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Strongly Clean Matrices Over Power Series

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Abstract. An $n \times n$ matrix A over a commutative ring is strongly clean provided that it can be written as the sum of an idempotent matrix and an invertible matrix that commute. Let R be an arbitrary commutative ring, and let $A(x) \in M_n\big(R[[x]]\big)$. We prove, in this note, that $A(x) \in M_n\big(R[[x]]\big)$ is strongly clean if and only if $A(0) \in M_n(R)$ is strongly clean. Strongly clean matrices over quotient rings of power series are also determined.

1. Introduction

An $n \times n$ matrix over a commutative ring is strongly clean provided that it can be written as the sum of an idempotent matrix and an invertible matrix. It is attractive to determine when a matrix over a commutative ring is strongly clean. In [8, Example 1], Wang and Chen constructed 2×2 matrices over a commutative local ring which are not strongly clean. In fact, it is hard to determine when a matrix is strongly clean. In [4, Theorem 2.3], Chen et al. discussed when every $n \times n$ matrix over a commutative local ring R, i.e., $M_n(R)$, is strongly clean($n = 2, 3$). In [7, Theorem 2.6], Li investigated when a single 2×2 matrix over a commutative local ring is strongly clean. In [11, Theorem 7], Yang and Zhou characterized a 2×2 matrix ring over a local ring in which every matrix is strongly clean. Strongly clean generalized 2×2 matrices over a local ring were also studied by Tang and Zhou (cf.

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[9, Theorem 15]. In [1, Theorem 12], Borooah et al. characterized when an $n \times n$ matrix over a commutative local ring R is strongly clean, in terms of factorization in the polynomial ring $R[t]$. A commutative ring R is projective free provided that every finitely generated R-module is free. So as to study strong cleanness of matrices over a projective free ring, Fan introduced the condition, and characterize strong cleanness of matrices over commutative projective free rings having. Recently, Diesl and Dorsey investigate strongly clean matrices over arbitrary rings. For a commutative ring R, they proved that every $\varphi \in M_n(R)$ with characteristic polynomial h is strongly clean, if and only if h has an SRC factorization (cf. [5, Theorem 4.6]).

Let $R[[x]]$ be the ring of power series over R. Then there exists a natural epimorphism $R[[x]] \to R, f(x) \mapsto f(0)$, which takes any matrix $A(x)$ over power series to the matrix $A(0)$ over R. The motivation of this note is to determine if the strong cleanness of matrices in $R[[x]]$ and the corresponding one in R coincide with each other. Let R be an arbitrary commutative ring, and let $A(x) \in M_n\big(R[[x]]\big)$. We shall prove that $A(x) \in M_n\big(R[[x]]\big)$ is strongly clean if and only if $A(0) \in M_n(R)$ is strongly clean. Strongly clean matrices over quotient rings of power series are also determined.

Throughout, all rings are commutative with an identity. Let $h(t) \in R[t]$. We say that $h(t)$ is a monic polynomial of degree n if $h(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$ where $a_{n-1}, \cdots, a_1, a_0 \in R$. Every square matrix $\varphi \in M_n(R)$ over a commutative ring R is associated with a characteristic polynomial $\chi(\varphi)$. Let $f, g \in R[t]$. The notation $(f, g) = 1$ means that there exist some $h, k \in R[t]$ such that $fh + gk = 1$. That is, the ideal generated by f, g is $R[t]$. We write $U(R)$ for the set of all invertible elements in R and $M_n(R)$ for the rings of all $n \times n$ matrices over a ring R. $R[t]$ and $R[[t]]$ always stand for the rings of polynomials and power series over a ring R, respectively.

