C$  in (4)

exists if and only if  $E$  majorizes  $D$  [5, 14], i.e. there exists some  $k > 0$  such that:

$$\|Dx\| \leq k\|Ex\| \quad \forall x \in X. \quad (5)$$

In this context the operator  $E$  need not to be surjective, and  $C$  is then defined on the range of  $E$ . However, in the lumping analysis one considers mostly surjective lumping operators  $M$ , because all the reduction operators used in the lumping literature, like averages or projections, are indeed surjective. It is worth noting that, if we relax our assumption on the range of  $M$ , we do not fall in the setting considered in [5, 14] and the analysis does not generalize in a straightforward way. On the other hand, we do consider unbounded operators  $A$ , without assuming the existence of a pseudoinverse for the lumping operator, but focusing on generators of strongly continuous semigroups, which may be unbounded but have some interesting spectral properties. Indeed, our aim is to study lumpability from the dynamical systems point of view, in order to obtain well-posed reduced dynamics. In passing, we mention the related notion of  $B$ -bounded semigroups [6, 4, 2] and the successive reflection method [7] for proving the existence of a semigroup.

Before proceeding to operators on generic Banach spaces, it is instructive to look at the situation in finite-dimensional Euclidean spaces. In the notation of diagram (2), let  $X = \mathbb{R}^n$  and  $Y = \mathbb{R}^k$ , let  $M$  be a matrix with full row rank and  $\hat{A}$  be a  $(k \times k)$  matrix. If  $k < n$ ,  $M$  represents a reduction of the state space dimension. Lumpability of finite-dimensional systems has been studied by, e.g., Li and Rabitz in application to chemical kinetics [22, 25] and by Gurvits and Ledoux in the setting Markov chains [19]. In this finite dimensional context the following result is known (e. g., [22]).

**Proposition 1.1.** *The following statements are equivalent:*

1.  $MA = \hat{A}M$ ;
2.  $\ker(M)$  is  $A$ -invariant;
3.  $\ker(M) \subseteq \ker(MA)$ .

We also note the relation of lumpability to the notion of observability in control theory. Indeed, the action of the lumping operator  $M$  can be viewed as yielding a *system observable*  $y = Mx$ , or the output of a linear time-invariant control system

$$\begin{cases} \dot{x}(t) = Ax(t), & A : \mathbb{R}^n \rightarrow \mathbb{R}^n, \\ y(t) = Mx(t), & M : \mathbb{R}^n \rightarrow \mathbb{R}^k, \end{cases} \quad (6)$$

where typically  $k < n$ . Recall that the system is called *observable* if every initial state  $x_0 \in \mathbb{R}^n$  can be uniquely reconstructed from the system output  $y$ . This happens if and only if the *observability matrix*

$$\mathcal{O} = \begin{pmatrix} M \\ MA \\ \vdots \\ MA^{n-1} \end{pmatrix}$$

has full rank  $n$ . It is easy to see that if the system is lumpable by  $M$ , then

$$\text{Rank}(\mathcal{O}) = \text{Rank}(M) = k < n.$$

Thus, in this case lumpability implies that the control system (6) is not observable [11].

Our aim in this paper is to extend these results to infinite-dimensional systems involving both bounded and unbounded operators. Previous work in this area

was carried out for bounded operators by Coxson [11], and by Zoltan and Toth in the context of Hilbert spaces [24], both requiring the existence of a continuous pseudoinverse of the lumping operator. We shall obtain more general conditions for lumpability in abstract Banach spaces that apply to dynamics generated by unbounded operators, such as partial and delay differential equations. In particular, the pseudoinverse of the lumping operator is not involved in our method, so we don't need additional hypotheses to guarantee its existence. Our approach is based on the theory of strongly continuous semigroups in Banach spaces and holds under quite general conditions, requiring only that the dynamics be well posed, in the sense of the Hille and Yosida theorem.

We prove that a necessary and sufficient condition for lumpability is the invariance of the kernel of the lumping operator under the whole semigroup of the solution operators. In particular, if this kernel is invariant under the semigroup, then one can construct a new strongly continuous semigroup on the reduced state space whose generator makes Diagram 2 commute. In case the semigroup of solutions is not known *a priori*, we give necessary and sufficient conditions for lumpability directly on the infinitesimal generator: here a condition is needed on the resolvent set of the generator to guarantee the invariance of the kernel of the lumping operator under the generated semigroup.

Furthermore, we complement the analysis by describing lumpability with respect to the dual space, dealing with the adjoints of the evolution operators. This represents an alternative view of the problem, allowing a different interpretation of lumpability and exploiting some interesting properties of adjoint operators. To obtain lumpability, the range of the lumping operator adjoint must be invariant under the adjoint semigroup. The adjoint of a strongly continuous semigroup need not be strongly continuous on the whole dual Banach space, but it is continuous with respect to the weak star topology. For this reason we use the notion of *weak star generator* and we analyze how the lumping operator adjoint acts on the *sun dual space* of the reduced state space, i.e. the closed subspace on which the reduced semigroup preserves strong continuity.

The paper is organized as follows. In Section 2 we discuss some known results about lumpability for bounded operators and indicate a relation to operator factorization. In Section 3 we extend the lumpability analysis to unbounded operators using semigroup theory. We also discuss the case of non-surjective lumping operators. Section 4 presents dual conditions for lumpability using sun dual spaces. We supplement the theory with several applications, including delay differential equations.

**2. Lumpability for bounded operators.** Let  $X$  be a Banach space, and let  $\mathcal{B}(X)$  denote the Banach algebra of linear bounded operators from  $X$  to itself with norm  $\|A\| = \sup_{\|x\| \leq 1} \|Ax\|$ . We first consider system (1) when  $A \in \mathcal{B}(X)$ . Since  $A$  is bounded, (1) is well defined and the solutions are given by  $x(t) = e^{At}x(0)$ , where  $e^{At} = \sum_{k=0}^{\infty} A^k t^k / k!$ , and the series is convergent in the topology of  $\mathcal{B}(X)$ . We consider the diagram (2) where  $M : X \rightarrow Y$  is a linear, bounded and surjective operator between Banach spaces  $X, Y$ . The main lumpability result in this setting is the following.

**Theorem 2.1.** *Let  $A \in \mathcal{B}(X)$ . There exists a linear, bounded operator  $\hat{A} \in \mathcal{B}(Y)$  satisfying  $MA = \hat{A}M$  if and only if  $\ker(M) \subseteq \ker(MA)$ .*

This result was proved by Barnes in [5] in the context of factorization of operators. A proof in the context of lumping can be found in [11], which uses the pseudoinverse of a bounded operator under the additional assumption that the kernel of  $M$  is topologically complemented in  $X$ .

**Remark 1.** A basic kind of lumping is obtained by the familiar quotient projection operation. Consider a closed subset  $\mathcal{C} \subset X$  such that  $A\mathcal{C} \subseteq \mathcal{C}$ , and take  $Y = \frac{X}{\mathcal{C}}$ . By the invariance of  $\mathcal{C}$ , we can define the bounded linear operator  $\widehat{A}[x] := [Ax]$ . Then, for  $x \in X$ ,

$$\pi Ax = [Ax] = \widehat{A}[x] = \widehat{A}\pi x,$$

so that the following diagram commutes:

$$\begin{array}{ccc} \frac{X}{\mathcal{C}} & \xrightarrow{\widehat{A}} & \frac{X}{\mathcal{C}} \\ \pi \uparrow & & \uparrow \pi \\ X & \xrightarrow{A} & X \end{array}$$

**Remark 2.** As in the finite-dimensional case, we can view the system

$$\begin{cases} \dot{x}(t) = Ax(t), & A \in \mathcal{B}(X), \\ y(t) = Mx(t) \end{cases} \quad (7)$$

as a control system with output  $y = Mx$ . In this context, (7) is said to be observable if [26]

$$\bigcap_{k=0}^{+\infty} \ker(MA^k) = \{0\}.$$

If the system is lumpable by  $M$ , then by definition  $\bigcap_{k=0}^{+\infty} \ker(MA^k) = \ker(M) \neq \{0\}$ , so that it is non-observable [11].

The following result related to factorization of operators can be seen as a connection to lumpability of bounded operators.

**Theorem 2.2** ([14], Thm. 1). *Let  $D$  and  $E$  be bounded linear operators from a Banach space  $X$  to itself. Then the following conditions are equivalent:*

- (i)  $D = CE$  for some bounded operator  $C$  on  $\text{ran}(E)$ ,
- (ii)  $\exists k > 0$  such that  $\|Dx\| \leq k\|Ex\|$ ,  $\forall x \in X$ ,
- (iii)  $\text{ran}(D^*) \subset \text{ran}(E^*)$ .

