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On Cohen–Macaulayness and depth of ideals in invariant rings

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ABSTRACT

We investigate the presence of Cohen–Macaulay ideals in invariant rings and show that an ideal of an invariant ring corresponding to a modular representation of a p-group is not Cohen–Macaulay unless the invariant ring itself is. As an intermediate result, we obtain that non-Cohen–Macaulay factorial rings cannot contain Cohen–Macaulay ideals. For modular cyclic groups of prime order, we show that the quotient of the invariant ring modulo the transfer ideal is always Cohen–Macaulay, extending a result of Fleischmann.

1. Introduction

The depth and Cohen–Macaulay property of invariant rings have always been among the major interests of invariant theorists, see the references below. In this paper, we consider ideals of invariant rings (as modules over the latter), and investigate their depth and Cohen–Macaulayness. The original goal of this paper was to find filtrations of the invariant rings with Cohen–Macaulay quotients (a weakening of being "sequentially Cohen–Macaulay" as introduced in [18, Section III.2]). However, the results of this paper show that in many cases, invariant rings fail to contain any Cohen–Macaulay ideal, so the goal is missed in the first step already. Before we go into more details, we fix our setup. Let V be a finite dimensional representation of a group G over a field K. The representation is called modular if the characteristic of K divides the order of G. Otherwise, it is called nonmodular. There is an induced action on the symmetric algebra $K[V] := S(V^*)$ given by $\sigma(f) = f \circ \sigma^{-1}$ for $\sigma \in G$ and $f \in K[V]$. We let

$$K[V]^G := \{f \in K[V] \mid \sigma(f) = f \ \text{ for all } \sigma \in G\}$$

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denote the subalgebra of invariant polynomials in $K[V]$. For any nonmodular representation, $K[V]^G$ is a Cohen–Macaulay ring [12]. In the modular case, on the other hand, $K[V]^G$ almost always fails to be Cohen–Macaulay, see [13]. The depth of $K[V]^G$ has attracted much attention and has been determined for various families of representations, see for example [4,9,11,14,17]. In this paper we consider ideals of $K[V]^G$ as modules over $K[V]^G$. We show that, if V is a modular representation of a p-group, then $K[V]^G$ does not contain a Cohen–Macaulay ideal unless $K[V]^G$ is Cohen–Macaulay itself. Combining this with a result of Broer allows us to show the equivalence of the transfer ideal being Cohen–Macaulay or principal and the invariant ring being a direct summand of the polynomial ring for these groups, see Corollary 10.

However, our results hold in a broader generality. We first show that a Cohen–Macaulay ideal in an affine domain can not have height bigger than one. If, in addition, the affine domain is factorial, then only principal ideals can be Cohen–Macaulay. So we get the desired implication for the groups and their representations whose invariants are factorial. We also include an example that shows that the condition that the affine domain is factorial can not be dropped. We then restrict to modular representations of a cyclic group of prime order. Our main result here is that the quotient $K[V]^G/I^G$ of the invariant ring modulo the transfer ideal I^G is Cohen–Macaulay. Note that this extends results of Fleischmann [10] in this case, namely that $K[V]^G/\sqrt{I^G}$ is Cohen–Macaulay, and that $\sqrt{I^G} = I^G$ if V is projective. This also allows us to compute the depth of the transfer ideal. We end with a reduction result that reduces computing the depth of a $K[V]^G$-module to computing a grade of the transfer ideal.

We refer the reader to [1,5,6] for more background in modular invariant theory.

2. Preliminaries

In this section we summarize our notation as well as some basic results that we use in our computations. Let $R = \bigoplus_{d=0}^{\infty} R_d$ be a graded affine K-algebra such that $R_0 = K$, and $M = \bigoplus_{d=0}^{\infty} M_d$ a finitely generated graded nonzero R-module. We call $R_+ := \bigoplus_{d=1}^{\infty} R_d$ the maximal homogeneous ideal of R. A sequence of homogeneous elements $a_1, \dots, a_k \in R_+$ is called M-regular if each a_i is a nonzero divisor on $M/(a_1, \dots, a_{i-1})M$ for $i = 1, \dots, k$. For a homogeneous ideal $I \subseteq R_+$, the maximal length of an M-regular sequence lying in I is called the grade of I on M, denoted by $\operatorname{grade}(I, M)$. Furthermore, one calls $\operatorname{depth}(M) := \operatorname{grade}(R_+, M)$ the depth of M. Recall that we have $\operatorname{depth}(M) \le \dim(M)$ (where $\dim(M) := \dim(R/\operatorname{Ann}_R(M))$ denotes the Krull dimension), and M is called *Cohen–Macaulay* if equality holds.

