Optimization Methods and Software

ISSN: 1055-6788 (Print) 1029-4937 (Online) Journal homepage: http://www.tandfonline.com/loi/goms20

Optimal allocation with costly inspection and discrete types under ambiguity

Halil I. Bayrak, Kemal Güler & Mustafa Ç. Pınar

To cite this article: Halil I. Bayrak, Kemal Güler & Mustafa Ç. Pınar (2017) Optimal allocation with costly inspection and discrete types under ambiguity, Optimization Methods and Software, 32:4, 699-718, DOI: 10.1080/10556788.2016.1277996

To link to this article: https://doi.org/10.1080/10556788.2016.1277996

Published online: 18 Jan 2017.

Optimization Methods & Software, 2017
Vol. 32, No. 4, 699–718, https://doi.org/10.1080/10556788.2016.1277996

Optimal allocation with costly inspection and discrete types under ambiguity

Halil I. Bayrak[a], Kemal Güler[b] and Mustafa Ç. Pınar[a*]

[a]*Department of Industrial Engineering, Bilkent University, 06800 Ankara, Turkey;* [b]*Department of Economics, Anadolu University, Eskişehir, Turkey*

(*Received 27 November 2015; accepted 22 December 2016*)

We consider the following problem: a principal has a good to allocate among a collection of agents who attach a private value to receiving the good. The principal, instead of using monetary transfers (i.e. charging the agents) to allocate the good, can check the truthfulness of the agents' value declaration at a cost. Under the assumption that the agents' valuations are drawn from a discrete set of values at random, we characterize the class of optimal Bayesian mechanisms which are symmetric, direct and maximizing the expected value of assigning the good to the principal minus the cost of verification using such standard finite-dimensional optimization tools as linear programming and submodular functions, thus extending the work of [R.V. Vohra, *Optimization and mechanism design*, Math. Program. 134 (2012), pp. 283–303]. Our results are discrete-type analogs of those of [E. Ben-Porath, E. Dekel, and B.L. LipmanBen-Porath, *Optimal allocation with costly verification*, Amer. Econ. Rev. 104 (2014), pp. 3779–3813]. When the distribution of valuations is not known but can be one of a set of distributions (the case referred to as *ambiguity*), we compute a robust allocation mechanism by maximizing the worst-case expected value of the principal in two cases amenable to solution with two suitable assumptions on the set of distributions.

Keywords: optimal allocation; costly inspection; ambiguity; linear programming; submodular functions; implementation

1. Introduction

The goal of the present paper is to contribute to the closed-form solution of an important problem at the core of mathematical economics, game theory and microeconomics using well-known tools of mathematical programming, namely linear programming and submodular functions, a line of research essentially initiated by Vohra [17]. To appreciate the problem in a stylized scenario, we consider the following application from [1]. Imagine the dilemma faced by the head of an organization with different kinds of departments. A typical example is the dean of a faculty/school facing the decision to allocate a job slot to one of the departments. He/she naturally wants to allocate the resource to the department which can put it to the best use in order to profit the entire organization. All departments privately know that they can derive some profit from the resource, and they all desire to claim the resource to be credited for the profit that would ensue from the allocation. It is not logical to use monetary transfers in this problem since any payment from departments lowers the profit that they can achieve for the organization. Therefore

*Corresponding author. Email: mustafap@bilkent.edu.tr

as Ben-Porath *et al.* [1] propose, he/she is given another tool, inspection of the claims made by the departments. We can describe the usage of this tool in a game structure as follows: he/she first asks departments to report the contribution that they can make if they claim the resource. Then, according to the allocation rule of the game, he/she chooses the winner, and according to the inspection rule he/she inspects the accuracy of its report. Inspection is error free but it carries some cost. Therefore, he/she would be better off if he/she skipped the inspection in some cases. The objective is to find the optimal mechanism consisting of the allocation and inspection rules to maximize the organization's total welfare. In other words, one wants to make an efficient allocation and minimize the inspection cost at the same time.

The purpose of the present paper is to derive the optimal allocation/inspection mechanism under discrete valuations and when the distribution of valuations is not known exactly but can be one of a finite set of distributions. As is common in the literature on decision theory (see e.g. the expected maxmin utility preferences of Gilboa and Schmeidler [10] or the more recent smooth ambiguity model of Klibanoff *et al.* [12]) we shall refer to this situation as *ambiguity*. While the main contribution of the present paper lies in the exploration of ambiguity aversion for optimal allocation with costly inspection, we shall look first into the problem under discrete valuations without ambiguity in Section 2 since the results derived in this section will be useful in the subsequent section incorporating ambiguity. We shall treat the case where agents are symmetric as in [17]. Our work is different from [17] in two ways: (1) we work on a model different from that of Vohra [17], (2) we give the exact optimal mechanism. Hence, our (more modest) contribution in the first part of the paper is to make precise the earlier results of Vohra [17], and derive using finite space optimization tools the recent results of Ben-Porath *et al.* [1]. An earlier version of the paper also contained the solution for agents with different costs of inspection. In the interest of keeping the paper at a reasonable length, and since we are only using the case of symmetric agents in the second part of the paper, we leave out the non-symmetric case.[1] In the second and main part of the paper in Section 4, we shall focus on dealing with ambiguity in the problem, namely instead of assuming the type distribution is known exactly to the head of the organization, we shall assume it can be any one of a finite number of distributions and look for a worst-case (max–min) optimal allocation/inspection mechanism. While we are not aware of a worst-case-oriented study directly on the optimal allocation without monetary transfers, robust mechanism and auction design is a subject treated in several papers; see, e.g. [4–6,8,14]. Our departure point is the growing literature on robust optimization that started with the pioneering papers of Ben-Tal and Nemirovski [2,3] where the constraints or the objective function of a linear/convex optimization problem were affected by uncertainty, and a robust, worst-case solution was sought while remaining in the domain of tractable convex optimization. Against this background, we investigate the structure of the optimal allocation/inspection mechanism in the face of multiple (potential) distributions faced by the principal about the valuations of the agents. Even in the simplest case of a finite number of distributions, it is challenging to reveal the form of the optimal policy. However, under a suitable assumption of stochastic dominance, we can infer the optimal policy. In the absence of the stochastic dominance assumption, we give an algorithm whose output is, for some specified cases, optimal. For other cases it is a close approximation that is easier to implement compared to the optimal solution. We give a numerical example illustrating the performance of the algorithm, and compare the resulting mechanism with the true optimal mechanism. The example shows that the optimal mechanism in the relatively simpler case when an ambiguity averse principal is facing two possible distributions may violate some desirable monotonicity properties, and hence would be difficult to implement. Our algorithm gives a simpler mechanism very close to the optimal and preserving such properties.

A brief review of the literature pertaining to the problem treated in the present paper can be found in [1]; see, e.g. the contributions in [7,9,11,15,16]. Since our paper focuses on the solution technique for the discrete valuations rather than the economic theory insights, we direct the reader interested in the related literature to the references in [1].

2. Model

We assume a risk-neutral principal with a single good and n risk-neutral agents that have non-negative private valuation for the good. These private valuations will be addressed as agents' types and valuation of the principal is assumed to be 0. Define $T = \{1, 2, \ldots, m\}$ as the type space whose discrete form will allow us to use linear programming. We have the common prior assumption which means that the types of agents are independent draws from a commonly known probability mass function f satisfying $f_i > 0$ for all $i \in T$. We invoke the Revelation Principle through which we focus on direct mechanisms only.