2. Results

Let R be a ring. Given polynomials $f(t) = t^m + a_1 t^{m-1} + \cdots + a_m, g(t) = b_0 t^n + b_1 t^{n-1} + \cdots + b_n (b_0 \neq 0) \in R[t]$, the resultant of f and g is defined by the determinant of the $(m+n) \times (m+n)$ matrix

$$res(f,g) = \begin{vmatrix} 1 & a_1 & \cdots & a_m & & & & \\ & 1 & a_1 & \cdots & a_m & & & \\ & & 1 & a_1 & \cdots & a_m & & \\ & & & \ddots & \ddots & \ddots & \ddots & \\ & & & & 1 & a_1 & \cdots & a_m \\ b_0 & b_1 & \cdots & b_n & & & & \\ & b_0 & b_1 & \cdots & b_n & & & \\ & & b_0 & b_1 & \cdots & b_n & & \\ & & & \ddots & \ddots & \ddots & \ddots & \\ & & & & b_0 & b_1 & \cdots & b_n \end{vmatrix}$$

where blank spaces consist of zeros. The following is called the Weyl Principal. Let f, g be polynomials in $\mathbb{Z}[x_1, x_2, \cdots, x_n]$ with $g \neq 0$ (i.e., g is not the zero polynomial). If for all $(\alpha_1, \cdots, \alpha_n) \in \mathbb{Q}^n$, $f(\alpha_1, \cdots, \alpha_n) = 0$ whenever $g(\alpha_1, \cdots, \alpha_n) \neq 0$, then $f = 0$ in any commutative ring R. We begin with the following results which are analogous to those over fields.

Lemma 1. *Let R be a ring, and let $f \in R[t]$ be monic and $g, h \in R[t]$. Then the following are equivalent:*

(1) $res(f, g) = res(f, g + fh)$.

(2) $res(f, gh) = res(f, g)res(f, h)$.

Proof. (1) Write $h = c_0t^s + \cdots + c_s \in R[t]$. It will suffice to show that $res(f, g) = res(f, g + c_{s-i}t^if)$. Since any determinant in which every entry in a row is a sum of two elements is the sum of two corresponding determinants, the result follows.

(2) Write $f = t^m + a_1t^{m-1} + \cdots + a_m, g = b_0t^n + b_1t^{n-1} + \cdots + b_n, h = c_0t^s + c_1t^{s-1} + \cdots + c_s$. Then

$$\begin{array}{rcl} \alpha(a_1, \cdots, a_m; b_0, \cdots, b_n; c_0, \cdots, c_s) : & = & res(f, gh) - res(f, g)res(f, h) \\ & \in & \mathbb{Z}[a_1, \cdots, a_m; b_0, \cdots, b_n; c_0, \cdots, c_s]. \end{array}$$

Consider

$$\alpha(x_1, \cdots, x_m; y_0, \cdots, y_n; z_0, \cdots, z_s) \in \mathbb{Z}[x_1, \cdots, x_m; y_0, \cdots, y_n;\ z_0, \cdots, z_s].$$

Clearly, the result holds if $R = \mathbb{Q}$.

For any $u_1, \cdots, u_m;\ v_0, \cdots, v_n;\ w_0, \cdots, w_s \in \mathbb{Q}$, we see that

$$\alpha(u_1, \cdots, u_m; v_0, \cdots, v_n; w_0, \cdots, w_s) = 0.$$

By the Weyl Principal that

$$\alpha(x_1, \cdots, x_m; y_0, \cdots, y_n; z_0, \cdots, z_s) = 0$$

in $\mathbb{Z}[x_1, \cdots, x_m; y_0, \cdots, y_n; z_0, \cdots, z_s]$. Therefore

$$\alpha(a_1, \cdots, a_m; b_0, \cdots, b_n;\ c_0, \cdots, c_s) = 0,$$

and so $res(f, gh) = res(f, g)res(f, h)$. □

Lemma 2. *Let R be a ring, and let $f, g \in R[t]$ be monic. Then the following are equivalent:*

(1) $(f, g) = 1$.

(2) $res(f, g) \in U(R)$.

Proof. (1) ⇒ (2) As $(f, g) = 1$, we can find some $u, v \in R[t]$ such that $uf + vg = 1$. By virtue of Lemma 1, one easily checks that $res(f, vg) = res(f, v)res(f, g) = res(f, vg + uf) = res(f, 1) = 1$. Accordingly, $res(f, g) \in U(R)$.