By Noether-Normalization, R contains a homogeneous system of parameters (h.s.o.p.), i.e., algebraically independent homogeneous elements $a_1, \dots, a_n \in R$ such that R is finitely generated as a module over the (polynomial) subalgebra $A := K[a_1, \dots, a_n]$. Note that $n = \dim(R)$ is uniquely determined. Any subset of an h.s.o.p. is called a partial h.s.o.p. (p.h.s.o.p.). If R is also a domain, then homogeneous elements $a_1, \dots, a_k \in R_+$ form a p.h.s.o.p. if and only if $\operatorname{height}(a_1, \dots, a_k) = k$, (see [13, Lemma 1.5], [3, Theorem A.16]). Note that M is also an A-module, and from the graded Auslander–Buchsbaum formula [7, Exercise 19.8] we get that M is free as an A-module if and only if its depth as an A-module is equal to $\dim(A)$. But since the depths of M as an A- and as an R-module are equal (see [6, Lemma 3.7.2] or [3, Exercise 1.2.26]), this is also equivalent to the condition that the depth of M as an R-module is $\dim(R) = \dim(A)$. In other words, M is free as an A-module if and only if M is Cohen–Macaulay and $\dim(M) = \dim(R)$, i.e., M is *maximal Cohen–Macaulay.*

We include the following standard facts about depth for the reader's convenience.

Lemma 1. *(See [3, Proposition 1.2.9].) Assume that I is a homogeneous nonzero proper ideal of the graded affine ring R. Then we have the following inequalities.*

(a) $\operatorname{depth}(R) \ge \min\{\operatorname{depth}(I), \operatorname{depth}(R/I)\}$.
(b) $\operatorname{depth}(I) \ge \min\{\operatorname{depth}(R), \operatorname{depth}(R/I) + 1\}$.
(c) $\operatorname{depth}(R/I) \ge \min\{\operatorname{depth}(I) - 1, \operatorname{depth}(R)\}$.

This lemma implies that $\operatorname{depth}(I)$ and $\operatorname{depth}(R/I)$ are often strongly related:

Lemma 2. *Assume that I is a homogeneous nonzero proper ideal of the graded affine ring R.*

(a) *If one of the following conditions*
  (i) $\operatorname{depth}(R) > \operatorname{depth}(I)$,
  (ii) $\operatorname{depth}(R) > \operatorname{depth}(R/I)$,
  (iii) *R is a Cohen–Macaulay domain*

holds, then

$$\operatorname{depth}(I) = \operatorname{depth}(R/I) + 1.$$

(b) *If* $\operatorname{depth}(I) > \operatorname{depth}(R)$, *then* $\operatorname{depth}(R/I) = \operatorname{depth}(R)$.
(c) *If* $\operatorname{depth}(R/I) > \operatorname{depth}(R)$, *then* $\operatorname{depth}(I) = \operatorname{depth}(R)$.

Proof. (a) Assume first $\operatorname{depth}(R) > \operatorname{depth}(I)$. From Lemma 1 (b) it follows that $\operatorname{depth}(I) \geq \operatorname{depth}(R/I)+1$, and from Lemma 1 (c) we get $\operatorname{depth}(R/I) \geq \operatorname{depth}(I) - 1$, implying the desired equality. Secondly, assume $\operatorname{depth}(R) > \operatorname{depth}(R/I)$. From Lemma 1 (b) it follows that $\operatorname{depth}(I) \geq \operatorname{depth}(R/I) + 1$, and from Lemma 1 (c) we have $\operatorname{depth}(R/I) \geq \operatorname{depth}(I) - 1$ so we obtain the result again. Finally assume R is a Cohen–Macaulay domain. As R is a domain and $I \neq \{0\}$, it follows that $\dim(R/I) < \dim(R)$. Hence we have the inequality $\operatorname{depth}(R/I) \leq \dim(R/I) < \dim(R) = \operatorname{depth}(R)$, so the assertion follows from (ii).

