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Parametric Identification of Hybrid Linear-Time-Periodic Systems

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Abstract:
In this paper, we present a state-space system identification technique for a class of hybrid LTP systems, formulated in the frequency domain based on input–output data. Other than a few notable exceptions, the majority of studies in the state-space system identification literature (e.g. subspace methods) focus only on LTI systems. Our goal in this study is to develop a technique for estimating time-periodic system and input matrices for a hybrid LTP system, assuming that full state measurements are available. To this end, we formulate our problem in a linear regression framework using Fourier transformations, and estimate Fourier series coefficients of the time-periodic system and input matrices using a least-squares solution. We illustrate the estimation accuracy of our method for LTP system dynamics using a hybrid damped Mathieu function as an example.

Keywords: Linear time periodic systems, state-space identification, hybrid systems.

1. INTRODUCTION

Our main focus in this paper is the identification of nonlinear, hybrid dynamical systems that operate near their periodic solutions. A wide variety of dynamical phenomena in biology and engineering include *oscillatory* and *hybrid* characteristics (Buehler et al., 1994; Chevallereau et al., 2009; Hurmuzlu and Basdogan, 1994). Thus, such dynamical behaviors are commonly modeled as nonlinear hybrid dynamical systems that operate near some isolated periodic orbits (a.k.a. limit-cycle). Though, there are remarkably fewer studies focusing on the problem of system identification for hybrid dynamical systems operating around limit-cycles than system identification studies on dynamical systems that operate near their point equilibria (e.g. LTI systems).

In the broadest sense, a hybrid dynamical system is one that both flows smoothly (defined by a set of differential equations) and jumps discretely (defined by a set of transition maps) (Guckenheimer and Holmes, 1991). These discrete jumps are often accompanied by a switch between different smooth flows, punctuating system trajectories with discontinuous jumps, sometimes even changing the dimension of the underlying state space (Burden et al., 2015). Despite the generality of this definition, we limit our scope to hybrid systems for which state trajectories are continuous, but possibly non-differentiable. In other words, we exclude systems that undergo discrete jumps in states as well as changes in the state dimensions.

Under certain assumptions, the linearization of smooth nonlinear systems around their periodic solutions (orbit), yields linear time-periodic (LTP) systems (Guckenheimer and Holmes, 1991), whereas the linearization of the class of nonlinear hybrid systems we consider around their periodic orbits yields hybrid LTP systems (DaCunha and Davis, 2011). Since we exclude hybrid transitions with discrete jumps in system state and dimension, the class of induced hybrid LTP systems that we study exhibit continuous but only piece-wise differentiable vector fields (Uyanik et al., 2015a, 2016). In Section 2.2, we formally define the general form of LTP systems that we focus on. Our main contribution in this paper is a parametric system identification method for hybrid LTP systems that we consider, using frequency domain representations of input-output data.

Unlike the literature on LTP and/or Hybrid system identification, the identification of LTI systems is a relatively mature field. There is a wide range of techniques for the identification of LTI systems, appropriate for widely differing needs of engineers and scientists (Ljung, 1998).

There are a number of methods that extend the LTI identification techniques to the identification of LTP systems. For example Shi et al. (2007) utilizes the subspace system identification method (Van Overschee and De Moor, 1996) to estimate physical parameters of smooth linear time-varying systems, whereas Verhaegen and Yu (1995) developed a different subspace system identification technique for discrete time periodically time-varying systems.

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In the context of piece-wise smooth system identification, Verdult and Verhaegen (2004) introduced a formulation to estimate state space models for piecewise LTI systems, which may be considered as a special case of our formulation in Section 2.2, when the switching time between the subsystems is known. Similarly, Buchan et al. (2013) utilizes a data-driven input–output system identification method to estimate piecewise affine models for approximating dynamics of a hexapedal robot. However, none of these methods completely cover our class of LTP systems and they all perform identification based on time domain input-output data.

In our formulation, we assume that switching times between different continuous LTP vector fields are known. This information is used to separately identify individual contributions from each LTP subsystem to the overall periodic system. In our approach, we obtain Fourier series coefficients for the state and input matrices and then formulate the problem in a linear regression framework. After estimating system matrices using a least squares solution, we use Fourier synthesis to construct time-periodic system and input matrices.

