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ORIGINAL PAPER

A finite dominating set of cardinality O(k) and a witness set of cardinality O(n) for 1.5D terrain guarding problem

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Abstract 1.5 dimensional (1.5D) terrain is characterized by a piecewise linear curve. Locating minimum number of guards on the terrain (T) to cover/guard the whole terrain is known as 1.5D terrain guarding problem. Approximation algorithms and a polynomial-time approximation scheme have been presented for the problem. The problem has been shown to be NP-Hard. In the problem, the set of possible guard locations and the set of points to be guarded are uncountable. To solve the problem to optimality, a finite dominating set (FDS) of size O(n^2) and a witness set of size O(n^3) have been presented, where n is the number of vertices on T. We show that there exists an even smaller FDS of cardinality O(k) and a witness set of cardinality O(n), where k is the number of convex points. Convex points are vertices with the additional property that between any two convex points the piecewise linear curve representing the terrain is convex. Since it is always true that $k \leq n$ for $n \geq 2$ and since it is possible to construct terrains such that $n = 2^k$, the existence of an FDS with cardinality O(k) and a witness set of cardinality of O(n) leads to the reduction of decision variables and constraints respectively in the zero-one integer programming formulation of the problem.

Keywords Finite dominating sets · Location · Terrain guarding problem · Zero-one integer programming

1 Introduction

Guarding a geographical terrain has several areas of application such as locating receivers to maintain communication (De Floriani et al. 1994), using watchtowers to protect forests from fires (Goodchild and Lee 1989) or securing a certain region for military purposes to name a few. A commonly used representation for terrains is triangulated irregular network (TIN)

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(De Floriani and Magillo 2003). Terrains represented as TIN are known as 2.5D terrains. Cole and Sharir (1989) showed that 2.5D terrain guarding problem is NP-Hard.

1.5D terrain (T) may be considered as an intersection of a vertical plane with 2.5D terrain and is characterized by a piecewise linear continuous curve (also referred to as an x-monotone polygonal chain). Guarding 1.5D terrains has applications where guarding a thin and long strip of land makes sense such as placing street lights or security sensors along roads, constructing communication networks (Ben-Moshe et al. 2007) or locating cameras along a borderline such that the cameras watch the border to prevent intruders from sneaking into homeland.

2 Related work

Chen et al. (1995) proposed an optimal polynomial-time algorithm for left-guarding of T. In left-guarding of T, guards are only allowed to guard those points to their left. Several constant-factor approximation algorithms were given for TGP and its variants (King 2006; Ben-Moshe et al. 2007; Clarkson and Varadarajan 2007; Elbassioni et al. 2011, 2012). Gibson et al. (2009) presented a PTAS for the discrete version of the problem in which the set of possible guard locations and the set of points to be guarded are both given finite sets. Later, Friedrichs et al. (2014) presented a PTAS for TGP. King and Krohn (2011) showed that 1.5D terrain guarding problem is NP-Hard.

An FDS is a finite set of points which contains an optimal solution to an optimization problem. There may be many FDS's for an optimization problem, and their existence is especially important for optimization problems with uncountable feasible sets since FDS's allow for a search for an optimal solution among a finite number of points rather than over an uncountable set. In linear programming, the set of extreme points is an example of an FDS. FDS's exist for several network location problems as well (Hooker et al. 1991; Fernández et al. 2005). Hamacher and Klamroth (2000), and Carrizosa et al. (2010) use finite dominating sets to solve planar location problems. Friedrichs et al. (2014) presented the first FDS for TGP. They showed that the vertices and the 'x-extremal points' of the viewshed of each vertex form an FDS, which has a size of $O(n^2)$. They also discretized the terrain to obtain a finite set, which they call 'witness set', such that guarding of all elements of the witness set implies guarding of T. The witness set presented in Friedrichs et al. (2014) is of size $O(n^3)$. Earlier, Ben-Moshe et al. (2007) showed that there exists a witness set of size $O(n^2)$ for the problem where guards are restricted to the vertices of the terrain. However, an argument similar to the one in their study (see the proof of Lemma 6.2 in Ben-Moshe et al. (2007)) shows that their witness set is a valid set for TGP with respect to the FDS's in Friedrichs et al. (2014) and in our paper.

