



Arithmetical rank of binomial ideals

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Abstract. In this paper, we investigate the arithmetical rank of a binomial ideal J . We provide lower bounds for the binomial arithmetical rank and the J -complete arithmetical rank of J . Special attention is paid to the case where J is the binomial edge ideal of a graph. We compute the arithmetical rank of such an ideal in various cases.

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1. Introduction. Consider the polynomial ring $K[x_1, \dots, x_m]$ in the variables x_1, \dots, x_m over a field K . For the sake of simplicity, we will denote by $\mathbf{x}^{\mathbf{u}}$ the monomial $x_1^{u_1} \cdots x_m^{u_m}$ of $K[x_1, \dots, x_m]$, with $\mathbf{u} = (u_1, \dots, u_m) \in \mathbb{N}^m$, where \mathbb{N} stands for the set of non-negative integers. A *binomial* in the sense of [12, Chapter 8] is a difference of two monomials, i.e. it is of the form $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$. A *binomial ideal* is an ideal generated by binomials.

Toric ideals serve as important examples of binomial ideals. Let $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ be a subset of \mathbb{Z}^n . The *toric ideal* $I_{\mathcal{A}}$ is the kernel of the K -algebra homomorphism $\phi: K[x_1, \dots, x_m] \rightarrow K[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ given by

$$\phi(x_i) = \mathbf{t}^{\mathbf{a}_i} = t_1^{a_{i,1}} \cdots t_n^{a_{i,n}} \quad \text{for all } i = 1, \dots, m,$$

where $\mathbf{a}_i = (a_{i,1}, \dots, a_{i,n})$.

We grade $K[x_1, \dots, x_m]$ by the semigroup $\mathbb{N}\mathcal{A} := \{l_1\mathbf{a}_1 + \cdots + l_m\mathbf{a}_m \mid l_i \in \mathbb{N}\}$ setting $\deg_{\mathcal{A}}(x_i) = \mathbf{a}_i$ for $i = 1, \dots, m$. The \mathcal{A} -degree of a monomial $\mathbf{x}^{\mathbf{u}}$ is defined by

$$\deg_{\mathcal{A}}(\mathbf{x}^{\mathbf{u}}) = u_1\mathbf{a}_1 + \cdots + u_m\mathbf{a}_m \in \mathbb{N}\mathcal{A}.$$

A polynomial $F \in K[x_1, \dots, x_m]$ is \mathcal{A} -homogeneous if the \mathcal{A} -degrees of all the monomials that occur in F are the same. An ideal is \mathcal{A} -homogeneous if it is generated by \mathcal{A} -homogeneous polynomials. The ideal $I_{\mathcal{A}}$ is generated by all

the binomials $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ such that $\deg_{\mathcal{A}}(\mathbf{x}^{\mathbf{u}}) = \deg_{\mathcal{A}}(\mathbf{x}^{\mathbf{v}})$ (see [11, Lemma 4.1]), thus $I_{\mathcal{A}}$ is \mathcal{A} -homogeneous.

Let $J \subset K[x_1, \dots, x_m]$ be a binomial ideal. There exist a positive integer n and a vector configuration $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_m\} \subset \mathbb{Z}^n$ such that $J \subset I_{\mathcal{A}}$, see for instance [7, Theorem 1.1]. We say that a polynomial $F = c_1M_1 + \dots + c_sM_s \in J$, where $c_i \in K$ and M_1, \dots, M_s are monomials, is J -complete if $M_i - M_l \in J$ for every $1 \leq i < l \leq s$. Clearly every J -complete polynomial F is also \mathcal{A} -homogeneous.

Computing the least number of polynomial equations defining an algebraic set is a classical problem in Algebraic Geometry which goes back to Kronecker [9]. This problem is equivalent, over an algebraically closed field, with the corresponding problem in Commutative Algebra of the determination of the smallest integer s for which there exist polynomials F_1, \dots, F_s in J such that $\text{rad}(J) = \text{rad}(F_1, \dots, F_s)$. The number s is commonly known as the *arithmetical rank* of J and will be denoted by $\text{ara}(J)$. Since J is generated by binomials, it is natural to define the *binomial arithmetical rank* of J , denoted by $\text{bar}(J)$, as the smallest integer s for which there exist binomials B_1, \dots, B_s in J such that $\text{rad}(J) = \text{rad}(B_1, \dots, B_s)$. Furthermore we can define the J -complete arithmetical rank of J , denoted by $\text{ara}_c(J)$, as the smallest integer s for which there exist J -complete polynomials F_1, \dots, F_s in J such that $\text{rad}(J) = \text{rad}(F_1, \dots, F_s)$. Finally we define the \mathcal{A} -homogeneous arithmetical rank of J , denoted by $\text{ara}_{\mathcal{A}}(J)$, as the smallest integer s for which there exist \mathcal{A} -homogeneous polynomials F_1, \dots, F_s in J such that $\text{rad}(J) = \text{rad}(F_1, \dots, F_s)$. From the definitions and [2, Corollary 3.3.3] we deduce the following inequalities:

$$\text{cd}(J) \leq \text{ara}(J) \leq \text{ara}_{\mathcal{A}}(J) \leq \text{ara}_c(J) \leq \text{bar}(J)$$

where $\text{cd}(J)$ is the cohomological dimension of J .

In Sect. 2 we introduce the simplicial complex Δ_J and use combinatorial invariants of the aforementioned complex to provide lower bounds for the binomial arithmetical rank and the J -complete arithmetical rank of J . In particular we prove that $\text{bar}(J) \geq \delta(\Delta_J)_{\{0,1\}}$ and $\text{ara}_c(J) \geq \delta(\Delta_J)_{\Omega}$, see Theorem 2.6.