With  $D = MA$  and  $E = M$ , (i) corresponds to the lumping relation (3), with  $C = \widehat{A}$ . Unlike the case of lumping, in the context of factorization the operator  $E$  need not be surjective, and the operator  $C$  is then defined on the range of  $E$ . Here in most cases we assume the surjectivity of  $M$ , but at the same time we relax (ii) and we only ask an invariance condition for the kernel of  $M$ . A condition for lumpability in the case of a non-surjective lumping operator  $M$  is discussed in Section 3.

**3. Lumpability for unbounded operators.** We now turn to the case when the operator generating the dynamics is unbounded; thus, we consider the abstract Cauchy problem (1) where  $A : \mathcal{D}(A) \subset X \rightarrow X$  is a linear unbounded operator. It is a classical result in semigroup theory [20, 15, 23] that the dynamics (1) is well posed if and only if  $A$  is the generator of a strongly continuous semigroup  $\{T(t)\}_{t \geq 0}$

on  $X$ , and in that case, for every  $x_0 \in \mathcal{D}(A)$ , the unique classical solution of (1) is given by  $t \mapsto T(t)x_0$ .

The lumpability problem in the unbounded case can be expressed as the commutativity of the diagram

$$\begin{array}{ccc} M(\mathcal{D}(A)) \subset Y & \xrightarrow{\widehat{A}} & Y \\ \uparrow M & & \uparrow M \\ \mathcal{D}(A) \subset X & \xrightarrow{A} & X \end{array} \quad (8)$$

We assume that the linear operator  $M : X \rightarrow Y$  is bounded and surjective, while  $A$  and  $\widehat{A}$  are defined on a proper subset of  $X$  and  $Y$ , respectively. Suppose that  $A$  generates a strongly continuous semigroup on  $X$ , which we denote by  $\{T(t)\}_{t \geq 0}$ . We want the operator  $\widehat{A}$  to be again the generator of a strongly continuous semigroup in order to obtain a well-defined dynamics on the upper level. Thus, we need the lumping relation  $MA = \widehat{A}M$  to hold on  $\mathcal{D}(A)$ .

**Theorem 3.1.** *The following statements are equivalent.*

1.  $\ker(M)$  is invariant under  $T(t)$  for every  $t \geq 0$ .
2. There exists a linear operator  $\widehat{A}$  on  $M(\mathcal{D}(A))$  such that  $\widehat{A}$  generates a strongly continuous semigroup on  $Y$ , and  $\widehat{A}M = MA$  (i.e., system (1) is lumpable by the operator  $M$ ).

*Proof.* 1  $\Rightarrow$  2. Suppose that  $\ker(M)$  is invariant under  $T(t)$ ,  $\forall t \geq 0$ . Consider the family of linear operators  $\{\widehat{T}(t)\}_{t \geq 0}$  on  $Y$  defined by

$$\widehat{T}(t)y = MT(t)x, \quad y = Mx. \quad (9)$$

For each  $t \geq 0$ ,  $\widehat{T}(t)$  is well defined due to the invariance of the kernel, and, applying theorem 2.1, one can see that it is bounded. Moreover, the family (9) is a strongly continuous semigroup on  $Y$  because:

1.  $\widehat{T}(0)y = \widehat{T}(0)Mx = MT(0)x = Mx = y$ ;
2. for all  $t, s \geq 0$ ,

$$\begin{aligned} \widehat{T}(t+s)y &= MT(t+s)x = MT(t)T(s)x \\ &= \widehat{T}(t)MT(s)x = \widehat{T}(t)\widehat{T}(s)Mx = \widehat{T}(t)\widehat{T}(s)y; \end{aligned}$$

3.  $\lim_{h \rightarrow 0^+} \widehat{T}(h)y - y = \lim_{h \rightarrow 0^+} \|MT(h)x - Mx\| \leq \lim_{h \rightarrow 0^+} \|M\| \|T(h)x - x\| = 0$ .

In particular, letting  $\widehat{\omega}$  denote the growth bound of  $\widehat{T}(t)$ , we will show that  $\widehat{\omega}$  is less or equal than the growth bound  $\omega$  of  $T(t)$ . To this end, we consider the quotient Banach space  $X/\ker(M)$  with the quotient norm

$$\|[x]\| = \inf_{m \in \ker(M)} \|x - m\|, \quad [x] = \{x + m, m \in \ker(M)\} \in X/\ker(M).$$

Define the following operators from  $X/\ker(M)$  to  $Y$ :

- (i)  $\widetilde{M}[x] := Mx$ ,
- (ii)  $\widetilde{MT}(t)[x] := MT(t)x, t \geq 0$ .

By the Banach-Schauder theorem,  $\widetilde{M}$  is a homeomorphism. By the boundedness of  $T(t)$ , it follows that  $\widetilde{MT}(t)$  is bounded:

$$\|\widetilde{MT}(t)[x]\| = \inf_{m \in \ker(M)} \|MT(t)(x - m)\| \leq \|M\| \|T(t)\| \|x\| \leq C \|M\| e^{\omega t} \|x\|.$$

It follows that

$$\|\widehat{T}(t)y\| = \|\widetilde{MT}(t)\widetilde{M}^{-1}y\| \leq C \|M\| \|\widetilde{M}^{-1}\| \cdot e^{\omega t} \|y\|,$$

showing that  $\widehat{\omega} \leq \omega$ .

Let  $\widehat{A}$  be the generator of the new semigroup  $\widehat{T}(t)$ . Consider an element  $y = Mx$  in  $M(\mathcal{D}(A))$ . By the definition of a generator and the continuity of  $M$  on  $X$ ,

$$\begin{aligned} \widehat{A}y &= \lim_{h \rightarrow 0^+} \frac{1}{h} (\widehat{T}(h)y - y) = \lim_{h \rightarrow 0^+} \frac{1}{h} (MT(h)x - Mx) \\ &= M \left( \lim_{h \rightarrow 0^+} \frac{1}{h} (T(h)x - x) \right) = MAx. \end{aligned}$$

Hence,  $\widehat{A}$  is defined on  $M(\mathcal{D}(A))$ , which is a dense subset of  $Y$  because  $A$  is densely defined and  $M$  is bounded and surjective. On this subset the lumping relation also holds between the two generators:  $\widehat{A}Mx = MAx$ . We have thus obtained the inclusion  $M(\mathcal{D}(A)) \subset \mathcal{D}(\widehat{A})$ . We next show that the domain of  $\widehat{A}$  is exactly  $M(\mathcal{D}(A))$ . For this purpose, we take  $\lambda \in \mathbb{C}$  that belongs both to the resolvent set of  $A$  and of  $\widehat{A}$ , and use the integral representation of the resolvent operator. Given an arbitrary element  $y$  for which  $\widehat{A}$  is defined, there exists  $s = Mx \in Y$  such that  $y = (\lambda I - \widehat{A})^{-1}s$ . Hence one can write

$$\begin{aligned} y &= \int_0^{+\infty} e^{-\lambda t} \widehat{T}(t)s dt = \int_0^{+\infty} e^{-\lambda t} \widehat{T}(t)Mx dt \\ &= \int_0^{+\infty} e^{-\lambda t} MT(t)x dt = M \int_0^{+\infty} e^{-\lambda t} T(t)x dt \\ &= M(\lambda I - A)^{-1}x = Mz, \end{aligned}$$

where  $z$  belongs to  $\mathcal{D}(A)$ . Therefore,  $\mathcal{D}(\widehat{A}) = M(\mathcal{D}(A))$ .