Our main contribution is to show that there exist a smaller finite dominating set and a smaller witness set than those given in Friedrichs et al. (2014) and Ben-Moshe et al. (2007) respectively. The FDS we construct is of cardinality $O(k)$, where k is the number of convex points, for which a formal definition is given in the next section. Since it is possible to construct terrains such that $n = 2^k$, the number of decision variables decreases considerably in the zero-one integer programming (ZOIP) formulation given in Friedrichs et al. (2014). Also, the witness set we present has a size of $O(n)$ compared to the witness set of size $O(n^2)$ presented in Ben-Moshe et al. (2007), which leads to the reduction of points to be covered by the guards.

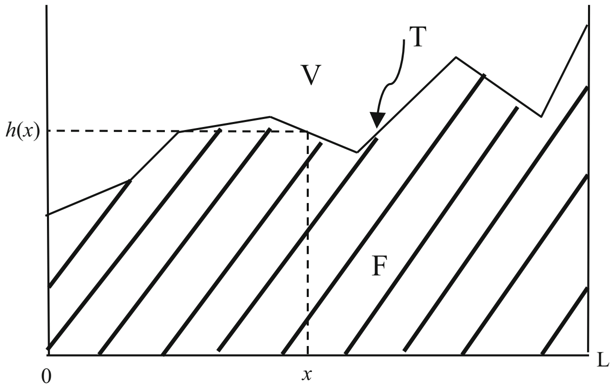

Fig. 1 A 1.5 dimensional terrain

3 Description of the problem, definitions and notation

3.1 Description of the problem

In this paper, we use a somewhat different terminology from that in the literature to be able to introduce the new concepts. Consider a piecewise linear continuous curve in the nonnegative orthant of the 2-dimensional space representing the surface of a 1.5 dimensional terrain (Fig. 1). The length of the region of interest is L and we assume without loss of generality that the one-dimensional region is situated between 0 and L. For any point x in the interval [0,L] let $h(x)$ denote the height of the point x. We assume h is a continuous real-valued function defined on [0,L]. We also assume $h(x) \geq 0\ \forall x \in [0, \mathrm{L}]$.

Let $T = \{(x, h(x)) : x \in [0, \mathrm{L}]\}$. We refer to T as the surface of the terrain of interest. T is the graph of the function $h : [0, L] \rightarrow \mathbb{R}_+$. The region below T, shown as a shaded region in Fig. 1, is denoted by F, that is, $F = \{(x, y) : x \in [0, L]$ and $0 \leq y < h(x)\}$. Let V be the visible region above T, i.e. $V = \{(x, y) : x \in [0, L]$ and $y \geq h(x)\}$. We note that T belongs to the visible region by definition.

We confine the discussion that follows to the half-strip $H(0, L)$ in the nonnegative orthant defined by the vertical lines passing through (0,0) and $(L, 0)$ (Fig. 1). The line-of-sight originating at a point in a given direction is the set of points of the form $\boldsymbol{x} + \lambda \boldsymbol{d}$, $\lambda \geq 0$, where $\mathbf{x}$ is a point in $\mathbb{R}^{\mathbf{2}}$, $\boldsymbol{d}$ is a nonzero direction in $\mathbb{R}^2$ and λ is a nonnegative real. Given the visible region V, region F, and the surface T that forms the border between V and F, let $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ in $\mathbb{R}^{\mathbf{2}}$ be two points in the half-strip $H(0, L)$. Consider the line segment $\mathrm{LS}(\boldsymbol{x_1}, \boldsymbol{x_2}) \equiv \{x_1 + \lambda(\boldsymbol{x_2} - \boldsymbol{x_1}) : \lambda \in [0, 1]\}$ connecting the points $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$. We say $\boldsymbol{x_2}$ is visible from $\boldsymbol{x_1}$ if $\mathrm{LS}(\boldsymbol{x_1}, \boldsymbol{x_2})$ is a subset of V, and $\boldsymbol{x_2}$ is not visible from $\boldsymbol{x_1}$ if $\mathrm{LS}(\boldsymbol{x_1}, \boldsymbol{x_2}) \cap F \neq \emptyset$.

Visibility is a symmetric concept, i.e. if $\boldsymbol{x_2}$ is visible from $\boldsymbol{x_1}$ then $\boldsymbol{x_1}$ is visible from $\boldsymbol{x_2}$. We define a visibility function as follows,

$$v(\boldsymbol{x}_1, \boldsymbol{x_2}) = v(\boldsymbol{x_2}, \boldsymbol{x_1}) = \begin{cases} 1 & \text{if LS } (\boldsymbol{x_1}, \boldsymbol{x_2}) \subseteq V \\ 0 & \text{otherwise} \end{cases}$$

