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# Chebyshev Polynomials on Generalized Julia Sets

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**Abstract** Let $(f_n)_{n=1}^{\infty}$ be a sequence of non-linear polynomials satisfying some mild conditions. Furthermore, let $F_m(z) := (f_m \circ f_{m-1} \cdots \circ f_1)(z)$ and $\rho_m$ be the leading coefficient of $F_m$. It is shown that on the Julia set $J_{(f_n)}$, the Chebyshev polynomial of degree deg $F_m$ is of the form $F_m(z)/\rho_m - \tau_m$ for all $m \in \mathbb{N}$ where $\tau_m \in \mathbb{C}$. This generalizes the result obtained for autonomous Julia sets in Kamo and Borodin (Mosc. Univ. Math. Bull. 49:44–45, 1994).

**Keywords** Chebyshev polynomials · Extremal polynomials · Julia sets · Widom factors

**Mathematics Subject Classification** 37F10 · 41A50

## 1 Introduction

Let $(f_n)_{n=1}^{\infty}$ be a sequence of rational functions in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Let us define the associated compositions by $F_m(z) := (f_m \circ \cdots f_1)(z)$ for each $m \in \mathbb{N}$. Then the set of points in $\overline{\mathbb{C}}$ for which $(F_n)_{n=1}^{\infty}$ is normal in the sense of Montel is called the *Fatou set* for $(f_n)_{n=1}^{\infty}$. The complement of the Fatou set is called the *Julia set* for $(f_n)_{n=1}^{\infty}$ and is denoted by $J_{(f_n)}$. The metric considered here is the chordal metric. Julia sets

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corresponding to a sequence of rational functions, to our knowledge, were considered first in [9]. Several papers that have appeared in the literature (see e.g. [3,6,8,18]) show the possibility of adapting the results on autonomous Julia sets to this more general setting with some minor changes. By an autonomous Julia set, we mean the set $J_{(f_n)}$ with $f_n(z) = f(z)$ for all $n \in \mathbb{N}$ where $f$ is a rational function.

The Julia set $J_{(f_n)}$ is never empty provided that $\deg f_n \geq 2$ for all $n$. If, in addition, we assume that $f_n = f$ for all $n$ then $f(J(f)) = f^{-1}(J(f)) = J(f)$ where $J(f) := J_{(f_n)}$. But without the last assumption, we only have $F_k^{-1}(F_k(J_{(f_n)})) = J_{(f_n)}$ and $J_{(f_n)} = F_k^{-1}(J_{(f_{k+n})})$ for all $k \in \mathbb{N}$ in general, where $(f_{k+n}) = (f_{k+1}, f_{k+2}, f_{k+3}, \ldots)$. That is the main reason why further techniques are needed in this framework.

Let $K \subset \mathbb{C}$ be a compact set with Card $K \geq m$ for some $m \in \mathbb{N}$. Recall that, for every $n \in \mathbb{N}$ with $n \leq m$, the unique monic polynomial $P_n$ of degree $n$ satisfying

$$\|P_n\|_K = \min\{\|Q_n\|_K : Q_n \text{ monic of degree } n\}$$

is called the $n$th *Chebyshev polynomial* on $K$ where $\|\cdot\|_K$ is the sup-norm on $K$.

If $f$ is a non-linear complex polynomial then $J(f) = \partial\{z \in \mathbb{C} : f^{(n)}(z) \to \infty\}$ and $J(f)$ is an infinite compact subset of $\mathbb{C}$ where $f^{(n)}$ is the $n$th iteration of $f$. The next result is due to Kamo and Borodin [12]:

**Theorem 1** *Let $f(z) = z^m + a_{m-1}z^{m-1} + \cdots + a_0$ be a non-linear complex polynomial and $T_k(z)$ be a Chebyshev polynomial on $J(f)$. Then $(T_k \circ f^{(n)})(z)$ is also a Chebyshev polynomial on $J(f)$ for each $n \in \mathbb{N}$. In particular, this implies that there exists a complex number $\tau$ such that $f^{(n)}(z) - \tau$ is a Chebyshev polynomial on $J(f)$ for all $n \in \mathbb{N}$.*

