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IFAC-PapersOnLine 49-10 (2016) 206–211

On the Delay Margin for Consensus in Directed Networks of Anticipatory Agents

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Abstract: We consider a linear consensus problem involving a time delay that arises from predicting the future states of agents based on their past history. In case the agents are coupled in a connected and undirected network, the exact condition for consensus is that the delay be less than a constant threshold that is independent of the network topology or size. In directed networks, however, the situation is quite different. We show that the allowable maximum delay for consensus depends on the network topology in a nontrivial way. We study this delay margin in several network constellations, including various circulant networks with directed links. We show that the delay margin depends not only on the number of neighbors, but also on the directionality of connections with those neighbors. Furthermore, the delay margin improves as the circulant networks are rewired en route to a small-world configuration.

Keywords: Consensus, prediction, anticipation, delay, network, graph theory, stability analysis.

1. INTRODUCTION

Consensus and coordination problems arise in a wide range of applications where multi-agent systems interact to agree on a common value of a certain quantity of interest. We can cite here, among others, Lynch (1996) in distributed computing, DeGroot (1974) in management science, Vicsek et al. (1995) in flocking and swarming theory, Fax and Murray (2004) in distributed control, and Olfati-Saber and Shamma (2005) in sensor networks. The classical linear consensus problem can be formulated in the form

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}(x_j(t) - x_i(t)), \quad i = 1, \ldots, n \tag{1}$$

where n is the number of agents in the network, $x_i \in \mathbb{R}$ is the state of the agent i at time t, which changes under the interaction with other agents, and a_{ij} are nonnegative numbers describing the interaction strength between agents i and j. Consensus can then be formally defined as follows.

Definition 1. The system (1) is said to *reach consensus* if for any set of initial conditions $\{x_i(0)\}$ there exists $c \in \mathbb{R}$ such that $\lim_{t\to\infty} x_i(t) = c$ for all i, in which case the number c is called the *consensus value.*

Under mild conditions related to the connectivity of the network, it can be shown that the system (1) reaches consensus from arbitrary initial conditions, and the consensus value equals the average of initial conditions of the agents. The problem becomes more interesting when the system involves a time delay τ, for example an information processing delay modeled by

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}\left(x_j(t-\tau) - x_i(t-\tau)\right) \tag{2}$$

which has been studied in Olfati-Saber and Murray (2004). In this case, it is known that there exists an upper limit $\tau_{\max}$ such that the system (2) reaches consensus from arbitrary initial conditions if and only if $\tau < \tau_{\max}$ (see, for instance, Olfati-Saber and Murray (2004)). Another model, which involves an information transmission delay, is given by (Moreau, 2004; Seuret et al., 2008; Atay, 2012, 2013)

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}\left(x_j(t-\tau) - x_i(t)\right). \tag{3}$$

It has been shown that such a system reaches consensus from arbitrary initial conditions regardless of the value of the delay τ as long as the network contains a spanning tree; however, the consensus value depends on the system's history in a nontrivial way (Atay, 2012, 2013).

In this paper we are concerned with a rather different source of delay, arising from the *anticipatory nature* of the agents. More precisely, we consider a network of intelligent agents who try to predict the future states of their neighbors in their interaction. Formulating in the context of system (1), agent i uses, instead of the current state $x_j(t)$ of its neighbor, a predicted value $\hat{x}_j(t+\tau)$ of its future state, yielding

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}(\hat{x}_j(t+\tau) - x_i(t)), \quad i = 1, \ldots, n \tag{4}$$

Using a *first order estimation* (Atay and Irofti, 2015), the prediction of the future can be done by a linear

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Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2016.07.530