In Sect. 3 we study the arithmetical rank of the binomial edge ideal J_G of a graph G . This class of ideals generalizes naturally the determinantal ideal generated by the 2-minors of the matrix

$$\begin{pmatrix} x_1 & x_2 & \dots & x_n \\ x_{n+1} & x_{n+2} & \dots & x_{2n} \end{pmatrix}.$$

We prove (see Theorem 3.3) that, for a binomial edge ideal J_G , both the binomial arithmetical rank and the J_G -complete arithmetical rank coincide with the number of edges of G . If G is the complete graph on the vertex set $\{1, \dots, n\}$, then, from [3, Theorem 2], the arithmetical rank of J_G equals $2n - 3$. It is still an open problem to compute $\text{ara}(J_G)$ when G is not the complete graph. We show that $\text{ara}(J_G) \geq n + l - 2$, where n is the number of vertices of G and l is the vertex connectivity of G . Furthermore we prove that in several cases $\text{ara}(J_G) = \text{cd}(J_G) = n + l - 2$, see Theorems 3.7, 3.9, Corollary 3.10, and Theorem 3.13.

2. Lower bounds. First we will use the notion of indispensability to introduce the simplicial complex Δ_J . Let $J \subset K[x_1, \dots, x_m]$ be a binomial ideal containing no binomials of the form $\mathbf{x}^{\mathbf{u}} - 1$, where $\mathbf{u} \neq \mathbf{0}$. A *binomial* $B = M - N \in J$ is called *indispensable* of J if every system of binomial generators of J contains B or $-B$, while a *monomial* M is called *indispensable* of J if every system of binomial generators of J contains a binomial B such that M is a monomial of B . Let \mathcal{M}_J be the ideal generated by all monomials M for which there exists a nonzero $M - N \in J$. By [7, Proposition 1.5] the set $G(\mathcal{M}_J)$ of indispensable monomials of J is the unique minimal generating set of \mathcal{M}_J .

The support of a monomial $\mathbf{x}^{\mathbf{u}}$ of $K[x_1, \dots, x_m]$ is $\text{supp}(\mathbf{x}^{\mathbf{u}}) := \{i | x_i \text{ divides } \mathbf{x}^{\mathbf{u}}\}$. Let \mathcal{T} be the set of all $E \subset \{1, \dots, m\}$ for which there exists an indispensable monomial M of J such that $E = \text{supp}(M)$. Let \mathcal{T}_{\min} denote the set of minimal elements of \mathcal{T} .

Definition 2.1. We associate to J a simplicial complex Δ_J with vertices the elements of \mathcal{T}_{\min} . Let $T = \{E_1, \dots, E_k\}$ be a subset of \mathcal{T}_{\min} , then $T \in \Delta_J$ if there exist $M_i, 1 \leq i \leq k$, such that $\text{supp}(M_i) = E_i$ and $M_i - M_l \in J$ for every $1 \leq i < l \leq k$.

Next we will study the connection between the radical of J and Δ_J . The *induced subcomplex* Δ' of Δ_J by certain vertices $\mathcal{V} \subset \mathcal{T}_{\min}$ is the subcomplex of Δ_J with vertices \mathcal{V} and $T \subset \mathcal{V}$ is a simplex of the subcomplex Δ' if T is a simplex of Δ_J . A subcomplex H of Δ_J is called a *spanning subcomplex* if both have exactly the same set of vertices.

Let F be a polynomial in $K[x_1, \dots, x_m]$. We associate to F the induced subcomplex $\Delta_J(F)$ of Δ_J consisting of those vertices $E_i \in \mathcal{T}_{\min}$ with the property: there exists a monomial M_i in F such that $E_i = \text{supp}(M_i)$. The next theorem provides a necessary condition under which a set of polynomials in the binomial ideal J generates the radical of J up to radical.

Proposition 2.2. *Let K be any field. If $\text{rad}(J) = \text{rad}(F_1, \dots, F_s)$ for some polynomials F_1, \dots, F_s in J , then $\cup_{i=1}^s \Delta_J(F_i)$ is a spanning subcomplex of Δ_J .*

Proof. Let $E = \text{supp}(\mathbf{x}^{\mathbf{u}}) \in \mathcal{T}_{\min}$, where $B = \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in J$ and $\mathbf{x}^{\mathbf{u}}$ is an indispensable monomial of J . We will show that there exists a monomial M in some $F_l, 1 \leq l \leq s$, such that $E = \text{supp}(M)$. Since $\text{rad}(J) = \text{rad}(F_1, \dots, F_s)$, there is a power $B^r, r \geq 1$, which belongs to the ideal generated by F_1, \dots, F_s . Thus there is a monomial M in some F_l dividing the monomial $(\mathbf{x}^{\mathbf{u}})^r$ and therefore $\text{supp}(M) \subseteq \text{supp}(\mathbf{x}^{\mathbf{u}})$. But $F_l \in J$ and J is generated by binomials, so there exists $\mathbf{x}^{\mathbf{z}} - \mathbf{x}^{\mathbf{w}} \in J$ such that $\mathbf{x}^{\mathbf{z}}$ divides M . Since $\mathbf{x}^{\mathbf{z}} \in \mathcal{M}_J$ and $G(\mathcal{M}_J)$ generates \mathcal{M}_J , there is an indispensable monomial N dividing $\mathbf{x}^{\mathbf{z}}$, thus

$$\text{supp}(N) \subseteq \text{supp}(\mathbf{x}^{\mathbf{z}}) \subseteq \text{supp}(M) \subseteq E.$$

Since $E \in \mathcal{T}_{\min}$, we deduce that $E = \text{supp}(N)$, and therefore $E = \text{supp}(M)$. □

- Remark 2.3.** (1) If F is a J -complete polynomial of J , then $\Delta_J(F)$ is a simplex. To see that $\Delta_J(F)$ is a simplex, suppose that $\Delta_J(F) \neq \emptyset$ and let $T = \{E_1, \dots, E_k\}$ be the set of vertices of $\Delta_J(F)$. For every $1 \leq i \leq k$ there exists a monomial M_i , $1 \leq i \leq k$, in F such that $E_i = \text{supp}(M_i)$. Since F is J -complete, we have that $M_i - M_l \in J$ for every $1 \leq i < l \leq k$. Thus $\Delta_J(F)$ is a simplex.
- (2) If B is a binomial of J , then $\Delta_J(B)$ is either a vertex, an edge, or the empty set.