This paper is organized as follows. Section 2 details the formulation of the problem as well as underlying models and assumptions. Section 3 describes the theory behind the estimation of time-periodic system matrices. Section 4 illustrates a case study on a simple example, time-switched damped Mathie function with associated estimation results.

2. PROBLEM FORMULATION

2.1 Linear Time-Periodic Systems

In this paper, we focus on linear time-periodic systems, whose state evaluation equation can be written as

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t) \ , \quad (1)$$

where both system matrices are periodic with a fixed, known period $T > 0$ such that $A(t) = A(t + nT)$ and $B(t) = B(t+nT)$, $\forall n \in \mathbb{Z}$. In this study, we further assume that we can measure all system states.

In an LTI system, a sinusoidal input signal with a frequency of ω, at steady state produces output only at the same frequency, possibly at a different phase and magnitude. This is the well-known frequency separation principle of LTI systems, allowing the use of transfer functions to characterize input–output relations for such systems in the frequency domain.

On the other hand, for an LTP system, a sinusoidal input at a specific frequency ω, produces not only an output at the same frequency, ω, but also components at frequencies that are the sum of ω and the harmonics of the pumping frequency $\omega_p = 2\pi/T$ of the system (i.e. at $\omega + k\omega_p, k \in \mathbb{Z}$), all with possibly different magnitudes and phases in steady state. Based on this property, the concept of Harmonic Transfer Functions (HTFs) were developed by Wereley (1991), where distinct transfer functions capture each of these harmonic responses.

Existing literature on frequency domain system identification of LTP systems concentrates mainly on non-parametric estimation of the harmonic transfer functions (HTFs) (Hwang, 1997; Siddiqi, 2001; Louarroudi et al., 2012; Uyanik et al., 2016), Even though a number of previous studies perform parametric identification by fitting parameterized transfer function models to the non-parametrically identified HTFs (Ankarali and Cowan, 2014; Uyanik et al., 2015a,b), the present study focuses on a direct state-space parametric identification method for the hybrid LTP system without dealing with computational details of HTFs.

In this formulation, the steady-state response of the system can be represented as

$$X(j\omega) = \sum_{n=-\infty}^{\infty} H_n(j\omega - jn\omega_p)U(j\omega - jn\omega_p) \ , \quad (2)$$

where $H_n(s)$ can be theoretically derived for certain special cases when the state space representation of the system is available, such as for systems with finite harmonic expansions or constant system matrices (Wereley, 1991; Möllerstedt, 2000).

2.2 Modeling and Assumptions

In this paper, we will work with systems in the form of (1), assumed to be driven by an observable input, $u(t)$, with measurements provided for all of its states. Moreover, we also require the following assumptions to hold.

Assumption 1. Our models of interest consist of M alternating "unknown" LTP sub-dynamics, $A^1(t)$, $A^2(t)$, $\cdots$, $A^M(t)$, whose activations are triggered by M complementary "known" rectangular switching functions, $s^1(t)$, $s^2(t)$, $\cdots$, $s^M(t)$, during each cycle of the system. Both $A^i(t)$ and $s^i(t)$ are T-periodic functions, with the switching functions taking the form

$$s^i(t) = \begin{cases} 1, & \text{if} \quad t_i + nT \le t < t_{i+1} + nT, \quad \forall n \in \mathbb{Z} \\ 0, & \text{otherwise,} \end{cases} \quad (3)$$

where t_i's denote the known switching times and satisfy the conditions $t_1 = 0$, $t_{M+1} = T$, and $t_i < t_{i+1}, \forall i \in 1, \cdots, M$. The state and input matrices of (1) can hence be written as

$$A(t) = \sum_{i=1}^{M} A^i(t)s^i(t), \qquad B(t) = \sum_{i=1}^{M} B^i(t)s^i(t). \quad (4)$$

□

Assumption 2. We assume that the system period T as well as the transition times between different sub-system dynamics can be measured and are known. This information is sufficient to construct the switching functions, $s^1(t)$, $s^2(t)$, $\cdots$, $s^M(t)$, that trigger the activation of alternating sub-systems. □

Based on the LTP framework and our assumptions listed above, the problem we are interested can be defined as:
Given

- a number of single-sine (or sums-of-sines) input measurements applied at different frequencies, $u(t)$,
- corresponding state measurements, $x(t)$,
- the system period, T, and the switching times between successive subsystems, $s^1(t)$, $s^2(t)$, $\cdots$, $s^M(t)$,

Estimate piecewise smooth, linear time-periodic state and input matrices, $A^1(t)$, $A^2(t)$, $\cdots$, $A^M(t)$ and $B^1(t)$, $B^2(t)$, $\cdots$, $B^M(t)$.