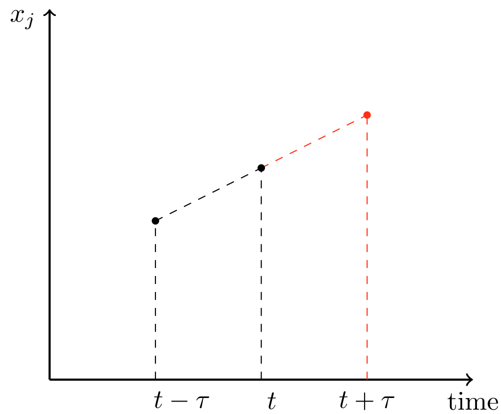


Fig. 1. Linear prediction of the future state $x_j(t+\tau)$ of an agent j using its present and past states.

extrapolation from past values, namely

$$\hat{x}_j(t+\tau) = x_j(t) + \frac{x_j(t) - x_j(t-\tau)}{\tau}\tau, \\ = 2x_j(t) - x_j(t-\tau). \quad (5)$$

(See Figure 1 for a graphical depiction.)

Using (5) in (4), we arrive at the main model that we will study in this paper:

$$\dot{x}_i(t) = \frac{1}{d_i}\sum_{j=1}^{n} a_{ij}(2x_j(t) - x_j(t-\tau) - x_i(t)), \quad (6)$$

Note that we have additionally normalized the summation term via dividing by the (generalized) degree d_i of node i, $d_i = \sum_{j=1}^{n} a_{ij}$. This normalization gives rise to a *normalized Laplace operator*, which is a natural choice in several applications and has some advantageous properties, as will be briefly reviewed in Section 2. In particular, the normalization bounds the spectrum of Laplacian regardless of the network size, thus allowing comparison of networks of very different sizes.

When the network is undirected (i.e. $a_{ij} = a_{ji}\ \forall i, j$) and connected, it has been shown (Atay and Irofti, 2015, 2016) that system (6) reaches consensus from arbitrary initial conditions if and only if

$$\tau < 1. \quad (7)$$

In other words, in the undirected case, the maximum allowable delay for consensus in (6) (the *delay margin*) equals 1 regardless of the network topology. The situation for directed networks is different, however, as we show in this paper. In particular, the network topology turns out to play an important role in affecting the delay margin.

In the following, we first prove that (7) is a necessary condition for consensus, but it is not sufficient. Moreover, as already mentioned above, the undelayed network ($\tau = 0$) always reaches consensus (as long as it contains a directed spanning tree). It follows by continuity that, sufficiently small delays will not destroy stability of the consensus. Hence, the delay margin for (6) is some positive number less than one. We calculate the locus of (complex) Laplacian eigenvalues that are detrimental for consensus. Just like undirected networks, many directed networks also turn out to enjoy (7) as the exact condition for consensus. However, we show that some specific networks that are actually frequently used in the literature do have much lower delay margins. We study these circulant networks in detail with respect to their Laplacian eigenvalues and determine their delay margins. We also consider random rewiring of circulant networks en route to small-world configuration and show that a few rewirings improve the delay margin already, although the improvement is not monotone with further rewirings.

2. DIRECTED NETWORKS AND CHARACTERISTIC ROOTS

A directed graph (or *digraph*) $G = (V, E)$ consists of a finite set V of vertices and a set of directed edges $E \subset V \times V$ consisting of ordered pairs of vertices. We consider only simple, non-trivial graphs without self-loops or multiple edges. We denote by $A = [a_{ij}]$ the (weighted) adjacency matrix of the graph, where $a_{ij} > 0$ if there is a directed link from node j to node i, and $a_{ij} = 0$ otherwise. The *in-degree* d_i of node i is defined as $d_i = \sum_{j=1}^{n} a_{ij}$, i.e., the sum of the elements of the i^{th} row of A, and $D = \mathrm{diag}(d_1, \ldots, d_n)$ denotes the diagonal matrix of in-degrees.

Assuming that $d_i \neq 0\ \forall i$, the normalized Laplacian matrix is defined as

$$L = I_n - D^{-1}A, \quad (8)$$

where n is the number of nodes in the network and I_n is the identity matrix of size n. The normalized Laplacian L naturally arises in a class of important problems, in particular in random walks on networks, as $D^{-1}A$ is the transition matrix for probability distributions arising from such walks (Chung, 1997).