In the proposed model, we define the decision variable $a(i, t^{-1})$ to be probability that agent 1 receives the good when he reports i and other agents report t^{-1} and $s(i, t^{-1})$ to be probability that agent 1 receives the good without inspection when he reports i and other agents report t^{-1}. Since we assume that all agents are symmetric, it is enough to consider only one agent. We will also use the notation $a_i(t)$ and $s_i(t)$ as allocation and inspection skipping probability for agents who reported i in the profile t. Let $\pi(t^{-1})$ denote the probability that agents other than 1 report types giving rise to profile t^{-1} and let $n_i(t)$ denote the number of type is in the profile t. Non-negative variables $A(i)$ and $S(i)$ represent the expected allocation to type i and expected inspection skipping variable for type i, respectively:

$$A(i) = \sum_{t^{-1} \in T^{n-1}} a_i(i, t^{-1})\pi(t^{-1}),$$

$$S(i) = \sum_{t^{-1} \in T^{n-1}} s_i(i, t^{-1})\pi(t^{-1}).$$

The objective of the problem is to maximize expected utility from the allocation with inspection in addition to the savings from not inspecting:

$$\max_{S,A,s,a} \quad \sum_{i \in T} f_i(i - K)A(i) + K \sum_{i \in T} f_i S(i) \tag{1}$$

$$\text{s.t.} \quad iA(i) \geq iS(j) \quad \forall i, j \in T, \tag{2}$$

$$A(i) = \sum_{t^{-1} \in T^{n-1}} a_i(i, t^{-1})\pi(t^{-1}) \quad \forall i \in T, \tag{3}$$

$$S(i) = \sum_{t^{-1} \in T^{n-1}} s_i(i, t^{-1})\pi(t^{-1}) \quad \forall i \in T, \tag{4}$$

$$a_i(t) \geq s_i(t) \quad \forall i \in T \ \forall t \in T^n, \tag{5}$$

$$\sum_{i \in T} n_i(t)a_i(t) \leq 1 \quad \forall t \in T^n, \tag{6}$$

$$a_i(t) \geq 0 \quad \forall i \in T \ \forall t \in T^n, \tag{7}$$

$$s_i(t) \geq 0 \quad \forall i \in T \ \forall t \in T^n. \tag{8}$$

We do not allow a negative amount of allocation in (7), and we only have one good to allocate in (6). By constraints (5) and (8), it is obvious that the probability of getting the object without any inspection should be between 0 and the actual allocation value. Constraints (3) and (4) simply ensure that expectation variables are consistent with the underlying mechanism.

Constraint (2) is the incentive compatibility constraint in which the left-hand side is the utility of agent if he reports his type truthfully. There is no effect of inspection since he will keep his object if we find out that he was truthful. However, on the right-hand side the utility of lying is affected by the inspection. The principal will reclaim the good if the type was misreported. Therefore, an agent can expect some utility from lying only if the principal chooses to skip inspection.

2.1 *Structure of the optimal solution*

We first simplify the formulation. We ignore the $s_i(i, t^{-1})$ variables for the time being, and focus on the expected skipping variable. In this manner we can leave out the constraints (4), (5) and (8). Besides, notice that in the constraint (2), multiplier i cancels out. We now apply Border's theorem to constraints (3), (6) and (7) to obtain

$$\begin{aligned} \max_{S,A} \quad & \sum_{i\in T} f_i(i-K)A(i) + K\sum_{i\in T} f_i S(i) \\ \text{s.t.} \quad & A(i) \geq S(j) \quad \forall i,j \in T, \\ & n\sum_{i\in\mathcal{S}} f_i A(i) \leq 1 - \left(\sum_{i\notin\mathcal{S}} f_i\right)^n \quad \forall\mathcal{S}\subseteq T. \end{aligned}$$

We conclude that the incentive compatibility constraint is satisfied if we have $\min_{k\in T} A(k) \geq S(j)$ for all j in T. Since there is no other constraint on $S(j)$, and it has a positive coefficient in the objective, we set it to its upper bound, $\min_{k\in T} A(k) = S(j)$. In the optimal solution, we should have either $A^*(i) > \min_{k\in T} A^*(k)$ or $A^*(i) = \min_{k\in T} A^*(k)$ for any $i \in T$. Therefore we will have a partition for our type set such as

$$L = \left\{ i \in T \mid A^*(i) = \min_{k\in T} A^*(k) \right\},$$

$$U = \left\{ i \in T \mid A^*(i) > \min_{k\in T} A^*(k) \right\}.$$

Now, we can make the following arrangements in the formulation. We set $\min_{k\in T} A(k) = S(j)$ for all j in T, which is the optimal solution. Besides, for all i in L, denote $A(i)$ by $\min_{k\in T} A(k)$ which are equal in the optimal solution. This gives the problem:

$$\begin{aligned} \max_{A} \quad & \sum_{i\in U} f_i(i-K)A(i) + \sum_{i\in L} f_i(i-K)\min_{k\in T} A(k) + K\min_{k\in T} A(k) \\ \text{s.t.} \quad & n\sum_{i\in\mathcal{S}} f_i A(i) \leq 1 - \left(\sum_{i\notin\mathcal{S}} f_i\right)^n \quad \forall\mathcal{S}\subseteq T. \end{aligned}$$

As a final arrangement, set $z_i = f_i A(i)$ for all i in U and set $z_l = \min_{k\in T} A(k)\sum_{i\in L} f_i$. Also define $f_i' = f_i$ when i in U and $f_l' = \sum_{i\in L} f_i$. Since we have a submodular function in our constraint, we can find the optimal solution by the Greedy Algorithm. We start from the variable with the highest coefficient and set it to its tightest upper bound:

$$\max_{z} \quad \sum_{i\in U}(i-K)z_i + \left(\frac{\sum_{i\in L} f_i i + (1-\sum_{i\in L} f_i)K}{\sum_{i\in L} f_i}\right) z_l \quad OPTi$$

$$\text{s.t.} \quad n\sum_{i\in S} z_i \le 1 - \left(\sum_{i\notin S} f_i'\right)^n \qquad \forall \mathcal{S} \subseteq U \cup \{l\}.$$

PROPOSITION 1 *In an optimal solution, one has* $\min_{k\in T} A(k) = A(1)$.

Proof Assume to the contrary that $A^*(1) > \min_{k\in T} A^*(k)$. Then, there exists $l \in T \setminus \{1\}$ such that $\min_{k\in T} A^*(k) = A^*(l)$. Then we should have $1 \in U$. This means that in *OPTi*, coefficient of z_1 should be greater than coefficient of z_l:

$$1 - K > \mathrm{E}[i \mid i \in L] + \frac{(1-\sum_{i\in L} f_i)K}{\sum_{i\in L} f_i}. \tag{9}$$

This is obviously not possible since $1 < \mathrm{E}[i \mid i \in L]$. This is a contradiction. ∎

In order to explain the reasoning behind the use of sets U and L, let us consider the Greedy Algorithm. Our formulation had incentive compatibility constraints, (IC), which prevent us from utilizing the characteristic of submodular functions. We know that any optimization problem constrained by submodular functions can be optimally solved by the Greedy Algorithm. Therefore we assume that sets U and L optimally partition the type set so that when we apply the Greedy Algorithm, IC constraints are automatically satisfied.

Now, we can describe more precisely the optimal solution by extending the result of Proposition 1. In order to decide optimally whether a type belongs to set U or set L, we compare its coefficients in both cases. The membership of a given type in L and U is decided according to the amount of profit derived from this membership by the Greedy Algorithm. Let us consider the following case for any type $j \in T$:

$$j - K > \left(\frac{\sum_{i=1}^{j} f_i i + (1-\sum_{i=1}^{j} f_i)K}{\sum_{i=1}^{j} f_i}\right).$$

From the above inequality one infers that coefficient of type j when it is in set U is bigger than its coefficient when it is in set L. Then it is more profitable to put it in set U. Since we have a maximization problem, optimal solution should have $j \in U$. We define a function $Q(i)$: $T \setminus \{1\} \to \mathbb{R}$ in order to optimally partition the type set:

$$Q(i) = i - \mathrm{E}[j \mid j \le i] - \frac{K}{\sum_{j\le i} f_j} = i - \left(\frac{\sum_{j\le i} f_j j + K}{\sum_{j\le i} f_j}\right).$$

We do not need to define function Q for type 1 since we know that type 1 is always in set L. Notice that any type i is in the set U if $Q(i)$ is strictly positive and it is in the set L otherwise. We now show that the optimal cut-off which separates sets U and L can be deduced from function Q.