Statement (b) follows similarly from Lemma 1 (a) and (c). Statement (c) follows from Lemma 1 (a) and (b). □

For example, if R has positive depth, then the homogeneous maximal ideal always has depth one by the above lemma, as its quotient is zero-dimensional. Now we note that for any given number $1 \leq k \leq \operatorname{depth}(R)$, there exists an ideal of depth k:

Lemma 3. *Assume that the homogeneous elements $a_1, \ldots, a_k$ of positive degree form a regular sequence of R. Then*

$$\operatorname{depth}((a_1, \ldots, a_k)R) = \operatorname{depth}(R) + 1 - k.$$

Proof. We have $\operatorname{depth}(R/(a_1, \ldots, a_k)R) = \operatorname{depth}(R) - k < \operatorname{depth}(R)$, and the result follows from the previous lemma. □

3. Cohen–Macaulay ideals in affine domains

The main result of this section is Theorem 5 where it is shown that only principal ideals can be Cohen–Macaulay in factorial affine domains. Nevertheless, even when the affine domain is not factorial, the height of a Cohen–Macaulay ideal can be at most one.

Lemma 4. *Assume that R is a graded affine domain, and $I \neq R$ a homogeneous ideal of height at least 2. Then I is not Cohen–Macaulay as an R-module.*

Proof. As I is homogeneous of height at least two, it contains a p.h.s.o.p. p, q of R. We extend this p.h.s.o.p. to an h.s.o.p. $h_1, \ldots, h_n$ with $h_1 = p$, $h_2 = q$ and consider the K-subalgebra A of R generated by this h.s.o.p., i.e., $A = K[h_1 \ldots, h_n]$. Then A is isomorphic to a polynomial ring over K in $\dim(R)$ variables. Assume by way of contradiction that I is Cohen–Macaulay as an R-module. Then I is free as an A-module, i.e., there exist elements $g_1, \dots, g_m \in I$ such that $I = \bigoplus_{i=1}^m Ag_i$. As p, q are elements of I, we can find unique elements $a_i, b_i \in A$ for $i = 1, \dots, m$ such that $p = \sum_{i=1}^m a_i g_i$ and $q = \sum_{i=1}^m b_i g_i$. Multiplying both equations with q and p respectively, we get $\sum_{i=1}^m (qa_i)g_i = pq = \sum_{i=1}^m (pb_i)g_i$. As $qa_i, pb_i \in A$ and $g_1, \dots, g_m$ is a free A-basis of I, we get $qa_i = pb_i$ for all $i = 1, \dots, m$. As p, q are different variables in the polynomial ring A, it follows $p|a_i$ and $q|b_i$ for all i. Therefore there exist $b_i' \in A$ such that $b_i = qb_i'$ for $i = 1, \dots, m$. Hence we get $q = \sum_{i=1}^m b_i g_i = \sum_{i=1}^m qb_i' g_i$, and as we are in a domain dividing by q yields $1 = \sum_{i=1}^m b_i' g_i \in \bigoplus_{i=1}^m Ag_i = I$. This implies $I = R$, contradicting the hypothesis $I \neq R$ of the lemma. □

Theorem 5. *Assume that R is a graded factorial affine domain. If $I \neq R$ is a homogeneous ideal which is not principal, then I is not Cohen–Macaulay as an R-module. Therefore, if R is not Cohen–Macaulay, then R does not contain any nonzero homogeneous Cohen–Macaulay ideal (as an R-module).*

Proof. Assume by way of contradiction that I is Cohen–Macaulay and not principal. Let $a_1, \dots, a_n$ denote a finite set of generators of I. As we are in a factorial ring, we can consider the greatest common divisor d of those elements. Then we have $I = (a_1, \dots, a_n) \subsetneq (d)$, where the inclusion is strict as I is not a principal ideal by assumption. As R is a domain and $d \mid a_i$ for all i, the elements $\frac{a_i}{d} \in R$ are well defined, and we can consider the ideal $J := (\frac{a_1}{d}, \dots, \frac{a_n}{d}) = \frac{1}{d}I$. Note that from $I \subsetneq (d)$ it follows that $J \subsetneq (1) = R$, so J is a proper ideal of R. Multiplication by d yields an R-module isomorphism from J to I, and therefore J is also Cohen–Macaulay as an R-module. From Lemma 4 it follows that the height of J is at most 1. But R is a domain and $J \neq 0$, so the height of J is 1. It follows that there exists a prime ideal $\wp$ of R of height one such that $I \subseteq \wp$. As R is factorial, height one primes are principal, and so $\wp$ is generated by a prime element p, so we have $J \subseteq \wp = (p)$, which implies that p is a common divisor of $\frac{a_1}{d}, \dots, \frac{a_n}{d}$. This is a contradiction, as d is the greatest common divisor of $a_1, \dots, a_n$.