Remark 2.4. If the equality $\text{rad}(J) = \text{rad}(F_1, \dots, F_s)$ holds for some J -complete polynomials F_1, \dots, F_s in J , then $\cup_{i=1}^s \Delta_J(F_i)$ is a spanning subcomplex of Δ_J and each $\Delta_J(F_i)$ is a simplex.

For a simplicial complex Δ we denote by r_Δ the smallest number s of simplices T_i of Δ , such that the subcomplex $\cup_{i=1}^s T_i$ is spanning and by b_Δ the smallest number s of simplices T_i of Δ , such that the subcomplex $\cup_{i=1}^s T_i$ is spanning and each T_i is either an edge, a vertex, or the empty set.

Theorem 2.5. *Let K be any field, then $b_{\Delta_J} \leq \text{bar}(J)$ and $r_{\Delta_J} \leq \text{ara}_c(J)$.*

It turns out that both b_{Δ_J} and r_{Δ_J} have a combinatorial interpretation in terms of matchings in Δ_J .

Let Δ be a simplicial complex on the vertex set \mathcal{T}_{\min} and Q be a subset of $\Omega := \{0, 1, \dots, \dim(\Delta)\}$. A set $\mathcal{N} = \{T_1, \dots, T_s\}$ of simplices of Δ is called a Q -matching in Δ if $T_k \cap T_l = \emptyset$ for every $1 \leq k, l \leq s$ and $\dim(T_k) \in Q$ for every $1 \leq k \leq s$; see also [8, Definition 2.1]. Let $\text{supp}(\mathcal{N}) = \cup_{i=1}^s T_i$, which is a subset of the vertices \mathcal{T}_{\min} . We denote by $\text{card}(\mathcal{N})$ the cardinality s of the set \mathcal{N} . A Q -matching \mathcal{N} in Δ is called a *maximal Q -matching* if $\text{supp}(\mathcal{N})$ has the maximum possible cardinality among all Q -matchings. By $\delta(\Delta)_Q$, we denote the minimum of the set

$$\{\text{card}(\mathcal{N}) \mid \mathcal{N} \text{ is a maximal } Q\text{-matching in } \Delta\}.$$

Theorem 2.6. *Let K be any field, then $\text{bar}(J) \geq \delta(\Delta_J)_{\{0,1\}}$ and $\text{ara}_c(J) \geq \delta(\Delta_J)_\Omega$.*

Proof. By [8, Proposition 3.3], $b_{\Delta_J} = \delta(\Delta_J)_{\{0,1\}}$ and $r_{\Delta_J} = \delta(\Delta_J)_\Omega$. Now the result follows from Theorem 2.5. □

Proposition 2.7. *Let J be a binomial ideal. Suppose that there exists a minimal generating set \mathcal{S} of J such that every element of \mathcal{S} is a difference of two squarefree monomials. Assume that J is generated by the indispensable binomials, namely \mathcal{S} consists precisely of the indispensable binomials (up to sign). Then $\text{bar}(J) = \text{card}(\mathcal{S})$.*

Proof. Let $\text{card}(\mathcal{S}) = t$. Since \mathcal{S} is a generating set of J , we have that $\text{bar}(J) \leq t$. It is enough to prove that $t \leq \text{bar}(J)$. Let $|\mathcal{T}_{\min}| = g$. By [4, Corollary 3.6] it holds that $\text{card}(G(\mathcal{M}_J)) = 2t$, so $g = 2t$. For every maximal $\{0, 1\}$ -matching \mathcal{M} in Δ_J , we have that $\text{supp}(\mathcal{M}) = \mathcal{T}_{\min}$, so $\delta(\Delta_J)_{\{0,1\}} \geq \lfloor \frac{g}{2} \rfloor$ and therefore $\delta(\Delta_J)_{\{0,1\}} \geq t$. Thus, from Theorem 2.6, $\text{bar}(J) \geq t$. □

Example 2.8. Let J be the binomial ideal generated by $f_1 = x_1x_6 - x_2x_5$, $f_2 = x_2x_7 - x_3x_6$, $f_3 = x_1x_8 - x_4x_5$, $f_4 = x_3x_8 - x_4x_7$, and $f_5 = x_1x_7 - x_3x_5$. Actually J is the binomial edge ideal of the graph G with edges $\{1, 2\}$, $\{2, 3\}$, $\{1, 4\}$, $\{3, 4\}$, and $\{1, 3\}$, see Sect. 3 for the definition of such an ideal. Note that J is \mathcal{A} -homogeneous where $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_8\}$ is the set of columns of the matrix

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

By [4, Theorem 3.3] every binomial f_i is indispensable of J . Thus

$$\begin{aligned} \mathcal{T}_{\min} &= \{E_1 = \{1, 6\}, E_2 = \{2, 5\}, E_3 = \{2, 7\}, E_4 = \{3, 6\}, E_5 = \{1, 8\}, \\ &E_6 = \{4, 5\}, E_7 = \{3, 8\}, E_8 = \{4, 7\}, E_9 = \{1, 7\}, E_{10} = \{3, 5\}\}. \end{aligned}$$