(2) $\Rightarrow$ (1) Let $m = deg(f)$ and $n = deg(g)$. Observing that

$$res(f,g)\begin{vmatrix} & I_{m+n-1} & & t^{m+n} \\ & & & \vdots \\ & & & t \\ 0 & \cdots & 0 & 1 \end{vmatrix} = \begin{vmatrix} & t^n f \\ & \vdots \\ & f \\ * & t^m g \\ & \vdots \\ & g \end{vmatrix},$$

therefore we can find some $u, v \in R[t]$ such that $res(f,g) = uf + vg$. Hence $\big(res(f,g)\big)^{-1}uf + \big(res(f,g)\big)^{-1}vg = 1$, as asserted. □

For any $r \in R$, set $S_r = \{f \in R[t] \mid f \text{ monic, and } f(r) \in U(R)\ \}$.

Lemma 3. ([5, Theorem 4.4 and Theorem 4.6]) *Let R be a ring, and let $h \in R[t]$ be a monic polynomial of degree n. Then the following are equivalent:*

(1) *Every $\varphi \in M_n(R)$ with $\chi(\varphi) = h$ is strongly clean.*

(2) *There exists a factorization $h = h_0h_1$ such that $h_0 \in S_0, h_1 \in S_1$ and $(h_0, h_1) = 1$.*

Let $A(x) = \big(a_{ij}(x)\big) \in M_n\big(R[[x]]\big)$, where each $a_{ij}(x) \in R[[x]]$. We use $A(0)$ to denote the matrix $\big(a_{ij}(0)\big) \in M_n(R)$. We now have at our disposal the information necessary to prove the following.

Theorem 4. *Let R be a ring, and let $A(x) \in M_n\big(R[[x]]\big)(n \geq 1)$. Then the following are equivalent:*

(1) *$A(0) \in M_n(R)$ is strongly clean.*

(2) *$A(x) \in M_n\big(R[[x]]\big)$ is strongly clean.*

Proof. (1) $\Rightarrow$ (2) Obviously, $R[[x]]$ is projective-free. Let $H(x,t) = \chi\big(A(x)\big) \in R[[x]][t]$. Then $H(0,t) = \chi\big(A(0)\big) \in R[t]$. By using Lemma 3, $H(0,t) = h_0h_1$, where $h_0 = t^m + \alpha_1 t^{m-1} + \cdots + \alpha_m \in S_0, h_1 = t^s + \beta_1 t^{s-1} + \cdots + \beta_s \in S_1$ and $(h_0, h_1) = 1$. Next, we will find a factorization $H(x,t) = H_0H_1$ where $H_0(x,t) = t^m + \sum\limits_{i=0}^{m-1} A_i(x)t^i \in S_0$ and $H_1(x,t) = t^s + \sum\limits_{i=0}^{s-1} B_i(x)t^i \in S_1$. Choose $H_0(0,t) \equiv h_0$ and $H_1(0,t) \equiv h_1$. Write $H(x,t) = \sum\limits_{i=0}^{n}\big(\sum\limits_{j=0}^{\infty} c_{ij}x^j\big)t^i$. Then

$$\begin{aligned} H(x,t) &= \sum_{j=0}^{\infty}\big(\sum_{i=0}^{n} c_{ij}t^i\big)x^j \\ &= H(0,t) + \sum_{j=1}^{\infty}\big(\sum_{i=0}^{n} c_{ij}t^i\big)x^j. \end{aligned}$$

Write $A_i(x) = \sum\limits_{j=0}^{\infty} a_{ij}x^j$ and $B_i(x) = \sum\limits_{j=0}^{\infty} b_{ij}x^j$. Then

$$\begin{aligned} H_0 &= t^m + \sum_{i=0}^{m-1}\big(\sum_{j=0}^{\infty} a_{ij}x^j\big)t^i \\ &= t^m + \sum_{j=0}^{\infty}\big(\sum_{i=0}^{m-1} a_{ij}t^i\big)x^j \\ &= h_0 + \sum_{j=1}^{\infty}\big(\sum_{i=0}^{m-1} a_{ij}t^i\big)x^j. \end{aligned}$$

Likewise,

$$H_1 = h_1 + \sum_{j=1}^{\infty}\big(\sum_{i=0}^{s-1} b_{ij}t^i\big)x^j.$$

Write $H_0H_1 = h_0h_1 + \sum\limits_{j=1}^{\infty} z_jx^j$. Thus, we should have

$$\begin{aligned} z_1 &= h_0\big(\sum_{i=0}^{s-1} b_{i1}t^i\big) + \big(\sum_{i=0}^{m-1} a_{i1}t^i\big)h_1 \\ &= \sum_{i=0}^{n-1} c_{i1}t^i. \end{aligned}$$