Now the second assertion of the theorem follows from the first and the fact that principal ideals of R are isomorphic to R as R-modules. □

We demonstrate two examples of affine domains with non-principal Cohen–Macaulay ideals. First one is a Cohen–Macaulay ring, the second one is not. Therefore, a non-Cohen–Macaulay ring may contain a Cohen–Macaulay ideal, and the hypothesis of R being factorial can not be dropped out in the previous theorem.

Example 6. Consider the subalgebra $R = K[x^2, y^2, xy]$ of the polynomial ring $K[x, y]$ in two variables. Note that R is not factorial as the equality $x^2 \cdot y^2 = (xy) \cdot (xy)$ shows. We claim that the ideal $I = (x^2, xy)$ of R is Cohen–Macaulay and not principal. Clearly I is not principal, because R is a graded ring that starts in degree 2. We now consider the h.s.o.p. x^2, y^2 of R and the subalgebra $A = K[x^2, y^2]$ generated by the h.s.o.p.. We claim that we have the direct sum decompositions $R = A \oplus Axy$ and $I = Ax^2 \oplus Axy$. In both cases, the sum is direct because in the first summands all x degrees are even, while in the second summands all x degrees are odd. As both sums contain the respective ideal generators, it only remains to show that both sums are invariant under multiplication with xy. For the sum for R, this follows from $xy \cdot xy = x^2y^2 \in A$. For the sum for I, we have $xy \cdot x^2 = x^2 \cdot xy \in Axy$ and $xy \cdot xy = y^2 \cdot x^2 \in Ax^2$. Therefore R and I are free A-modules, so R and I are Cohen–Macaulay as R-modules. Also note that $I \subseteq \sqrt{(x^2)}$, as $(xy)^2 = y^2 \cdot x^2 \in (x^2)$. Thus, $\text{height}(I) \leq \text{height}((x^2)) = 1$, as predicted by Lemma 4.

We state the following example as a proposition.

Proposition 7. *Consider the subalgebra $R := K[x^4, x^3y, xy^3, y^4]$ of the polynomial ring $K[x, y]$. Then the ideal $I := (x^4, x^3y)$ of R is of height one and Cohen–Macaulay as an R-module. (While R is not Cohen–Macaulay and not factorial.)*

Proof. First note that R is well known to be non-Cohen–Macaulay, see [6, Example 2.5.4], and as $x^4 \cdot y^4 = (x^3y) \cdot (xy^3)$, R is also not factorial. Also since $(x^3y)^4 = y^4x^8 \cdot x^4 \in (x^4)$ we have $I \subseteq \sqrt{(x^4)}$, which shows that the height of I is 1. We consider the subalgebra $A := K[x^4, y^4]$ of R generated by an h.s.o.p., and we will show that I is a free A-module, which implies that I is Cohen–Macaulay as an R-module. We set

$$a := x^4, \quad p := x^3y, \quad q := xy^3, \quad b := y^4$$

and claim that

$$I = (a, p) = Aa \oplus Aaq \oplus Ap \oplus Ap^2.$$

The inclusion "$\supseteq$" is clear. We first show that the sum on the right hand side is indeed direct. Let ϵ denote the map from the set of monomials of $K[x, y]$ to $\mathbb{N}_0^2$ given by $\epsilon(x^iy^j) := (i, j)$. We compute the epsilon values of the A-module generators modulo 4:

$$\epsilon(a) = (4, 0), \quad \epsilon(aq) = (5, 3) \equiv (1, 3), \quad \epsilon(p) = (3, 1), \quad \epsilon(p^2) = (6, 2) \equiv (2, 2).$$

As $\epsilon(m) \equiv (0, 0)$ for any monomial m in A, it follows that the ϵ-values of monomials in Aa, Aaq, Ap, Ap^2 fall into different congruence classes modulo 4, so the sum is indeed direct.