PROPOSITION 2 *Either function $Q(i)$ is non-positive for all types or there is a unique $i^* > K$ such that the function $Q(i)$ is strictly positive when $i \ge i^*$ and non-positive otherwise.*

Proof

$$Q(i+1) = i+1-\left(\frac{\sum_{j\le i+1} f_j j + K}{F(i+1)}\right) = i+1-\left(\frac{\sum_{j\le i} f_j j + f_{i+1}(i+1) + K}{F(i+1)}\right)$$
$$= i+1-\left(\frac{f_{i+1}(i+1)}{F(i+1)} + \frac{\sum_{j\le i} f_j j + K}{F(i+1)}\right) = \frac{F(i)(i+1)}{F(i+1)} - \frac{F(i)(i-Q(i))}{F(i+1)}$$
$$= \frac{F(i)(1+Q(i))}{F(i+1)}.$$

In the end, we have:

$$Q(i+1) = \frac{F(i)(1+Q(i))}{F(i+1)}, \quad Q(i) = \frac{F(i+1)Q(i+1)}{F(i)} - 1.$$

This means that if there exists i such that $Q(i) \ge 0$, then $Q(i+1)$ is also positive. If $Q(i+1) < 0$, then $Q(i)$ is also negative. Therefore, if we have a type $i^* \in T \setminus \{1\}$ where $Q(i^*) \ge 0$ and $Q(i^*-1) < 0$, then it is unique since for the types above i^*, $Q(i)$ is positive, and it is negative otherwise. Note that $Q(i)$ is not necessarily monotone.

It is easy to see that cut-off i^* is greater than K:

$$Q(i^*) = i^* - \left(\frac{\sum_{j<i^*} f_j j + K}{\sum_{j<i^*} f_j}\right) > 0$$
$$i^* > \left(\frac{\sum_{j<i^*} f_j j + K}{\sum_{j<i^*} f_j}\right) > \frac{K}{\sum_{j<i^*} f_j} > K.$$

Note that for large inspection cost K, it is possible to have negative $Q(i)$ for all i in T. ∎

In order to find the optimal cut-off for the continuous-type space, we should find the type where $Q(i) = 0$. This result is consistent with the results from [1].

We now use the Greedy Algorithm to find the optimal interim allocations of the formulation *OPTi*. Obviously, above the cut-off, coefficient $i-K$ is monotone. Setting variable z_i to its tightest constraint will give the optimal solution as optimal interim allocations. Recall that we will treat the variables that are under the cut-off as one variable, z_l. Since type 1 is the minimum, let us denote the variable which represents set L as z_1. Consider the constraint when $S = T$ which is the tightest constraint on z_1 in order to find the optimal solution:

$$z_1 + \sum_{i=i^*}^{m} \frac{F(i)^n - F(i-1)^n}{n} \le \frac{1}{n} \Rightarrow z_1 \le \frac{1}{n} - \frac{1-F(i^*-1)^n}{n} = \frac{F(i^*-1)^n}{n}.$$

Therefore, we can give the optimal solution for the interim allocations whose monotonicity is easy to prove:

$$A(i) = \begin{cases} \dfrac{F(i)^n - F(i-1)^n}{f_i n} & \text{if } i \ge i^* \\ \dfrac{F(i^*-1)^{n-1}}{n} & \text{otherwise.} \end{cases}$$

This solution is unique if we have $Q(i^*)$ strictly positive. If it is equal to zero, then we are indifferent between whether or not inspecting the type i^*. This case yields infinitely many optimal solutions where $\varphi \in [F(i^*-1)^{n-1}/n, F(i^*)^{n-1}/n]$:

$$A(i) = \begin{cases} \dfrac{F(i)^n - F(i-1)^n}{f_i n} & \text{if } i > i^*, \\ \dfrac{F(i)^n - n\varphi F(i-1)}{f_i n} & \text{if } i = i^*, \\ \varphi & \text{otherwise.} \end{cases}$$

Since the first introduced expected allocations are optimal for both cases, and $Q(i^*)$ is less likely to be exactly zero, we will consider the previous solution for the rest of the paper.

We know that allocating the good to the highest bidder is an allocation rule consistent with this solution when the maximum bid is over the cut-off. Otherwise it is optimal to equally distribute the good to all agents, or in other words, choose the winner with a lottery where all agents have $1/n$ winning probability:

$$A(1) = \frac{1}{n} \cdot P(\text{other agents bid under the cut-off}) = \frac{F(i^*-1)^{n-1}}{n}.$$

In order to completely characterize an optimal mechanism, we should also define an optimal inspection skipping rule. We know that $A(1) = S(j)$ for all j in T, which allows us to set $a_i(1, t^{-1}) = s_i(j, t^{-1})$ for all j in T and t^{-1} in T^{n-1}. It means that we can skip allocation with probability $1/n$ if all other agents bid under the cut-off. Note that optimal inspection skipping rule is only dependent on the profile of other agents:

$$a_i^*(i, t^{-1}) = \begin{cases} \dfrac{1}{n_i(t)} & \text{if } i \geq \max_{k \in t^{-1}} k \text{ and } i \geq i^*, \\ \dfrac{1}{n} & \text{if } \max_{k \in t^{-1}} k < i^* \text{and } i < i^*, \\ 0 & \text{otherwise,} \end{cases}$$

$$s_i^*(i, t^{-1}) = \begin{cases} \dfrac{1}{n} & \text{if } \max_{k \in t^{-1}} k < i^*, \\ 0 & \text{otherwise.} \end{cases}$$

In this mechanism, we allocate the good to the maximum bid if it is above the cut-off and we randomize the allocation if everyone bid under the cut-off. The inspection certainly takes place if there is more than one bid over the cut-off. If there was one bid over the cut-off, then we skip the inspection with probability $1/n$. If there is no bid above the cut-off, we allocate everyone $1/n$ and skip the inspection for all with probability $1/n$. Since there is no inspection skipping when there is no allocation, conditional probability says: do not inspect at all.

With the above derivations we gave the exact structure of the optimal mechanism. In [17], it is stated that there exists a cut-off, above which we award the object to the highest bidder and inspect with positive probability. Under the cut-off, randomize the object and do not inspect. However, Vohra [17] does not give the optimal cut-off nor the optimal inspection probability while our description is complete.

Our result is consistent with the results of Ben-Porath *et al.* [1]. They do not assume symmetric agents so that as the optimal mechanism, they have a *favoured-agent* mechanism which we also derive but omit here since in this paper we focus on the symmetric case. One can generalize the results of this section by introducing allocation and inspection skipping variables for all agents, and then use the Greedy Algorithm. In favoured-agent mechanism, there is a favourite agent who gets the good whenever no agent bids over the cut-off. The choice of favourite agent is dependent on a function which has distribution f and inspection cost K as its inputs. Note that this function is the continuous version of $Q(i)$ from Proposition 2. Since we assume that these inputs are identical for all agents, it is optimal to choose any of them as favourite agent or to allocate them equally.

3. Optimal mechanism under ambiguity

In optimal auction mechanism literature, it is a common assumption that bidders' valuations are independently drawn from a unique distribution. We now relax this common priors assumption and investigate the form of optimal allocation/inspection mechanism for an ambiguity averse principal who believes that valuation distribution is a random outcome from a discrete set of distributions $\mathcal{P}$. The principle is given the information that there are n ambiguity neutral agents of whom each has a unique prior from the same distribution set $\mathcal{P}$. Here we study the problem from the previous section but this time the principle needs to design a mechanism that is incentive compatible with respect to all distributions in $\mathcal{P}$ and maximizing the worst-case expected profit over all distributions in $\mathcal{P}$. We will continue to make use of linear programming and submodular optimization.