3. ESTIMATION OF LINEAR TIME-PERIODIC SYSTEM MATRICES

Our analysis begins with obtaining Fourier series expansions for the state and input matrices, $A(t)$ and $B(t)$ as

$$A(t) = \sum_{n=-\infty}^{\infty} A_n e^{jn\omega_p t}, \quad B(t) = \sum_{n=-\infty}^{\infty} B_n e^{jn\omega_p t}, \tag{5}$$

and transforming (1) into

$$\dot{\mathbf{x}}(t) = \sum_{n=-\infty}^{\infty} A_n e^{jn\omega_p t}\mathbf{x}(t) + \sum_{n=-\infty}^{\infty} B_n e^{jn\omega_p t}\mathbf{u}(t). \tag{6}$$

Assuming that the system is stable and that oscillations reach steady-state, we can then switch to the frequency domain through the Fourier transformation to yield

$$(j\omega)X(j\omega) = \sum_{n=-\infty}^{\infty} A_n X(j\omega - jn\omega_p) + \sum_{n=-\infty}^{\infty} B_n U(j\omega - jn\omega_p). \tag{7}$$

Fourier series coefficients of the multiplication of two periodic signals with the same period can be obtained as the convolution of the Fourier coefficients of the each individual signal. Considering the Fourier series coefficients for the rectangular switching functions as

$$s^i(t) = \sum_{n=-\infty}^{\infty} S_n^i e^{jn\omega_p t} \tag{8}$$

and using (4), Fourier series coefficients of $A(t)$ can then be obtained as

$$A_n = \sum_{k=-\infty}^{\infty} A_k^1 S_{n-k}^1 + \cdots + \sum_{k=-\infty}^{\infty} A_k^M S_{n-k}^M. \tag{9}$$

Substituting (9) and a similar expansion for B_n into (7), we obtain

$$\begin{aligned}(j\omega)X(j\omega) = &\sum_{n=-\infty}^{\infty}\Big\{\sum_{k=-\infty}^{\infty} A_k^1 S_{n-k}^1\Big\}X(j\omega - jn\omega_p) + \\ &\vdots \\ &\sum_{n=-\infty}^{\infty}\Big\{\sum_{k=-\infty}^{\infty} A_k^M S_{n-k}^M\Big\}X(j\omega - jn\omega_p) + \\ &\sum_{n=-\infty}^{\infty}\Big\{\sum_{k=-\infty}^{\infty} B_k^1 S_{n-k}^1\Big\}U(j\omega - jn\omega_p) + \\ &\vdots \\ &\sum_{n=-\infty}^{\infty}\Big\{\sum_{k=-\infty}^{\infty} B_k^M S_{n-k}^M\Big\}U(j\omega - jn\omega_p). \end{aligned}\tag{10}$$

After reorganizing the terms, we obtain

$$\begin{aligned}(j\omega)X(j\omega) = &\sum_{k=-\infty}^{\infty} A_k^1 \underbrace{\Big\{\sum_{n=-\infty}^{\infty} S_{n-k}^1 X(j\omega - jn\omega_p)\Big\}}_{X_k^1(j\omega)} + \\ &\vdots \\ &\sum_{k=-\infty}^{\infty} A_k^M \underbrace{\Big\{\sum_{n=-\infty}^{\infty} S_{n-k}^M X(j\omega - jn\omega_p)\Big\}}_{X_k^M(j\omega)} + \\ &\sum_{k=-\infty}^{\infty} B_k^1 \underbrace{\Big\{\sum_{n=-\infty}^{\infty} S_{n-k}^1 U(j\omega - jn\omega_p)\Big\}}_{U_k^1(j\omega)} + \\ &\vdots \\ &\sum_{k=-\infty}^{\infty} B_k^M \underbrace{\Big\{\sum_{n=-\infty}^{\infty} S_{n-k}^M U(j\omega - jn\omega_p)\Big\}}_{U_k^M(j\omega)}. \end{aligned}\tag{11}$$

