Universitat Autònoma de Barcelona

Albanese varieties of non-Archimedean uniformized varieties

Iago Giné Vázquez

PhD Thesis

Albanese varieties of non-Archimedean uniformized varieties

by
Iago Giné Vázquez

Advisor:
Francesc-Xavier Xarles Ribas

Doctorat en Matemàtiques
Departament de Matemàtiques
Universitat Autònoma de Barcelona

Barcelona, 2017

Programa de doctorat en Matemàtiques.
Memòria presentada per aspirar al grau de Doctor en Matemàtiques per la Universitat Autònoma de Barcelona.

Certifico que la present memòria per a optar al grau de Doctor en Matemàtiques ha estat realitzada per en Iago Giné Vázquez sota la meva direcció.

Dr. Francesc-Xavier Xarles Ribas,
a Bellaterra (Cerdanyola del Vallès),
maig de 2017.

Acknowledgements

Primer de tot, vull agrair al Xavier Xarles haver acceptat dirigir-me la tesi -a més a més sense conèixer-me de res-, donar-me el projecte, i la seva infatigable tasca de guia intel·lectual, a la manera d'un psicopomp matemàtic. Sovint, la idea bona o la manera millor d'arribar a algun resultat ha sigut seva, sempre m'ha explicat una i una altra vegada tot allò que jo li preguntava fins que ho comprenia i també ha posat ordre al "totum revolutum" que hi havia al meu cap en moltes ocasions. Així vaig arribar a entendre moltes coses i a avançar quan em quedava encallat. Gràcies també per totes les signatures i els informes que ha omplert per fer el projecte de tesi, per demanar les beques, per les successives memòries que havia d'entregar així com ara per depositar, a més de l'ajut que m'ha donat en general amb tots els tràmits per viatges i demés.

En segundo lugar, gracias a mi padre y a mi madre, porque es por quienes yo he seguido este camino y he llegado hasta aquí. Y gracias a ellos y también a mi hermana, abuelo/as y tíos/as, tanto por la paciencia, como por creer en mí, como por el soporte anímico y material durante estos años. Moitas grazas! Moltes gràcies!

Gracias a Celeste por el impulso que me ha dado este último año de tesis y por la gran motivación que ha supuesto. Gracias por esa energía que me ha transmitido y que me ha permitido acabar la tesis. Muito obrigado!

Grazie mille a Piermarco Milione, compañero de doctorado en buena medida, por los ánimos y buenos consejos que me ha dado desde el comienzo, por ser también una fuente de motivación a menudo, y por todas sus aportaciones. Gracias a él he trabajado más para preparar seminarios y explicaciones mutuas que me han llevado a comprender mejor muchas cuestiones referentes a la tesis.

He podido hacer la tesis por los profesores que he tenido en la UB y por lo que me enseñaron en la licenciatura, y también, claro, por los que tuve en el instituto y lograron transmitirme el interés por las matemáticas que me llevó a estudiar esta carrera. Gracias a los que más me han animado y motivado, que también son con los que más he aprendido. Gràcies també als professors del STNB i del Grup de Recerca a la UAB (o sigui, del Seminari tropical) per acollir-me. Vull reconèixer també l'aportació que van suposar els comentaris i preguntes de la Pilar Bayer a les meves xerrades als STNB de 2013 i 2016.

Gràcies a la Roser Homs per l'ajut que m'ha donat sempre que li he demanat, especialment referent a dubtes de català, però també en altres temes. Gràcies també pels ànims que en tot moment m'ha donat. Gràcies al Narcís Baños per l'ajut amb el LaTeXi per les converses i intercanvis frikies que hem mantingut durant aquests anys. Gràcies també pels ànims i converses al Dani (i la Nuria), l'Adrià, la Kaouthar, en Kiko, la Xesca (i l'Àlex), la Zaira, en Víctor, la Julia, etc.