As in the previous section, we assume symmetry among the agents so that they have the same inspection cost. Again we focus on direct mechanisms as in [8]. Let us consider the formulation where our objective is to maximize the worst-case social welfare:

$$
\begin{aligned}
\max_{S,A,s,a} \quad & \left\{ \min_{f \in \mathcal{P}} \sum_{i \in T} f_i (i - K) A_f(i) + K \sum_{i \in T} f_i S_f(i) \right\} \\
\text{s.t.} \quad & A_f(i) \geq S_f(j) \quad \forall i, j \in T,\ \forall f \in \mathcal{P} \\
& A_f(i) = \sum_{t^{-1} \in T^{n-1}} a_i(i, t^{-1}) \pi_f(t^{-1}) \quad \forall i \in T,\ \forall f \in \mathcal{P} \\
& S_f(i) = \sum_{t^{-1} \in T^{n-1}} s_i(i, t^{-1}) \pi_f(t^{-1}) \quad \forall i \in T,\ \forall f \in \mathcal{P} \\
& a_i(t) \geq s_i(t) \quad \forall i \in T\ \forall t \in T^n \\
& \sum_{i \in T} n_i(t) a_i(t) \leq 1 \quad \forall t \in T^n \\
& a_i(t) \geq 0 \quad \forall i \in T\ \forall t \in T^n \\
& s_i(t) \geq 0 \quad \forall i \in T\ \forall t \in T^n.
\end{aligned}
$$

We drop the $s_i(t)$ variables and introduce a variable y in order to linearize the formulation:

$$
\max_{S,A,a,y} \quad y \quad OPTa \tag{10}
$$

$$
\text{s.t.} \quad y \leq \sum_{i \in T} f_i (i - K) A_f(i) + K \sum_{i \in T} f_i S_f(i) \quad \forall f \in \mathcal{P} \tag{11}
$$

$$
A_f(i) \geq S_f(j) \quad \forall i, j \in T,\ \forall f \in \mathcal{P} \tag{12}
$$

$$
A_f(i) = \sum_{t^{-1} \in T^{n-1}} a_i(i, t^{-1}) \pi_f(t^{-1}) \quad \forall i \in T,\ \forall f \in \mathcal{P} \tag{13}
$$

$$\sum_{i \in T} n_i(t)a_i(t) \leq 1 \quad \forall t \in T^n \tag{14}$$

$$a_i(t) \geq 0 \quad \forall i \in T, \ \forall t \in T^n. \tag{15}$$

3.1 *The solution approach*

It is obvious that if the optimal mechanism for an $f \in \mathcal{P}$, when it is the common prior, has the same cut-off as all other distributions in $\mathcal{P}$, then the same mechanism is optimal for the ambiguity averse principal. Again consider the optimal mechanism for f when it is the common prior. If distribution f attains worst case when constraint (11) is checked, then this solution is optimal. Since these results were proved in a previous paper [13] for the robust auction design problem, we omit them here.

Regarding the effect of the ambiguity aversion on the likelihood of inspection, one can say that it may increase the likelihood of inspection with respect to one distribution in $\mathcal{P}$ and decrease it with respect to another. This is because the optimal mechanism of an ambiguity averse decision-maker should have a threshold that is between the thresholds of the distributions in $\mathcal{P}$. Besides, optimal mechanism may have expected allocation variables violating monotonicity for some prior.

We will now show that if there is a distribution in $\mathcal{P}$ which is first-order stochastically dominated by all other distributions, we can solve the problem considering only that dominated distribution. Recall that f first-order stochastically dominates g if and only if we have $F(i) \leq G(i)$ for all i and with a strict inequality at some i. This means that the probability distribution for f is more concentrated on the highest types.

THEOREM 1 *If $g \in \mathcal{P}$ is first-order stochastically dominated by all $f \in \mathcal{P} \setminus \{g\}$, then the following is the optimal mechanism where t_g^* is the optimal cut-off for distribution g:*

$$a_i^*(i, t^{-1}) = \begin{cases} \dfrac{1}{n_i(t)} & \text{if } i \geq \max\limits_{k \in t^{-1}} k \text{ and } i \geq t_g^*, \\ \dfrac{1}{n} & \text{if } \max\limits_{k \in t^{-1}} k < t_g^* \text{ and } i < t_g^*, \\ 0 & \text{otherwise}, \end{cases}$$

$$s_i^*(i, t^{-1}) = \begin{cases} \dfrac{1}{n} & \text{if } \max\limits_{k \in t^{-1}} k < t_g^*, \\ 0 & \text{otherwise}. \end{cases}$$

Proof Firstly, this mechanism is feasible for all $f \in \mathcal{P}$. Note that it is the optimal solution if we have g as the unique prior. Therefore, if we show that $g \in \mathcal{P}$ is the worst case then we complete the proof because the given mechanism yields the maximum utility for g and it cannot be improved.

Let us consider the constraint (11) with expected allocations obtained from this mechanism. This is for all f in $\mathcal{P}$:

$$y \leq \sum_{i=t_g^*}^{m} (i-K)\frac{F(i)^n - F(i-1)^n}{n} + \sum_{i=1}^{t_g^*-1} f_i(i-K)\frac{F(t_g^*-1)^{n-1}}{n} + K\frac{F(t_g^*-1)^{n-1}}{n}.$$

We define a parameter $v_f^i = \sum_{j=1}^{i-1} f_j j + K/F(i-1)$ which can be described as a valuation for the set of types that are strictly below i if they were not to be inspected when the distribution is f.

Then we rearrange the right-hand side:

$$\sum_{i=t_g^*}^{m}(i-K)\frac{F(i)^n-F(i-1)^n}{n}+\sum_{i=1}^{t_g^*-1}f_i i\frac{F(t_g^*-1)^{n-1}}{n}+(1-F(t_g^*-1))K\frac{F(t_g^*-1)^{n-1}}{n}$$
$$=\sum_{i=t_g^*}^{m}(i-K)\frac{F(i)^n-F(i-1)^n}{n}+(\nu_f^{t_g^*}-K)\frac{F(t_g^*-1)^n}{n}$$
$$=(m-K)\frac{1^n-F(m-1)^n}{n}+\cdots(t_g^*-K)\frac{F(t_g^*)^n-F(t_g^*-1)^n}{n}+(\nu_f^{t_g^*}-K)\frac{F(t_g^*-1)^n}{n}$$
$$=\frac{(m-K)}{n}-\sum_{i=t_g^*}^{m-1}\frac{F(i)^n}{n}-(t_g^*-\nu_f^{t_g^*})\frac{F(t_g^*-1)^n}{n}. \tag{16}$$

Now we can compare this value for g and any f that first-order stochastically dominates g. If we have $\nu_f^{t_g^*}\geq\nu_g^{t_g^*}$ then the following is obvious since we have $F(i)\leq G(i)$ for all i in T and $F(j)<G(j)$ for some j in T. The right-hand side of constraint (11) for f is bigger than the right-hand side of constraint (11) for g:

$$-\sum_{i=t_g^*}^{m-1}\frac{G(i)^n}{n}-(t_g^*-\nu_g^{t_g^*})\frac{G(t_g^*-1)^n}{n}<-\sum_{i=t_g^*}^{m-1}\frac{F(i)^n}{n}-(t_g^*-\nu_f^{t_g^*})\frac{F(t_g^*-1)^n}{n}. \tag{17}$$

When $\nu_f^{t_g^*}<\nu_g^{t_g^*}$, we distinguish two cases. The first one is if $F(t_g^*-1)=G(t_g^*-1)$. Now, check the ν values:

$$\frac{\sum_{i=1}^{t_g^*-1}f_i i+K}{F(t_g^*-1)}<\frac{\sum_{i=1}^{t_g^*-1}g_i i+K}{G(t_g^*-1)}\Rightarrow\frac{\sum_{i=1}^{t_g^*-1}f_i i}{F(t_g^*-1)}<\frac{\sum_{i=1}^{t_g^*-1}g_i i}{G(t_g^*-1)},$$
$$\sum_{i=1}^{t_g^*-1}\frac{f_i}{F(t_g^*-1)}i<\sum_{i=1}^{t_g^*-1}\frac{g_i}{G(t_g^*-1)}i.$$

Define $f_i/F(t_g^*-1)$ as a new distribution which will also dominate $g_i/G(t_g^*-1)$ or they will be exactly the same. In both cases, this expectation violates the assumption that f first-order stochastically dominates g. We cannot have $\nu_f^{t_g^*}<\nu_g^{t_g^*}$ and $F(t_g^*-1)=G(t_g^*-1)$ at the same time.