In Sect. 2, we review some facts about generalized Julia sets and Chebyshev polynomials. In the last section, we present a result which can be seen as a generalization of Theorem 1. Polynomials considered in these sections are always non-linear complex polynomials unless stated otherwise. For a deeper discussion of Chebyshev polynomials, we refer the reader to [15,16,19]. For different aspects of the theory of Julia sets, see [2,4,13] among others.

## 2 Preliminaries

Autonomous polynomial Julia sets enjoy plenty of nice properties. These sets are non-polar compact sets which are regular with respect to the Dirichlet problem. Moreover, there are a couple of equivalent ways to describe these sets. For further details, see [13]. In order to have similar features for the generalized case, we need to put some restrictions on the given polynomials. The conditions used in the following definition are from [4, Sec. 4].

**Definition 1** Let $f_n(z) = \sum_{j=0}^{d_n} a_{n,j} \cdot z^j$ where $d_n \geq 2$ and $a_{n,d_n} \neq 0$ for all $n \in \mathbb{N}$. We say that $(f_n)$ is a *regular polynomial sequence* if the following properties are satisfied:

- There exists a real number $A_1 > 0$ such that $|a_{n,d_n}| \geq A_1$, for all $n \in \mathbb{N}$.
- There exists a real number $A_2 \geq 0$ such that $|a_{n,j}| \leq A_2|a_{n,d_n}|$ for $j = 0, 1, \ldots, d_n - 1$ and $n \in \mathbb{N}$.
- There exists a real number $A_3$ such that

$$\log |a_{n,d_n}| \leq A_3 \cdot d_n,$$

for all $n \in \mathbb{N}$.

If $(f_n)$ is a regular polynomial sequence then we use the notation $(f_n) \in \mathcal{R}$. Here and in the rest of this paper, $F_l(z) := (f_l \circ \cdots \circ f_1)(z)$ and $\rho_l$ is the leading coefficient of $F_l$. Let $\mathcal{A}_{(f_n)}(\infty) := \{z \in \overline{\mathbb{C}} : (F_n(z))_{n=1}^{\infty}$ goes locally uniformly to $\infty\}$ and $\mathcal{K}_{(f_n)} := \{z \in \mathbb{C}\colon (F_n(z))_{n=1}^{\infty}$ is bounded$\}$. In the next theorem, we list some facts that will be necessary for the subsequent results.

**Theorem 2** [4,6] *Let $(f_n) \in \mathcal{R}$. Then the following hold:*

(a) *$J_{(f_n)}$ is a compact set in $\mathbb{C}$ with positive logarithmic capacity.*
(b) *For each $R > 1$ satisfying*

$$A_1 R \left(1 - \frac{A_2}{R-1}\right) > 2, \tag{1}$$

*we have $\mathcal{A}_{(f_n)}(\infty) = \cup_{k=1}^{\infty} F_k{}^{-1}(\triangle_R)$ and $f_n(\overline{\triangle_R}) \subset \triangle_R$ where*

$$\triangle_R = \{z \in \overline{\mathbb{C}}\colon |z| > R\}$$

*Furthermore, $\mathcal{A}_{(f_n)}(\infty)$ is a domain in $\overline{\mathbb{C}}$ containing $\triangle_R$.*
(c) $\triangle_R \subset \overline{F_k^{-1}(\triangle_R)} \subset F_{k+1}^{-1}(\triangle_R) \subset \mathcal{A}_{(f_n)}(\infty)$ *for all $k \in \mathbb{N}$ and each $R > 1$ satisfying* (1).
(d) $\partial\mathcal{A}_{(f_n)}(\infty) = J_{(f_n)} = \partial\mathcal{K}_{(f_n)}$ *and* $\mathcal{K}_{(f_n)} = \overline{\mathbb{C}} \backslash \mathcal{A}_{(f_n)}(\infty)$. *Thus, $\mathcal{K}_{(f_n)}$ is a compact subset of $\mathbb{C}$ and $J_{(f_n)}$ has no interior points.*

The next result is an immediate consequence of Theorem 2.