This implies that

$$\big(b_{(s-1)1}, \cdots, b_{01}, a_{(m-1)1}, \cdots, a_{01}\big)A = \big(c_{(n-1)1}, \cdots, c_{01}\big),$$

where $A = \begin{pmatrix} 1 & \alpha_1 & \cdots & \alpha_m & & & & \\ & 1 & \alpha_1 & \cdots & \alpha_m & & & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & & & 1 & \alpha_1 & \cdots & \alpha_m & \\ 1 & \beta_1 & \cdots & \beta_s & & & & \\ & 1 & \beta_1 & \cdots & \beta_s & & & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & & & 1 & \beta_1 & \cdots & \beta_s & \end{pmatrix}$. As $(h_0, h_1) = 1$, it follows from Lemma 2 that $res(h_0, h_1) \in U(R)$. Thus, $det(A) \in U(R)$, and so we can find $a_{i1}, b_{j1} \in R$.

$$\begin{aligned} z_2 &= h_0\big(\sum_{i=0}^{s-1} b_{i2}t^i\big) + \big(\sum_{i=0}^{m-1} a_{i1}t^i\big)\big(\sum_{i=0}^{s-1} b_{i1}t^i\big) + \big(\sum_{i=0}^{m-1} a_{i2}t^i\big)h_1 \\ &= \sum_{i=0}^{n-1} c_{i2}t^i. \end{aligned}$$

Hence

$$\begin{aligned} h_0\big(\sum_{i=0}^{s-1} b_{i2}t^i\big)+\big(\sum_{i=0}^{m-1} a_{i2}t^i\big)h_1 &= \sum_{i=0}^{n-1} c_{i2}t^i-\big(\sum_{i=0}^{m-1} a_{i1}t^i\big)\big(\sum_{i=0}^{s-1} b_{i1}t^i\big)\\ &= \sum_{i=0}^{n-1} d_{i2}t^i. \end{aligned}$$

Thus,

$$\big((b_{(s-1)2},\cdots,b_{02},a_{(m-1)2},\cdots,a_{02})\big)A=\big(d_{(n-1)2},\cdots,d_{02}\big),$$

whence we can find $a_{i2}, b_{j2}\in R$. By iteration of this process, we can find $a_{ij}, b_{ij}\in R, j=3,4,\cdots$. Therefore we have H_0 and H_1 such that $H(x,t)=H_0H_1$. Further, $H_0(x,0)=H_0(0,0)+xf(x)=h_0(0)+xf(x)\in U\big(R[[x]]\big)$ and $H_1(x,1)=H_1(0,1)+xg(x)=h_1(1)+xg(x)\in U\big(R[[x]]\big)$. Thus, $H_0(x,t)\in S_0$ and $H_1(x,t)\in S_1$. As $(h_0,h_1)=1$, we get $\big(H_0,H_1\big)\equiv 1\big(mod\ (xR[[x]])[t]\big)$, and so $\big(H_0,H_1\big)+J\big(R[[x]]\big)R[[x]][t]=R[[x]][t]$. Set $M=R[[x]][t]/\big(H_0,H_1\big)$. Then M is a finitely generated $R[[x]]$-module, and that $J\big(R[[x]]\big)M=M$. By Nakayama's Lemma, $M=0$, and so $\big(H_0,H_1\big)=1$. Accordingly, $A(x)\in M_n\big(R[[x]]\big)$ is strongly clean by Lemma 3.