Consider the case when $\nu_f^{t_g^*}<\nu_g^{t_g^*}$ and $F(t_g^*-1)<G(t_g^*-1)$. Again inspect the ν values:

$$\frac{\sum_{i=1}^{t_g^*-1}f_i i+K}{F(t_g^*-1)}<\frac{\sum_{i=1}^{t_g^*-1}g_i i+K}{G(t_g^*-1)}.$$

For this to be true the following inequality should hold:

$$K<\frac{F(t_g^*-1)\sum_{i=1}^{t_g^*-1}g_i i-G(t_g^*-1)\sum_{i=1}^{t_g^*-1}f_i i}{G(t_g^*-1)-F(t_g^*-1)}. \tag{18}$$

We will use this inequality in comparing constraints (11) of distributions. We wanted to show the following where denominator n is cancelled out from inequality (17):

$$-\sum_{i=t_g^*}^{m-1} G(i)^n - (t_g^* - \nu_g^{t_g^*})G(t_g^* - 1)^n < -\sum_{i=t_g^*}^{m-1} F(i)^n - (t_g^* - \nu_f^{t_g^*})F(t_g^* - 1)^n,$$

$$\nu_g^{t_g^*} G(t_g^* - 1)^n - \nu_f^{t_g^*} F(t_g^* - 1)^n - t_g^*(G(t_g^* - 1)^n - F(t_g^* - 1)^n) < \sum_{i=t_g^*}^{m-1} G(i)^n - F(i)^n.$$

We know that the right-hand side is non-negative. Consider the left-hand side which is equal to following if we replace $\nu_f^{t_g^*}$ with $\sum_{i=1}^{t_g^*-1} f_i i + K/F(t_g^* - 1)$ and $\nu_g^{t_g^*}$ with its respective value:

$$G(t_g^* - 1)^{n-1} \sum_{i=1}^{t_g^*-1} g_i i - F(t_g^* - 1)^{n-1} \sum_{i=1}^{t_g^*-1} f_i i + (G(t_g^* - 1)^{n-1} - F(t_g^* - 1)^{n-1})K$$
$$- t_g^*(G(t_g^* - 1)^n - F(t_g^* - 1)^n).$$

Since we know that K is bounded from inequality (18), the expression above is less than the following: (this follows by using the bound on K and making some simplifications)

$$(G(t_g^* - 1)^n - F(t_g^* - 1)^n) \frac{\sum_{i=1}^{t_g^*-1} (g_i - f_i)(i - t_g^*)}{G(t_g^* - 1) - F(t_g^* - 1)}.$$

Recall that we are studying the case where $F(t_g^* - 1) < G(t_g^* - 1)$. Therefore, $G(t_g^* - 1)^n - F(t_g^* - 1)^n$ and $G(t_g^* - 1) - F(t_g^* - 1)$ are positive. Let us consider $\sum_{i=1}^{t_g^*-1} (g_i - f_i)(i - t_g^*)$:

$$\begin{aligned}\sum_{i=1}^{t_g^*-1} (g_i - f_i)(i - t_g^*) &= (G(1) - F(1))(1 - t_g^*) + (G(2) - F(2) - (G(1) - F(1)))(2 - t_g^*) + \cdots \\ &\quad + (G(t_g^*) - F(t_g^*) - (G(t_g^* - 1) - F(t_g^* - 1)))(t_g^* - 1 - t_g^*), \\ &= -\sum_{i=1}^{t_g^*-1} G(i) - F(i) < 0.\end{aligned}$$

This bound turns out to be negative. Therefore, the left-hand side we investigated is definitely less than the non-negative right-hand side $\sum_{i=t_g^*}^{m-1} G(i)^n - F(i)^n$. It means that inequality (17) is satisfied for this case as well. There is no other case so we proved that the dominated distribution g is the worst case and the given mechanism is optimal. ∎

When stochastic dominance does not hold, we shall see below on a numerical example that the optimal mechanism may not resemble any well-known mechanism. A critical issue is that the optimal mechanism may violate monotonicity. As a partial remedy, we propose an algorithm below for the case $\mathcal{P} = \{h, g\}$ where an approximately optimal (in the sense that we bound the error in the objective function), and monotonicity preserving mechanism is calculated. The algorithm works as follows. Since we can easily deduce the optimal cut-offs for each distribution by Proposition 2, Algorithm 1 starts with the optimal mechanism for the distribution with the maximum cut-off. Then it calculates the corresponding utilities for each distribution, and finds the worst case. Then it improves the worst case by increasing the allocation to types that are above the optimal cut-off for the worst-case distribution. If the algorithm stops at the first **if clause**, it means that it increased the worst case to its optimal value and arrived at the optimal cut-off of the worst-case distribution, which means that worst-case utility cannot be improved further. Otherwise, it defines a mechanism in the second **if clause** so that the expected utilities for both distributions are equal to each other.

Before moving forward, recall the definition of the notation ν_f^i which is a valuation for the set of types that are strictly below i if they were not to be inspected when the distribution is f. We also define $\nu_f^* = \nu_f^{t_f^*}$ where t_f^* is the optimal cut-off for the common prior f.

$$\nu_f^i = \frac{\sum_{j=1}^{i-1} f_j j + K}{F(i-1)}. \tag{19}$$

In order to clarify the meaning of this valuation, we can consider its relation with the function $Q(i)$ from Proposition 2:

$$Q(i) = i - \left(\frac{\sum_{j<i} f_j j + K}{\sum_{j<i} f_j} \right) > 0 \quad \text{iff } i - \nu_f^i > 0. \tag{20}$$

THEOREM 2 *When $\mathcal{P} = \{h, g\}$, Algorithm 1 gives the optimal solution if it stops at the first **if clause**. It gives an approximate solution with value not less than $(1 - 1/(\min_{f\in\mathcal{P}} \sum_{j=1}^m f_j j))$ times the optimal value if it stops at the second **if clause**.*

Proof If the algorithm stopped at the first **if clause** where $\nu^* = \nu_f^*$, then we have a mechanism which is the optimal solution to the worst-case distribution. Therefore, the given mechanism is optimal for maximizing the worst-case utility.

The plan for the rest of proof is as follows: we will first show that the allocation rule from the second **if clause** is feasible. Then we will prove that it gives an approximate solution.

For the second **if clause**, without getting into the **while clause** it is not possible to satisfy $\nu^* < \nu_f^*$. In the **while clause**, ν^* at least gets the value of ν_f^* which gives the optimum cut-off of the minimum objective at that time, and then the mechanism changes, which means that the condition $\nu^* < \nu_f^*$ is satisfied only if the minimum obj_f also changes after the change of mechanism. Then we can assume that the last update on ν^* changes the worst case from g to h or vice versa. Therefore, using the rearranged version of constraint (11) in (16), the following holds. Without loss of generality assume that obj_h is the worst case for the allocation with cut-off i^*:

$$-\sum_{i=i^*+1}^{m-1} \frac{H(i)^n}{n} - (i^* + 1 - \nu_h^{i^*+1})\frac{H(i^*)^n}{n} > -\sum_{i=i^*+1}^{m-1} \frac{G(i)^n}{n} - (i^* + 1 - \nu_g^{i^*+1})\frac{G(i^*)^n}{n}, \tag{21}$$

$$-\sum_{i=i^*}^{m-1} \frac{H(i)^n}{n} - (i^* - \nu_h^{i^*})\frac{H(i^*-1)^n}{n} < -\sum_{i=i^*}^{m-1} \frac{G(i)^n}{n} - (i^* - \nu_g^{i^*})\frac{G(i^*-1)^n}{n}.$$

The first inequality is the relation between constraint (11)'s value for each distribution when allocation with cut-off $i^* + 1$ is undertaken. Since our algorithm did not give this mechanism as a solution then the distribution g achieving the worst case can be improved. In other words, $\nu^* > \nu_f^*$ is satisfied. Since g attains the worst case we have $\nu_f^* = \nu_g^*$. After an update in **while clause**, we have $\nu^* \leq i^*$ and ν^* is not less than ν_g^*. Since the algorithm goes through with second 'if' condition, our worst-case objective is not g anymore but h. The second inequality above