**Proposition 1** *Let $(f_n) \in \mathcal{R}$. Then*

$$\lim_{k\to\infty} \left( \sup_{a \in \overline{\mathbb{C}} \backslash F_k^{-1}(\triangle_R)} \operatorname{dist}(a, \mathcal{K}_{(f_n)}) \right) = 0,$$

*where $R$ be a real number satisfying* (1).

*Proof* Using the part (c) of Theorem 2, we have $\overline{\mathbb{C}} \backslash F_{k+1}^{-1}(\triangle_R) \subset \overline{\mathbb{C}} \backslash F_k^{-1}(\triangle_R)$ which implies that

$$(a_k) := \left( \sup_{a \in \overline{\mathbb{C}} \backslash F_k^{-1}(\triangle_R)} \operatorname{dist}(a, \mathcal{K}_{(f_n)}) \right)$$

is a decreasing sequence.

Suppose that $a_k \to \epsilon$ as $k \to \infty$ for some $\epsilon > 0$. Then, by compactness of the set $\overline{\mathbb{C}} \backslash F_k^{-1}(\triangle_R)$, there exists a number $b_k \in \overline{\mathbb{C}} \backslash F_k^{-1}(\triangle_R)$ for each $k$ such that $\text{dist}(b_k, \mathcal{K}_{(f_n)}) \geq \epsilon$. But since $\cap_{k=1}^{\infty} \overline{\mathbb{C}} \backslash F_k^{-1}(\triangle_R) = \mathcal{K}_{(f_n)}$ by parts (*b*) and (*d*) of Theorem 2, $(b_k)$ should have an accumulation point $b$ in $\mathcal{K}_{(f_n)}$ with $\text{dist}(b, \mathcal{K}_{(f_n)}) > \epsilon/2$ which is clearly impossible. This completes the proof. □

For a compact set $K \subset \mathbb{C}$, the smallest closed disk $\overline{D(a, r)}$ containing $K$ is called the *Chebyshev disk* for $K$. The center $a$ of this disk is called the *Chebyshev center* of $K$. These concepts were crucial and widely used in the paper [14]. The next result which is vital for the proof of Lemma 1 is from [14]:

**Theorem 3** *Let $L \subset \mathbb{C}$ be a compact set with* card $L \geq 2$ *having the origin as its Chebyshev center. Let $L_p = p^{-1}(L)$ for some monic complex polynomial $p$ with* $\deg p = n$. *Then $p$ is the unique Chebyshev polynomial of degree $n$ on $L_p$.*

## 3 Results

First, we begin with a lemma which is also interesting in its own right.

**Lemma 1** *Let $f$ and $g$ be two non-constant complex polynomials and $K$ be a compact subset of $\mathbb{C}$ with* card $K \geq 2$. *Furthermore, let $\alpha$ be the leading coefficient of $f$. Then the following propositions hold.*

(a) *The Chebyshev polynomial of degree* $\deg f$ *on the set $(g \circ f)^{-1}(K)$ is of the form $f(z)/\alpha - \tau$ where $\tau \in \mathbb{C}$.*
(b) *If $g$ is given as a linear combination of monomials of even degree and $K = \overline{D(0, R)}$ for some $R > 0$ then the* $\deg f$*th Chebyshev polynomial on $(g \circ f)^{-1}(K)$ is $f(z)/\alpha$.*