Gràcies a les altres amistats (no matemàtiques) que he fet al llarg d'aquests anys i que m'han ajudat a alliberar-me del tancament que suposa fer una tesi, i més en aquest camp, escoltant, mirant, debatent i, en resum, aprenent.

I, de nou, moltes gràcies, Xavier, per tot el suport, paciència i optimisme que sempre has tingut i que ha permès que arribéssim fins aquí.

Formal issues

- Esta tesis ha sido posible gracias a la beca de Formación del Profesorado Universitario AP2010-5558 del Ministerio de Educación, Cultura y Deporte y los proyectos MTM2013-40680-P y MTM2016-75980-P del Ministerio de Economía y Competitividad del Gobierno de España, y gracias al proyecto número 2014 SGR 206 del Departament d'Economia i Coneixement de la Generalitat de Catalunya.

- All the figures in this thesis were elaborated by the author.

Contents

Introduction

Along this thesis we give a conjectural construction of the Albanese variety of a non-Archimedean uniformized analytic variety. The main idea we use is that analytic varieties contain a certain topological structure called skeleton over which one can develope a parallel theory in an easier way, and then one can rise several results and constructions to the analytic variety.

For example, in dimension 1, the skeleton of an analytic curve is a metric graph, and the Jacobian of the curve can be built filling the Jacobian of the graph, as we show for Mumford curves in the chapters 2 and 3 (and as Baker and Rabinoff show for more general curves in [BR15]). Further, the skeleton of the uniformizing space of a Mumford curve coincides with the uniformizing tree of the skeleton of the curve, and several results on the analytic space can be reduced to results on its skeleton.

In general dimension and over a discrete field, the given variety degenerates to a certain simplicial complex which behaves as its skeleton, and the uniformization space has a building by degeneration complex. In certain cases there is a reduction map from the analytic space to its degeneration, which generalizes the retraction to the skeleton.

Thus, some constructions and proofs on analytic varieties can be reduced to constructions and proofs on combinatorial objects (or metric spaces, when there are weights). With this frame, we did this thesis trying to get the construction of the Albanese varieties from a study of the skeletons of the given varieties.

Next, we present an historical account of the developments which lead to this framework and to the objectives of this thesis.

A well known result on complex algebraic curves states that the uniformizing space of a hyperbolic curve is the upper half-plane with an structure of analytic space. At the begining of the 1970s, Mumford proved in [Mum72a] an analogue result for a class of algebraic curves (those with totally degenerate reduction) defined over a p-adic field K changing the complex upper half-plane by a rigid analytic space called p-adic upper half plane. In fact, he proved that giving the curve is equivalent to give the uniformizing group (up to isomorphism in the respective categories). Thus, he starts with a Schottky group $\Gamma \subset \mathrm{PGL}_2(K)$ (which generalizes the complex groups studied previously by Schottky to the non-Archimedean setting), he builds the corresponding p-adic upper half plane $\Omega_{\mathcal{L}_\Gamma}$ as the complement in $\mathbb{P}^1_K$ of the closure $\mathcal{L}_\Gamma$ of the set of fixed points of elements of Γ, as the analytification of a formal scheme built through the subtree $\mathcal{T}_\Gamma$ of the Bruhat-Tits tree having $\mathcal{L}_\Gamma$ as ends, and then, he obtains the curve as the quotient $C_\Gamma = \Gamma \backslash \Omega_{\mathcal{L}_\Gamma}$.

Then, Mumford proved in [Mum72b] that the Jacobians of these curves are abelian varieties that can be expressed as rigid analytic tori. After those works, Manin, Drinfeld, Gerritzen, van der Put and other authors found explicit methods to build such Jacobians using p-adic theta functions and a relation between these constructions and the theory of graphs, mainly involving the Bruhat-Tits trees and their quotients $G_\Gamma = \Gamma \backslash \mathcal{T}_\Gamma$ in which degenerate the Mumford curves. In the paper by Manin and Drinfeld [MD73] already appear two pairings