Algorithm 1 An approximate mechanism when $\mathcal{P} = \{h, g\}$

Initialize:

$$\nu^* \leftarrow \max_{k\in\mathcal{P}}\{\nu_k^*\}$$

$$\nu_f^* \leftarrow 1$$

while $\nu^* > \nu_f^*$ **do**

$$\nu^* = \begin{cases} \nu^* & \text{if } \nu_f^* = 1 \\ \max\{\nu^* - 1, \nu_f^*\} & \text{otherwise} \end{cases}$$

$$A_f(i) = \begin{cases} \dfrac{F(i)^n - F(i-1)^n}{f_i n} & \text{if } i \geq \nu^* \\ \dfrac{F(i^*-1)^{n-1}}{n} & \text{where } i^* = \min\limits_{j\in T, j\geq \nu^*} j \end{cases}$$

$$obj_h \leftarrow \sum_{i\in T} h_i(i-K)A_h(i) + KA_h(1)$$

$$obj_g \leftarrow \sum_{i\in T} g_i(i-K)A_g(i) + KA_g(1)$$

$$\nu_f^* = \begin{cases} \nu_h^* & \text{if } obj_h = \min\limits_{k\in\mathcal{P}} obj_k \\ \nu_g^* & \text{otherwise} \end{cases}$$

end while

if $\nu^* = \nu_f^*$ **then**

Stop at the current solution

else

if $\nu^* < \nu_f^*$ **then**

$i^* \leftarrow \min\limits_{j\in T, j\geq\nu^*} j$

$x \leftarrow \frac{n(obj_h - obj_g)}{G(i^*-1)(G(i^*)^{n-1} - G(i^*-1)^{n-1})(\nu_g^{i^*} - i^*) - H(i^*-1)(H(i^*)^{n-1} - H(i^*-1)^{n-1})(\nu_h^{i^*} - i^*)}$

$$a_i(i, t^{-1}) = \begin{cases} \dfrac{1}{n_i(t)} & \text{if } i \geq \max_{k\in t^{-1}} k \text{ and } i > i^* \\ (1 - \dfrac{n - n_i(t)}{n} x)/n_i(t) & \text{if } i = \max\limits_{k\in t} k = i^* \\ \dfrac{x}{n} & \text{if } i < \max_{k\in t^{-1}} k = i^* \\ \dfrac{1}{n} & \text{if } \max\limits_{k\in t} k < i^* \\ 0 & \text{otherwise} \end{cases}$$

Stop at the current solution

end if

end if

shows this relation when we have allocation with cut-off i^*. Note that $\nu_g^{i^*+1}$ and $\nu_h^{i^*}$ values are given by Equation (19).

The first inequality above can be achieved by the algorithm's allocation when $x = 1$ and the second inequality is given by the same allocation when $x = 0$. Next we will show that the value given to x by Algorithm 1 results in $obj_h^* = obj_g^*$. Since these functions are linear, we can say that $0 \leq x \leq 1$ and the given allocation rule is feasible. Let

$$A_g^*(i) = \begin{cases} \dfrac{G(i)^n - G(i-1)^n}{ng_i} & \text{if } i \geq i^* + 1, \\ \dfrac{G(i^*)^n - G(i^*-1)^n}{ng_{i^*}} - \dfrac{G(i^*)^{n-1}G(i^*-1) - G(i^*-1)^n}{ng_{i^*}} x & \text{if } i = i^*, \\ \dfrac{G(i^*-1)^{n-1}}{n} + \dfrac{G(i^*)^{n-1} - G(i^*-1)^{n-1}}{n} x & \text{otherwise.} \end{cases}$$

This interim allocation is consistent with the allocation given by the algorithm. In order to check, set $x = 1$ and $x = 0$ to get the values $G(i^*)^{n-1}/n$ and $G(i^*)^n - G(i^*-1)^n/ng_{i^*}$ for $A_g^*(i^*)$, respectively. Let us consider the obj_g^* that resulted from the algorithm's allocation: (We will also use the last given values to obj_g and obj_h from the algorithm.)

$$\begin{aligned} obj_g^* = {} & \frac{(m-K)}{n} - \sum_{i=i^*+1}^{m-1} \frac{G(i)^n}{n} - (i^*+1-K)\frac{G(i^*)^n}{n} \\ & + g_{i^*}(i^*-K)\left(\frac{G(i^*)^n - G(i^*-1)^n}{ng_{i^*}} - \frac{G(i^*)^{n-1}G(i^*-1) - G(i^*-1)^n}{ng_{i^*}} x\right) \\ & + G(i^*-1)(\nu_g^{i^*} - K)\left(\frac{G(i^*-1)^{n-1}}{n} + \frac{G(i^*)^{n-1} - G(i^*-1)^{n-1}}{n} x\right) \\ = {} & \frac{(m-K)}{n} - \sum_{i=i^*+1}^{m-1} \frac{G(i)^n}{n} - (i^*+1-K)\frac{G(i^*)^n}{n} + g_{i^*}(i^*-K)\frac{G(i^*)^n - G(i^*-1)^n}{ng_{i^*}} \\ & + (\nu_g^{i^*} - K)\frac{G(i^*-1)^n}{n} + x\left(G(i^*-1)\frac{G(i^*)^{n-1} - G(i^*-1)^{n-1}}{n}(\nu_g^{i^*} - i^*)\right) \\ = {} & \frac{(m-K)}{n} - \sum_{i=i^*}^{m-1} \frac{G(i)^n}{n} - (i^* - \nu_g^{i^*})\frac{G(i^*-1)^n}{n} \\ & + x\left(G(i^*-1)\frac{G(i^*)^{n-1} - G(i^*-1)^{n-1}}{n}(\nu_g^{i^*} - i^*)\right) \\ = {} & obj_g + x\left(G(i^*-1)\frac{G(i^*)^{n-1} - G(i^*-1)^{n-1}}{n}(\nu_g^{i^*} - i^*)\right). \end{aligned}$$

This equality is also true for obj_h^*. Let us find the value of x when $obj_g^* = obj_h^*$:

$$\begin{aligned} & obj_g + x\left(G(i^*-1)\frac{G(i^*)^{n-1} - G(i^*-1)^{n-1}}{n}(\nu_g^{i^*} - i^*)\right) \\ & \quad = obj_h + x\left(H(i^*-1)\frac{H(i^*)^{n-1} - H(i^*-1)^{n-1}}{n}(\nu_h^{i^*} - i^*)\right), \end{aligned}$$

$$x\left(G(i^*-1)\frac{G(i^*)^{n-1}-G(i^*-1)^{n-1}}{n}(\nu_g^{i^*}-i^*)-H(i^*-1)\right.$$
$$\left.\times\frac{H(i^*)^{n-1}-H(i^*-1)^{n-1}}{n}(\nu_h^{i^*}-i^*)\right)=obj_h-obj_g,$$
$$x=\frac{n(obj_h-obj_g)}{\begin{array}{c}G(i^*-1)(G(i^*)^{n-1}-G(i^*-1)^{n-1})(\nu_g^{i^*}-i^*)\\-H(i^*-1)(H(i^*)^{n-1}-H(i^*-1)^{n-1})(\nu_h^{i^*}-i^*)\end{array}}.$$

This is the exact value given to x. Therefore, we have a feasible allocation and $obj_h^*=obj_g^*$.

In order to prove the performance of the mechanism given in the second **if clause**, we will show some upper bounds on the optimal solution. Without loss of generality assume that $\nu_h^*=\max_{f\in\mathcal{P}}\{\nu_f^*\}$ and recall the arguments regarding the inequalities in (21).