By Proposition 2.7 the binomial arithmetical rank of J equals 5. The simplicial complex Δ_J has 5 connected components and all of them are 1-simplices, namely $\Delta_1 = \{E_1, E_2\}$, $\Delta_2 = \{E_3, E_4\}$, $\Delta_3 = \{E_5, E_6\}$, $\Delta_4 = \{E_7, E_8\}$, and $\Delta_5 = \{E_9, E_{10}\}$. Consequently

$$\delta(\Delta_J)_\Omega = \sum_{i=1}^5 \delta(\Delta_i)_\Omega = 1 + 1 + 1 + 1 + 1 = 5,$$

and therefore $5 \leq \text{ara}_c(J)$. Since $\text{ara}_c(J) \leq \text{bar}(J)$, we get that $\text{ara}_c(J) = 5$. We will show that $\text{ara}_\mathcal{A}(J) = 5$. Suppose that $\text{ara}_\mathcal{A}(J) = s < 5$, and let F_1, \dots, F_s be \mathcal{A} -homogeneous polynomials in J such that $\text{rad}(J) = \text{rad}(F_1, \dots, F_s)$. For every vertex $E_i \in \mathcal{T}_{\min}$ there exists, from Proposition 2.2, a monomial M_i in F_k such that $E_i = \text{supp}(M_i)$. But $s < 5$, so there exist $E_i \in \mathcal{T}_{\min}$ and $E_j \in \mathcal{T}_{\min}$ such that

- (1) $\{E_i, E_j\}$ is not a 1-simplex of Δ_J ,
- (2) $E_i = \text{supp}(M_i)$, $E_j = \text{supp}(M_j)$, and
- (3) M_i and M_j are monomials of some F_k .

Since F_k is \mathcal{A} -homogeneous, it holds that $\deg_\mathcal{A}(M_i) = \deg_\mathcal{A}(M_j)$. Considering all possible combinations of E_i and E_j , we finally arrive at a contradiction. Thus $\text{ara}_\mathcal{A}(J) = 5$. Note that J is \mathcal{B} -homogeneous where \mathcal{B} is the set of columns of the matrix

$$N = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since every row of D is a row of N , we deduce that every \mathcal{B} -homogeneous polynomial in J is also \mathcal{A} -homogeneous. So $\text{ara}_\mathcal{B}(J)$ is an upper bound for $\text{ara}_\mathcal{A}(J)$, therefore $\text{ara}_\mathcal{B}(J) = 5$. We have that $\text{rad}(J) = \text{rad}(f_1, f_2 + f_3, f_4, f_5)$, since the second power of both binomials f_2 and f_3 belongs to the ideal generated

by the polynomials $f_1, f_2 + f_3, f_4, f_5$. Remark that the polynomials $f_1, f_2 + f_3, f_4$, and f_5 are \mathcal{C} -homogeneous, where \mathcal{C} is the set of columns of the matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 5 & 6 & 7 & 8 \end{pmatrix}.$$

Thus $\text{ara}_{\mathcal{C}}(J) \leq 4$, so $\text{ara}(J) \leq 4$. A primary decomposition of J is

$$J = (f_1, f_2, f_3, f_4, f_5, x_2x_8 - x_4x_6) \cap (x_1, x_3, x_5, x_7).$$

Hence, by [2, Proposition 19.2.7], it follows that $\text{ara}(J) \geq 4$. Thus

$$\text{ara}(J) = \text{ara}_{\mathcal{C}}(J) = 4 < 5 = \text{ara}_{\mathcal{A}}(J) = \text{ara}_{\mathcal{B}}(J) = \text{ara}_{\mathcal{C}}(J) = \text{bar}(J).$$

3. Binomial edge ideals of graphs. In this section we consider a special class of binomial ideals, namely binomial edge ideals of graphs. This ideal was introduced in [6] and independently at the same time in [10].

Let G be an undirected connected simple graph on the vertex set $[n] := \{1, \dots, n\}$ and with edge set $E(G)$. Consider the polynomial ring

$$R := K[x_1, \dots, x_n, x_{n+1}, \dots, x_{2n}]$$

in $2n$ variables, $x_1, \dots, x_n, x_{n+1}, \dots, x_{2n}$, over K .

Definition 3.1. The binomial edge ideal $J_G \subset R$ associated to the graph G is the ideal generated by the binomials $f_{ij} = x_i x_{n+j} - x_j x_{n+i}$, with $i < j$, such that $\{i, j\}$ is an edge of G .

Remark 3.2. From [7, Corollary 1.13] every binomial f_{ij} , where $\{i, j\}$ is an edge of G , is indispensable of J_G . Thus

$$\mathcal{T}_{\min} = \{E_{ij}^1 = \{i, n + j\}, E_{ij}^2 = \{j, n + i\} \mid \{i, j\} \in E(G)\}.$$

We recall some fundamental material from [6]. Let G be a connected graph on $[n]$ and let $S \subset [n]$. By $G \setminus S$, we denote the graph that results from deleting all vertices in S and their incident edges from G . Let $c(S)$ be the number of connected components of $G \setminus S$, and let $G_1, \dots, G_{c(S)}$ denote the connected components of $G \setminus S$. Also let \tilde{G}_i denote the complete graph on the vertices of G_i . We set

$$P_S(G) = \left(\cup_{i \in S} \{x_i, x_{n+i}\}, J_{\tilde{G}_1}, \dots, J_{\tilde{G}_{c(S)}} \right) R.$$

Then $P_S(G)$ is a prime ideal for every $S \subset [n]$. The ring $R/P_{\emptyset}(G)$ has Krull dimension $n+1$. For $S \neq \emptyset$ the ring $R/P_S(G)$ has Krull dimension $n - \text{card}(S) + c(S)$. The ideal $P_S(G)$ is a minimal prime of J_G if and only if $S = \emptyset$ or $S \neq \emptyset$, and for each $i \in S$ one has $c(S \setminus \{i\}) < c(S)$. Moreover J_G is a radical ideal and it admits the minimal primary decomposition $J_G = \cap_{S \in \mathcal{M}(G)} P_S(G)$, where $\mathcal{M}(G) = \{S \subset [n] : P_S(G) \text{ is a minimal prime of } J_G\}$.