We now verify the inclusion "$\subseteq$". Clearly, $a, p \in S := Aa \oplus Aaq \oplus Ap \oplus Ap^2$, and S is closed under multiplication with a and b. It remains to show that S is closed under multiplication with p and q, which follows from

$$\begin{aligned} p(Aa) &= Aap \subseteq Ap, & q(Aa) &= Aaq, \\ p(Aaq) &= Aax^4y^4 \subseteq Aa, & q(Aaq) &= Ax^6y^6 = Abp^2 \subseteq Ap^2, \\ p(Ap) &= Ap^2, & q(Ap) &= Apq = Ax^4y^4 \subseteq Aa, \\ p(Ap^2) &= Ax^9y^3 = Aa^2q \subseteq Aaq, & q(Ap^2) &= Ax^7y^5 = Ax^4y^4p \subseteq Ap. \qquad \square \end{aligned}$$

Remark 8. We learned from Roger Wiegand that there are theorems that say that, for some special classes of rings, non-free maximal Cohen–Macaulay modules have high ranks. Since the rank of an ideal in a domain is one, and non-principal ideals are non-free, Lemma 4 and Theorem 5 readily follow for such rings whose non-free maximal Cohen–Macaulay modules are known to have a high rank. But we can not expect that a non-free maximal Cohen–Macaulay module will always have rank > 1 as the previous two examples demonstrate.

We note two applications of Theorem 5 to modular invariant rings.

Corollary 9. *Assume that K is of positive characteristic p and G is a finite p-group. For any finite dimensional linear representation V of G over K such that the invariant ring $K[V]^G$ is not Cohen–Macaulay, no nonzero homogeneous ideal of $K[V]^G$ is Cohen–Macaulay (as a $K[V]^G$-module).*

Proof. It is well known that $K[V]^G$ is factorial, see for instance [5, Theorem 3.8.1]. The claim now follows from Theorem 5. $\square$

The transfer ideal I^G is defined as the image of the transfer map Tr, i.e.

$$I^G = \mathrm{Tr}(K[V]), \quad \text{with } \mathrm{Tr} : K[V] \to K[V]^G, \qquad f \mapsto \mathrm{Tr}(f) = \sum_{\sigma \in G} \sigma(f).$$

Corollary 10. *Assume that K is of positive characteristic p and G is a finite p-group. Then the following are equivalent.*

(1) *$K[V]^G$ is a direct summand of $K[V]$ as a graded $K[V]^G$-module.*
(2) *I^G is a principal ideal of $K[V]^G$.*
(3) *I^G is Cohen–Macaulay.*

Proof. The equivalence of the first two statements is established in [2, Corollary 4]. Assume now that one and hence both of them hold. It is well known that from (1) it follows that $K[V]^G$ is Cohen–Macaulay. The ideal I^G is principal by (2), hence I^G is also Cohen–Macaulay. Conversely, assume that I^G is Cohen–Macaulay. Then since $K[V]^G$ is factorial, Theorem 5 applies and so I^G is principal. □

4. Depth of ideals and quotient of the transfer in invariant rings for a cyclic group of prime order

In this section we specialize to a cyclic group G of prime order p equal to the characteristic of the field K, which we assume to be algebraically closed. Fix a generator σ of G. There are exactly p indecomposable G-modules $V_1, \dots, V_p$ over K and each indecomposable module V_i is afforded by a Jordan block of dimension i with 1's on the diagonal. Let V be an arbitrary G-module over K. Assume that V has l summands and so we can write $V = \sum_{1\leq j\leq l} V_{n_j}$. Notice that $l = \dim V^G$. We also assume that none of these summands is trivial, i.e., $n_j > 1$ for $1 \leq j \leq l$. We set $K[V] = K[x_{i,j} \mid 1 \leq i \leq n_j,\ 1 \leq j \leq l]$ and the action of σ is given by $\sigma(x_{i,j}) = x_{i,j} + x_{i-1,j}$ for $1 < i \leq n_j$ and $\sigma(x_{1,j}) = x_{1,j}$. We define the norm