Here, $X_k^i(j\omega)$ and $U_k^j(j\omega)$ correspond to the convolution of the Fourier coefficients for the switching functions with the Fourier coefficients of the state and input functions, respectively. Now, truncating the infinite Fourier series to only K components in either direction and converting (11) into matrix form yields

$$\begin{aligned}(j\omega)X(j\omega) = &\underbrace{\begin{bmatrix} A_{-K}^1 & \cdots & A_0^1 & \cdots & A_K^1 \end{bmatrix}}_{\mathcal{A}^1} \underbrace{\begin{bmatrix} X_{-K}^1(j\omega) \\ \vdots \\ X_0^1(j\omega) \\ \vdots \\ X_K^1(j\omega) \end{bmatrix}}_{\mathcal{X}^1(j\omega)} + \\ &\vdots \\ &\underbrace{\begin{bmatrix} A_{-K}^M & \cdots & A_0^M & \cdots & A_K^M \end{bmatrix}}_{\mathcal{A}^M} \underbrace{\begin{bmatrix} X_{-K}^M(j\omega) \\ \vdots \\ X_0^M(j\omega) \\ \vdots \\ X_K^M(j\omega) \end{bmatrix}}_{\mathcal{X}^M(j\omega)} + \\ &\underbrace{\begin{bmatrix} B_{-K}^1 & \cdots & B_0^1 & \cdots & B_K^1 \end{bmatrix}}_{\mathcal{B}^1} \underbrace{\begin{bmatrix} U_{-K}^1(j\omega) \\ \vdots \\ U_0^1(j\omega) \\ \vdots \\ U_K^1(j\omega) \end{bmatrix}}_{\mathcal{U}^1(j\omega)} + \\ &\vdots \\ &\underbrace{\begin{bmatrix} B_{-K}^M & \cdots & B_0^M & \cdots & B_K^M \end{bmatrix}}_{\mathcal{B}^M} \underbrace{\begin{bmatrix} U_{-K}^M(j\omega) \\ \vdots \\ U_0^M(j\omega) \\ \vdots \\ U_K^M(j\omega) \end{bmatrix}}_{\mathcal{U}^M(j\omega)}. \end{aligned}\tag{12}$$

Remark 1. In the identification of real systems and even for most simulated non-linear systems, prior information on the proper choice of K, the limit on the number of Fourier series to be estimated, is unavailable. Moreover, the "true" value of K can be even infinity. Since we currently focus on the identification of deterministic systems, an ad-hoc, yet acceptable solution is to choose a sufficiently big value for K and disregard Fourier series coefficients that are less then a certain threshold. □

Remark 2. Note that our choice of K does not require truncating infinite summations for computing $X_k^1(j\omega), \cdots, X_k^M(j\omega)$, $U_k^1(j\omega)$, $\cdots$, $U_k^M(j\omega)$ in (11). On the other hand, computing infinite summations in computerized environments is of course not generally possible and hence another, possibly larger truncation $n = N$ can be used for these summations involving known quantities. □

Before proceeding with a least-squares solution, we add an additional constraint to (12) to capture the requirements that system matrices, states and inputs are real valued. Let

$$A_K^i = A_{K,Re}^i + jA_{K,Im}^i \ , \tag{13}$$

where $A_{K,Re}^i$ and $A_{K,Im}^i$ denote real and imaginary parts of the K^{th} Fourier coefficient of the i^{th} system matrix. We must then have

$$A_{-K}^i = A_{K,Re}^i - jA_{K,Im}^i \tag{14}$$

to ensure that the system matrix is real-valued in the time domain. This yields

$$\mathcal{A}^i = \begin{bmatrix} A_{K,Re}^i - jA_{K,Im}^i & \cdots & A_0^i & \cdots & A_{K,Re}^i + jA_{K,Im}^i \cdot \end{bmatrix}$$

In order to simplify the formulation of our least-squares solution, we re-organize the terms in $\mathcal{A}^i$ to eliminate repetitions. More formally, we define

$$\bar{\mathcal{A}}^i := \begin{bmatrix} A_{K,Re}^i & \cdots & A_0^i & \cdots & A_{K,Im}^i \end{bmatrix} \tag{15}$$

and

$$P := \begin{bmatrix} \mathcal{I} & & & & & & \mathcal{I} \\ & \ddots & & & & \cdot^{\cdot^{\cdot}} & \\ & & \mathcal{I} & 0 & \mathcal{I} & & \\ & & 0 & \mathcal{I} & 0 & & \\ & & -j\mathcal{I} & 0 & j\mathcal{I} & & \\ & \cdot^{\cdot^{\cdot}} & & & & \ddots & \\ -j\mathcal{I} & & & & & & j\mathcal{I} \end{bmatrix}, \tag{16}$$