We also say that if $v(\boldsymbol{x_1}, \boldsymbol{x_2}) = 1$ then $\boldsymbol{x_1}$ *guards/covers* $\boldsymbol{x_2}$. Let $\boldsymbol{x}$ be a point on T and $VS(\boldsymbol{x})$ denote the "viewshed" of x, i.e. $VS(\boldsymbol{x}) = \{\boldsymbol{y} \in T : v(\boldsymbol{x}, \boldsymbol{y}) = 1\}$. Let $\boldsymbol{X} = \{\boldsymbol{x_1}, \ldots, \boldsymbol{x_k}\}$ be a set of points on T. $\boldsymbol{X}$ *guards* or *covers* T if every point on T is guarded by at least one of the guards located at points in $\boldsymbol{X}$. We express guarding of a point '$\boldsymbol{y}$' by a set $\boldsymbol{X}$ by the function;

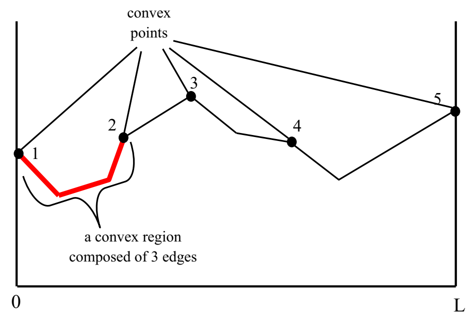

Fig. 2 Convex points and convex regions on T

$$VIS(\boldsymbol{y}, \boldsymbol{X}) = \begin{cases} 1, & \text{if } \exists \boldsymbol{x_j} \in \boldsymbol{X} \text{ such that } v(\boldsymbol{y}, \boldsymbol{x_j}) = 1 \\ 0, & \text{otherwise} \end{cases}$$

In TGP, we seek to find the minimum cardinality set $\boldsymbol{X}$ whose elements belong to T such that $\boldsymbol{X}$ guards T. Formally;

$$\begin{aligned} &\text{(TGP)} \\ &\text{Minimize } |\boldsymbol{X}| \\ &\text{Subject to } VIS\,(\boldsymbol{y},\boldsymbol{X}) = 1, \forall \boldsymbol{y} \in \text{T} \\ &\boldsymbol{X} \subseteq T \end{aligned}$$

3.2 Additional definitions and notation

Vertex: A vertex is a point where $h(x)$ changes slope. Endpoints $(0, h(0))$ and $(L, h(L))$ are also considered as vertices. The number of vertices is given by 'n'. The leftmost vertex $(0, h(0))$ is the first and $(L, h(L))$ is the nth vertex. Given 'n' vertices, the terrain is constructed by connecting vertices by line segments.

Edge: The line segment between vertex i and $i + 1$, $i = 1, \ldots, n - 1$. The number of edges is $n - 1$.

Convex region: A convex portion of T (i.e. of $h(x)$) which is composed of maximally connected edges. We denote the set of convex regions by '*CR*.'

Convex point: A vertex where two convex regions intersect. We also consider the two end points of T, i.e. $((0, h(0)$ and $(L, h(L))$, as convex points. We denote the set of convex points by 'C', $|C| = k$. $((0, h(0))$ is the first convex point and $(L, h(L))$ is the kth convex point. The convex region between the ith and $(i+1)$st convex points is the ith convex region. Obviously, if k is the number of convex points then the number of convex regions is $k-1$ and vice versa. There are nine vertices, eight edges, five convex points and four convex regions in Fig. 2. The part of T that is between convex points 1 and 2 is a convex region which has 3 edges. The other three convex regions in the figure lie between convex points 2 and 3, 3 and 4, and 4 and 5.

Q_{IJ}: A point p on T "partially covers" convex region M if only a proper subset of M is covered by p and "fully covers" M if $M \subseteq VS(p)$. For our purposes, we desire to find points on T that fully cover convex regions to their left and right. One set of such points is the set of convex points. To find other points, we define a set Q_{IJ}, which consists of nonconvex points that fully cover as many convex regions as possible to their left and right (Fig. 3).