$$N(f) := \prod_{\tau\in G} \tau(f) \qquad \text{for all } f \in K[V].$$

Notice that for $1 \leq i \leq n_j$, $N(x_{i,j})$ is monic of degree p as a polynomial in $x_{i,j}$. For simplicity we set $N_j := N(x_{n_j,j})$ for $1 \leq j \leq l$. By a famous theorem of Ellingsrud and Skjelbred [8], $\operatorname{depth}(K[V]^G) = \min\{\dim_K(V^G) + 2, \dim_K(V)\} = \min\{l + 2, \dim_K(V)\}$. In [4], this result is extended to some other classes of groups, and the proof is also made more elementary and explicit. Restricting the results of [4] to our case, we get the following description of a maximal $K[V]^G$-regular sequence, which allows to explicitly construct an ideal of a given depth at most that of the invariant ring.

Proposition 11. *A maximal $K[V]^G$-regular sequence is given by*

$$\begin{aligned} &x_{1,1}, x_{1,2}, N_1, \dots, N_l && \text{if } l > 1;\\ &x_{1,1}, N(x_{2,1}), N_1 && \text{if } l = 1,\ n_1 > 2;\\ &x_{1,1}, N_1 && \text{if } l = 1,\ n_1 = 2. \end{aligned}$$

Let I_k denote the ideal of $K[V]^G$ generated by the first k elements of the sequence. Then we have $\operatorname{depth} I_k = \operatorname{depth}(K[V])^G + 1 - k$ *for* $1 \leq k \leq \operatorname{depth}(K[V]^G)$.

Proof. Let b denote the second element of the sequence, i.e., $x_{1,2}$ or $N(x_{2,1}) = x_{2,1}^p - x_{2,1}x_{1,1}^{p-1}$. As $x_{1,1}$ and b are coprime in $K[V]$, and both are invariant, they form a regular sequence in $K[V]^G$. Proceeding by induction, we assume that the elements $x_{1,1}, b, N_1, \dots, N_{k-1}$ form a regular sequence for some $k < l$. Consider the standard basis vector $e_{n_k,k} \in V$ corresponding to the variable $x_{n_k,k}$. Then $e_{n_k,k}$ is a fixed point, and $U := Ke_{n_k,k}$ is a 1-dimensional submodule of V. Since no element of the regular sequence $x_{1,1}, b, N_1, \dots, N_{k-1}$ contains the variable $x_{n_k,k}$, [4, Corollary 17] applies to U and $x_{n_k,k}$, so the regular sequence can be extended by the element N_k. Since the length of the given sequence equals $\operatorname{depth}(K[V]^G)$ in each case, we are done. The final statement now follows from Lemma 3. □

The transfer ideal often plays an important role in computing the invariant ring and its various aspects have been subject to research. The vanishing set of I^G equals the fixed point space V^G (see [5, Theorem 9.0.10]), in particular we have $\dim(K[V]^G/I^G) = \dim(V^G) = l$. We will show that $K[V]^G/I^G$ is Cohen–Macaulay, which also allows us to compute the depth of the transfer ideal. To do this we prove that $N_1, \dots, N_l$ is a $K[V]^G/I^G$-regular sequence. Let $f \in K[V]$ and $1 \le j_1 < j_2 < \cdots < j_t \le l$ be arbitrary. We denote the degree of f as a polynomial in $x_{n_j,j}$ by $\deg_j f$. Since N_{j_1} is a monic polynomial of degree p in $x_{n_{j_1},j_1}$, we can write $f = q_1 N_{j_1} + r_1$, where $\deg_{j_1} r_1 < p$. Next we divide r_1 by N_{j_2} and we get a decomposition $f = q_1 N_{j_1} + q_2 N_{j_2} + r_2$, where $\deg_{j_1} r_2, \deg_{j_2} r_2 < p$ and $\deg_{j_1} q_2 < p$. In this way we get a decomposition

$$f = q_1 N_{j_1} + \cdots + q_t N_{j_t} + r,$$

where $\deg_{j_i} r < p$ for $1 \le i \le t$ and $\deg_{j_i} q_{i'} < p$ for $i < i'$. This is called the norm decomposition and r is called the remainder of f with respect to $N_{j_1}, \dots, N_{j_t}$. Notice that r is unique. If $f \in K[V]^G$ is an invariant, then the quotients q_i for $1 \le i \le t$ and the remainder r are also invariant, see [16, Proposition 2.1]. We denote the coset of an element $f \in K[V]^G$ in $K[V]^G/I^G$ by $\overline{f}$.