Theorem 3.3. *Let G be a connected graph on the vertex set $[n]$ with m edges. Then $\text{bar}(J_G) = \text{ara}_{\mathcal{C}}(J_G) = m$.*

Proof. Every binomial f_{ij} , where $\{i, j\}$ is an edge of G , is indispensable of J_G , thus, from Proposition 2.7, $\text{bar}(J_G) = m$. Note that, for every edge $\{i, j\}$ of G , $\{E_{ij}^1, E_{ij}^2\}$ is a 1-simplex of Δ_{J_G} . Furthermore Δ_{J_G} has exactly m connected components and all of them are 1-simplices. Thus $\delta(\Delta_{J_G})_\Omega = m$ and therefore, from Theorem 2.6, $\text{ara}_c(J_G) \geq m$. Consequently $\text{ara}_c(J_G) = m$. \square

Theorem 3.4. *Let G be a connected graph on the vertex set $[n]$ with m edges. Consider the canonical basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ of \mathbb{Z}^n and the canonical basis $\{\mathbf{w}_1, \dots, \mathbf{w}_{n+1}\}$ of \mathbb{Z}^{n+1} . Let $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_{2n}\} \subset \mathbb{N}^n$ be the set of vectors where $\mathbf{a}_i = \mathbf{e}_i$, $1 \leq i \leq n$, and $\mathbf{a}_{n+i} = \mathbf{e}_i$ for $1 \leq i \leq n$. Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_{2n}\} \subset \mathbb{N}^{n+1}$ be the set of vectors where $\mathbf{b}_i = \mathbf{w}_1 + \mathbf{w}_{i+1}$, $1 \leq i \leq n$, and $\mathbf{b}_{n+i} = \mathbf{w}_{i+1}$ for $1 \leq i \leq n$. Then $\text{ara}_{\mathcal{A}}(J_G) = \text{ara}_{\mathcal{B}}(J_G) = m$.*

Proof. Suppose that $\text{ara}_{\mathcal{A}}(J_G) = t < m$, and let F_1, \dots, F_t be \mathcal{A} -homogeneous polynomials in J_G such that $J_G = \text{rad}(F_1, \dots, F_t)$. For every edge $\{i, j\}$ of G with $i < j$ there exist, from Proposition 2.2, monomials M_{ij}^k and N_{ij}^l in F_k and F_l , respectively, such that $E_{ij}^1 = \text{supp}(M_{ij}^k)$ and $E_{ij}^2 = \text{supp}(N_{ij}^l)$. But $t < m$, so there exists $E_{rs}^1 \in \mathcal{T}_{\min}$, where $\{r, s\}$ is an edge of G with $r < s$, such that

- (1) $\{E_{ij}^1, E_{rs}^1\}$ is not a 1-simplex of Δ_{J_G} ,
- (2) $E_{ij}^1 = \text{supp}(M_{ij}^k)$, $E_{rs}^1 = \text{supp}(M_{rs}^k)$, and
- (3) M_{ij}^k and M_{rs}^k are monomials of some F_k .

Let $M_{ij}^k = x_i^{g_i} x_{n+j}^{g_j}$ and $M_{rs}^k = x_r^{g_r} x_{n+s}^{g_s}$. Since F_k is \mathcal{A} -homogeneous, we deduce that $\text{deg}_{\mathcal{A}}(M_{ij}^k) = \text{deg}_{\mathcal{A}}(M_{rs}^k)$, and therefore $g_i \mathbf{e}_i + g_j \mathbf{e}_j = g_r \mathbf{e}_r + g_s \mathbf{e}_s$. Consequently $i = r$, $j = s$, and also $M_{ij}^k = M_{rs}^k$ is a contradiction. Let D and Q be the matrices with columns \mathcal{A} and \mathcal{B} , respectively. Since every row of D is a row of Q , we deduce that every \mathcal{B} -homogeneous polynomial in J_G is also \mathcal{A} -homogeneous. Thus $\text{ara}_{\mathcal{B}}(J_G)$ is an upper bound for $\text{ara}_{\mathcal{A}}(J_G)$, so $m \leq \text{ara}_{\mathcal{B}}(J_G)$ and therefore $\text{ara}_{\mathcal{B}}(J_G) = m$. \square

The graph G is called l -vertex-connected if $l < n$ and $G \setminus S$ is connected for every subset S of $[n]$ with $\text{card}(S) < l$. The vertex connectivity of G is defined as the maximum integer l such that G is l -vertex-connected.

In [1] the authors study the relationship between algebraic properties of a binomial edge ideal J_G , such as the dimension and the depth of R/J_G , and the vertex connectivity of the graph. It turns out that this notion is also useful for the computation of the arithmetical rank of a binomial edge ideal.