For the terrain in Fig. 3, Q_{IJ} is the thick line segment in convex region 3. Points in Q_{IJ} fully cover convex regions 1 and 2 to their left and regions 4 and 5 to their right. For a convex

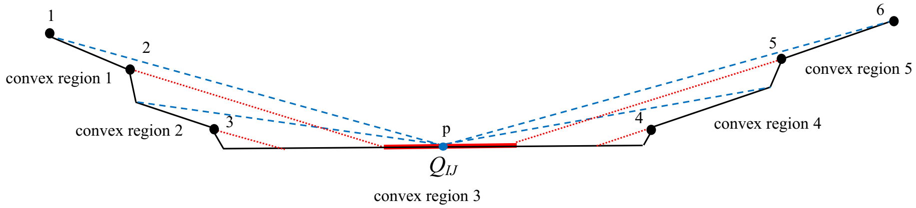

Fig. 3 Illustration of a Q_{IJ}

region M, $CS(M) = \{y \in T : v(y, x) = 1$ for all $x \in M\}$, i.e. $CS(M)$ is the set of points on T that are covered by all points in convex region M. Let M and N be two convex regions. It is true that if $N \subseteq CS(M)$ then $M \subseteq CS(N)$ due to the symmetric property of visibility. For the formal definition of this set, let I and J be index sets that are used to index convex regions. For $i \in I$, M_i is the ith convex region in CR. Let L_M and R_M denote the left and right convex point of the convex region M respectively. xc(p) denotes the x-coordinate of a point 'p' $\in T$. Similarly, yc(p) denotes the y-coordinate of p. We give the formal definition of Q_{IJ} as follows,

$Q_{IJ} = \{p : \exists$ convex regions $M_i, i \in I$ and $N_j, j \in J$ s.t. $I \neq \emptyset, J \neq \emptyset, CS(M_i) \cap CS(N_j) \cap C = \emptyset$, $p \in CS(M_i) \cap CS(N_j)\ \forall i \in I, j \in J$, $\text{xc}(L_{N_j}) > \text{xc}(p) > \text{xc}(R_{M_i})\forall i \in I$ and $j \in J$, $I \cup J$is maximal$\}$. In the definition we assume, without loss of generality, that convex regions indexed by I are to the left and those indexed by J are to the right of the points in Q_{IJ}. For the terrain in Fig. 3, $I = \{1, 2\}$ and $J = \{4, 5\}$. We choose the leftmost point in Q_{IJ}'s (for our purposes any points in Q_{IJ} would work) to form a set $DP = \{p : p \in Q_{IJ}$ for some index sets I and J, xc $(p) <$ xc$(q) \forall q \in Q_{IJ}$ s.t. $p \neq q\}$. We call an element of DP a "dip point." Let $CP = C \cup DP$. CP is the set of "critical points" containing all convex points and dip points.

4 A finite dominating set of critical points

Observe that a guard located at a convex point can guard the convex regions on both sides. We can obtain a feasible solution to TGP by placing a guard at each convex point where two convex regions meet such that the first guard covers the first and second convex regions, second guard covers the third and fourth regions and so on, with a total number of $\left\lceil \frac{(k-1)}{2} \right\rceil$ guards. Let 'p' be a point on T. L_p is defined to be the left convex point in a convex region in which p exists. R_p denotes similarly the right convex point. H_p is the hyperplane represented by the vertical line that passes through p. H_p^- and H_p^+ denote the halfspaces to the left and right of H_p respectively that consist of points on T. For $x \in T$, if $x \in H_p^-$ or H_p^+ we assume $x \neq p$.

Lemma 1 *Let p be a nonconvex point on T. Then $VS(p) \cap H_p^+ \subseteq VS(L_p) \cap H_p^+$ and $VS(p) \cap H_p^- \subseteq VS(R_p) \cap H_p^-$. In words, L_p covers all points that p covers in H_p^+ and R_p covers all points that p covers in H_p^-.*

Proof Let q be a point in H_p^+ such that $v(p, q) = 1$. Since $h(x)$ is convex in the convex region where L_p and p exist, the line segment between L_p and q, LS(L_p, q), lies on or above LS(p, q) in $[\text{xc}(L_p),\ \text{xc}(q)]$ (see Fig. 4). This implies that if q is visible from p then it is