(2) ⇒ (1) This is obvious. □

Corollary 5. *Let R be a ring, and let $A(x)\in M_n\big(R[x]/(x^n)\big)(n\geq 1)$. Then the following are equivalent:*

(1) *$A(0)\in M_n(R)$ is strongly clean.*

(2) *$A(x)\in M_n\big(R[x]/(x^n)\big)$ is strongly clean.*

Proof. (1) ⇒ (2) Write $\overline{A(x)}=\sum\limits_{i=0}^{n-1}\overline{a_i}x^i\in M_n\big(R[x]/(x^n)\big)$. Then $A(x)\in M_n\big(R[[x]]\big)$. In view of Theorem 4, there exist $E^2=E=\big(\sum\limits_{k=0}^{\infty}e_k^{ij}x^k\big), U=\big(\sum\limits_{k=0}^{\infty}u_k^{ij}x^k\big)\in GL_n\big(R[[x]]\big)$ such that $A(x)=E+U$ and $EU=UE$. As $R[[x]]/(x^n)\cong R[x]/(x^n)$, we see that $\overline{A(x)}=\overline{E}+\overline{U}$ and $\overline{EU}=\overline{UE}$, where $\overline{E}^2=\overline{E}=\big(\sum\limits_{k=0}^{n-1}e_k^{ij}x^k\big)\in M_n\big(R[x]/(x^n)\big)$ and $\overline{U}=\big(\sum\limits_{k=0}^{n-1}u_k^{ij}x^k\big)\in GL_n\big(R[[x]]/(x^n)\big)$, as desired.

(2) ⇒ (1) This is clear. □

We now extend [6, Theorem 2.10] and [10, Theorem 2.7] as follows.

Corollary 6. *Let R be a ring, and let $n\geq 1$. Then the following are equivalent:*

(1) *$M_n(R)$ is strongly clean.*

(2) $M_n\big(R[[x]]\big)$ *is strongly clean.*

(3) $M_n\big(R[x]/(x^m)\big)(m \geq 1)$ *is strongly clean.*

(3) $M_n\big(R[[x_1, \cdots, x_m]]\big)(m \geq 1)$ *is strongly clean.*

(3) $M_n\big(R[[x_1, \cdots, x_m]]/(x_1^{n_1}, \cdots, x_m^{n_m})\big)(m \geq 1)$ *is strongly clean.*

Proof. These are obvious by induction, Theorem 4 and Corollary 5. □

Example 7. Let $A(x) \in M_2\big(\mathbb{Z}[[x]]\big)$. Then $A(x) \in M_2\big(\mathbb{Z}[[x]]\big)$ is strongly clean if and only if $A(0) \in GL_2(\mathbb{Z})$, or $I_2 - A(0) \in GL_2(\mathbb{Z})$, or $A(0)$ is similar to one of the matrices in the set $\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \}$.

Proof. In light of Theorem 4, $A(x) \in M_2\big(\mathbb{Z}[[x]]\big)$ is strongly clean if and only if so is $A(0)$. Therefore we complete the proof, by [2, Example 16.4.9]. □

Example 8. Let $A(x) = \begin{pmatrix} x & 3+x^2 \\ 1+\sum\limits_{i=1}^{\infty} x^i & 2-x \end{pmatrix} \in M_2(\mathbb{Z}[[x]])$. Then $\chi(A(0)) = t^2 - 2t - 3$. It is easy to verify that there are no any $h_0 \in S_0$ and $h_1 \in S_1$ such that $\chi(A) = h_0h_1$. Accordingly, $A(0) \in M_2(\mathbb{Z})$ is not strongly clean by Lemma 3. Therefore $A(x) \in M_2(\mathbb{Z}[[x]])$ is not strongly clean, in terms of Theorem 4.

Lemma 9. *Let R be a ring, $char(R) = 2$, and let $G = \{1, g\}$ be a group. Then the following hold:*

(1) $R[x]/(x^2 - 1) \cong RG$.

(2) $a + bg \in U(RG)$ *if and only if* $a + b \in U(R)$.

Proof. (1) is proved in [3, Lemma 2.1].

(2) Obviously, $(a + bg)(a - bg) = a^2 - b^2 = (a + b)(a - b)$. Hence, $(a + bg)^2 = (a + b)^2$, as $char(R) = 2$. If $a + bg \in U(RG)$, then $(a + bg)(x + yg) = 1$ for some $x, y \in R$. This implies that $(a + bg)^2(x + yg)^2 = 1$, hence that $(a + b)^2(x + y)^2 = 1$. Accordingly, $a + b \in U(R)$. The converse is analogous. □

Let $A(x) = \big(\overline{a_{ij}(x)}\big) \in M_n\big(R[x]/(x^2-1)\big)$ where $deg\big(a_{ij}(x)\big) \leq 1$, and let $r \in R$. We use $A(r)$ to stand for the matrix $\big(a_{ij}(r)\big) \in M_n(R)$.