*Proof* Let $K_1 := g^{-1}(K)$. Then $(g \circ f)^{-1}(K) = f^{-1}(K_1) = (f/\alpha)^{-1}(K_1/\alpha)$ where $K_1/\alpha - \tau = \{z : z = z_1/\alpha - \tau \text{ for some } z_1 \in K_1\}$. By the fundamental theorem of algebra, $\text{card}(K_1/\alpha) = \text{card}\, K_1 \geq \text{card}\, K$ and $K_1$ is compact by the continuity of $g(z)$. The set $K_1/\alpha$ is also compact since the compactness of a set is preserved under a linear transformation. Let $\tau$ be the Chebyshev center for $K_1/\alpha$. Then $K_1/\alpha - \tau$ is a compact set with the Chebyshev center as the origin. Note that, $\text{card}(K_1/\alpha - \tau) = \text{card}(K_1/\alpha)$ and $(f/\alpha)^{-1}(K_1/\alpha) = (f/\alpha - \tau)^{-1}(K_1/\alpha - \tau)$. Using Theorem 3, for $p(z) = f(z)/\alpha - \tau$ and $L = K_1/\alpha - \tau$, we see that $p(z)$ is the $\deg f$th Chebyshev polynomial on $L_p = (g \circ f)^{-1}(K)$. This proves the first part of the lemma.

Suppose further that $g(z) = \sum_{j=0}^{n} a_j \cdot z^{2j}$ for some $n \geq 1$ and $(a_0, \ldots, a_n) \in \mathbb{C}^{n+1}$ with $a_n \neq 0$. Let $K = \overline{D(0, R)}$ for some $R > 0$. Then the Chebyshev center for $K_1/\alpha = g^{-1}(K)/\alpha = g^{-1}(\overline{D(0, R)})/\alpha$ is the origin since $g(z)/\alpha = g(-z)/\alpha$ for all $z \in \mathbb{C}$. Thus, $f(z)/\alpha$ is the $\deg f$th Chebyshev polynomial for $(g \circ f)^{-1}(K)$ under these extra assumptions. □

The next theorem shows that it is possible to obtain similar results to Theorem 1 in a richer setting.

**Theorem 4** *Let $(f_n) \in \mathcal{R}$. Then the following hold:*

(a) *For each* $m \in \mathbb{N}$*, the* $\deg F_m$*th Chebyshev polynomial on* $J_{(f_n)}$ *is of the form* $F_m(z)/\rho_m - \tau_m$ *where* $\tau_m \in \mathbb{C}$.
(b) *If, in addition, each* $f_n$ *is given as a linear combination of monomials of even degree then* $F_m(z)/\rho_m$ *is the* $\deg F_m$*th Chebyshev polynomial on* $J_{(f_n)}$ *for all* $m$.

*Proof* Let $m \in \mathbb{N}$ be given and $R > 1$ satisfy (1). For each natural number $l > m$, define $g_l := f_l \circ \cdots \circ f_{m+1}$. Then $F_l = g_l \circ F_m$ for each such $l$. Using part (a) of Lemma 1 for $g = g_l$, $f = F_m$ and $K = \overline{D(0,R)}$, we see that the $(d_1 \cdots d_m)$th Chebyshev polynomial on $(g_l \circ F_m)^{-1}(\overline{D(0,R)})$ is of the form $F_m(z)/\rho_m - \tau_l$ where $\tau_l \in \mathbb{C}$. Let $C_l := \|F_m/\rho_m - \tau_l\|_{(g_l \circ F_m)^{-1}(K)}$. Note that, by part (c) of Theorem 2,

$$F_t^{-1}(\overline{D(0,R)}) \subset F_s^{-1}(\overline{D(0,R)}) \subset \overline{D(0,R)} \tag{2}$$

provided that $s < t$. This implies that $(C_j)_{j=m+1}^{\infty}$ is a decreasing sequence of positive numbers and hence has a limit $C$. The last follows from the observation that the norms of the Chebyshev polynomials of same degree on a decreasing sequence of compact sets constitute a decreasing sequence on $\mathbb{R}$.