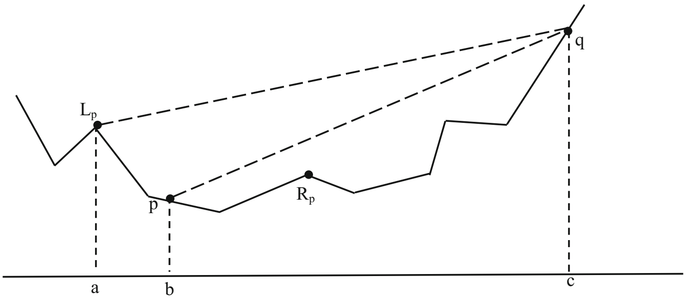

Fig. 4 L_p covers all points covered by p to its right

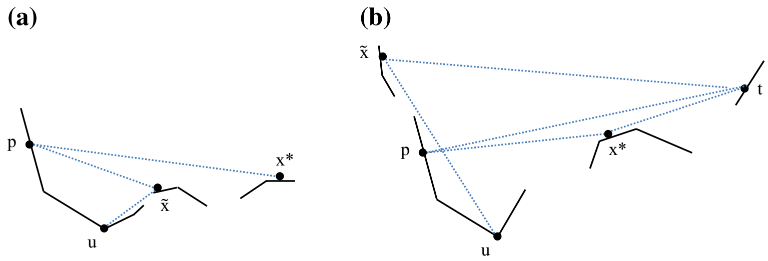

Fig. 5 **a** $\mathrm{xc}(\tilde{x}) > \mathrm{xc}(R_N)$, **b** $\mathrm{xc}(\tilde{x}) < \mathrm{xc}(L_N)$

visible from L_p. Since q is arbitrary the result follows. The same holds true for the other case □

Theorem 1 proves that if a nonconvex point covers only a proper subset of a convex region then, at an optimal solution, that part will be covered by another optimal point. Let x^* be a point in an optimal solution X^*. We define $OG(x^*)$ as the set of points on T guarded only by x^* and not by other points in X^*, that is, $OG(x^*) = \{p \in T : v(p, x^*) = 1, v(p, x) = 0 \text{ for } x \neq x^*, x \in X^*\}$.

Theorem 1 *Let x^* be a nonconvex point in an optimal solution X^* and let x^* be in convex region M. Suppose that $OG(x^*) \cap H^+_{R_M} \neq \emptyset$ and $OG(x^*) \cap H^-_{L_M} \neq \emptyset$. Suppose x^* covers a proper subset S of a convex region N, i.e. $S \subseteq VS(x^*) \cap N$ and $\exists u \in N$ such that $u \notin VS(x^*)$. Then it is true that $S \cap OG(x^*) = \emptyset$.*

Proof Suppose to the contrary that $S \cap OG(x^*) \neq \emptyset$. Suppose, without loss of generality, that $N \subseteq H^-_{L_M}$. Let $p \in S \cap OG(x^*) \cap H^-_{L_M}$ and $t \in OG(x^*) \cap H^+_{R_M}$. Let u be the first vertex in N, which is to the right of p and is not guarded by x^*. Note that such a vertex exists since otherwise N would be fully covered by x^*. u must be covered by another optimal point $\tilde{x}$. $\tilde{x}$ can not be in N since $\tilde{x}$ would also cover p. If $\mathrm{xc}(\tilde{x}) > \mathrm{xc}(R_N)$ then since N is convex the assumption that $\tilde{x}$ covers u implies $\tilde{x}$ also covers p, a contradiction (Fig. 5 (a)). We note that, for a nonconvex point x^*, the $\mathrm{LS}(p, t)$ can not be blocked by the terrain. If $\mathrm{xc}(\tilde{x}) < \mathrm{xc}(L_N)$ then since $\mathrm{LS}(\tilde{x}, u)$ and $\mathrm{LS}(p, t)$ are not blocked by the terrain, 'order claim' discussed in Ben-Moshe et al. (2007) implies $\mathrm{LS}(\tilde{x}, t)$ is not blocked either, which contradicts $t \in OG(x^*)$ (Fig. 5b) □

Theorem 2 is the main result of this paper. It states that it suffices to seek an optimal solution to an instance of TGP among the critical points.

Theorem 2 *The set of critical points CP is a finite dominating set. In other words, there exists an optimal solution to TGP whose elements consist of critical points.*

Proof Let X^* be an optimal solution to an instance of TGP and x^* be a noncritical optimal point in X^*. We assume that x^* is in convex region M. If T is convex, i.e $h(x)$ is convex in [0, L], then optimal solution value is 1 since any point on T guards T. Hence, convex point L_{x^*} (or R_{x^*}) is also an optimal point and the claim follows. In the following, we assume T is not convex.

Case *(I)* $OG(x^*) \subseteq H^+_{R_M} \cup M$ (or $OG(x^*) \subseteq H^-_{L_M} \cup M$): then L_{x^*} (or R_{x^*}) can replace x^* in the optimal solution due to Lemma 1.

Case *(II)* $OG(x^*) \cap H^+_{R_M} \neq \emptyset$ and $OG(x^*) \cap H^-_{L_M} \neq \emptyset$.