$$(\ , \ )_{\mathcal{L}_\Gamma} : \Gamma^{ab} \times \Gamma^{ab} \longrightarrow K^*$$

and

$$(\ , \ )_\Gamma : H_1(G_\Gamma, \mathbb{Z}) \times H_1(G_\Gamma, \mathbb{Z}) \longrightarrow \mathbb{Z}$$

and the formula

$$v_K((\gamma, \gamma')_{\mathcal{L}_\Gamma}) = (\gamma, \gamma')_\Gamma$$

which equals the valuation of the pairing on the Schottky group, which defines the Jacobian of the curve and gives the called monodromy pairing by Grothendieck, to the natural pairing on the edges of the graph. It is an important step to show that the analytic object defined previously has a polarization and therefore, it is algebraizable.

The construction by Mumford was taken up by Drinfeld in his study of the moduly varieties of elliptic modules [Dri74], where he showed that the space $\Omega_{\mathbb{P}^d(K)}$ which arise from the projective space of arbitrary dimension d over a local field by removing all rational hyperplanes has a natural rigid analytic structure of p-adic symmetric space by means of a reduction map to the Bruhat-Tits building

$$\mathrm{r} : \Omega_{\mathbb{P}^d(K)} \longrightarrow \mathscr{B}(\mathrm{PGL}_d(K)).$$

Later Mustafin generalized this result and described a class of varieties of any dimension generalizing the Mumford curves, built as the quotient of a more general rigid analytic symmetric space $\Omega_{\mathcal{L}_\Gamma}$ by a suitable hyperbolic subgroup $\Gamma \subset \mathrm{PGL}_d(K)$. The construction of the symmetric space was done again as a formal scheme with the structure given by a subbuilding $\mathscr{B}_{\mathcal{L}_\Gamma}$ of the Bruhat-Tits building of $\mathrm{PGL}_d(K)$. Later, first Schneider and Stuhler in [SS91], and then de Shalit and Alon in [dS01],[AdS02] and [AdS03] computed the rigid de Rham cohomology of the Drinfeld p-adic symmetric spaces and their quotient varieties $\Gamma\backslash\Omega_{\mathbb{P}^d(K)}$ by torsion-free, discrete, cocompact subgroups $\Gamma \subset \mathrm{PGL}_d(K)$, for which they are algebraic, in terms of the Bruhat-Tits building. They introduce harmonic cochains on the Bruhat-Tits building and harmonic measures on the spherical building at infinity (that, as a set also can be seen as the K-points of a flag variety) to give different descriptions of the rigid de Rham cohomology groups.

Further, Raskind and Xarles in [RX07a] and [RX07b] proved for these p-adically uniformized varieties and other under the assumption of having totally degenerate reduction that their cohomology verifies certain properties which allow to develope a p-adic theory of intermediate Jacobians analogue to the complex theory by Griffiths by means of a cohomological construction of those, but not analytic as in the case of the Jacobians of Mumfrod curves. The authors expected that such expected analytic construction could be done by means of the description of the cohomology with harmonic measures previously mentioned. More recently, Wilke in [Wil11] has given an analytic construction of the Picard variety of certain rigid analytic varieties, which he called totally degenerated, and that also generalize the Mumford curves to any dimension.

In addition to that, new analytic tools for non-Archimedean geometry have been devoleped through the last decades.

On one hand, new analytic theories have appeared, like the Berkovich analytic geometry, related to the tropical geometry, or the adic geometry introduced by Huber. These have better properties than rigid analytic theory and, as a consequence, they produce stronger results. For example, while rigid analytic spaces have a G-topology (since the original topology is totally disconnected, so that it is almost useless), but not a ("good") topology in the usual sense, Berkovich spaces did, since they have "more points".

On the other hand, Bertolini, Darmon, Dasgupta, Green and Longhi among others (cf. [BDG04], [Dar01],[Dar06], [Das04], [Das05], [Lon02]) have developed a theory of multiplicative integrals which allow to give an analytic construction of Stark-Heegner points (a certain class of p-adic points on modular elliptic curves), which are conjecturally global algebraic points, with the hope of proving several cases of the Birch and Swinnerton-Dyer conjecture (cf. [Dar06, §4]).