which are specifically constructed to satisfy

$$\mathcal{A}^i = \bar{\mathcal{A}}^i P \ . \tag{17}$$

Using the decomposition above, (12) can be simplified by reorganizing terms and grouping known and unknown quantities in separate matrices as

$$\underbrace{(j\omega)X(j\omega)}_{\mathbf{y}^T(j\omega)} = \underbrace{\begin{bmatrix} \bar{\mathcal{A}}^1 & \cdots & \bar{\mathcal{A}}^M & \bar{\mathcal{B}}^1 & \cdots & \bar{\mathcal{B}}^M \end{bmatrix}}_{\mathbf{v}^T} \underbrace{\begin{bmatrix} P\bar{\mathcal{X}}^1(j\omega) \\ \vdots \\ P\bar{\mathcal{X}}^M(j\omega) \\ P\bar{\mathcal{U}}^1(j\omega) \\ \vdots \\ P\bar{\mathcal{U}}^M(j\omega) \end{bmatrix}}_{\mathbf{n}^T(j\omega)} .$$

Transposing both sides yields a linear equation as

$$\mathbf{n}(j\omega)\ \mathbf{v} = \mathbf{y}(j\omega) \tag{18}$$

As explained in Section 2.1, LTP system outputs contain components not only in the input frequency but also at frequencies shifted by the harmonics of the pumping frequency. Consequently, we will evaluate (7) both at the input frequency ω as well as the shifted harmonics $\omega \pm h\omega_p, h \in \mathbb{Z}$ in order to capture time-periodic system as

$$\underbrace{\begin{bmatrix} \mathbf{n}(j\omega + h\omega_p) \\ \vdots \\ \mathbf{n}(j\omega) \\ \vdots \\ \mathbf{n}(j\omega - h\omega_p) \end{bmatrix}}_{N(j\omega)} \mathbf{v} = \underbrace{\begin{bmatrix} \mathbf{y}(j\omega + h\omega_p) \\ \vdots \\ \mathbf{y}(j\omega) \\ \vdots \\ \mathbf{y}(j\omega - h\omega_p) \end{bmatrix}}_{Y(j\omega)} . \tag{19}$$

In order to ensure that the solutions are real-valued, we separate the real and imaginary parts of the possibly complex-valued components computed from out test data as

$$\underbrace{\begin{bmatrix} Re\{N(j\omega)\} \\ Im\{N(j\omega)\} \end{bmatrix}}_{N_w} \mathbf{v} = \underbrace{\begin{bmatrix} Re\{Y(j\omega)\} \\ Im\{Y(j\omega)\} \end{bmatrix}}_{Y_w} . \tag{20}$$

Remark 3. Separating real and imaginary components of complex-valued components computed from data doubles the number of tests used for the least squares solution. □

Subsequently, collecting together multiple measurements from different frequencies yields

$$\underbrace{\begin{bmatrix} \vdots \\ N_w \\ \vdots \end{bmatrix}}_{\mathcal{N}} \mathbf{v} = \underbrace{\begin{bmatrix} \vdots \\ Y_w \\ \vdots \end{bmatrix}}_{\mathcal{Y}} \tag{21}$$

Now, the least squares error solution can be found as

$$\mathbf{v} = (\mathcal{N}^H \mathcal{N})^{-1} \mathcal{N}^H \mathcal{Y}. \tag{22}$$

We can extract Fourier series coefficient matrices from $\mathbf{v}$ and then construct $A^1(t)$, $A^2(t)$, $\cdots$, $A^M(t)$ and $B^1(t)$, $B^2(t)$, $\cdots$, $B^M(t)$ using Fourier series synthesis as

$$A^i(t) = \sum_{n=-K}^{K} A_n^i e^{jn\omega_p t}, \quad B^i(t) = \sum_{n=-K}^{K} B_n^i e^{jn\omega_p t}. \tag{23}$$

4. APPLICATION: A SWITCHING DAMPED MATHIEU FUNCTION

In this section, we present an example system, a piecewise smooth linear time-periodic function, and evaluate the performance of the proposed algorithm on this example.