CLAIM 1 *Optimal solution of the OPTa in* (10) *has the following upper bound:*

$$y^*\le\max\left\{obj_h+\left(H(i^*-1)\frac{H(i^*)^{n-1}-H(i^*-1)^{n-1}}{n}(\nu_h^{i^*}-i^*)\right),obj_g\right\}.$$

Proof We can rewrite the inequalities in (21) as:

$$obj_h+\left(H(i^*-1)\frac{H(i^*)^{n-1}-H(i^*-1)^{n-1}}{n}(\nu_h^{i^*}-i^*)\right)>obj_g$$
$$+\left(G(i^*-1)\frac{G(i^*)^{n-1}-G(i^*-1)^{n-1}}{n}(\nu_g^{i^*}-i^*)\right),$$
$$obj_h<obj_g.$$

First inequality gives the relation between expected utilities for h and g when we allocate according to cut-off i^*+1 and the second inequality is similarly calculated with respect to cut-off i^*. In order to get these values one can also set x to 1 and 0, respectively. Now we compare the allocation rule that gives the second inequality and the optimal allocation rule when h is the common prior:

$$a_i(i,t^{-1})=\begin{cases}\frac{1}{n_i(t)} & \text{if } i\ge\max_{k\in t}k\ge i^*,\\ \frac{1}{n} & \text{if } i\le\max_{k\in t}k<i^*,\\ 0 & \text{otherwise,}\end{cases}\qquad a_i^{*h}(i,t^{-1})=\begin{cases}\frac{1}{n_i(t)} & \text{if } i\ge\max_{k\in t}k\ge t_h^*,\\ \frac{1}{n} & \text{if } i\le\max_{k\in t}k<t_h^*,\\ 0 & \text{otherwise.}\end{cases}$$

Recall our assumption, $\nu_h^*=\max_{f\in\mathcal{P}}\{\nu_f^*\}$, and the definition of i^* from Algorithm 1, $i^*\le\nu_h^*$. Notice that above allocations have the same values for $i\ge\max_{k\in t}k\ge t_h^*$ and $i\le\max_{k\in t}k<i^*$. Therefore, it is not possible to improve the worst case, obj_h, by changing these variables. This is because of the fact that when we consider only distribution h, objective becomes additively separable in allocation rule variables. Then obj_h can be improved only by altering $a_i(i,t^{-1})$ variables where $t_h^*>i\ge\max_{k\in t}k\ge i^*$. However, these variables have the same values as the in optimal allocation for common prior g since $i^* \geq \nu_g^*$ holds by definition:

$$a_i^{*g}(i, t^{-1}) = \begin{cases} \dfrac{1}{n_i(t)} & \text{if } i \geq \max_{k \in t} k = t_g^*, \\ \dfrac{1}{n} & \max_{k \in t} k < t_g^*, \\ 0 & \text{otherwise,} \end{cases}$$

which means that improving the worst case will decrease obj_g. Now suppose that we switched to allocation rule with cut-off $i^* + 1$ which will result in the first inequality above. Then using the same reasoning worst case for prior g cannot be improved without decreasing the utility for h. Therefore, we conclude that

$$y^* \leq \max \left\{ obj_g, obj_h + \left(H(i^* - 1) \frac{H(i^*)^{n-1} - H(i^* - 1)^{n-1}}{n} (\nu_h^{i^*} - i^*) \right) \right\}. \qquad \blacksquare$$

Now we can prove the performance of the mechanism given by Algorithm 1 in the second **if clause**. We know that the following inequalities hold:

$$obj_g + x \left(G(i^* - 1) \frac{G(i^*)^{n-1} - G(i^* - 1)^{n-1}}{n} (\nu_g^{i^*} - i^*) \right) > obj_g$$
$$+ \left(G(i^* - 1) \frac{G(i^*)^{n-1} - G(i^* - 1)^{n-1}}{n} (\nu_g^{i^*} - i^*) \right),$$
$$obj_h + x \left(H(i^* - 1) \frac{H(i^*)^{n-1} - H(i^* - 1)^{n-1}}{n} (\nu_h^{i^*} - i^*) \right) > obj_h,$$
$$y^* \leq \max \left\{ obj_g, obj_h + \left(H(i^* - 1) \frac{H(i^*)^{n-1} - H(i^* - 1)^{n-1}}{n} (\nu_h^{i^*} - i^*) \right) \right\}.$$

Let X denote the left-hand side of the first two inequalities, i.e. utility of mechanism given by the second **if clause** of the algorithm. Then the following is also true:

$$X > \max \left\{ obj_g + \left(G(i^* - 1) \frac{G(i^*)^{n-1} - G(i^* - 1)^{n-1}}{n} (\nu_g^{i^*} - i^*) \right), obj_h \right\}.$$

If we divide the left-hand side of this inequality by y^* and divide the right-hand side with the upper bound of y^*, direction of the inequality would be the same:

$$\frac{X}{y^*} > \frac{\max \left\{ obj_g + \left(G(i^* - 1) \frac{G(i^*)^{n-1} - G(i^* - 1)^{n-1}}{n} (\nu_g^{i^*} - i^*) \right), obj_h \right\}}{\max \left\{ obj_g, obj_h + \left(H(i^* - 1) \frac{H(i^*)^{n-1} - H(i^* - 1)^{n-1}}{n} (\nu_h^{i^*} - i^*) \right) \right\}},$$

$$X > \min\{Y, Z\} y^*, \text{ where,}$$

$$Y = \frac{obj_g + \left(G(i^* - 1) \frac{G(i^*)^{n-1} - G(i^* - 1)^{n-1}}{n} (\nu_g^{i^*} - i^*) \right)}{obj_g},$$

$$Z = \frac{obj_h}{obj_h + \left(H(i^* - 1) \frac{H(i^*)^{n-1} - H(i^* - 1)^{n-1}}{n} (\nu_h^{i^*} - i^*) \right)}.$$

Recall our assumption $v_h^* \geq i^* \geq v_g^*$ which also refers to $t_h^* \geq i^* \geq t_g^*$. Then we can say $v_h^{i^*} \geq i^* \geq v_g^{i^*}$ because of the property of function $Q(i)$ explained in Proposition 2. Therefore both Y and Z are less than 1. Now we will simplify the approximation ratio. Y and Z can be interpreted as the ratio of the utility for one agent to his improved utility when the strategy for one type is changed. If we consider the highest possible change, i.e. from no allocation to full allocation to some type i, it is easy to see that expected improvement to one agent's expected utility can be at most $1/n$: (illustrating with Y)

$$Y = \frac{obj_g + \left(G(i^* - 1)\frac{G(i^*)^{n-1} - G(i^*-1)^{n-1}}{n}(v_g^{i^*} - i^*)\right)}{obj_g} > \frac{obj_g - 1/n}{obj_g}.$$

Besides obj_g value, expected utility for one agent cannot be worst than utility given by a randomized mechanism: $obj_g \geq (\sum_{j=1}^m g_j j)/n$. Therefore the following holds:

$$Y > \frac{obj_g - 1/n}{obj_g} > 1 - 1/\left(\sum_{j=1}^m g_j j\right).$$

Using the same method on Z, one can deduce that the given mechanism has value at least $(1 - 1/(\min_{f\in\mathcal{P}} \sum_{j=1}^m f_j j))$ times the optimal value. ■

In the second mechanism, the algorithm allocates the good to the highest bid if it is strictly above cut-off i^*. If the highest bid is exactly i^*, then we decide the allocation after a lottery where the highest bidder has a bigger chance of winning. Under the cut-off, we allocate the good randomly.

Both mechanisms offered by Algorithm 1 give simple, applicable mechanisms with monotone interim allocations. Algorithm 1 computes the cut-off and the allocation probabilities. We again deduce the inspection skipping rule from the equality $a_i(1, t^{-1}) = s_i(j, t^{-1})$ for all j in T and t^{-1} in T^{n-1}, which returns expected inspection skipping rules equal to $A_f(1)$ for all $f \in \mathcal{P}$.

We now consider a numerical example to illustrate the performance of the second mechanism.