2  $\Rightarrow$  1. We will show that the invariance of  $\ker(M)$  under the semigroup is a necessary condition to have a well-defined dynamics on  $Y$ . Suppose that the operator  $\widehat{A}y := MAx$  defined on  $M(\mathcal{D}(A))$  generates a strongly continuous semigroup on  $Y$ . Consider the following maps from  $\mathbb{R}^+$  to  $Y$ :

1.  $t \mapsto \widehat{T}(t)y_0$ ,
2.  $t \mapsto MT(t)x_0$ ,

where  $y_0 = Mx_0$ ,  $x_0 \in \mathcal{D}(A)$ . These two maps are both solutions of the abstract Cauchy problem

$$\begin{cases} \dot{y}(t) = \widehat{A}y(t), \\ y(0) = y_0. \end{cases} \quad (10)$$

In fact, the first map is a solution by definition, while for the second map we have

$$\frac{d}{dt} MT(t)x_0 = M \frac{d}{dt} T(t)x_0 = MAT(t)x_0 = \widehat{A}MT(t)x_0,$$

and  $MT(0)x_0 = Mx_0 = y_0$ , where we have used the continuity of  $M$  to interchange with the differentiation. Since the solution of the Cauchy problem (10) is unique, for all  $t > 0$  we have

$$\widehat{T}(t)Mx_0 = MT(t)x_0,$$

and this equality holds for every  $x_0 \in \mathcal{D}(A)$ . The operators  $MT(t)$  and  $\widehat{T}(t)M$  are equal on a dense subspace of  $Y$ , so they coincide on the whole space. The invariance of  $\ker(M)$  under the semigroup follows then from the relation  $MT(t) = \widehat{T}(t)M$ , which proves the statement above.  $\square$

We note that if a closed subspace is invariant under  $T(t)$  for all  $t \geq 0$ , then by definition it is invariant under the infinitesimal generator  $A$ ; however, the converse is not true. As a simple counterexample, let  $X$  be the Banach space  $C_0(\mathbb{R})$  of all continuous functions on  $\mathbb{R}$  that tend to zero at infinity, endowed with the supremum norm. The differentiation operator

$$Af = f', \quad \mathcal{D}(A) = \{f \in C_0^1(\mathbb{R}) : f' \in C_0(\mathbb{R})\}, \quad (11)$$

generates the strongly continuous semigroup of left translations

$$T(t)f(x) := f(x+t), \quad x \in \mathbb{R}, t \geq 0. \quad (12)$$

Clearly, the closed subspace  $\mathcal{C} = \{f \in X : f(s) = 0, \forall s \leq 0\}$  is invariant under  $A$  but not invariant under translations. We mention the following characterization of closed invariant subspaces (see, e.g., [30]), which will be used in subsequent proofs.

**Proposition 3.2** ( *$T(t)$ -invariance of a closed subspace*). *Let  $A$  be the infinitesimal generator of a strongly continuous semigroup  $\{T(t)\}_{t \geq 0}$  having growth bound  $\omega$ . Let  $\mathcal{V} \subset X$  be a closed subspace such that  $A(\mathcal{D}(A) \cap \mathcal{V}) \subseteq \mathcal{V}$ , and let  $A|_{\mathcal{V}} : \mathcal{D}(A) \cap \mathcal{V} \rightarrow \mathcal{V}$  be the restriction of  $A$  to  $\mathcal{V}$ . Then the following are equivalent:*

1.  $\mathcal{V}$  is invariant under  $T(t)$ .
2. There exists  $\lambda > \omega$  such that  $\lambda \in \rho(A) \cap \rho(A|_{\mathcal{V}})$ .

It is typically the case in applications that one knows the generator  $A$  but not the associated semigroup. Therefore, it is necessary to find conditions on  $M$  that give the invariance of its kernel under the semigroup without knowing the semigroup itself. The next result gives conditions on the operator  $A$  for lumpability.

**Theorem 3.3.** *System (1) is lumpable by the linear, bounded, and surjective operator  $M : X \rightarrow Y$  if and only if the following two conditions hold.*

1.  $A(\ker(M) \cap \mathcal{D}(A)) \subset \ker(M)$ , and
2. there exists  $\lambda > \omega$  such that  $(\lambda I - A)$  is surjective from  $\ker(M) \cap \mathcal{D}(A)$  to  $\ker(M)$ .

*Proof of Theorem 3.3.* If (1) is lumpable by  $M$ , by definition there exists a linear operator  $\widehat{A}$  such that  $MA = \widehat{A}M$  on  $\mathcal{D}(A)$  and  $\widehat{A}$  generates a strongly continuous semigroup on  $Y$ . By Theorem 3.1,  $\ker(M)$  is  $T(t)$ -invariant, and so  $\ker(M)$  is also  $A$ -invariant; i.e. condition 1 holds. By Proposition 3.2, there exists  $\lambda > \omega$  such that  $\lambda \in \rho(A) \cap \rho(A|_{\ker(M)})$ . Thus,  $(\lambda I - A)$  must be surjective from  $\ker(M) \cap \mathcal{D}(A)$  onto  $\ker(M)$ ; i.e. condition 2 holds.

Conversely, condition 1 gives that  $\ker(M)$  is invariant under  $A$ . Since the injectivity of  $(\lambda I - A)$  on the whole domain  $\mathcal{D}(A)$  guarantees the injectivity on the subspace  $\ker(M) \cap \mathcal{D}(A)$ , condition 2 implies that statement 2 of Proposition 3.2 holds with  $\mathcal{V} = \ker(M)$ . Hence,  $\ker(M)$  is invariant under the semigroup  $\{T(t)\}_{t \geq 0}$  generated by  $A$ . Lumpability then follows by Theorem 3.1.  $\square$

**Remark 3.** As a special case of condition 1 in Theorem 3.3, consider the case when

$$\ker(M) \subset \mathcal{D}(A) \text{ and } A(\ker(M)) \subset \ker(M). \quad (13)$$



If (13) holds, then the restricted operator  $A|_{\ker(M)} : \ker(M) \rightarrow \ker(M)$  is bounded by the closed graph theorem; so, its spectrum is compact in the complex plane and one can find a  $\lambda > \omega$  such that  $\lambda \in \rho(A) \cap \rho(A|_{\ker(M)})$ . It follows that there exists  $\lambda > \omega$  such that  $(\lambda I - A)$  is surjective from  $\ker(M) \cap \mathcal{D}(A)$  to  $\ker(M)$ , so that  $M$  makes a lumping by Theorem 3.3. However, condition (13) is usually too strong and generally not satisfied.

We can also give an equivalent version of Theorem 3.3 where condition 2 is formulated in terms of the spectra of  $A$  and  $\widehat{A}$ . As usual, when possible, we define the reduced operator by  $\widehat{A}y := MAx$ ,  $y = Mx$ . Let  $\rho_\infty(A)$  denote the largest connected component of  $\rho(A)$  containing an interval of the form  $[r, +\infty)$ , for some  $r \in \mathbb{R}$ . We know that if  $A$  is the infinitesimal generator of a strongly continuous semigroup, then  $\rho_\infty(A) \neq \emptyset$ . (Indeed,  $(\omega, +\infty) \subset \rho_\infty(A)$ , where  $\omega$  is the growth bound of the semigroup  $T(t)$  generated by  $A$ .)

**Proposition 3.4.** *Let  $A$  be the generator of a strongly continuous semigroup on  $X$ . The system associated with  $A$  is lumpable by  $M$  if and only if the following hold:*

1.  $A(\ker(M) \cap \mathcal{D}(A)) \subset \ker(M)$ , and
2.  $\sigma(\widehat{A}) \subseteq \mathbb{C} \setminus \rho_\infty(A)$ , i.e.  $\rho_\infty(A) \subseteq \rho(\widehat{A})$ .

*Proof.* Suppose that 1 and 2 hold. By 1, the operator  $\widehat{A}$  is well-defined. By 2,  $(\lambda I - \widehat{A})$  is invertible for every  $\lambda \in \rho_\infty(A)$ . Let  $x \in \ker(M)$ . Since  $(\lambda I - A)$  is surjective,  $x = (\lambda I - A)x_0$  for some  $x_0 \in \mathcal{D}(A)$ . Then  $x_0 \in \ker(M)$  since

$$0 = Mx = M(\lambda I - A)x_0 = (\lambda I - \widehat{A})Mx_0,$$

and  $(\lambda I - \widehat{A})$  is injective by assumption. We have proved that for every  $\lambda \in \rho_\infty(A)$  (in particular, for  $\lambda > \omega$ ),  $(\lambda I - A)$  is surjective from  $\ker(M) \cap \mathcal{D}(A)$  to  $\ker(M)$ . By Theorem 3.3, system (1) is lumpable by  $M$ .