Case *(II)(a)* There are fully covered convex regions in both $H^+_{R_M}$ and $H^-_{L_M}$ by x^*: Since x^* is not a critical point it is not a convex point. If there is a convex point $\bar{x}$ such that $\bar{x}$ fully covers all convex regions that x^* fully covers then $\bar{x}$ can replace x^* in the optimal solution. To see this, let N be a partially covered convex region in $H^+_{R_M}$ (or in $H^-_{L_M}$) by x^* and S be the covered part of N. Theorem 1 implies that S can not be in $OG(x^*)$. This implies $OG(x^*)$ consist only of convex regions fully covered by x^*, which in turn implies $\tilde{x}$ can replace x^* in the optimal solution. If there is no convex point that covers the convex regions x^* covers then it must be true that $x^* \in Q_{IJ}$ for some index sets I and J by definition of Q_{IJ}. But this implies there is a point $\tilde{x}$ in *DP* such that $\mathrm{xc}(\tilde{x}) < \mathrm{xc}(x^*)$ and $\tilde{x} \in Q_{IJ}$ by construction of *DP*. A reasoning similar to the one discussed above shows that $\tilde{x}$ can replace x^*.

Case *(II) (b)* In at least one of $H^+_{R_M}$ and $H^-_{L_M}$, all convex regions in which there are points covered by x^* are partially covered by x^*: Suppose, without loss of generality, that all convex regions partially covered by x^* is in $H^+_{R_M}$. As in case *(II) (a)*, let N be a convex region in $H^+_{R_M}$ and S be the proper subset of N that is covered by x^*. Theorem 1 implies that $S \cap OG(x^*) = \emptyset$. Since this is true for all convex regions in $H^+_{R_M}$, it contradicts the standing assumption that $OG(x^*) \cap H^+_{R_M} \neq \emptyset$. □

5 Construction of a witness set

Let $CP = \{cp_1, \ldots cp_m\}$ denote the set of critical points and $E = \{e_1, \ldots, e_{n-1}\}$ denote the set of edges. A critical point may cover all of an edge, only a proper subset of an edge or can not see the edge. If only a proper subset of an edge is covered by a convex point then that part of the edge is considered an element of the witness set that needs guarding. However, if a subset of an edge is covered by a dip point then that part of the edge is not considered an element of the witness set since theorem 1 implies that in case a nonconvex point covers only a proper subset of a convex region that subset of the convex region will also be guarded by another guard in an optimal solution. If no proper subset of an edge is covered then that edge, as a whole, is considered an element of the witness set.

Let d_{ij} be the proper subset of e_i visible from convex point c_j, if one exists, such that j is not one of the two convex points within the convex region containing e_i. Let $d'_i = e_i \setminus \bigcup_j d_{ij}$. We note that d'_i is covered by the two convex points on both ends of the convex region in which e_i exists. If guards on T cover d_{ij} and d'_i, for $i = 1, \ldots, n-1$, $j = 1, \ldots, k$ and/or each edge then T is covered. If, for an edge 'i', it is true that $d_{ij} \subset d_{ik}$ for some convex points 'j' and 'k' then we can further eliminate d_{ij} from consideration as a line segment to be covered since, at optimality, guarding of d_{ik} implies guarding of d_{ij}. We renumber each remaining line segment (and possibly the edges) to create the witness set.

Let $D = \{d_1, \ldots, d_l\}$ be the final set of line segments to be covered and $\mathbf{A}$ be the visibility matrix whose rows correspond to the elements of D and coloumns correspond to the critical points. Let a_{ij} denote the value at the ith row and the jth coloumn of $\mathbf{A}$. Then $a_{ij} = 1$ if d_i is visible by the critical point cp_j and 0 otherwise. Let x_j be a binary decision variable corresponding to cp_j. x_j is 1 if cp_j is in the optimal solution and 0 otherwise. ZOIP formulation (TGP_{ZOIP}) is given below;

$$
\begin{aligned}
&(\text{TGP}_{\text{ZOIP}}) \\
&\text{Minimize} \quad \sum_{j \in CP} x_j \\
&\text{Subject to} \quad \sum_{j \in CP} a_{ij} x_j \geq 1, \forall i = 1, \ldots, l \\
&\quad x_j \in \{0, 1\},\ j = 1, \ldots, m
\end{aligned}
$$

We illustrate the approach with an example. Consider the terrain in Fig. 6. There are 18 vertices, 17 edges, 9 convex points and one dip point (with a total of 10 critical points) on the terrain. Since $d_{13} \subset d_{14}, d_{72} \subset d_{71}, d_{13,10} \subset d_{13,9}, d_{13,5} \subset d_{13,4}$ we need not consider $d_{13}, d_{72}, d_{13,10}$, and $d_{13,5}$ as elements of the witness set to be covered (Fig. 6a). After eliminating these redundant segments there are 23 line segments that need to be covered (Fig. 6b). Thus, $\mathbf{A}$ will have 23 rows and 10 coloumns. 7th critical point, which is a dip point, does not see the line segments from 1 to 10 and see all line segments from 11 to 23. Then, $a_{i7} = 0$ for i = 1,…,10 and $a_{i7} = 1$ for i = 11,…,23.