Example 1 We have two agents and $\mathcal{P} = \{h, g\}$. Inspection cost is $K = 0.1$, type set is $T = \{1, 2, \ldots, 10\}$ and density functions are as follows:

$$h_i = \begin{cases} 0.1001 & \text{if } i = 1 \\ 0.0001 & \text{if } i = 2 \\ 0.0001 & \text{if } i = 3 \\ 0.0001 & \text{if } i = 4 \\ 0.0001 & \text{if } i = 5 \\ 0.0001 & \text{if } i = 6 \\ 0.4001 & \text{if } i = 7 \\ 0.4991 & \text{if } i = 8 \\ 0.0001 & \text{if } i = 9 \\ 0.0001 & \text{if } i = 10, \end{cases} \qquad g_i = \begin{cases} 0.005 & \text{if } i = 1 \\ 0.005 & \text{if } i = 2 \\ 0.005 & \text{if } i = 3 \\ 0.005 & \text{if } i = 4 \\ 0.005 & \text{if } i = 5 \\ 0.33 & \text{if } i = 6 \\ 0.27 & \text{if } i = 7 \\ 0.27 & \text{if } i = 8 \\ 0.1 & \text{if } i = 9 \\ 0.005 & \text{if } i = 10. \end{cases}$$

If we solve for distributions separately, we get optimal cut-offs $t_h^* = 2$ and $t_g^* = 7$. Recall that optimal mechanism allocates the good to the highest type if it is above the cut-off, otherwise it will randomize. Optimal inspection policy is to skip inspection if all types fall below the cut-off and with probability $1/n$ skip inspection if there is only one type above the cut-off. Inspection takes place for all other cases.

Returning to the ambiguity averse principal's problem with two distributions, the optimal solution for interim allocation variables (obtained by solving numerically the linear program) are given below:

$$A_h(i) = \begin{cases} \dfrac{H(i)^2 - H(i-1)^2}{2h_i} & \text{if } i \geq 7, \\ 0.058 & \text{if } i = 6, \\ 0.1 & \text{if } i = 5, \\ 0.1 & \text{if } i = 4, \\ 0.1 & \text{if } i = 3, \\ 0.05 & \text{if } i = 2, \\ 0.05 & \text{if } i = 1, \end{cases} \quad A_g(i) = \begin{cases} \dfrac{G(i)^2 - G(i-1)^2}{2g_i} & \text{if } i \geq 7, \\ 0.18 & \text{if } i = 6, \\ 0.144 & \text{if } i = 5, \\ 0.144 & \text{if } i = 4, \\ 0.144 & \text{if } i = 3, \\ 0.144 & \text{if } i = 2, \\ 0.144 & \text{if } i = 1. \end{cases}$$

Optimal values of expected inspection skipping variables are

$$S_h(i) = 0.05 \quad \forall i \in T, \quad S_g(i) = \begin{cases} 0.144 & \text{if } i \geq 6 \text{ or } i = 1, \\ 0.141 & \text{if } i = 5, \\ 0.136 & \text{if } i = 4, \\ 0.131 & \text{if } i = 3, \\ 0.129 & \text{if } i = 2. \end{cases}$$

The optimal objective value for the worst-case utility is 3.799727. First of all, optimal inspection skipping for g is not always equal to $A_g(1)$. Secondly, underlying allocation rule is designed in a way that expected allocation for type 6 breaks the monotonicity for distribution f but not for g. We can conclude that implementing the underlying allocation and inspection policy is not easy.

For this example, Algorithm 1 cannot find the optimal solution but promises a mechanism whose objective value is not less than 0.85502 times the optimal solution. However, its actual performance is much better. Below we give the algorithm's allocation and the inspection skipping rules whose actual value is 3.799698, i.e. 0.999992 multiple of the optimal objective:

$$a_i^*(i,j) = \begin{cases} 1 & \text{if } i > j \text{ and } i \geq 7, \\ \frac{1}{2} & \text{if } i = j \text{ and } i \geq 7, \\ 0.815 & \text{if } i = 6 > j, \\ \frac{1}{2} & \text{if } i = j = 6, \\ 0.185 & \text{if } i < j = 6, \\ \frac{1}{2} & \text{if } i < 6 \text{ and } j < 6, \\ 0 & \text{otherwise}, \end{cases}$$

$$s_i^*(i,j) = \begin{cases} 0.185 & \text{if } j = 6, \\ \frac{1}{2} & \text{if } j < 6, \\ 0 & \text{otherwise}. \end{cases}$$

For ease of notation, the value given to $x = 0.392234$ by Algorithm 1 is changed to 0.39. In order to understand the allocation rule above, consider a mechanism with cut-off equal to 6.

Algorithm 1 only changes the way type 6 is allocated so that the worst case is increased. In this example, for profile with a maximum type equal to 6, allocation to 6 is decided by a lottery with probability 0.815 so that the inspection skipping rule for those profiles and the utility for g is increased. Recall that optimal cut-offs were $t_h^* = 2$ and $t_g^* = 7$.

4. Concluding remarks

In this paper, we considered the problem of optimal allocation under costly inspection (without monetary transfers) of an ambiguity averse principal facing a number of potential distributions that may govern the agents' discrete valuations. We first formulated the problem without ambiguity in the valuation distribution, and derived the optimal mechanism for the case where the inspection cost is identical for all agents. Then, we investigated the optimal mechanism when ambiguity in the valuation distribution is present, and the principal maximizes his/her worst-case expected utility. Under a stochastic dominance condition, we are able to give the optimal mechanism for an arbitrary number of potential distributions. When the stochastic dominance condition does not hold, we gave an algorithm for an arbitrary number of agents, but two potential distributions. The algorithm either gives the optimal mechanism or a very close approximation with desirable monotonicity properties. The approximation ratio depends on the distributions in the prior set. Therefore, we consider providing a constant approximation ratio as a future work as well as extending the results (Theorem 2) of the present paper to an arbitrary number of distributions, allowing ambiguity averse agents and other ambiguity models such as the smooth ambiguity model of [12].

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

The research of Kemal Güler was supported by a fellowship grant from TUBITAK BIDEP 2236 Co-Circulation Fellowship Program Project Number 114C020.

Note

1. The details can be obtained from the authors upon request.

References

[1] E. Ben-Porath, E. Dekel, and B.L. Lipman, *Optimal allocation with costly verification*, Amer. Econ. Rev. 104 (2014), pp. 3779–3813.
[2] A. Ben-Tal and A. Nemirovski, *Robust convex optimization*, Math. Oper. Res. 23 (1998), pp. 769–805.
[3] A. Ben-Tal and A. Nemirovski, *Robust solutions of uncertain linear programs*, Oper. Res. Lett. 25 (1999), pp. 1–13.
[4] D. Bergemann and K. Schlag, *Pricing without priors*, J. Eur. Econ. Assoc. 6 (2008), pp. 560–569.
[5] D. Bergemann and K. Schlag, *Robust monopoly pricing*, J.f. Econ. Theory 146 (2011), pp. 2527–2543.
[6] A.L. Bodoh-Creed, *Ambiguous beliefs and mechanism design*, Games Econ. Behav. 75 (2012), pp. 518–537.
[7] K.C. Border and J. Sobel, *Samurai accountant: A theory of auditing and plunder*, Rev. Econ. Stud. 54 (1987), pp. 525–540.
[8] S. Bose, E. Ozdenoren, and A. Pape, *Optimal auctions with ambiguity*, Theoret. Econ. 1 (2006), pp. 411–438.
[9] D. Gale and M. Hellwig, *Incentive compatible debt contracts: The one-period problem*, Rev. Econ. Stud. 52 (1985), pp. 647–663.
[10] I. Gilboa and D. Schmeidler, *Maxmin expected utility with non-unique prior*, J. Math. Econ. 18 (1989), pp. 141–153.

[11] J. Glazer and A. Rubinstein, *On optimal rules of persuasion*, Econometrica 72 (2004), pp. 1715–1736.
[12] P. Klibanoff, M. Marinacci, and S. Mukerji, *A smooth model of decision making under ambiguity*, Econometrica 73 (2005), pp. 1849–1892.
[13] C. Koçyigit, H. Bayrak, and M. Pınar, *Robust auction design under multiple priors by linear and integer programming*, Tech. Rep., Bilkent University, Ankara, July 2015.
[14] G. Lopomo, L. Rigotti, and C. Shannon, *Uncertainty in mechanism design*, Tech. Rep., Fuqua School of Business, Duke University, Durham, NC, 2009.
[15] D. Mookherjee and I. Png, *Optimal auditing, insurance and redistribution*, J. Econ. 104 (1989), pp. 399–415.
[16] R.M. Townsend, *Optimal contracts and competitive markets with costly state verification*, J. Econ. Theory 21 (1979), pp. 265–293.
[17] R.V. Vohra, *Optimization and mechanism design*, Math. Program. 134 (2012), pp. 283–303.