Theorem 12. *The algebra $K[V]^G/I^G$ is Cohen–Macaulay, and an h.s.o.p. is given by the set $\{\overline{N_j} \mid 1 \le j \le l\}$. In particular, we have* $\operatorname{depth}(I^G) = l + 1$.

Proof. We show that $\overline{N_1}, \dots, \overline{N_l}$ forms a regular sequence for $K[V]^G/I^G$. As its length l equals the dimension of $K[V]^G/I^G$, it follows that this ring is Cohen–Macaulay. First, we show that $\overline{N_i}$ is a $K[V]^G/I^G$-regular element for $1 \le i \le l$. Assume $fN_i \in I^G$ for some invariant f. Then $fN_i = \operatorname{Tr}(g)$ for some $g \in K[V]$. Consider the norm decomposition $g = qN_i + r$ of g with respect to N_i. We have

$$fN_i = \operatorname{Tr}(qN_i + r) = \operatorname{Tr}(q)N_i + \operatorname{Tr}(r).$$

Hence,

$$0 = (f - \operatorname{Tr}(q))N_i + \operatorname{Tr}(r).$$

Note that the group action preserves the $x_{n_i,i}$-degree, so we have

$$\deg_i \operatorname{Tr}(r) \le \deg_i r < p = \deg_i N_i.$$

So, we get that $f - \operatorname{Tr}(q) = 0$ and $\operatorname{Tr}(r) = 0$. Therefore $f \in I^G$, and $\overline{N_i}$ is a $K[V]^G/I^G$-regular element. Assume now by induction that $\overline{N_1}, \dots, \overline{N_{j-1}}$ is a $K[V]^G/I^G$-regular sequence, and we have

$$fN_j = f_1 N_1 + \cdots + f_{j-1} N_{j-1} + \operatorname{Tr}(t), \tag{1}$$

where $f, f_i \in K[V]^G$ for $1 \le i \le j-1$ and $t \in K[V]$. Consider the norm decompositions of f and t with respect to $N_1, \dots, N_{j-1}$. Since the quotients and the remainder in the decomposition of f are invariants, we can replace f by its remainder. As for $\operatorname{Tr}(t)$, notice that $\operatorname{Tr}(t)$ and the transfer of the remainder of t differ by a $K[V]^G$-linear combination of $N_1, \dots, N_{j-1}$. Therefore, we can replace $\operatorname{Tr}(t)$ with the transfer of the remainder of t. Moreover, by considering the norm decomposition of f_i with respect to $N_1, \dots, N_{i-1}$ for $1 \le i \le j-1$, we can replace f_i with its corresponding remainder. Therefore, we may assume that $\deg_{i'} f_i < p$ for $1 \le i' < i$ and $1 \le i \le j-1$. Notice also that the degree of f and $\operatorname{Tr}(t)$ with respect to any variable $x_{n_{i'},i'}$ is $< p$ for $1 \le i' \le j-1$. Now, considering Equation (1) as a polynomial equation in the variable $x_{n_1,1}$ gives that $f_1 = 0$. Then comparing the coefficients of $x_{n_2,2}$ gives $f_2 = 0$. In the same way we get $f_1 = f_2 = \cdots = f_{j-1} = 0$. So, Equation (1) becomes $fN_j = \mathrm{Tr}(t)$. However, since $\overline{N_j}$ is a $K[V]^G/I^G$-regular element, we have $f \in I^G$ as desired. This shows that $\overline{N_1}, \ldots, \overline{N_l}$ is a regular sequence. From $\mathrm{depth}(K[V]^G/I^G) = l < \mathrm{depth}(K[V]^G) = \min\{l+2, \dim_K(V)\}$ (we assume a non-trivial action) and Lemma 2, it now follows that $\mathrm{depth}(I^G) = l+1$. □

We also prove a reduction result for the depth of a module over the invariant ring, which is based on the following lemma. The statement is probably folklore, but for the convenience of the reader and the lack of a reference, we provide a proof.