An application of Gershgorin's theorem to the definition of L shows that the Laplacian eigenvalues λ_k are complex numbers satisfying

$$|1 - \lambda_k| \leq 1, \quad k = 1, 2, \ldots, n. \quad (9)$$

Furthermore, the first eigenvalue λ_1 is always zero and corresponds to the eigenvector $(1, 1, \ldots, 1)^\top$. In the special case of undirected networks the eigenvalues are all real, because $D^{-1}A$ is similar to a symmetric matrix, $D^{-1}A = D^{-1/2}(D^{-1/2}AD^{-1/2})D^{1/2}$, as A is symmetric and D is diagonal.

In matrix form, (6) becomes

$$\dot{x}(t) = D^{-1}A(2x(t) - x(t-\tau)) - x(t), \quad (10)$$

with $x = (x_1, x_2, \ldots, x_n)^\top$. Suppose that L has a complete set of eigenvectors $\{\mathbf{v}_k\}$ corresponding to the eigenvalues $\{\lambda_k\}$. Then one can write $x(t) = \sum_{k=1}^{n} u_k(t)\mathbf{v}_k$ for some scalar coefficients u_k, which transforms (10) into a system of n decoupled scalar equations

$$\dot{u}_k(t) = (1 - 2\lambda_k)\, u_k(t) - (1 - \lambda_k)u_k(t-\tau), \quad (11)$$

for $k = 1, \ldots, n$. The characteristic equation corresponding to the eigenmode (11) is

$$\psi_k(s) := s - 2(1 - \lambda_k) + 1 + (1 - \lambda_k)e^{-s\tau} = 0, \quad (12)$$

and the characteristic equation for the whole system (10) can be written as

$$\Psi(s) := \prod_{k=1}^{n} \psi_k(s) = 0. \quad (13)$$

Note that $s = 0$ is always a characteristic root for the first factor

$$\psi_1(s) = s - 1 + e^{-s\tau}$$

corresponding to the first eigenmode, $\lambda_1 = 0$. Thus, points on the synchronization subspace spanned by $\mathbf{v}_1 = (1, 1, \ldots, 1)^\top$ can be at best neutrally stable. If that is the case, and in addition $\lim_{t\to\infty} u_k(t) = 0$ for all $k \geq 2$, then the system converges to a point on $\mathbf{v}_1$, i.e. it reaches consensus, from arbitrary initial conditions. This clearly happens if and only if zero is a simple root of Ψ and all other roots of Ψ have negative real parts.

3. UPPER BOUND FOR THE DELAY MARGIN

Based on the considerations of the previous section, we can state a necessary condition for consensus, which serves as an upper bound for the delay margin.

Theorem 1. The inequality

$$\tau < 1 \tag{14}$$

is a necessary condition for (6) to reach consensus from arbitrary initial conditions. In other words, the delay margin for the stability of consensus in (6) is at most 1.

Proof. Consider the first factor $\psi_1(s) = s - 1 + e^{-s\tau}$ in (13). Since $\psi_1(0) = 0$ and $\psi_1'(0) = 1 - \tau$, zero is always a root of ψ_1 and is a simple root unless $\tau = 1$. For $\tau > 1$, on the other hand, ψ_1 is unstable; in fact, it has a real and positive root, which can easily be seen by plotting the real functions $1-s$ and $e^{-s\tau}$ and observing that they must intersect at a positive value of s if $\tau > 1$. Therefore, (14) is a necessary condition that ψ_1, and hence the characteristic equation (13), has a simple zero root and all the remaining roots have strictly negative real parts.