Both developments have dealt with Bruhat-Tits buildings and Jacobians of Mumford curves or objects related to them. Indeed, the equality of pairings described above have been studied and generalized in the context of Berkovich geometry in [BR15, Thm.'s 2.3 and 2.9], and the reduction map by Drinfeld has been generalized and reinterpreted as a retraction and as a tropicalization. Multiplicative integrals have been used to describe Jacobians of Mumford curves over local fields, after and equality relating these ones and theta functions in [Das04], and taking into account subtrees of the Bruhat-Tits tree.

We make our thesis in this context. Its first goal is to build the Jacobian of a Berkovich-analytic Mumford curve over any complete non-Archimedean field using these recently introduced techniques. After introducing the theory of harmonic measures, the Schottky groups and the Mumford curves with the tools provided by Berkovich geometry we give the defining morphism of the Jacobian

$$\begin{aligned} \Gamma^{ab} &\xrightarrow{\;\times\!\!\!\!\int_{\bullet} d\;} \mathrm{Hom}(\mathscr{M}(\mathcal{L},\mathbb{Z})_0^{\Gamma}, \mathbb{G}_{m,K}) := T \\ \gamma &\longmapsto \times\!\!\!\!\!\int_{\gamma p - p} d : \mu \mapsto \times\!\!\!\!\!\int_{\gamma p - p} d\mu \end{aligned}$$

and the prove that it is an abelian variety rests, mainly, in the equality between the pairings and that they are positive definite. To do that, we prove the isomorphism between the harmonic cochains on the tree associated to Γ and the harmonic measures on the ends of the tree and we build a retraction map from the Berkovich upper half-plane to the tree, which is also its skeleton.

On the way, we prove in all its generality some old claims of which we did not find any other rigurous proof in the literature; we generalize several definitions and results which were only stated for rigid analytic geometry or when the base field is local. In particular, some of these results are important theorems given in [GvdP80] or in [vdP92]. Moreover, we also reprove for Mumford curves recent results of Berkovich geometry of curves mainly appeared in [BPR13] and [BR15], using the theory of harmonic measures, related to the theories of harmonic cochains and of multiplicative integrals by means of the Bruhat-Tits $\mathbb{R}$-trees and subtrees of those. In particular, we relate the Jacobian of the Mumford curve with the Jacobian of the skeleton of the curve (that when the valuation is discrete it is known as its degeneration graph, and that it is a tropical curve), the last Jacobian playing the role of tropicalization of the first one, thanks, again, to the equality between the graph and the monodromy pairings through the valuation map.

$$
\begin{array}{ccccc}
\Omega_{\mathcal{L}_\Gamma} & \twoheadrightarrow & C_\Gamma & \xrightarrow{\;i_{z_0}\;} & \dfrac{\mathrm{Hom}(\mathscr{M}(\mathcal{L},\mathbb{Z})_0^\Gamma,\mathbb{G}_{m,K})}{\times\!\!\!\!\!\!\int d\left(\Gamma^{ab}\right)} \cong \mathrm{Jac}(C_\Gamma) \\
\Big\downarrow{\scriptstyle \mathrm{r}_{\mathcal{L}_\Gamma}} & & \Big\downarrow{\scriptstyle \mathrm{r}_{\mathcal{L}_\Gamma,\Gamma}} & & \Big\downarrow{\scriptstyle v_K} \\
\mathcal{T}_K(\mathcal{L}_\Gamma) & \twoheadrightarrow & G_\Gamma & \xrightarrow{\;i_p\;} & \dfrac{\mathrm{Hom}(\mathscr{M}(\mathcal{E}(\mathcal{T}_K(\mathcal{L}_\Gamma)),\mathbb{Z})_0^\Gamma,\mathbb{R})}{\int d\left(\Gamma^{ab}\right)} \cong \mathrm{Jac}(G_\Gamma).
\end{array}
$$

Our construction of the Jacobian of a Berkovich-analytic Mumford curve can be useful to make even more explicit the description of such Jacobians by trying to compute the defining lattices (that is, the periods) using the multiplicative integrals (this is part of a common project with Piermarco Milione). It also provides some hints to generalize it to higher dimension. More specifically, a similar way can be followed to construct the Albanese variety of a non-Archimedean uniformized variety.