For the inverse implication, we first show that  $(\lambda I - \widehat{A})$  is invertible whenever  $\ker(M)$  is invariant under  $(\lambda I - A)^{-1}$ . If  $\ker(M)$  is  $(\lambda I - A)^{-1}$ -invariant, the following operator from  $Y$  to  $\mathcal{D}(A)$  is well-defined:

$$\widehat{\mathcal{R}}(\lambda)y := M\mathcal{R}(\lambda)x, \quad y = Mx.$$

We know that  $\widehat{\mathcal{R}}(\lambda)$  is bounded. Moreover,  $\widehat{\mathcal{R}}(\lambda)$  is the inverse operator of  $(\widehat{A} - \lambda I)$ ; indeed, for  $y = Mx$ ,

1.  $(\lambda I - \widehat{A})\widehat{\mathcal{R}}(\lambda)y = (\lambda I - \widehat{A})M\mathcal{R}(\lambda)x = M(\lambda I - A)\mathcal{R}(\lambda)x = y$ ;
2.  $\widehat{\mathcal{R}}(\lambda)(\lambda I - \widehat{A})y = M\mathcal{R}(\lambda)(\lambda I - A)x = Mx = y$ .

Therefore,  $\lambda I - \widehat{A}$  has a bounded inverse  $\widehat{\mathcal{R}}(\lambda) = (\lambda I - \widehat{A})^{-1}$ , i.e.  $\lambda \in \rho(\widehat{A})$ . Suppose that (1) is lumpable by  $M$ . By Theorem 3.3,  $\ker(M)$  is  $A$ -invariant (i.e. condition 1 holds), and there exists  $\lambda_0 > \omega$  such that  $(\lambda_0 I - A)$  is surjective from  $\ker(M) \cap \mathcal{D}(A)$  to  $\ker(M)$ . Let  $x \in \ker(M) \cap \mathcal{D}(A)$ . Then, for some  $z \in \ker(M)$ :

$$M(\lambda_0 I - A)^{-1}x = M(\lambda_0 I - A)^{-1}(\lambda_0 I - A)z = Mz = 0,$$

showing that  $(\lambda_0 I - A)^{-1} \ker(M) \subseteq \ker(M)$ . We verify that  $\ker(M)$  is  $(\lambda I - A)^{-1}$ -invariant for every  $\lambda \in \rho_\infty(A)$ , following the idea given in [12, Lemma 2.5.6] for generators in Hilbert spaces. It is known that the resolvent function  $s \mapsto (sI - A)^{-1}$  is analytic in  $\rho_\infty(A)$ . Recall that the *annihilator* of  $\ker(M)$  is

$$\ker(M)^\perp := \{f \in X^* : f(m) = 0 \quad \forall m \in \ker(M)\}.$$

For fixed  $m \in \ker(M)$  and  $f \in \ker(M)^\perp$ , we define the map

$$G(s) := f((A - sI)^{-1}m),$$

which is an holomorphic function from  $\rho(A)$  to  $\mathbb{C}$ . It is known that for  $|\lambda - \lambda_0|$  sufficiently small (to be precise,  $|\lambda - \lambda_0| < \|\mathcal{R}(\lambda_0)\|^{-1}$ ), one has  $\mathcal{R}(\lambda) = \sum_{n=0}^{\infty} \mathcal{R}(\lambda_0)^{n+1}(\lambda - \lambda_0)^n$ . In particular, all the derivatives of the holomorphic function  $G$  vanish at the point  $\lambda_0$ , so  $G$  vanishes in a neighborhood of  $\lambda_0$ . Since  $\rho_\infty(A)$  is a connected component of  $\rho(A)$ ,  $G$  must be identically zero on  $\rho_\infty(A)$ . Since  $f$  is arbitrary, we conclude that every functional in  $\ker(M)^\perp$  vanishes on  $(sI - A)^{-1}m$ . It follows by the Hahn-Banach theorem that  $(sI - A)^{-1}m \in \ker(M)$  for all  $s \in \rho_\infty(A)$ . Since  $m \in \ker(M)$  is also arbitrary, we have  $(sI - A)^{-1}\ker(M) \subseteq \ker(M)$  for all  $s \in \rho_\infty(A)$ , from which condition 2 follows.  $\square$

**Remark 4.** Observe that in the finite dimensional case lumpability implies

$$\sigma(\widehat{A}) \subset \sigma(A). \quad (14)$$

Indeed, if  $(\lambda I - A)\ker(M) \subseteq \ker(M)$ , then also  $(\lambda I - A)^{-1}\ker(M) \subseteq \ker(M)$ . Moreover, (14) holds when  $\rho_\infty(A) = \rho(A)$ . This is the case for, e.g., infinitesimal generators with discrete spectrum having a connected resolvent set.

**Remark 5** (Observability with unbounded operators). Let  $A$  be the unbounded generator of a strongly continuous semigroup with growth bound  $\omega$ . It can be shown that the system

$$\begin{cases} \dot{x}(t) = Ax(t), \\ y(t) = Mx(t) \end{cases}$$

is observable if and only if, for any  $\mu \in \rho(A)$  satisfying  $\operatorname{Re}(\mu) > \omega$ , the following system is observable:

$$\begin{cases} \dot{x}(t) = \mathcal{R}(\mu, A)x(t), \\ y(t) = Mx(t), \end{cases}$$

where the resolvent operator  $\mathcal{R}(\mu, A) = (\mu I - A)^{-1}$  is indeed bounded [16, 27]. Hence the condition for observability is reduced to

$$\bigcap_{k=0}^{\infty} \ker(M\mathcal{R}(\mu, A)^k) = 0. \quad (15)$$

If the system is lumpable by  $M$  then  $\ker(M)$  is invariant under the semigroup, and hence also invariant under the resolvent operators for  $\operatorname{Re}(\mu) > \omega$  [30]. Since  $\ker(M) \neq 0$ , this implies that (15) is not satisfied and the system is non-observable. Hence, the observation stated in [11] for bounded operators holds also in the unbounded case.

**Example 1** (Quotient semigroup). Let  $\mathcal{C}$  be a closed subspace that is invariant under a semigroup  $\{T(t)\}_{t \geq 0}$  (or, equivalently, satisfying statement 2 of Proposition 3.2). As in the bounded case, the quotient projection

$$\pi : X \rightarrow \frac{X}{\mathcal{C}}, \quad x \mapsto [x]$$

yields a lumping on the system associated with the generator  $A$ . The semigroup induced on the quotient space is

$$\widehat{T}(t)[x] = [T(t)x], \quad t \geq 0, x \in X,$$

generated by  $\widehat{A}[x] = [Ax]$ . (See [1] for more details on quotient semigroups).

**Example 2.** Consider the space  $X = C_0(\mathbb{R})$ , and let  $h : \mathbb{R} \rightarrow \mathbb{C}$  be a continuous function. Define the multiplicative operator

$$Af(x) = h(x)f(x), \quad \mathcal{D}(A) = \{f \in X : hf \in X\},$$

(which is bounded if and only if  $h$  is a bounded function). One can show that  $A$  generates a strongly continuous semigroup if and only if  $\sup_{x \in \mathbb{R}} \operatorname{Re}(h(x)) < \infty$ , and in this case the semigroup is given by  $T(t)f(x) = e^{th(x)}f(x)$ ,  $\forall t \geq 0$ . If  $h$  is nonzero, then for any positive integer  $k$  there exist  $k$  points  $\{x_1, \dots, x_k\}$  on the real line at which  $h$  does not vanish. Consider the linear bounded operator  $M : C_0(\mathbb{R}) \rightarrow \mathbb{C}^k$  defined by  $Mf = (f(x_1), \dots, f(x_k))^\top$ , which simply evaluates a given function at the  $k$  points. We can write

$$\begin{aligned} MAf &= M(hf) = (h(x_1)f(x_1), \dots, h(x_k)f(x_k))^\top \\ &= \operatorname{diag}(h(x_1), \dots, h(x_k)) \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_k) \end{pmatrix} := \widehat{A}Mf, \end{aligned}$$

where “diag” denotes a diagonal matrix. Thus  $M$  yields a lumping on the system associated with  $A$ . Note that the kernel of  $M$  is invariant under  $A$ , but not fully contained in  $\mathcal{D}(A)$ ; hence (13) is not satisfied. Since the new operator  $\widehat{A}$  is a diagonal matrix, we pass from an infinite dimensional dynamical system to a system defined on a  $k$ -dimensional space. On the other hand, the resolvent condition given in statement 2 of Proposition 3.2 is satisfied. This can be easily seen considering that the resolvent set of  $A$  is the complementary set of

$$\sigma(A) = \{\lambda \in \mathbb{C} : h(x) = \lambda \text{ for some } x \in \mathbb{R}\}.$$

Taking  $\lambda \in \rho(A)$ , the operator  $\lambda I - A$  is surjective from  $\mathcal{D}(A) \cap \ker(M)$  to  $\ker(M)$  if and only if for every  $g \in \ker(M)$  the function  $f$  defined by  $f(x) = \frac{g(x)}{\lambda - h(x)}$  belongs to  $\mathcal{D}(A) \cap \ker(M)$ . This is indeed verified because:

1. since  $\lambda \in \rho(A)$ ,  $\frac{h(x)}{\lambda - h(x)}$  is bounded, so that  $h(x)f(x)$  tends to zero at infinity;
2. since  $g$  vanishes at the points  $x_i$  and the previous property holds,  $f$  also vanishes on this set of points. Hence, we can take every element in  $\rho(A)$  that is greater than  $\omega$  as  $\lambda$  of statement 2 of Proposition 3.2.