Let $P_{d_1 \cdots d_m}(z) = \sum_{j=0}^{d_1 \cdots d_m} a_j z^j$ be the $(d_1 \cdots d_m)$th Chebyshev polynomial on $\mathcal{K}_{(f_n)}$. Since $\mathcal{K}_{(f_n)} \subset (g_l \circ F_m)^{-1}(\overline{D(0,R)})$ for each $l$, we have $C_0 := \|P_{d_1 \cdots d_m}\|_{\mathcal{K}_{(f_n)}} \le C$. Suppose that $C_0 < C$.

Let $\epsilon = \min\{C - C_0, 1\}$. Using the compactness of $\overline{D(0,R)}$ let us choose a $\delta > 0$ such that for all $|z_1 - z_2| < \delta$ and $z_1, z_2 \in \overline{D(0,R)}$ we have

$$|P_{d_1 \cdots d_m}(z_1) - P_{d_1 \cdots d_m}(z_2)| < \frac{\epsilon}{2}$$

By Proposition 1, there exists a real number $N_0 > m$ such that $N > N_0$ with $N \in \mathbb{N}$ implies that

$$\sup_{z \in \overline{\mathbb{C}} \setminus F_N^{-1}(\triangle_R)} \operatorname{dist}(z, \mathcal{K}_{(f_n)}) < \delta.$$

Therefore, for any $z \in F_{N_0+1}^{-1}(\overline{D(0,R)})$, there exists a $z' \in \mathcal{K}_{(f_n)}$ with $|z - z'| < \delta$. Hence, for each $z \in F_{N_0+1}^{-1}(\overline{D(0,R)})$, we have

$$|P_{d_1 \cdots d_m}(z)| < |P_{d_1 \cdots d_m}(z')| + \frac{\epsilon}{2} < C \le \left\| \frac{F_m}{\rho_m} - \tau_{N_0+1} \right\|_{F_{N_0+1}^{-1}(\overline{D(0,R)})},$$

where in the first inequality, we use $z, z' \in \overline{D(0,R)}$. This contradicts with the fact that $F_m(z)/\rho_m + \tau_{N_0+1}$ is the $(d_1 \cdots d_m)$th Chebyshev polynomial on $F_{N_0+1}^{-1}(\overline{D(0,R)})$. Thus, $C_0 = C$.

Using the triangle inequality in (4) and (5), the monotonicity of $(C_l)_{l=m+1}^{\infty}$ in (6) and (2) in (7), we have

$$|\tau_l| = \left\| -\frac{F_m}{\rho_m} + \frac{F_m}{\rho_m} - \tau_l \right\|_{F_l^{-1}(\overline{D(0,R)})} \tag{3}$$

$$\leq \left\| \frac{F_m}{\rho_m} - \tau_l \right\|_{F_l^{-1}(\overline{D(0,R)})} + \left\| \frac{F_m}{\rho_m} \right\|_{F_l^{-1}(\overline{D(0,R)})} \tag{4}$$

$$\leq C_l + |\tau_{m+1}| + \left\| \frac{F_m}{\rho_m} - \tau_{m+1} \right\|_{F_l^{-1}(\overline{D(0,R)})} \tag{5}$$

$$\leq C_{m+1} + |\tau_{m+1}| + \left\| \frac{F_m}{\rho_m} - \tau_{m+1} \right\|_{F_l^{-1}(\overline{D(0,R)})} \tag{6}$$

$$\leq 2C_{m+1} + |\tau_{m+1}|. \tag{7}$$

for $l \geq m+1$. This shows that $(\tau_l)_{l=m+1}^{\infty}$ is a bounded sequence. Thus, $(\tau_l)_{l=m+1}^{\infty}$ has at least one convergent subsequence $(\tau_{l_k})_{k=1}^{\infty}$ with a limit $\tau_m$. Therefore,

$$C \leq \lim_{k\to\infty} \left\| \frac{F_m}{\rho_m} - \tau_m \right\|_{F_{l_k}^{-1}(\overline{D(0,R)})} \leq \lim_{k\to\infty} (C_{l_k} + |\tau_{l_k} - \tau_m|) = C. \tag{8}$$