Theorem 3.5. *Let K be a field of any characteristic and G be a connected graph on the vertex set $[n]$. Suppose that the vertex connectivity of G is l . Then $\text{ara}(J_G) \geq n + l - 2$.*

Proof. If G is the complete graph on the vertex set $[n]$, its vertex connectivity is $n - 1$, then $\text{ara}(J_G) = 2n - 3 = n + l - 2$ by [3, Theorem 2]. Assume now that G is not the complete graph. Let $P_\emptyset(G), W_1, \dots, W_t$ be the minimal primes of J_G . It holds that $J_G = P_\emptyset(G) \cap L$ where $L = \bigcap_{i=1}^t W_i$. First we will prove that $\dim(R/(P_\emptyset(G) + L)) \leq n - l + 1$. For every prime ideal Q such that $P_\emptyset(G) + L \subseteq Q$, we have that $L \subseteq Q$, so there is $1 \leq i \leq t$ such that $W_i \subseteq Q$. Thus

$P_\emptyset(G) + W_i \subseteq Q$ and therefore $\dim(R/(P_\emptyset(G) + L)) \leq \dim(R/(P_\emptyset(G) + W_i))$. It is enough to show that $\dim(R/(P_\emptyset(G) + W_i)) \leq n - l + 1$. Let $W_i = P_S(G)$ for $\emptyset \neq S \subseteq [n]$. We have that $P_\emptyset(G) + P_S(G)$ is generated by

$$\{x_i x_{n+j} - x_j x_{n+i} : i, j \in [n] \setminus S\} \cup \{x_i, x_{n+i} : i \in S\}.$$

Then $\dim(R/(P_\emptyset(G) + P_S(G))) = n - \text{card}(S) + 1$. If $l = 1$, then $\text{card}(S) \geq 1$ since $S \neq \emptyset$, and therefore $\dim(R/(P_\emptyset(G) + W_i)) \leq n$. Suppose that $l \geq 2$ and also that $\text{card}(S) < l$. Since $P_S(G)$ is a minimal prime, for every $i \in S$ we have that $c(S \setminus \{i\}) < c(S)$. But G is l -vertex-connected, namely $G \setminus S$ is connected, so $P_\emptyset(G) \subset P_S(G)$, a contradiction to the fact that $P_S(G)$ is a minimal prime. Thus $\dim(R/(P_\emptyset(G) + W_i)) \leq n - l + 1$ and therefore $\dim(R/(P_\emptyset(G) + L)) \leq n - l + 1$. Next we will show that $\min\{\dim(R/P_\emptyset(G)), \dim(R/L)\} > \dim(R/(P_\emptyset(G) + L))$. Recall that $\dim(R/P_\emptyset(G)) = n + 1$, so $\dim(R/(P_\emptyset(G) + L)) < \dim(R/P_\emptyset(G))$. Since $L \subset P_\emptyset(G) + L$, we deduce that $\dim(R/(P_\emptyset(G) + L)) \leq \dim(R/L)$. Suppose that $\dim(R/(P_\emptyset(G) + L)) = \dim(R/L)$, say equal to s , and let $Q_1 \subsetneq Q_2 \subsetneq \dots \subsetneq Q_s$ be a chain of prime ideals containing $P_\emptyset(G) + L$. Then there is $1 \leq j \leq t$ such that $Q_1 = W_j$. So $P_\emptyset(G) \subset W_j$, a contradiction. By [2, Proposition 19.2.7] it holds that

$$\begin{aligned} \text{cd}(J_G) &\geq \dim(R) - \dim(R/(P_\emptyset(G) + L)) - 1 = 2n - \dim(R/(P_\emptyset(G) + L)) \\ &- 1 \geq 2n - (n - l + 1) - 1 = n + l - 2. \end{aligned}$$

Consequently $\text{ara}(J_G) \geq n + l - 2$. □

Example 3.6. Let G be the graph on the vertex set $[5]$ with edges $\{1, 2\}$, $\{2, 3\}$, $\{1, 3\}$, $\{2, 4\}$, $\{4, 5\}$, and $\{3, 5\}$. Here the vertex connectivity is $l = 2$. By Theorem 3.5, $\text{ara}(J_G) \geq 5$. The ideal J_G is generated up to radical by the polynomials $f_{12}, f_{23}, f_{13} + f_{24}, f_{35}$, and f_{45} , since both f_{13}^2 and f_{24}^2 belong to the ideal generated by $f_{12}, f_{23}, f_{13} + f_{24}, f_{35}$, and f_{45} . Thus $\text{ara}(J_G) = 5 < 6 = \text{bar}(J_G)$.

Theorem 3.7. *If G is a cycle of length $n \geq 3$, then $\text{ara}(J_G) = \text{bar}(J_G) = n$.*

Proof. The vertex connectivity of G is 2, so, from Theorem 3.5, the inequality $n \leq \text{ara}(J_G)$ holds. Since G has n edges, we have that $\text{ara}(J_G) \leq \text{bar}(J_G) = n$ and therefore $\text{ara}(J_G) = n$. □

Proposition 3.8. *Let G be a connected graph on $[n]$, with m edges and $n \geq 4$. If G contains an odd cycle of length 3, then $\text{ara}(J_G) \leq m - 1$.*

Proof. Let C be an odd cycle of G of length 3, with edge set $\{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$. Since G is connected, without loss of generality, there is a vertex $4 \leq i \leq n$ such that $\{1, i\}$ is an edge of G . We will show that $(x_1 x_{n+i} - x_i x_{n+1})^2$ belongs to the ideal L generated by the polynomials $f_{12}, f_{13}, f_{1i} + f_{23}$. We have that

$$\begin{aligned} x_1^2 x_{n+i}^2 &\equiv x_1 x_{n+i} x_i x_{n+1} - x_1 x_2 x_{n+i} x_{n+3} + x_1 x_3 x_{n+i} x_{n+2} \equiv x_1 x_i x_{n+i} x_{n+1} \\ &- x_2 x_{n+i} x_3 x_{n+1} + x_2 x_3 x_{n+1} x_{n+i} \equiv x_1 x_i x_{n+i} x_{n+1} \pmod{L}. \end{aligned}$$