**Example 3** (Delay differential equations). Given  $r \geq 0$ , let  $X = C([-r, 0], \mathbb{R}^n)$  be the Banach space of continuous vector-valued functions on the compact interval  $[-r, 0]$  equipped with the supremum norm, and let  $L : X \rightarrow \mathbb{R}^n$  be linear and continuous. A linear *delay differential equation* (DDE) is an equation of the form

$$\dot{x}(t) = Lx_t,$$

where  $x_t \in X$  is the function given by

$$x_t(s) = x(t + s), \quad s \in [-r, 0].$$

The unbounded linear operator  $A$  defined by

$$Af = f', \quad \mathcal{D}(A) = \{f \in C^1([-r, 0], \mathbb{R}^n) : f'(0) = Lf\}$$

generates a strongly continuous semigroup  $\{T(t)\}_{t \geq 0}$  that gives the solutions of the DDE. In other words, the unique solution  $x(t)$  of the Cauchy problem

$$\begin{cases} \dot{x}(t) = Lx_t & t \geq 0, \\ x(t) = f(t) & t \in [-r, 0], \end{cases} \quad (16)$$

with initial condition  $f \in X$ , satisfies

$$x_t(s) = T(t)f(s), \quad s \in [-r, 0], t \geq 0.$$

Given a set of non-zero real numbers  $a_i, i = 1, \dots, n$ , we define a linear, bounded and surjective operator  $M : X \rightarrow Y := C([-r, 0], \mathbb{R})$  by

$$M(f)(s) = a_1 f_1(s) + \dots + a_n f_n(s), \quad \forall f \in X.$$

Keeping the notation as above, we have the following result.

**Proposition 3.5.** *If there exists a linear and bounded functional  $\widehat{L} : Y \rightarrow \mathbb{R}$  such that  $ML = \widehat{L}M$ , then system (16) is lumpable by the operator  $M$ . The upper level dynamics is described by a DDE on the space of scalar-valued functions  $C([-r, 0], \mathbb{R})$ :*

$$\begin{cases} \dot{y}(t) = \widehat{L}y_t & t \geq 0, \\ y(t) = g(t) & t \in [-r, 0]. \end{cases} \quad (17)$$

*Proof.* It is easy to verify that  $\ker(M) \cap \mathcal{D}(A)$  is invariant under  $A$ . (Note that  $\ker(M)$  is not fully contained in the domain of  $A$ ; so condition (13) does not hold). Letting  $\omega$  denote the growth bound of the semigroup generated by  $A$ , we shall prove that there exists  $\lambda > \omega$  such that  $(\lambda I - A)$  is surjective from  $\ker(M) \cap \mathcal{D}(A)$  to  $\ker(M)$ . To this end, we take  $\lambda > 0$  in  $\rho(A) \cap \rho(\widehat{L})$  (this number always exists because  $A$  is a generator and  $\widehat{L}$  is bounded; so its spectrum is closed and bounded in  $\mathbb{C}$ ). For every  $g \in \ker(M)$  there exists  $f \in \mathcal{D}(A)$  such that  $(\lambda I - A)f = g$ ; that is  $f'(x) = \lambda f(x) - g(x)$ . Solving this differential equation,  $f$  can be written as

$$f(x) = \left( c_0 - \int_0^x g(s)e^{-\lambda s} ds \right) e^{\lambda x}$$

for  $c_0 = f(0) \in \mathbb{R}^n$ . We will show that  $f \in \ker(M)$ . Since  $g \in \ker(M)$  and  $M$  is linear,

$$Mf(x) = e^{\lambda x} M c_0.$$

Therefore  $Mf = 0$  if and only if  $M c_0 = 0$ . We need to show that  $c_0 \in \ker(M)$ . Since  $f \in \mathcal{D}(A)$ , we have  $f'(0) = Lf$ ; i.e.,

$$\lambda c_0 - g(0) = Lf.$$

Applying  $M$  on both sides gives  $\lambda M c_0 = MLf$ . Using the hypothesis, one can write  $\lambda M c_0 = \widehat{L}Mf$ , which leads to

$$\lambda M c_0 = e^{\lambda x} \widehat{L}M c_0, \quad \forall x \in [-r, 0]. \quad (18)$$

Evaluating at  $x = 0$  yields

$$\widehat{L}M c_0 = \lambda M c_0. \quad (19)$$

Since  $\lambda \in \rho(\widehat{L})$ , (19) holds iff  $M c_0 = 0$ , i.e,  $c_0 \in \ker(M)$ .

We have proved that system (16) is lumpable by  $M$ . For every  $h = Mf$ ,  $f \in \mathcal{D}(A)$ , the generator of the semigroup on the upper level is

$$\widehat{A}h(x) = MAf(x) = a_1 f_1'(x) + \dots + a_n f_n'(x) = h'(x);$$

which is again the differentiation operator, but defined on the set

$$M\mathcal{D}(A) = \{h \in Y : h' \in Y \text{ and } h'(0) = \widehat{L}h\}.$$

This operator is exactly the generator of the semigroup associated with the delayed system (17).  $\square$

To give an example of functionals  $L$  on  $X$  which satisfy the hypothesis of the previous proposition, take

$$Lf(x) := \sum_{i=1}^k q_i f(-\alpha_i)$$

where  $q_i \in \mathbb{R}$  and  $\alpha_i \in (0, r)$ . It is easy to verify that  $\widehat{L}$  acts the same way as  $L$  but on a space of scalar-valued functions,

$$\widehat{L}h(x) = \sum_{i=1}^k q_i h(-\alpha_i), \quad h \in C([-r, 0], \mathbb{R}).$$

**Example 4.** The following example illustrates that  $\sigma(\widehat{A})$  is generally not contained in  $\sigma(A)$ . Consider again the Banach space  $C_0(\mathbb{R})$  and the semigroup of left translations (12) generated by the derivative operator  $Af = f'$ , as given in (11). The spectrum of  $A$  is the imaginary axis;  $\sigma(A) = i\mathbb{R}$ ,  $i$  being the imaginary unit [1, A-III,2.4]. Indeed, for every  $\lambda = i\alpha$ ,  $\alpha \in \mathbb{R}$ , there exists a sequence  $f_n(x) := e^{-|x|/n} e^{i\alpha x}$  such that  $\|f_n\| = 1$  and  $\lim_{n \rightarrow +\infty} \|Af_n - \lambda f_n\| = 0$ . A sequence of this kind is called an *approximated eigenvector* and its existence implies that  $(A - \lambda I)$  is not bounded below, i.e. not invertible. It follows that  $\rho(A)$  is a disconnected subset of the complex plane. Consider now the lumping operator

$$M : C_0(\mathbb{R}) \rightarrow C_0(\mathbb{R}^+), \quad Mf := f|_{\mathbb{R}^+},$$

which acts as the restriction to  $\mathbb{R}^+$ . The operator  $M$  linear, bounded, and surjective by the Tietze extension theorem. Furthermore,  $\ker(M)$  is the ideal of functions vanishing on  $\mathbb{R}^+$  and it is invariant under  $A$ . If  $f \in \mathcal{D}(A)$ , it is clear that  $MAf = f'|_{\mathbb{R}^+} = (f|_{\mathbb{R}^+})'$ . It follows that the reduced operator  $\widehat{A}$  is again a derivative generating the semigroup of left translations on  $C_0(\mathbb{R}^+)$ :

$$\widehat{T}(t)g(s) = g(s+t), \quad s \in \mathbb{R}^+, t \geq 0, g \in C_0(\mathbb{R}^+).$$

It is known that the spectrum of  $\widehat{A}$  is

$$\sigma(\widehat{A}) = \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) \leq 0\}.$$

Indeed, the functions  $e^{\lambda x}$  are eigenfunctions for  $\operatorname{Re}(\lambda) < 0$ , and  $f_n(x) := e^{-x/n} e^{i\alpha x}$  is an approximated eigenfunction for  $\operatorname{Re}(\lambda) = 0$  [1]. In this case  $\sigma(\widehat{A})$  is larger than the spectrum of the original operator  $A$ . Note that the growth bound of the semigroup  $T(t)$  is  $\omega = 0$  (indeed,  $T(t)$  is a contraction semigroup). In this case  $\sup_{\lambda \in \sigma(A)} \{\operatorname{Re}(\lambda)\} = \omega(T) = 0$ . The largest connected component of  $\rho(A)$  containing an interval  $[r, +\infty)$  is

$$\rho_\infty(A) = \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) > 0\}.$$

Hence,  $(\widehat{A} - \lambda I)$  is invertible for all  $\lambda \in \rho_\infty(A)$  by Proposition 3.4.