Lemma 13. *Assume that R is a graded affine ring and M is a finitely generated graded nonzero R-module. If $h_1, \ldots, h_r \in R_+$ form a homogeneous M-regular sequence and I is a homogeneous ideal of R such that $\sqrt{I + (h_1, \ldots, h_r)R} = R_+$, then*

$$\mathrm{depth}(M) = \mathrm{grade}(I, M/(h_1, \ldots, h_r)M) + r.$$

Proof. As the homogeneous elements $h_1, \ldots, h_r \in R_+$ form an M-regular sequence, we have that $\mathrm{depth}(M) = \mathrm{depth}(M/(h_1, \ldots, h_r)M) + r$. We show that

$$\mathrm{grade}(I, M/(h_1, \ldots, h_r)M) \geq \mathrm{grade}(R_+, M/(h_1, \ldots, h_r)M),$$

as the reverse inequality is obvious. Let $f_1, \ldots, f_d \in R_+$ be a maximal homogeneous $M/(h_1, \ldots, h_r)M$-regular sequence. Since taking powers does not change the property of being a regular sequence, we can assume that all elements in the sequence are contained in $I + (h_1, \ldots, h_r)R$. Therefore for $1 \leq i \leq d$ we can write $f_i = g_i + b_i$ with homogeneous elements $g_i \in I$ and $b_i \in (h_1, \ldots, h_r)R$. Since b_i is in the annihilator of

$$M/(h_1, \ldots, h_r, f_1, \ldots, f_{i-1})M = M/(h_1, \ldots, h_r, g_1, \ldots, g_{i-1})M$$

it follows that g_i is regular on $M/(h_1, \ldots, h_r, g_1, \ldots, g_{i-1})M$ as well. Hence the elements $g_1, \ldots, g_d$ of I form an $M/(h_1, \ldots, h_r)M$-regular sequence. □

We recall that for any ideal I of the invariant ring $K[V]^G$, we have $\sqrt{I} = \sqrt{IK[V]} \cap K[V]^G$. This holds generally when G is a reductive group [15, Lemma 3.4.2], and an elementary proof for finite groups can be found in [5, Lemma 12.1.1].

Proposition 14. *Let M be a finitely generated graded $K[V]^G$-module on which the norms $N_1, \ldots, N_l$ form an M-regular sequence. Then*

$$\mathrm{depth}(M) = \mathrm{grade}(I^G, M/(N_1, \ldots, N_l)M) + l.$$

Proof. We have already mentioned that the zero set of I^G is given by $V^G = \bigoplus_{i=1}^{l} Ke_{n_i,i}$. As for an element $v = \sum_{i=1}^{l} \lambda_i e_{n_i,i} \in V^G$ with $\lambda_i \in K$, we have $N_i(v) = \lambda_i^p$, the common zero set of $I^G + (N_1, \ldots, N_l)$ is zero, hence by the Nullstellensatz $\sqrt{(I^G + (N_1, \ldots, N_l))K[V]} = K[V]_+$. From the paragraph before the proposition we obtain that the radical ideal of $I^G + (N_1, \ldots, N_l)$ equals $K[V]^G_+$, and the lemma above applies. □

Examples where the proposition applies include the case $l = 1$ and $M = I$ a nonzero homogeneous ideal of $K[V]^G$. The corollary also applies for arbitrary l and $M = K[V]^G$ by Proposition 11. In the "non-trivial" cases where $\mathrm{depth}(K[V]^G) = l+2$, it follows from $\mathrm{depth}(K[V]^G) = \mathrm{grade}(I^G, K[V]^G/(N_1, \ldots, N_l)) + l$, that

$$\mathrm{grade}(I^G, K[V]^G/(N_1, \ldots, N_l)) = 2.$$

Therefore, there is a maximal $K[V]^G$-regular sequence consisting of the l norms and two transfers. Also compare with the known fact that $\mathrm{grade}(I^G, K[V]^G) = 2$ in these cases, see [4, Propositions 20 and 22]. As $\mathrm{depth}(K[V]^G) = l + 2$, this also shows that $\mathrm{depth}(M) \neq \mathrm{grade}(I^G, M)$ in general.

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