Lemma 2 *The number of dip points is bounded by* $k - 1$.

Proof There can be at most one dip point in a convex region by definition of a dip point. Since there are $k - 1$ convex regions the number of dip points is bounded by $k - 1$. □

Total number of constraints, i.e. the size of the witness set, is O(n) since on each edge there are at most three line segments (after elimination). The number of variables, i.e. the size of the FDS, is the number of convex points plus the number of dip points, which is O(k).

Convex points and convex regions can be found in O(n) time. An edge e_i is visible from convex point c_j if both vertices of the edge are visible to c_j, which is computed in O(n). If only a part of e_i (d_{ij}) is visible from a convex point c_j then this implies that the line-of-sight from c_j to e_i is blocked by another convex point c_k. Thus, d_{ij} can be found in O(n) as well. When the same operation is done for all edges and for all convex points the total effort to find the visible regions on edges takes O(kn^2) time.

Whether a vertex v_i can see another vertex v_j can be computed in O(n) by checking the intersection of the line-of-sight between v_i and v_j and the edges of the terrain. Let M and N be two convex regions such that $\text{xc}(R_M) < \text{xc}(L_N)$. Then N is in $CS(M)$ if both the first vertex to the left of R_M and R_M see both L_N and the first vertex to the right of L_N, which can be computed in O(n) by the preceding argument. To find dip points, we look for maximal number of convex regions such that their intersection is nonempty. Since for each convex region M there are O(k) convex regions in $CS(M)$ and O(n) edge intersections are needed to find their intersection, the effort to find a dip point is O(kn). Finding the entries of $\mathbf{A}$ requires checking whether the line segments in the witness set are visible to the critical points. For each critical point this amounts to O(n) time and since there are O(k) critical points the total time is O(kn). Hence, the overall effort to find the critical points and to create the visibility matrix $\mathbf{A}$ is O(kn^2).

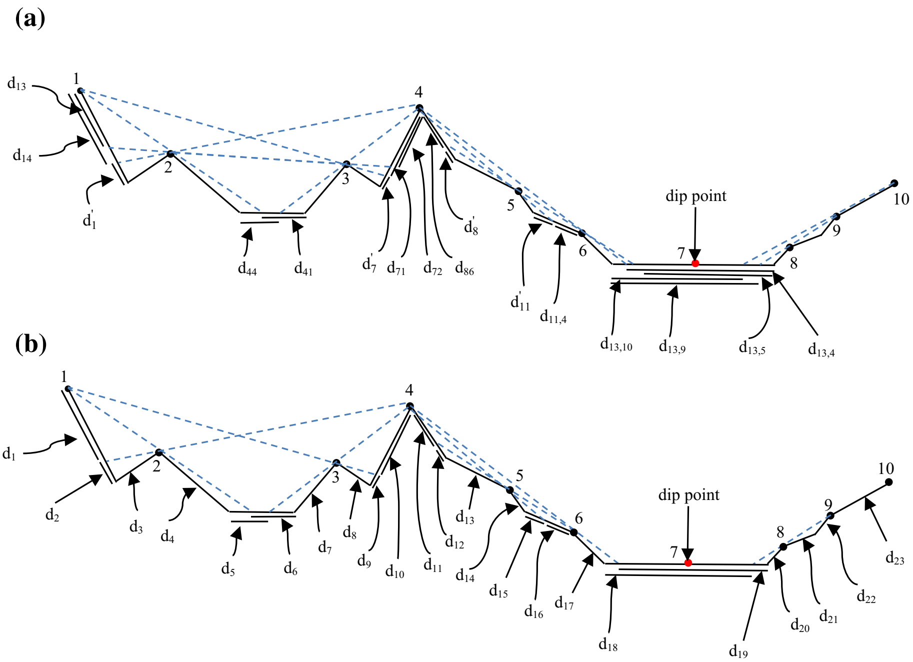

Fig. 6 **a** The visible parts of edges by convex points, **b** the final set of line segments that need to be covered

6 Conclusions and directions for future research

We have proved the existence of a finite dominating set and a witness set of cardinalities O(k) and O(n) respectively. Since, in the worst case n could be as large as 2^k, our results eliminate a large number of decision variables in the ZOIP formulation given in Friedrichs et al. (2014). Also, the witness set constructed in this paper is smaller than found in Ben-Moshe et al. (2007). Whether a witness set of cardinality O(k) can be constructed is a topic for further research.