Although the lumping operators in the literature are surjective, it is interesting to discuss lumpability in the case when  $\operatorname{ran}(M) \neq Y$ . A condition for the existence of a reduced operator in the bounded case is given by Theorem 2.2 in the context of operator factorization. Here we prove the following result.

**Proposition 3.6.** *Let  $T(t)$  be a strongly continuous semigroup on  $X$  generated by  $A$ . Let  $M$  be linear and continuous from  $X$  to  $Y$  such that the following condition holds:*

- (j) For  $\{x_n\} \subset X$ ,  $\|Mx_n\| \rightarrow 0$  implies  $\|MT(t)x_n\| \rightarrow 0 \forall t \geq 0$ .

Then there exists a strongly continuous semigroup  $\widehat{T}(t)$  on  $\overline{\text{ran}(M)}$  such that  $MT(t) = \widehat{T}(t)M$ . Moreover,  $\widehat{T}(t)$  is generated by the closure  $\widehat{A}$ , where  $\widehat{A}$  is the operator defined by

$$\widehat{A}y = MAx, \quad y = Mx \in M\mathcal{D}(A).$$

**Remark 6.** Note that condition (j) is stronger than assuming that

$$T(t)\ker(M) \subset \ker(M) \quad \forall t \geq 0. \quad (20)$$

If, in addition to (20),  $M$  has closed range, then condition (j) follows (see [14]). However, (j) does not follow from (20) if the range of  $M$  is not closed. Hence, Proposition 3.6 does not generalize Theorem 3.1, but rather gives another version of lumpability with a different assumption.

*Proof of Proposition 3.6.* It is not hard to see that (j) is equivalent to the following statement:

$$\text{For all } t \geq 0, \text{ there exists } k_t > 0 \text{ such that } \|MT(t)x\| \leq k_t\|Mx\|. \quad (21)$$

Indeed, (21) clearly implies (j), and the converse follows since  $\|M\cdot\|$  is a seminorm. Now, by (21) and Theorem 2.2, for every  $t \geq 0$  one can construct a family of linear and bounded operators on  $\text{ran}(M)$ :

$$\widehat{T}(y) := MT(t)x, \quad y = Mx. \quad (22)$$

By the boundedness of  $\widehat{T}(y)$ , these operators can be extended to  $\overline{\text{ran}(M)}$  in the following way:

$$\widehat{T}(t)y := \lim_{n \rightarrow \infty} MT(t)x_n \text{ for } Mx_n \rightarrow y.$$

It can be verified that  $\widehat{T}(t)$  is a strongly continuous semigroup of operators on  $\overline{\text{ran}(M)}$ . Note also that the value of  $k_t$  in (21) can be controlled by an exponential function. Indeed,  $\widehat{\omega}$  being the growth bound of  $\widehat{T}(t)$ , there exists  $K > 0$  such that  $\|MT(t)x\| \leq Ke^{\widehat{\omega}t}\|Mx\|$ ,  $\forall x \in X$ .

Let  $\widetilde{A}$  denote the infinitesimal generator of  $\widehat{T}(t)$ . Given  $y = Mx \in M\mathcal{D}(A)$ , one can write

$$\lim_{h \rightarrow 0} \frac{1}{h}(\widehat{T}(t)y - y) = M \lim_{h \rightarrow 0} \frac{1}{h}(T(t)x - x) = MAx.$$

Hence  $M\mathcal{D}(A) \subset \mathcal{D}(\widetilde{A})$  and  $\widetilde{A}y = \widehat{A}y$  on  $M\mathcal{D}(A)$ , where  $\widehat{A}y := MAx$ . Now, consider  $y \in \mathcal{D}(\widetilde{A})$ . Since both  $A$  and  $\widetilde{A}$  are infinitesimal generators, we can find some  $\lambda > 0$  in  $\rho(A) \cap \rho(\widetilde{A})$  such that  $\lambda > \widehat{\omega}$ , where  $\widehat{\omega}$  is the growth bound of  $\widehat{T}(t)$ . By the integral representation of the resolvent operator, for some  $y_0 \in Y$  with  $Mx_n \rightarrow y_0$  we have

$$\begin{aligned} y &= (\lambda I - \widetilde{A})^{-1}y_0 = \int_{-\infty}^{+\infty} e^{-\lambda s} \lim_{n \rightarrow \infty} MT(s)x_n ds = \lim_{n \rightarrow \infty} \int_{-\infty}^{+\infty} e^{-\lambda s} MT(s)x_n ds \\ &= \lim_{n \rightarrow \infty} M \int_{-\infty}^{+\infty} e^{-\lambda s} T(s)x_n ds = \lim_{n \rightarrow \infty} M(\lambda I - A)^{-1}x_n. \end{aligned}$$

Note that we have applied the Lebesgue theorem in the following passage:

$$\int_{-\infty}^{+\infty} e^{-\lambda s} \lim_{n \rightarrow \infty} MT(s)x_n ds = \lim_{n \rightarrow \infty} \int_{-\infty}^{+\infty} e^{-\lambda s} MT(s)x_n ds.$$

This is possible because  $\|MT(t)x_n\| \leq Ke^{\widehat{\omega}t}\|Mx_n\|$  and  $Mx_n$  is convergent. Moreover,  $\lambda > \widehat{\omega}$  by assumption. Note that  $M(\lambda I - A)^{-1}x_n$  belongs to  $M\mathcal{D}(A)$ . To

prove that  $y \in \mathcal{D}(\widehat{A})$ , we need to show that also  $\widehat{A}(M(\lambda I - A)^{-1}x_n)$  is convergent to some element in  $\overline{\text{ran}(M)}$ . To this end, we write  $\widetilde{A}$  as

$$\widetilde{A}x = \lambda x - (\lambda I - \widetilde{A})x. \quad (23)$$

Then by (23),

$$\begin{aligned} \widehat{A}(M(\lambda I - A)^{-1}x_n) &= \widetilde{A}(M(\lambda I - A)^{-1}x_n) \\ &= \lambda M(\lambda I - A)^{-1}x_n - (\lambda I - \widetilde{A})M(\lambda I - A)^{-1}x_n \\ &= \lambda M(\lambda I - A)^{-1}x_n - M(\lambda I - A)(\lambda I - A)^{-1}x_n = \lambda M(\lambda I - A)^{-1}x_n - Mx_n. \end{aligned}$$

It follows that

$$\lim_{n \rightarrow \infty} \widehat{A}(M(\lambda I - A)^{-1}x_n) = \lambda y - y_0.$$

This proves that  $\mathcal{D}(\widetilde{A}) \subset \mathcal{D}(\widehat{A})$ . Since  $\widetilde{A}$  is closed,  $\mathcal{D}(\widetilde{A}) = \mathcal{D}(\widehat{A})$  by definition of the closure of a linear operator.  $\square$

**Remark 7.** Consider the following condition on the generator  $A$ :

(jj) For every  $x_n \in \mathcal{D}(A)$ ,  $\|Mx_n\| \rightarrow 0$  implies  $\|MAx_n\| \rightarrow 0$ .

Condition (jj) implies that a reduced operator  $\widehat{A}$  can be constructed on the dense subspace  $M\mathcal{D}(A)$  in such a way that  $\widehat{A}M = MA$ . But from (jj) it follows that there exists  $k > 0$  such that  $\|MAx\| \leq k\|Mx\|$ ,  $\forall x \in \mathcal{D}(A)$  (this fact can be proved in the same way as for bounded operators; see e.g. [5]). Therefore, the reduced operator  $\widehat{A}$  can be extended to a bounded operator on  $\overline{\text{ran}(M)}$ . Condition (jj) is stronger than the hypotheses of Theorem 3.3. Indeed, not every lumping leads to a bounded reduced operator. Note also that,  $T(t)$  being the semigroup generated by  $A$ , condition (jj) cannot be obtained from the analogous condition (j), unless one assumes stronger hypotheses such as the boundedness of  $A$ .