4.1 System Dynamics and Parameters

The piecewise smooth LTP system we consider in this example consists of two switching damped Mathieu functions with the form

$$\ddot{x}(t) + \underbrace{2\zeta\omega_n}_{c} \dot{x}(t) + \underbrace{(1 + 2\beta\cos\omega_p t)\omega_n^2}_{\kappa(t)} x(t) = u(t) \tag{24}$$

where c represents piecewise constant damping term, while $\kappa(t)$ represents piecewise smooth time-periodic compliance term in the Mathieu function.

Piecewise smooth LTP equations of motions can now be written as

$$\ddot{x}(t) = \begin{cases} u(t) - c_1\dot{x}(t) - \kappa_1(t)x(t), & \text{if } Tn \leq t \leq Tn + T/2 \\ u(t) - c_2\dot{x}(t) - \kappa_2(t)x(t), & \text{otherwise.} \end{cases} \quad (25)$$

Using (25), state and input matrices can be obtained as

$$A^1(t) = \begin{bmatrix} 0 & 1 \\ -(1+2\beta_1\cos\omega_p t)\omega_n^2 & -2\zeta_1\omega_n \end{bmatrix} \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A^2(t) = \begin{bmatrix} 0 & 1 \\ -(1+2\beta_2\cos\omega_p t)\omega_n^2 & -2\zeta_2\omega_n \end{bmatrix}$$

where the input matrix is time-invariant for this example. Based on the parameters specified in Table 1, *true* values of the Fourier series coefficients can be found as

$$A_0^1 = \begin{bmatrix} 0 & 1 \\ -39.4784 & -3.7699 \end{bmatrix}, \; A_1^1 = \begin{bmatrix} 0 & 0 \\ -3.9478 & 0 \end{bmatrix}$$

$$A_0^2 = \begin{bmatrix} 0 & 1 \\ -39.4784 & -1.2566 \end{bmatrix}, \; A_1^2 = \begin{bmatrix} 0 & 0 \\ -7.8957 & 0 \end{bmatrix} \quad (26)$$

Table 1. Mathieu Function Parameters

T	ω_p	ω_n	ζ_1	β_1	ζ_2	β_2
0.5	4π	2π	0.3	0.1	0.1	0.2

4.2 Estimation Results with Single Sine Excitations

In order to illustrate our estimation method, we first simulated the system to collect input-output data for the identification process. In this example, we simulated the piecewise smooth LTP dynamics of (25) by applying single sine inputs, $u(t) = \sin(2\pi f t)$, at each experiment/simulation. We performed simulations using 20 diffrent frequencies equally spaced in the frequency band [0.2, 4] Hz, and record the state measurements.

Our input signals are 10 s. long and all data are sampled at 100 Hz. Note that we currently use single sine inputs in our tests but it is also posible to use sums-of-sines type input stimuli to decrease the number of tests required for system identification.

Note that previously, Hwang (1997) showed that in order to estimate HTF components uniquely using single sinusoidal signals (i.e. one experimental data per each frequency), the input signal must not be equal to the harmonics of the half of the pumping frequency, i.e. $\omega \neq k\omega_p/2, \; k \in \mathbb{Z}$. Since we are attempting to compute the parametric LTP matrices from input-output data, we no longer need to satisfy this constraint for the case of pure piece-wise smooth LTP systems. However, Ankarali (2015) also showed that for the identification of non-linear systems that operate around a limit-cycle, input frequencies that are equal to the harmonics of the pumping frequency should also be avoided in order to isolate the frequency components of the limit-cycle and response around the limit-cycle. For this reason we also do not include the harmonics of the pumping frequency in our input signals.

Once we have collected the input-output data from our simulations, we need to make three implementation choices before building our least squares estimation matrices. First, we consider $N = 20$, yielding 41 Fourier series coefficients for the computation of $X_k^i(j\omega)$ and $U_k^i(j\omega)$ in (11) (our bound here originates from the signal length,see Remark 2). Secondly, we choose $K = 1$ for each sub LTP dynamics, which yields exact number of Fourier series coefficients, 3, for the simulated example. Finally, we consider $h = 2$, total numer of computed harmonics of the whole LTP system for (19), so that we evaluate each input at 5 different *output* frequencies.