Closely related to TGP is the 2.5D terrain guarding problem and this problem is generally approached by restricting the guard locations either to vertices and edges or only to vertices of the triangles forming the terrain (Bose et al. 1997; Eidenbenz 2002). However, to the best of our knowledge, since the set of vertices and/or edges has not been proved to be an FDS, the solutions obtained through models which locate minimum number of guards on vertices and/or edges are only an approximation for the general problem. Therefore, it is still an open question whether any dominating set exists, or in that respect, what the optimal solution is for 2.5D terrain guarding problem. We believe our results in this paper may be useful for obtaining an FDS for the 2.5D terrain guarding problem.

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References

Ben-Moshe, B., Katz, M. J., & Mitchell, J. S. B. (2007). A constant-factor approximation algorithm for optimal 1.5D terrain guarding. *SIAM Journal on Computing*, *36*(6), 1631–1647.

Bose, P., Shermer, T., Toussaint, G., & Zhu, B. (1997). Guarding polyhedral terrains. *Computational Geometry*, *7*, 173–185.

Carrizosa, E., Hamacher, H. W., Klein, R., & Nickel, S. (2010). Solving nonconvex planar location problems by finite dominating sets. *Journal of Global Optimization*, *18*, 195–210.

Chen, D.Z., Estivill-Castro, V., & Urrutia, J. (1995). Optimal guarding of polygons and monotone chains. In *Proceedings of the 7th Annual Canadian Conference on Computational Geometry*, Quebec City, QB, (pp. 133–138).

Clarkson, K. L., & Varadarajan, K. (2007). Improved approximation algorithms for geometric set cover. *Discrete Computational Geometry*, *37*, 43–58.

Cole, R., & Sharir, M. (1989). Visibility problems for polyhedral terrains. *Journal of Symbolic Computation*, *7*, 11–30.

De Floriani, L., & Magillo, P. (2003). Algorithms for visibility computation on terrains: A survey. *Environment and Planning B: Planning and Design*, *30*(5), 709–728.

De Floriani, L., Marzano, P., & Puppo, E. (1994). Line-of-sight communication on terrain models. *International Journal of Geographical Information Science*, *8*(4), 329–342.

Eidenbenz, S. (2002). Approximation algorithms for terrain guarding. *Information Processing Letters*, *82*, 99–105.

Elbassioni, K., Krohn, E., Matijević, D., Mestre, J., & Ševerdija, D. (2011). Improved approximations for guarding 1.5-dimensional terrains. *Algorithmica*, *60*, 451–463.

Elbassioni, K., Matijević, D., & Ševerdija, D. (2012). Guarding 1.5D terrains with demands. *International Journal of Computer Mathematics*, *89*(16), 2143–2151.

Fernández, E., Hınojosa, Y., & Puerto, J. (2005). Filtering policies in loss queuing network location problems. *Annals of Operations Research*, *136*, 259–283.

Friedrichs, S., Hemmer, M., & Schmidt, C. (2014). A PTAS for the continuous 1.5D Terrain Guarding Problem. In *26th Canadian Conference on Computational Geometry (CCCG)*, Halifax, Nova Scotia.

Gibson, M., Kanade, G., Krohn, E., & Varadarajan, K. (2009). An approximation scheme for terrain guarding. *Proceedings of the 12th international workshop and 13th international workshop on approximation, randomization, and combinatorial optimization: algorithms and techniques* (pp. 140–148). New York: Springer.

Goodchild, M. F., & Lee, J. (1989). Coverage problems and visibility regions on topographic surfaces. *Annals of Operations Research*, *18*, 175–186.

Hamacher, H. W., & Klamroth, K. (2000). Planar Weber location problems with barriers and block norms. *Annals of Operations Research*, *96*, 191–208.

Hooker, J. N., Garfinkel, R. S., & Chen, C. K. (1991). Finite dominating sets for network location problems. *Operations Research*, *39*(1), 100–118.

King, J. (2006). A 4-approximation algorithm for guarding 1.5-dimensional terrains. In *LATIN 2006. Theoretical informatics, Lecture Notes in Computer Science* (Vol. 3887, pp. 629–640). Berlin: Springer

King, J., & Krohn, E. (2011). Terrain guarding is NP-hard. *SIAM Journal on Computing*, *40*(5), 1316–1339.