The second goal of this thesis is to present a conjectural analytic construction of the Albanese variety of a Mustafin uniformized variety (that is, those algebraic non-Archimedean uniformized varieties by Mustafin) and to give some steps in the proof when the dimension of this one is 2, following an analogue process to the one employed for Mumford curves. Very little work has been done until now on Mustafin varieties, beyond the uniformized by the Drinfeld p-adic symmetric spaces and torsion-free, discrete, cocompact subgroups $\Gamma \subset \mathrm{PGL}_d(K)$ on dimension greater than 1, and the same occurs with the theory of harmonic cochains. They appear, almost always, with a local base field and the total Bruhat-Tits building, as in the papers cited above by Schneider, Stuhler, de Shalit and Alon. We introduce the construction identically to the given in dimension 1 with the generality that Mustafin results allow, that is, under a complete, discrete valued field, and we show in dimension 2 that the isomorphism between harmonic cochains and harmonic measures, known in dimension 1 for any compact $\mathcal{L} \subset \mathbb{P}^1(K)$ and in any dimension d when $\mathcal{L}$ are $d+1$ points not contained in a hyperplane or when $\mathcal{L} = \mathbb{P}^d(K)$ and K is local, can be generalized. To get this, first we associate to a compact set $\mathcal{L} \subset \mathbb{P}^d(K)$ a chamber subcomplex $\mathscr{B}_{\mathcal{L}}$ of the Bruhat-Tits building, which is a building for the construction of Mustafin, and we study in detail the structure of the minimal 1-skeleton of $\mathscr{B}_{\mathcal{L}}$ in relation with $\mathcal{L}$ by means of the apartments of $\mathscr{B}_{\mathcal{L}}$ and of the open sets associated to the edges as defined in [AdS02]. This structure let us to consider three important maps

$$\mathrm{St}_{\mathcal{L}}^{min} : \mathbb{Z}[\mathscr{B}_{\mathcal{L}0}] \longrightarrow \mathbb{Z}[\widehat{\mathscr{B}_{\mathcal{L}1}}^{min}],$$

$$\partial^{min} : \mathbb{Z}[\mathscr{B}_{\mathcal{L}d}] \longrightarrow \mathbb{Z}[\widehat{\mathscr{B}_{\mathcal{L}1}}^{min}]$$

and

$$\mathrm{Flow}_{\mathcal{L}} : \mathbb{Z}[\widehat{\mathscr{B}_{\mathcal{L}1}}^{min}] \longrightarrow \mathbb{Z}[\widehat{\mathscr{B}_{\mathcal{L}1}}^{min}],$$

which have a key importance in such study.

In particular, we note that we introduce in dimension 2 a definition of harmonic cochain on $\mathscr{B}_{\mathcal{L}}$ -through these three maps presented just above-, which generalizes the harmonic cochains defined by de Shalit in the local, cocompact case. This is one of the difficult points, since we are interested in harmonic cochains on the minimal 1-skeleton of $\mathscr{B}_{\mathcal{L}}$ (more generally, we should take into account the q-skeleton to relate to the corresponding intermediate Jacobian), which is more directly related to the construction by Schneider and Stuhler than to the given by de Shalit (cf. [SS91, Cor. 17 Rem. (2)] and [dS01, §8.3]), but in a very different language. In addition, there is a third construction by Garland in [Gar73] on the quotients of the building which differs slightly with respect to the other (except for the harmonic cochains on the chambers).