Similarly we have that $x_i^2 x_{n+1}^2 \equiv x_1 x_i x_{n+i} x_{n+1} \pmod{L}$. Thus $x_1^2 x_{n+i}^2 + x_i^2 x_{n+1}^2 \equiv 2x_1 x_i x_{n+i} x_{n+1} \pmod{L}$, so $(x_1 x_{n+i} - x_i x_{n+1})^2$ belongs to L . Next we prove

that $(x_2x_{n+3} - x_3x_{n+2})^2$ belongs to L . We have that

$$\begin{aligned} x_2^2x_{n+3}^2 &\equiv x_2x_{n+3}x_3x_{n+2} - x_2x_{n+3}x_1x_{n+i} + x_2x_{n+3}x_ix_{n+1} \\ &\equiv x_2x_{n+3}x_3x_{n+2} - x_2x_{n+i}x_3x_{n+1} + x_{n+3}x_ix_1x_{n+2} \\ &\equiv x_2x_{n+3}x_3x_{n+2} - x_1x_{n+2}x_{n+i}x_3 + x_ix_{n+2}x_3x_{n+1} \pmod L. \end{aligned}$$

Furthermore

$$x_3^2x_{n+2}^2 \equiv x_2x_{n+3}x_3x_{n+2} - x_3x_{n+2}x_ix_{n+1} + x_3x_{n+2}x_1x_{n+i} \pmod L.$$

Thus $x_2^2x_{n+3}^2 + x_3^2x_{n+2}^2 \equiv 2x_2x_{n+3}x_3x_{n+2} \pmod L$, so $(x_2x_{n+3} - x_3x_{n+2})^2 \in L$. Let H be the subgraph of G consisting of the cycle C and the edge $\{1, i\}$. Then J_G is generated up to radical by the following set of $m - 1$ binomials:

$$\{f_{kl} \mid \{k, l\} \in E(G) \setminus E(H)\} \cup \{f_{12}, f_{13}, f_{1i} + f_{23}\}.$$

Therefore $\text{ara}(J_G) \leq m - 1$. □

Let $G_1 = (V(G_1), E(G_1))$, $G_2 = (V(G_2), E(G_2))$ be graphs such that $G_1 \cap G_2$ is a complete graph. The new graph $G = G_1 \oplus G_2$ with the vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2)$ is called the *clique sum* of G_1 and G_2 in $G_1 \cap G_2$. If the cardinality of $V(G_1) \cap V(G_2)$ is $k + 1$, then this operation is called a k -clique sum of the graphs G_1 and G_2 . We write $G = G_1 \oplus_{\hat{v}} G_2$ to indicate that G is the clique sum of G_1 and G_2 and that $V(G_1) \cap V(G_2) = \hat{v}$.

Theorem 3.9. *Let G be a connected graph on the vertex set $[n]$. Suppose that G has exactly one cycle C . If $n \geq 4$ and C is odd of length 3, then $\text{ara}(J_G) = n - 1$.*

Proof. The graph G can be written as the 0-clique sum of the cycle C and some trees. More precisely,

$$G = C \bigoplus_{v_1} T_1 \bigoplus_{v_2} \cdots \bigoplus_{v_s} T_s$$

for some vertices v_1, \dots, v_s of C . The vertex connectivity of G is 1. By Theorem 3.5, the inequality $n - 1 \leq \text{ara}(J_G)$ holds. Since G has exactly one cycle, we have that $\text{card}(E(G)) = n$. From Proposition 3.8, $\text{ara}(J_G) \leq n - 1$, and therefore $\text{ara}(J_G) = n - 1$. □

Let $\text{ht}(J_G)$ be the height of J_G , then we have, from the generalized Krull’s principal ideal theorem, that $\text{ht}(J_G) \leq \text{ara}(J_G)$. We say that J_G is a *set-theoretic complete intersection* if $\text{ara}(J_G) = \text{ht}(J_G)$.

Corollary 3.10. *Let G be a connected graph on the vertex set $[n]$ with $n \geq 4$. Suppose that G has exactly one cycle C and its length is 3. Then the following properties are equivalent:*

- (a) J_G is unmixed,
- (b) J_G is Cohen–Macaulay,
- (c) J_G is a set-theoretic complete intersection,
- (d) $G = C \bigoplus_{v_1} T_1 \bigoplus_{v_2} \cdots \bigoplus_{v_s} T_s$, where $\{v_1, \dots, v_s\} \subset V(C)$, $s \geq 1$, v_h are pairwise distinct and T_h are paths.

In particular, if one of the above conditions is true, then $\text{ara}(J_G) = \text{ht}(J_G) = n - 1$.

Proof. The implication (b) \Rightarrow (a) is well known. If J_G is a set-theoretic complete intersection, then, from Theorem 3.9, $\text{ht}(J_G) = n - 1$ and $\dim(R/J_G) = n + 1$. Also $\text{depth}(R/J_G) = n + 1$ by [5, Theorem 1.1], so J_G is Cohen–Macaulay, whence (c) \Rightarrow (b). Recall that $\mathcal{M}(G) = \{S \subset [n] : P_S(G) \text{ is a minimal prime of } J_G\}$. If J_G is unmixed, then every vertex v of T_h , $v \neq v_h$, has degree at most 2. In fact, $\{v\} \in \mathcal{M}(G)$ and, if $\deg_G(v) \geq 3$, then by [6, Lemma 3.1], one has $\text{ht}(P_{\{v\}}(G)) = n + \text{card}(\{v\}) - c(\{v\}) = n + 1 - \deg_G(v) \leq n - 2 < n - 1 = \text{ht}(P_{\emptyset}(G))$, a contradiction. Moreover, v_h has degree at most 3 for every h . In fact, $\{v_h\} \in \mathcal{M}(G)$ and, if $\deg_G(v_h) \geq 4$, then by [6, Lemma 3.1], one has $\text{ht}(P_{\{v_h\}}(G)) = n + \text{card}(\{v_h\}) - c(\{v_h\}) = n + 1 - (\deg_G(v_h) - 1) \leq n - 2 < n - 1 = \text{ht}(P_{\emptyset}(G))$, a contradiction. Thus, (d) follows. Finally, assuming (d), J_G is unmixed by [5, Theorem 1.1] and $\text{ht}(J_G) = n - 1$. By Theorem 3.9, it follows that

$$\text{ara}(J_G) = n - 1 = \text{ht}(J_G).$$

□

If C_1 and C_2 are cycles of G having no common vertex, then a *bridge* between C_1 and C_2 is an edge $\{i, j\}$ of G with $i \in V(C_1)$ and $j \in V(C_2)$.