**4. Dual conditions for lumpability.** We now consider the lumpability problem from a dual perspective. As a motivation, first consider the problem in finite dimensions. Let  $X = \mathbb{R}^n$  and  $Y = \mathbb{R}^k$  with  $k < n$ . Transposing both sides of (3) yields

$$MA = \widehat{A}M \iff A^\top M^\top = M^\top \widehat{A}^\top.$$

Moreover,

$$\ker(M) \subseteq \ker(MA) \iff \text{ran}(A^\top M^\top) \subseteq \text{ran}(M^\top).$$

Since the matrix  $\widehat{A}$  exists if and only if  $\ker(M)$  is  $A$ -invariant [11], an equivalent condition for lumpability is the invariance of  $\text{ran}(M^\top)$  under  $A^\top$ . This dual characterization has been utilized for studying lumpability in finite-dimensional systems and Markov chains, e. g., in [19] and [22]. Our aim is to generalize these results to infinite-dimensional systems, for both bounded and unbounded operators.

**4.1. Background in adjoint operators and semigroups.** Before going into details of lumping analysis, we briefly describe the setting and introduce the notation; for further details we refer to [28] and [18].

Let  $X^*$  denote the dual space of a Banach space  $X$ , namely the set of linear and bounded functionals from  $X$  to  $\mathbb{C}$ . Let  $j$  denote the canonical inclusion in the double dual  $X^{**}$  defined by

$$j : X \rightarrow X^{**}, \quad j(x)(x^*) := x^*(x), \quad \forall x^* \in X^*, x \in X. \quad (24)$$

For two subspaces  $\mathcal{C}$  and  $\mathcal{S}$  of  $X$  and  $X^*$ , respectively, we denote the annihilators

$$\mathcal{C}^\perp = \{x^* \in X^* : x^*(x) = 0 \forall x \in \mathcal{C}\}, \quad \mathcal{S}^\perp = \{x \in X : x^*(x) = 0 \forall x^* \in \mathcal{S}\}.$$

If  $\mathcal{C}$  is closed then  $\mathcal{C} = \mathcal{C}^{\perp\perp}$ , while  $\mathcal{S}^{\perp\perp}$  coincides with the weak\* closure of  $\mathcal{S}$ .

For a linear operator  $A$  between two Banach spaces  $X$  and  $Y$  whose domain  $\mathcal{D}(A)$  is dense in  $X$ , we also consider the adjoint operator  $A^* : Y^* \rightarrow X^*$  defined by  $A^*(y^*)(x) = y^*(Ax)$  on the domain

$$\mathcal{D}(A^*) = \{y^* \in Y^* : \text{the composition } y^*A \text{ is continuous on } \mathcal{D}(A)\}.$$

Let  $\{T(t)\}_{t \geq 0}$  be a strongly continuous semigroup on  $X$  generated by  $A$ . The family of the adjoint operators  $T^*(t) : X^* \rightarrow X^*$  is again a semigroup of bounded operators on  $X^*$  and is a continuous semigroup with respect to the weak star topology. In fact, it is the semigroup generated by the operator  $A^*$ , which is closed and densely defined with respect to the weak\* topology, and is given by

$$A^*x^* = \text{weak}^* \text{-} \lim_{h \rightarrow 0^+} \left( \frac{T^*(h)x^* - x^*}{h} \right).$$

Although the semigroup  $\{T^*\}_{t \geq 0}$  may fail to be strongly continuous, one can find a closed subspace of  $X^*$  in which strong continuity holds. Thus, the *sun dual* of  $X$  is the closed subspace  $X^\odot \subset X^*$  defined by

$$X^\odot = \{x^* \in X^* \text{ such that } \lim_{h \rightarrow 0^+} \|T^*(h)x^* - x^*\| = 0\}. \quad (25)$$

The *sun dual semigroup* of  $\{T(t)\}_{t \geq 0}$  is the strongly continuous semigroup obtained by restricting the adjoint semigroup to the sun dual space,

$$T^\odot(t)x^* := T^*(t)x^*, \quad x^* \in X^\odot, t \geq 0. \quad (26)$$

We denote the generator of the sun dual semigroup by  $A^\odot$ . It is the restriction of the adjoint operator  $A^*$  to the domain

$$\mathcal{D}(A^\odot) = \{x^* \in \mathcal{D}(A^*) : A^*x^* \in X^\odot\}.$$

It is known that  $A^*$  is the weak\* closure of  $A^\odot$  and  $\overline{\mathcal{D}(A^*)} = X^\odot$  [20]. As an example of a sun dual space we mention that, for the semigroup of left translations (12) on  $X = L^1(\mathbb{R})$ ,  $X^\odot$  is the space  $C_{\text{ub}}(\mathbb{R})$  of uniformly continuous and bounded functions on the real line [28].

One can iterate the construction of the sun dual space and define the double sun dual  $X^{\odot\odot}$  as the closed subspace of  $X^{\odot*}$  on which the adjoint semigroup  $T^{\odot*}(t)$  is strongly continuous. We call  $X$  *sun-reflexive* if  $X$  is isomorphic to  $X^{\odot\odot}$ .

Finally, we recall that there are some cases in which the passage to the adjoint semigroup preserves strong continuity. This always happens when  $X$  is a reflexive space: since in this case the weak and the weak\* topologies on the dual space coincide, the adjoint semigroup is weakly continuous and thus strongly continuous [20]. Similarly, if the semigroup is uniformly continuous, then its adjoint will also be uniformly continuous, because

$$\lim_{h \rightarrow 0^+} \|T^*(h)x^* - x^*\| \leq \lim_{h \rightarrow 0^+} \sup_{\|x\| \leq 1} \|T(h)x - x\| \|x^*\| = 0.$$



**4.2. Dual lumpability for bounded operators.** Consider system (1) generated by a bounded operator  $A \in \mathcal{B}(X)$ . We have seen in Theorem 2.1 that a lumping of this system through a bounded and surjective map  $M : X \rightarrow Y$  can be obtained if and only if  $\ker(M)$  is invariant under  $A$ . Similarly to the finite-dimensional case, we give an equivalent condition for lumpability in terms of adjoint operators.

**Proposition 4.1.** *Consider system (1) with  $A \in \mathcal{B}(X)$  and a surjective map  $M \in \mathcal{B}(X, Y)$ . Then the following statements are equivalent.*

1. *There exists  $\widehat{A} \in \mathcal{B}(Y)$  such that  $MA = \widehat{A}M$ , so that system (1) is lumpable by the operator  $M$ .*
2.  *$\text{ran}(M^*)$  is invariant under  $A^*$ .*

*Proof.*  $1 \Rightarrow 2$ . By the properties of the adjoint of a bounded operator, we have the implication  $(MA = \widehat{A}M) \Rightarrow (A^*M^* = M^*\widehat{A}^*)$ . Given  $x^* = M^*y^*$ , we have  $A^*x^* = A^*M^*y^* = M^*\widehat{A}^*y^* \in \text{ran}(M^*)$ ; i.e., statement 2 holds.

$2 \Rightarrow 1$ . Note that statement 2 is equivalent to  $\text{ran}(A^*M^*) \subseteq \text{ran}(M^*)$ . Thus,

$$\begin{aligned} \text{ran}(A^*M^*) \subseteq \text{ran}(M^*) &\Rightarrow \text{ran}(M^*)^\perp \subseteq \text{ran}(A^*M^*)^\perp \\ &\Rightarrow \ker(M) \subseteq \ker(MA), \end{aligned}$$

which is the condition for lumpability.  $\square$

**Example 5.** Consider the lumping operation corresponding to the quotient projection of Remark 1. By definition, the adjoint of the quotient projection is

$$\pi^* : \left( \frac{X}{\mathcal{C}} \right)^* \rightarrow X^*, \quad \pi^*\phi(x) := \phi([x]).$$

It is known that the range of  $\pi^*$  can be identified with the annihilator  $\mathcal{C}^\perp$ , which is invariant under  $A^*$ . (This can be seen by taking  $\phi \in \mathcal{C}^\perp$ , applying  $A^*$ , and using the invariance of  $\mathcal{C}$  to obtain  $A^*\phi(x) = \phi(Ax) = 0 \forall x \in \mathcal{C}$ .) The reduction of  $A$  to  $\widehat{A}$  through  $\pi$  can indeed be identified with the restriction of  $A^*$  to the closed subspace  $\mathcal{C}^\perp$ .