As remarked in the introduction, unlike the case of undirected networks, the condition (14) is not sufficient for consensus in directed networks. The distinction clearly lies in the fact that the Laplacian eigenvalues are real for undirected networks but not necessarily for directed ones. It is therefore of interest to have some understanding of the network structures and the corresponding Laplacian eigenvalues that give rise to reduced delay margins. We take up this task in the next section.

4. LAPLACIAN EIGENVALUES RESPONSIBLE FOR INSTABILITY

To understand the role of Laplacian eigenvalues on the stability of consensus, we consider a generic factor of the characteristic equation (12)–(13), namely

$$\psi(s) := s - 2(1-\lambda) + 1 + (1-\lambda)e^{-s\tau} = 0 \tag{15}$$

for an arbitrary $\lambda \in \mathbb{C}$ that might have come from the spectrum of the Laplacian of some network. The latter condition requires

$$|1 - \lambda| \leq 1 \tag{16}$$

in view of (9). The stability of (15) can be conveniently established using the Lambert W function, as follows.

Recall that the Lambert W function is defined as the inverse function of the mapping $z \mapsto ze^z$ for $z \in \mathbb{C}$ (see Corless et al. (1996)). In other words, the (multi-valued) function $W(z)$ satisfies

$$W(z)e^{W(z)} = z. \tag{17}$$

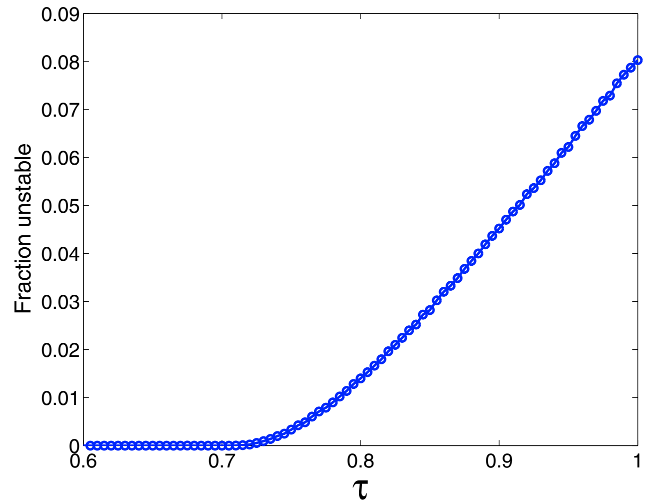


Fig. 2. Fraction of complex numbers λ that make the characteristic equation (15) unstable, from among 10^6 randomly selected complex numbers inside the shifted unit circle (16).

Letting $a = 2(1-\lambda) - 1$ and $b = -(1-\lambda)$, the roots of ψ satisfy $s = a + be^{-s\tau}$, which can be re-written as

$$s - a = be^{-(s-a)\tau}e^{-a\tau}.$$

A change of variables $s' \to (s-a)\tau$ gives

$$s' = \tau b e^{-a\tau}e^{-s'},$$

from which, by the definition (17), one has $s' = W(\tau b e^{-a\tau})$. In the original variable s, the roots of ψ satisfy

$$s = \frac{1}{\tau}W(\tau b e^{-a\tau}) + a.$$

Furthermore, by a result of Shinozaki and Mori (2006), the root having the largest real part is given by the principal branch W_0 of the Lambert function. Hence, for the stability of (15) it suffices to calculate