Proposition 3.11. *Let G be a connected graph on the vertex set $[n]$ with m edges. Suppose that G contains a subgraph H consisting of two vertex disjoint odd cycles of length 3, namely C_1 and C_2 , and also two bridges between the cycles C_1 and C_2 . Then $\text{ara}(J_G) \leq m - 2$.*

Proof. Let $E(C_1) = \{\{1, 2\}, \{2, 3\}, \{3, 1\}\}$ and $E(C_2) = \{\{4, 5\}, \{5, 6\}, \{4, 6\}\}$. Suppose first that the bridges have no common vertex. Let $e_1 = \{1, 4\}$ and $e_2 = \{3, 6\}$ be the bridges of the two cycles. Then f_{14}^2 belongs to the ideal generated by the polynomials $f_{12}, f_{13}, f_{14} + f_{23}$. Furthermore f_{36}^2 belongs to the ideal generated by the polynomials $f_{46}, f_{56}, f_{36} + f_{45}$. Thus J_G is generated up to radical by the union of $\{f_{12}, f_{13}, f_{14} + f_{23}, f_{46}, f_{56}, f_{36} + f_{45}\}$ and $\{f_{ij} \mid \{i, j\} \in E(G) \text{ and } \{i, j\} \notin E(H)\}$. If the bridges have a common vertex, then without loss of generality, we can assume that $e_1 = \{1, 4\}$ and $e_2 = \{3, 4\}$ are the bridges of the two cycles. Applying similar arguments as before, we deduce that $\text{ara}(J_G) \leq m - 2$. □

Example 3.12. Suppose that G is a graph with 6 vertices and 8 edges consisting of two vertex disjoint odd cycles of length 3, namely C_1 and C_2 , and also two vertex disjoint bridges between the cycles C_1 and C_2 . Here the vertex connectivity is $l = 2$. Thus $\text{ara}(J_G) \geq 6$. By Proposition 3.11, $\text{ara}(J_G) \leq 6$ and therefore $\text{ara}(J_G) = 6$.

Theorem 3.13. *Let G_k be a graph containing k odd cycles C_1, \dots, C_k of length 3 such that the cycles C_i and C_j have disjoint vertex sets, for every $1 \leq i < j \leq k$. Suppose that there exists exactly one path $P_{i, i+1}$ of length $r_i \geq 2$ connecting a vertex of C_i with a vertex of C_{i+1} , $1 \leq i \leq k - 1$. If G_k has no more vertices*

or edges, then $\text{ara}(J_{G_k}) = \text{ht}(J_{G_k}) = 2k + \sum_{i=1}^{r-1} r_i$. In particular, J_{G_k} is a set-theoretic complete intersection.

Proof. The graph G_k has $3k + \sum_{i=1}^{k-1} (r_i - 1)$ vertices. Here the vertex connectivity is $l = 1$, so

$$2k + \sum_{i=1}^{k-1} r_i = 3k + \sum_{i=1}^{k-1} (r_i - 1) + 1 - 2 \leq \text{ara}(J_{G_k}).$$

We will prove that $\text{ara}(J_{G_k}) \leq 2k + \sum_{i=1}^{k-1} r_i$ by induction on $k \geq 2$. Suppose that $k = 2$ and let $E(C_1) = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$, $P_{1,2} = \{\{3, 4\}, \{4, 5\}, \dots, \{r + 2, r + 3\}\}$, and $C_2 = \{\{r + 3, r + 4\}, \{r + 4, r + 5\}, \{r + 3, r + 5\}\}$. Then J_{G_2} is generated up to radical by the union of

$$\{f_{12} + f_{34}, x_{r+2}x_{n+r+3} - x_{r+3}x_{n+r+2} + x_{r+4}x_{n+r+5} - x_{r+5}x_{n+r+4}\}$$

and

$$\{f_{ij} | \{i, j\} \in E(G_2) \setminus \{\{1, 2\}, \{3, 4\}, \{r + 2, r + 3\}, \{r + 4, r + 5\}\}\}.$$

Thus $\text{ara}(J_{G_2}) \leq 4 + r$. Assume that the inequality $\text{ara}(J_{G_k}) \leq 2k + \sum_{i=1}^{k-1} r_i$ holds for k , and we will prove that $\text{ara}(J_{G_{k+1}}) \leq 2(k + 1) + \sum_{i=1}^k r_i$. We have that $J_{G_{k+1}} = J_{G_k} + J_H$ where H is the graph consisting of the path $P_{k,k+1}$ and the cycle C_{k+1} . By Theorem 3.9, $\text{ara}(J_H) = r_k + 2$. Then, from the induction hypothesis,

$$\text{ara}(J_{G_{k+1}}) \leq \text{ara}(J_{G_k}) + \text{ara}(J_H) \leq 2k + \sum_{i=1}^{k-1} r_i + r_k + 2 = 2(k + 1) + \sum_{i=1}^k r_i.$$

Since J_{G_k} is unmixed by [5, Theorem 1.1], we have that

$$\text{ht}(J_{G_k}) = \text{card}(V(G_k)) - 1 = 2k + \sum_{i=1}^{r-1} r_i.$$

□

Remark 3.14. All the results presented are independent of the field K .

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