Based on our implementation choices explained above, Fourier series coefficients are estimated as

$$\hat{A}_0^1 = \begin{bmatrix} 0.0002 & 0.9999 \\ -39.4404 & -3.7786 \end{bmatrix} \quad (27)$$

$$\hat{A}_1^1 = \begin{bmatrix} -0.0001 - j0.0003 & -0.0001 - j0.0000 \\ -3.9253 + j0.0095 & 0.0090 - j0.0012 \end{bmatrix}$$

$$\hat{A}_0^2 = \begin{bmatrix} 0.0009 & 1.0003 \\ -40.0038 & -1.2824 \end{bmatrix}$$

$$\hat{A}_1^2 = \begin{bmatrix} 0.0002 - j0.0008 & -0.0001 - j0.0002 \\ -7.9860 + j0.0242 & 0.0017 + j0.0084 \end{bmatrix}$$

with $A_{-1}^1 = A_1^{1*}$ and $A_{-1}^2 = A_1^{2*}$. We then reconstruct $\hat{A}^1(t)$ and $\hat{A}^2(t)$ using (23). Similarly, B is estimated as $\hat{B} = \begin{bmatrix} 0 \\ 0.9965 \end{bmatrix}$.

In order to better express our estimation results, we plot time-domain graphs of $\kappa(t)$ and $c(t)$ defined in (24) both using the actual and estimated system matrices as illustrated in Fig. 1. $\kappa(t)$ represents the piecewise time-periodic compliance behavior in our Mathieu function, while $c(t)$ represents piecewise time-invariant damping loss. Both of these variables switch to another parameter at the half of the period, which brings a time-periodic nature to both functions.

As illustrated in Fig. 1, our estimations with $K = 1$ fits well to the actual system parameters, since we didn't contaminate our simulation data with noise for experiments. Apart from that, $K = 1$ is the exact number of Fourier series coefficients for our system as seen in (26). Actually, it gives an exact fit for $\kappa(t)$ but it is an overfit for $c(t)$, since $K = 0$ would be sufficient to represent its piecewise LTI nature. However, due to deterministic

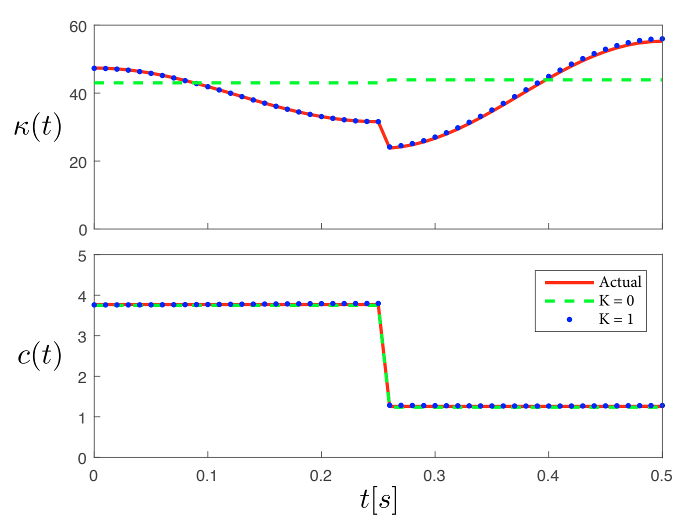

Fig. 1. Estimation results for compliance and damping term for a single period.

nature of our simulations overfitting does not generate an error in estimation of $c(t)$.

In order to investigate the effects of under-fitting, we repeated our estimations for $K = 0$ and re-estimated our system parameters as illustrated in Fig. 1. In this case, it can be observed that $\kappa(t)$ can not be estimated accurately, although we can still obtain accurate estimations for $c(t)$.

5. DISCUSSION

A huge class of physical physical dynamical systems exhibit quasi-periodic trajectories and hybrid characteristics. However, it is fair to assume that only a few of the system identification studies in literature concentrate on the identification of hybrid dynamic system that operate around some periodic orbits, which is the main goal of this paper. Specifically, we limit our attention to the hybrid systems that has continuous state trajectories but potentially discontinuous vector fields. Under some assumptions, the local flow around the periodic orbit of such a system can be approximated with a hybrid LTP system.

Based on these motivations, we introduced a state space parametric identification framework for hybrid LTP systems for which the periodic switching times are assumed to be known. We formulated the problem in a linear regression framework in frequency domain, where we estimated Fourier series coefficients of the time-periodic system and input matrices. Then, we re-constructed the time domain system and input matrices using Fourier synthesis after a least squares solution. Currently, our formulations assume full state measurements which is the main limitation of our method. As a future work, we will attempt to improve our method such that we can relax this assumption also including process and measurement noise.

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