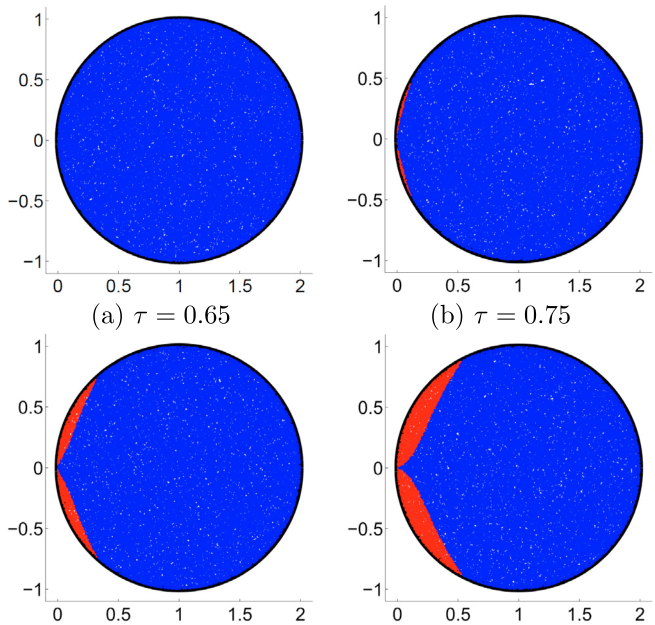
$$s = \frac{1}{\tau}W_0(\tau b e^{-a\tau}) + a, \tag{18}$$

and check whether its real part is negative.

As a first step in numerical investigations, we randomly generate complex numbers λ satisfying (16) and check the stability of (15) using (18) for the range of delay values $\tau \in (0, 1)$. The results of the experiment are shown in Figure 2 for one million randomly generated λ. It can be seen that only a small fraction of λ actually yield an unstable ψ.

To gain further insight into the nature of instability, we plot in Figure 3 the location of the stable and unstable Laplacian eigenvalues λ in the complex plane. As expected, for sufficiently small delays consensus is stable, and as the delay increases, two regions of unstable eigenvalues (shown with red color) grow from the circle boundary, eventually meeting at $\lambda = 0$ as $\tau \uparrow 1$ (Figure 3 (a) through (d)). The value $\tau = 1$ is the upper value of allowable delay for any network, since at this value the characteristic equation (15) has a double zero root (see Theorem 1 and its proof).

At first it might appear from Figures 2–3 that most networks would have a delay margin equal to 1. While this is true in a certain sense, a class of networks commonly used in the literature does turn out to belong to the



(a) $\tau = 0.65$ (b) $\tau = 0.75$

(c) $\tau = 0.85$ (d) $\tau = 0.99$

Fig. 3. Values of random complex numbers λ inside the shifted unit circle, colored according to the stability of the characteristic equation (15). Red points represent unstable values and blue points represent stable values of λ, shown for four different values of the delay.

exceptional unstable category. In the next section we investigate in detail these networks that have a poor delay tolerance.

5. DIRECTED CIRCULANT NETWORKS AS PROTOTYPES FOR WORST DELAY TOLERANCE

An important observation from Figure 3 is that the red regions for unstable Laplacian eigenvalues border on the shifted unit circle (16), and in fact disappear by merging into the circle border near the origin as τ is decreased. Therefore, networks having the worst delay tolerance must be those that have a Laplacian eigenvalue λ on the circle near the origin. Therefore, as the next step in our investigation, we consider a network configuration that has *all* its eigenvalues on the circle, namely the directed cycle shown in Figure 4(a). We refer to this configuration as Network A. Indeed, it is easy to calculate that the characteristic polynomial for the Laplacian for Network A is

$$(1 - \lambda)^n = 1;$$

thus the eigenvalues λ_k are roots of unity shifted by one:

$$\lambda_k = 1 - \exp(2\pi i(k-1)/n), \quad k = 1, \ldots, n;$$

see Figure 4(e). It follows that for very small network sizes n, the eigenvalues are outside the red region of Figure 3 (although they are still all on the circle); so Network A will have a delay margin equal to 1. As the network size n becomes larger, the eigenvalues move into the red region and the delay margin decreases, as shown in Figure 5.

We next consider circulant networks with in-degree equal to two. Here we have two natural variations as extensions of Network A: either all connections follow the same direction, or connections of distance one and two have opposite directions. We refer to these networks as Network B and Network C, respectively; see Figure 4 (b)–(c).

For directed circulant networks with three connections per node, there are more possibilities for the choice of connection directions. We only show one possibility in Figure 4(d), where all connections point in the same direction. We refer to this connection topology as Network D.