By the uniqueness of Chebyshev polynomials and (8), $F_m(z)/\rho_m - \tau_m$ is the $(d_1 \cdots d_m)$th Chebyshev polynomial on $\mathcal{K}_{(f_n)}$. By the maximum principle, for any polynomial $Q$, we have

$$\|Q\|_{\mathcal{K}_{(f_n)}} = \|Q\|_{\partial\mathcal{K}_{(f_n)}} = \|Q\|_{J_{(f_n)}}.$$

Hence, the Chebyshev polynomials on $\mathcal{K}_{(f_n)}$ and $J_{(f_n)}$ should coincide. This proves the first assertion.

Suppose that the assumption given in part (b) is satisfied. Then by the part (b) of Lemma 1, for $g = g_l$, $f = F_m$ and $K = \overline{D(0,R)}$, the $(d_1 \cdots d_m)$th Chebyshev polynomial on $(g_l \circ F_m)^{-1}(\overline{D(0,R)})$ is of the form $F_m(z)/\rho_m - \tau_l$ where $\tau_l = 0$ for $l > m$. Thus, arguing as above, we can reach the conclusion that $F_m(z)/\rho_m$ is the $(d_1 \cdots d_m)$th Chebyshev polynomial for $J_{(f_n)}$ provided that the assumption in the part (b) holds. This completes the proof. □

This theorem gives the total description of $2^n$ degree Chebyshev polynomials for the most studied case, i.e., $f_n(z) = z^2 + c_n$ with $c_n \in \mathbb{C}$ for all $n$. If $(c_n)_{n=1}^{\infty}$ is bounded then the logarithmic capacity of $J_{(f_n)}$ is 1. Moreover, by [5], we know that if $|c_n| \leq 1/4$ for all $n$ then $J_{(f_n)}$ is connected. If $|c_n| < c < 1/4$, then $J_{(f_n)}$ is a quasicircle and hence a Jordan curve. See [3], for the definition of a quasicircle and proof of the above fact.

For a non-polar compact set $K \subset \mathbb{C}$, let us define the sequence $(W_n(K))_{n=1}^{\infty}$ by $W_n(K) = \|P_n\|/(\text{Cap}(K))^n$ for all $n \in \mathbb{N}$. There are recent studies on the asymptotic behavior of these sequences on several occasions. See e.g. [1,10,20].

In [1,20], sufficent conditions are given for $(W_n(K))_{n=1}^{\infty}$ to be bounded in terms of the smoothness of the outer boundary of $K$. There is also an old and open question (we consider this as an open problem since we could not find any concrete examples in the literature although in [17], Pommerenke says that “D. Wrase in Karlsruhe has shown that an example constructed by J. Clunie [Ann. of Math., 69 (1959), 511–519] for a different purpose has the required property.”) proposed by Pommerenke [17] which is in the inverse direction: Find (if possible) a continuum $K$ with $\text{Cap}(K) = 1$ such that $(W_n(K))_{n=1}^{\infty}$ is unbounded. To answer this question positively, it is very natural to consider a continuum with a non-rectifiable outer boundary. Thus, we make the following conjecture:

**Conjecture 1** *Let* $f(z) = z^2 + 1/4$. *Then,* $(W_n(J(f))_{n=1}^{\infty}$ *is unbounded.*

By [11, Thm. 1], for $f(z) = z^2 + 1/4$, $J(f)$ has Hausdorff dimension greater than 1 and in this case (see e.g. [7, p. 130]) $J(f)$ is not a quasicircle. Hence, [1, Thm. 2] is not applicable for $J(f)$ since it requires even stronger assumptions on the outer boundary.

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