Theorem 10. *Let R be a ring with $char(R) = 2$ and let $A(x) \in M_n\big(R[[x]]/(x^2 - 1)\big)(n \geq 1)$. Then the following are equivalent:*

(1) $A(1) \in M_n(R)$ *is strongly clean.*

(2) $A(x) \in M_n\big(R[[x]]/(x^2 - 1)\big)$ *is strongly clean.*

Proof. (1) ⇒ (2) Let $H(g, t) = \chi\big(A(g)\big) \in (RG)[t]$. Then $H(1, t) = \chi\big(A(1)\big) \in R[t]$. In light of Lemma 3, $H(1, t) = h_0h_1$, where $h_0 = t^m + \alpha_{m-1}t^{m-1} + \cdots + \alpha_0 \in$

$S_0, h_1 = t^s + \beta_{s-1}t^{s-1} + \cdots + \beta_0 \in S_1$ and $(h_0, h_1) = 1$. We shall find a factorization $H(g,t) = H_0H_1$ where $H_0(g,t) = t^m + \sum\limits_{i=0}^{m-1}\big(y_i + (\alpha_i - y_i)g\big)t^i \in S_0$ and $H_1(g,t) = t^s + \sum\limits_{i=0}^{s-1}\big(z_i + (\beta_i - z_i)g\big)t^i \in S_1$. Clearly, $H_0(1,t) \equiv h_0$ and $H_1(1,t) \equiv h_1$. We will suffice to find $y_i's$ and $z_i's$. Write $H(g,t) = \sum\limits_{i=0}^{n}\big(r_i + s_ig\big)t^i$. The equality $H(g,t) = H_0H_1$ is equivalent to

$$t^n + \sum_{i=0}^{n-1} r_it^i = \big(t^m + \sum_{i=0}^{m-1} y_it^i\big)\big(t^s + \sum_{i=0}^{s-1} z_it^i\big) + \big(\sum_{i=0}^{m-1}(\alpha_i - y_i)t^i\big)\big(\sum_{i=0}^{s-1}(\beta_i - z_i)t^i\big)(*)$$
$$\sum_{i=0}^{n-1} s_it^i = \big(t^m + \sum_{i=0}^{m-1} y_it^i\big)\big(\sum_{i=0}^{s-1}(\beta_i - z_i)t^i\big) + \big(t^s + \sum_{i=0}^{s-1} z_it^i\big)\big(\sum_{i=0}^{m-1}(\alpha_i - y_i)t^i\big)(**).$$

$(**)$ holds from $H(1,t) = h_0h_1 = H_0(1,t)H_1(1,t)$. $(*)$ is equivalent to

$$\begin{gathered} y_0z_0 + (\alpha_0 - y_0)(\beta_0 - z_0) = r_0, \\ y_0z_1 + y_1z_0 + (\alpha_0 - y_0)(\beta_1 - z_1) + (\alpha_1 - y_1)(\beta_0 - z_0) = r_1, \\ \vdots \\ y_{m-2} + y_{m-1}z_{s-1} + z_{s-2} + (\alpha_{m-1} - y_{m-1})(\beta_{s-1} - z_{s-1}) = r_{n-2}, \\ y_{m-1} + z_{s-1} = r_{n-1}. \end{gathered}$$

As $char(R) = 2$, we have

$$\begin{gathered} \beta_0y_0 + \alpha_0z_0 = r_0 + \alpha_0\beta_0, \\ \beta_0y_1 + \beta_1y_0 + \alpha_0z_1 + \alpha_1z_0 = r_1 + \alpha_0\beta_1 + \alpha_1\beta_0, \\ \vdots \\ \beta_{s-1}y_{m-1} + y_{m-2} + \alpha_{m-1}z_{s-1} + z_{s-2} = r_{n-2} + \alpha_{m-1}\beta_{s-1}, \\ y_{m-1} + z_{s-1} = r_{n-1}. \end{gathered}$$