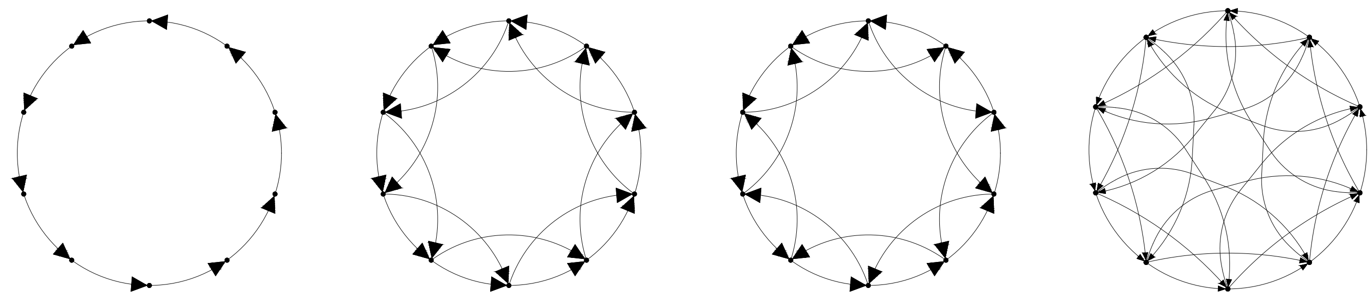
The Laplacian eigenvalues of network types B, C and D no longer fall on the shifted unit circle (16), as those of Network A did. Nevertheless, the locus of eigenvalues still comes close to the circle near the origin, as shown in Figure 4(f)–(h). Hence, we have a similar conclusion: up to a certain network size, these networks have a delay margin of 1, and for larger sizes the delay tolerance of the networks decreases. This critical network size, for which the delay margin equals 1, is smallest for Network A and largest for Network C, as confirmed by the horizontal segments of the curves in Figure 5. Interestingly, Network C has a much higher delay tolerance than B, although all nodes in both networks have the same in-degrees.

We also test the accuracy of the curves in Figure 5 by direct simulation of the system (6). We pick $\tau = 0.9$ and choose network sizes close to the curves in Figure 5. Starting from random initial conditions, the time evolution of (6) is depicted in Figure 6. The simulations confirm that there exists a critical network size, whose value agrees with the information in Figure 5, below which the systems reaches consensus and above which it diverges.

As a final step in our investigation, we consider random re-wiring of directed circulant networks, in a well-known process that produces small-world networks starting from regular ones (Watts and Strogatz, 1998). In a directed version of this re-wiring algorithm, we start with a network of type B or C, and at each step we randomly pick one node and reconnect one of existing outgoing links to a different, randomly-selected node. Figure 7 shows that the delay margin improves with re-wiring (although the effect is not always monotone, due to the randomness of the process and the relative sparsity of the network). This shows from yet another viewpoint that directed circulant networks, especially at large sizes, have poor delay tolerance, which can be improved significantly by a few re-wiring of links.

ACKNOWLEDGEMENTS

The authors thank the ZiF (Center for Interdisciplinary Research) of Bielefeld University for their support within the cooperation program *Discrete and Continuous Models in the Theory of Networks*, where this problem was initiated. Part of the research was carried out when DI visited the Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany, in Summer 2015 under financial support from DAAD Short-Term Grants 2015 (57130097) and from L2S Laboratory, France. FMA acknowledges support from the European Union's Seventh Framework Programme (FP7/2007–2013) under grant agreement no. 318723 (MATHEMACS).