**4.3. Dual lumpability for unbounded operators.** We will obtain the dual conditions for lumpability in the general case of dynamics generated by an unbounded operator  $A$ . Since the family  $\{T(t)\}_{t \geq 0}$  is made up of bounded operators, we have the following result.

**Proposition 4.2.** *The following statements are equivalent:*

1. *There exists an operator  $\widehat{A}$  defined on  $M(\mathcal{D}(A))$  such that  $\widehat{A}$  generates a strongly continuous semigroup on  $Y$  and  $\widehat{A}M = MA$  (i.e. the system is lumpable by the operator  $M$ );*
2.  *$\text{ran}(M^*)$  is invariant under  $T^*(t)$  for every  $t \geq 0$ .*

*Proof.*  $1 \Rightarrow 2$ . Let  $\widehat{T}(t)$  be the strongly continuous semigroup generated by  $\widehat{A}$ . We have shown (see proof of Theorem 3.1) that  $\widehat{T}$  satisfies the lumping relation  $\widehat{T}(t)Mx = MT(t)x$ ,  $x \in X$ . This implies that the kernel of  $M$  is  $T(t)$ -invariant. Statement 2 then follows through the following implications (considering that the surjectivity of  $M$  implies that the range of its adjoint is star-weakly closed):

$$\begin{aligned} \ker(M) \subseteq \ker(MT(t)) &\implies (\ker(MT(t)))^\perp \subseteq \ker(M)^\perp \\ \implies \text{ran}(T(t)^*M^*) \subseteq \ker(MT(t))^\perp &\subseteq \ker(M)^\perp = \text{ran}(M^*). \end{aligned}$$

2  $\Rightarrow$  1. From the invariance of  $\text{ran}(M^*)$  under  $T^*(t)$  we can write

$$\text{ran}(T(t)^* M^*) \subseteq \text{ran}(M^*) \implies \ker(M) \subseteq \ker(MT(t)),$$

which is the necessary and sufficient condition for lumpability.  $\square$

**Example 6.** We give a dual interpretation of the lumping through the evaluation operator described in Example 2. Let  $h : \mathbb{R} \rightarrow \mathbb{C}$  be a continuous function such that  $\sup_{x \in \mathbb{R}} \text{Re}(h(x)) < \infty$ . Then the family of bounded operators  $T(t)$  given by  $T(t)f(x) = e^{th(x)}f(x)$  is a strongly continuous semigroup on the Banach space  $X = C_0(\mathbb{R})$ , with generator  $Af(x) = h(x)f(x)$ . We consider the lumping operator  $M : C_0(\mathbb{R}) \rightarrow \mathbb{C}^k$  defined by  $Mf = (f(x_1), \dots, f(x_k))^\top$ , which evaluates a given function at the  $k$  points  $x_1, \dots, x_k \in \mathbb{R}$ . By the Riesz-Markov theorem,  $C_0(\mathbb{R})^*$  can be identified with the Banach space  $\mathcal{M}(\mathbb{R})$  of all complex, regular, Borel measures on the real line. If  $\phi \in C_0(\mathbb{R})^*$  and  $\mu_\phi$  is the measure associated with  $\phi$ ,  $\forall f \in C_0(\mathbb{R})$ , then  $\phi(f) = \int f(x) d\mu_\phi(x)$ . Consider now the adjoint of the lumping operator  $M$ ,

$$M^* : (\mathbb{C}^k)^* \rightarrow \mathcal{M}(\mathbb{R}), \quad M^*(\alpha_1, \dots, \alpha_k) = \alpha_1 \delta(x_1) + \dots + \alpha_k \delta(x_k).$$

This is an injective operator whose range is the closed subspace of all linear combinations of  $\delta(x_1), \dots, \delta(x_k)$  with complex coefficients (which is clearly isomorphic to  $\mathbb{C}^k$ ). It is easy to obtain

$$T^*(t)M^*(\alpha_1, \dots, \alpha_k) = \alpha_1 e^{th(x_1)} \delta(x_1) + \dots + \alpha_k e^{th(x_k)} \delta(x_k),$$

which implies that the range of  $M^*$  is invariant under  $T^*(t)$ . In particular:

$$T^*(t)M^* = M^* \widehat{T}(t)^*,$$

where  $\widehat{T}(t)$  is the reduced semigroup on  $\mathbb{C}^k$  given by

$$\widehat{T}(t)(\alpha_1, \dots, \alpha_k)^T = \text{diag} \left( e^{th(x_1)}, \dots, e^{th(x_k)} \right) \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{pmatrix}.$$

This construction shows the advantages of the dual approach, because  $M^*$  is indeed an invertible operator on a finite dimensional space.

We now establish a dual condition for lumpability in terms of the adjoint of the generator. To this end, recall that the adjoint operator  $A^*$  is a generator only in the sense of the weak\* topology. Fortunately, many properties of strongly continuous semigroups hold also for the adjoint semigroup where the same limits are considered in the weak\* topology; in fact, it is easy to verify (see [28] for more details) that

- 1) for every  $x^* \in X^*$ ,  $t > 0$ ,

$$T^*(t)x^* := \text{weak}^* \lim_{k \rightarrow \infty} \left[ \frac{k}{t} \mathcal{R} \left( \frac{k}{t}; A^* \right) \right]^k x^*,$$

where  $\mathcal{R}(\lambda; A^*)$ ,  $\lambda \in \rho(A^*)$ , is the resolvent operator of  $A^*$ , and

- 2)  $\mathcal{R}(\lambda; A^*) = \text{weak}^* \int_0^\infty e^{-\lambda s} T^*(s)x^* ds$ , where the right side is the *weak\* integral*, defined as the unique element such that for every  $x \in X$

$$\int_0^\infty e^{-\lambda s} T^*(s)x^* ds (x) = \lim_{k \rightarrow \infty} \int_0^k e^{-\lambda s} T^*(s)x^*(x) ds.$$

Since the range of  $M^*$  is weak\* closed, by the above results it is easy to verify that  $\text{ran}(M^*)$  is invariant under the adjoint semigroup  $T^*(t)$  if and only if it is

invariant under the resolvent operators  $\mathcal{R}(\lambda; A^*)$  for all  $\lambda > \omega(T)$ . Moreover,  $A$  being closed and densely defined, we have  $\mathcal{R}(\lambda; A^*) = \mathcal{R}(\lambda; A)^*$ , and

$$\text{ran}(\mathcal{R}(\lambda; A^*) M^*) \subseteq \text{ran}(M^*) \iff \ker(M) \subseteq \ker(M\mathcal{R}(\lambda; A)).$$

These facts allow us to write the dual condition of (3.3).

**Proposition 4.3.** *System (1) is lumpable by the bounded, surjective, linear map  $M$  if and only if both the following conditions hold:*

1.  $A^*(\text{ran}(M^*) \cap \mathcal{D}(A^*)) \subset \text{ran}(M^*)$ , and
2. *there exists  $\lambda > \omega$  such that  $(\lambda I - A^*)$  is surjective from  $\text{ran}(M^*) \cap \mathcal{D}(A^*)$  to  $\text{ran}(M^*)$ .*

Suppose that the lumping operator  $M$  is bounded but not surjective. Applying Theorem 2.2 to strongly continuous semigroups, the following statement can be proved.

**Proposition 4.4.** *Given a strongly continuous semigroup  $T(t)$  generated by  $A$ , there exists another strongly continuous semigroup  $\widehat{T}(t)$  on  $\overline{\text{ran}(M)}$  such that  $MT(t) = \widehat{T}(t)M$  if and only if  $\text{ran}(T^*(t)M^*) \subset \text{ran}(M)^*$  for all  $t \geq 0$ .*

Using Proposition 3.6, we can show that  $\widehat{T}(t)$  is generated by the closure  $\widehat{A}$ , where  $\widehat{A}$  is the operator given by

$$\widehat{A}y = MAx, \quad y = Mx \in M\mathcal{D}(A).$$

Note that the inclusion  $\text{ran}(T^*(t)M^*) \subset \text{ran}(M)^*$  does not imply  $\ker(M) \subset \ker(MT(t))$ , unless  $\text{ran}(M)$  is closed. Thus, Proposition 4.4 does not generalize Proposition 4.2, but rather gives a different version of dual lumpability.

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