This implies that

$$\big(y_{m-1}, \cdots, y_0, z_{s-1}, \cdots, z_0\big)A = \big(*, \cdots, *\big),$$

where

$$A = \begin{pmatrix} 1 & \beta_{m-1} & \cdots & \beta_0 & & & & \\ & 1 & \beta_{m-1} & \cdots & \beta_0 & & & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & & & 1 & \beta_{m-1} & \cdots & \beta_0 \\ 1 & \alpha_{s-1} & \cdots & \alpha_0 & & & \\ & 1 & \alpha_{s-1} & \cdots & \alpha_0 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & \alpha_{s-1} & \cdots & \alpha_0 \end{pmatrix}.$$

As $(h_1, h_0) = 1$, it follows from Lemma 2, that $res(h_1, h_0) \in U(R)$. Thus, $det(A) \in U(R)$, and so we can find $y_i, z_j \in R$ such that $(*)$ and $(**)$ hold. In other words, we have H_0 and H_1 such that $H(g,t) = H_0H_1$. Obviously, $H_0(g, 0) = y_0 + (\alpha_0 - y_0)g$. As $y_0 + (\alpha_0 - y_0) = \alpha_0 = h_0(0) \in U(R)$, it follows by Lemma 9 that $H_0(g, 0) \in U(RG)$, i.e., $H_0 \in S_0$. Further, $H_1(g, 1) = 1 + \sum\limits_{i=1}^{s-1} \big(z_i + (\beta_i - z_i)g\big) = 1 + \sum\limits_{i=1}^{s-1} z_i + \big(\sum\limits_{i=1}^{s-1} (\beta_i - z_i)\big)g$.

It is easy to check that $1 + \sum\limits_{i=1}^{s-1} z_i + \big(\sum\limits_{i=1}^{s-1} (\beta_i - z_i)\big) = 1 + \sum\limits_{i=1}^{s-1} \beta_i = h_1(1) \in U(R)$.

In view of Lemma 9, $H_1 \in S_1$. Clearly, $\varphi(g) := res(H_0, H_1) \in RG$. As $\varphi(1) = res\big(H_0(1,t), H_1(1,t)\big) = res(h_0, h_1) \in U(R)$. By using Lemma 9 again, $\varphi(g) \in U(RG)$, i.e., $res(H_0, H_1) \in U(RG)$. In light of Lemma 2, we get $(H_0, H_1) = 1$.

Therefore, $A(g) \in M_n\big(RG\big)$ is strongly clean, as required.

$(2) \Rightarrow (1)$ Let $\psi : RG \to R, a + bg \mapsto a + b$. Then we get a corresponding ring morphism $\mu : M_n(RG) \to M_n(R), \big(a_{ij}(g)\big) \mapsto \big(\psi(a_{ij}(g))\big)$. As $A(g)$ is strongly clean, we can find an idempotent $E \in M_n(RG)$ such that $A(g) - E \in GL_n(RG)$ and $EA = AE$. Applying μ, we get $A(1) - \mu(E) \in GL_n(R)$, where $\mu(E) \in M_n(R)$ is an idempotent, hence the result follows. □

Example 11. Let $S = \{0, 1, a, b\}$ be a set. Define operations by the following tables:

+	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

,

×	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	a

.

Then S is a finite field with $\mid S \mid = 4$. Let

$$R = \{s_1 + s_2 z \mid s_1, s_2 \in S, z \text{ is an indeterminant satisfying } z^2 = 0\}.$$

Then R is a commutative local ring with $charR = 2$. We claim that

$$A(x) = \begin{pmatrix} \overline{az} & \overline{z+x} \\ \overline{1+x} & \overline{b+zx} \end{pmatrix} \in M_2\big(R[x]/(x^2-1)\big)$$

is strongly clean. Clearly, $A(1) = \begin{pmatrix} az & 1+z \\ 0 & 1+bz \end{pmatrix} \in M_2(R)$. As $\chi\big(A(1)\big)$ has a root $az \in J(R)$ and a root $1 + bz \in 1 + J(R)$, $A(1)$ is strongly clean, and we are through by Theorem 10.

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