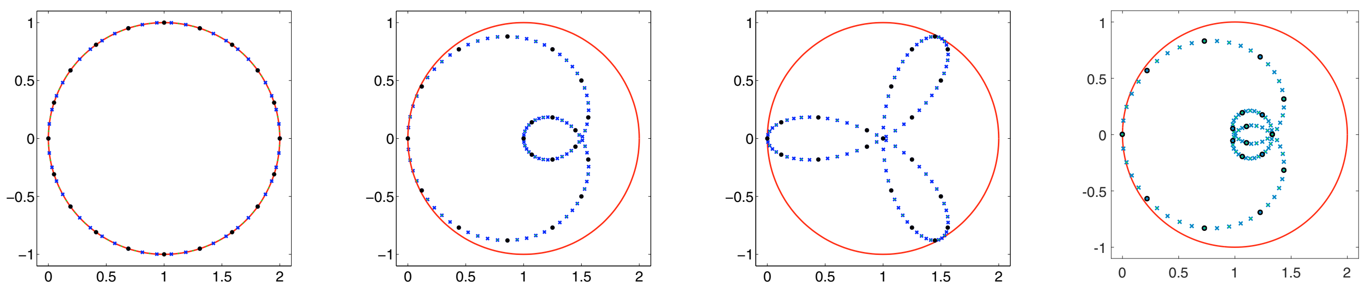


(a) Circulant network A: every node is connected to the next one counterclockwise.

(b) Circulant network B: every node is connected to the next two nodes counterclockwise.

(c) Circulant network C: similar to B, except the outer connection direction is counterclockwise and the inner direction is clockwise.

(d) Circular network D: every node is connected to the next three nodes counterclockwise.



(e) Laplacian eigenvalues of circulant networks of type A.

(f) Laplacian eigenvalues of circulant networks of type B.

(g) Laplacian eigenvalues of circulant networks of type C.

(h) Laplacian eigenvalues of circulant networks of type D.

Fig. 4. Example of directed circulant networks (upper row) and their Laplacian eigenvalue patterns (lower row). The eigenvalues are plotted in the complex plane for the network sizes $n = 20$ (black dots), and $n = 100$ (blue crosses); the red circle depicts the shifted unit circle (16) in the complex plane.

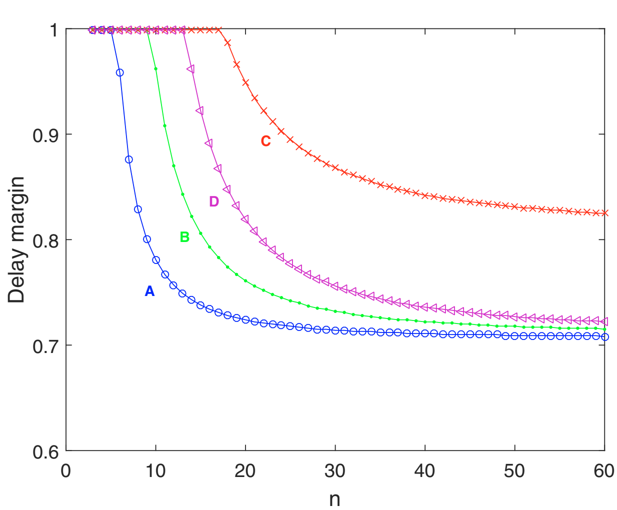


Fig. 5. Delay margin as a function of the network size n for various types of circulant networks: type A (blue circles), type B (green points), type C (red crosses), and type D (purple triangles).

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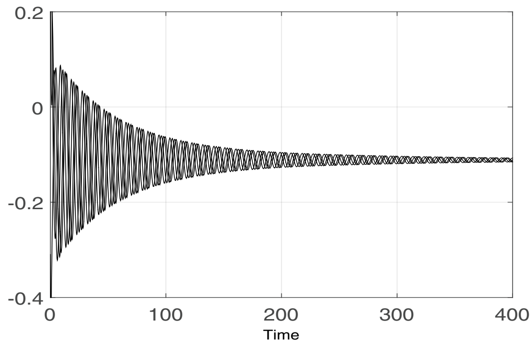
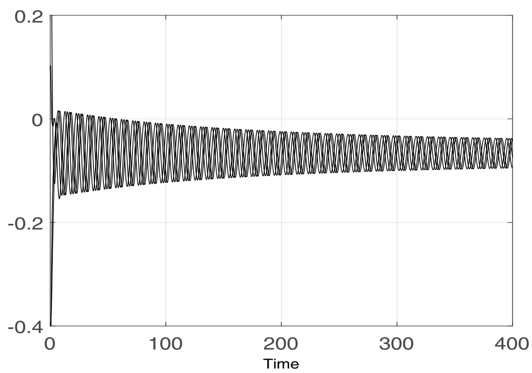
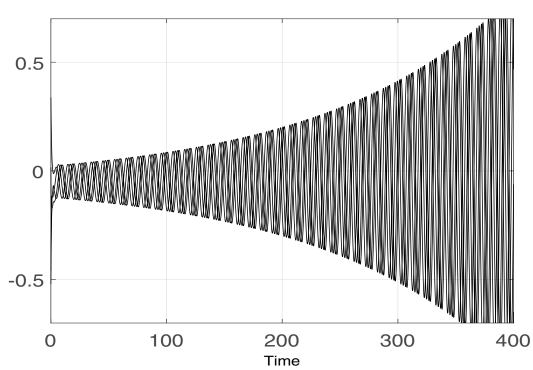


(a) $n = 23$

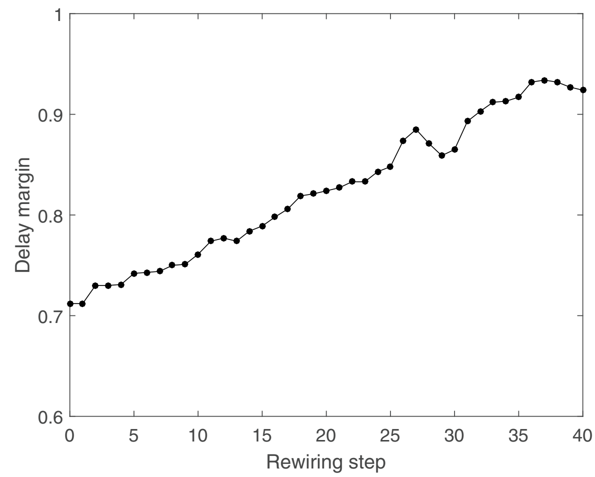


(b) $n = 24$

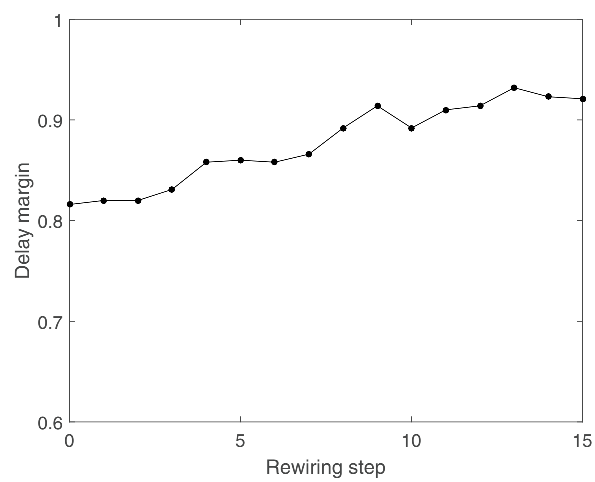


(c) $n = 25$

Fig. 6. Time evolution of system (6) for directed circulant networks of type C of various sizes n. At a chosen delay value $\tau = 0.9$, the critical network size for consensus equals 24, as read off from Figure 5. For smaller sizes Network C reaches consensus, while for larger sizes it diverges. The trajectories of only 5 nodes are shown for clarity.



(a) Network of type B with $n = 100$.



(b) Network of type C with $n = 100$.

Fig. 7. Improving the delay margin of circulant networks by random rewiring of the links.

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