The contents of this thesis are organized as follows:

Chapter 1 is devoted to compute the discrete Jacobian of a graph using concepts, tools or just ideas that are of great importance through the next chapters, like are the Jacobian of a graph itself, the harmonic cochains on a graph (cf. remark 1.4.4) and one of the versions of the star map.

We start by giving the concept of graph and other related that appear along all the thesis, like the orientation of a graph and the opposite edge. Then we introduce a suitable definition of principally polarized discrete abelian variety which adapts the classical analytic definition, and we study equivalent formulations. With it, we give the construction of the Jacobian, for which we follow [BdlHN97], we relate it to the Jacobian of a graph as presented in [BN07] and we also mention the isomorphic dual construction of the Albanese torus in [KS08]. Finally, we show two different ways to compute such Jacobians following the works of de Shalit and Alon [dS01], [AdS03], and Infante [Inf06].

In the chapter 2 we introduce finite metric graphs, which are non other thing than tropical curves, and we compute their Jacobian by means of their universal coverings and the theory of harmonic measures and integration. Further, we define harmonic cochains and relate them to harmonic measures in such construction.

After generalize the definition of graph given in the first chapter to weight and metric graphs, we study the structure of the covering tree and its ends, and then we introduce harmonic cochains, measures and integrals. In the final section we start by giving the definition of the Albanese torus of a metric graph by Caporaso and Viviani in [CV10] in relation with the definition of the discrete Albanese torus of a graph, and we compute it and the Abel-Jacobi map by using the ideas of [BF11], which relate them to a generalization of the discrete Jacobian of a graph which we give in our chapter 1 for weighted graphs. Finally, we show that we can compute the Albanese torus by means of integration on the ends of the universal covering tree.

In chapter 3 we present Mumford curves, their Jacobians and their Abel-Jacobi maps over any complete non-Archimedean field in the setting of Berkovich analytic geometry, we reprove or also generalize in an original way known results about them, as we remarked above in the explanation of our first goal.

First, we construct the Bruhat-Tits $\mathbb{R}$-tree as the skeleton of the Berkovich projective line, we define the subtree $\mathcal{T}_K(\mathcal{L})$ associated to a compact set $\mathcal{L} \subset \mathbb{P}^1(K)$ and we construct the retraction map $\mathrm{r}_{\mathcal{L}} : \Omega_{\mathcal{L}} \longrightarrow \mathcal{T}_K(\mathcal{L})$. Then, we define multiplicative integrals following Longhi in [Lon02] and we relate them to the integration introduced in the previous chapter over the corresponding tree, which allows that in the final section we prove the equality between the monodromy pairing and the natural pairing on the edges of the skeleton of the Mumford curve. To build the Mumford curve we also introduce Schottky groups and reprove the results given by [GvdP80] in an original way. Further, we define the map

$$\tilde{\mu} : \mathcal{O}(\Omega_{\mathcal{L}})^* :\longrightarrow \mathcal{M}(\mathcal{L}, \mathbb{Z})_0$$

and we prove some properties of it, going further that in [vdP92], in addition to the Poisson formula, which lead to the proof of the symmetry of the pairing defining the analytic torus associated to the curve, and thus, to the fact that it is indeed an abelian variety, and it is also useful to develope briefly a theory of theta function with the objective of obtaining that the analytic torus is indeed the Jacobian of the curve. Moreover, for this introduction of theta function we also use and reprove some recent results of Berkovich geometry appeared in [BPR13].

The last chapter is devoted to the second main objective of this thesis, which consists, on one hand, in the conjectural construction of the Albanes variety of a Mustafin uniformized variety over any dimension, and on the other hand, on the proof of the isomorphism between harmonic measures and harmonic cochains related to an arbitrary compact $\mathcal{L} \subset \mathbb{P}^2(K)$.

In fact, the construction of the Albanese variety is done in the last section, from a generalitazion to higher dimension of Schottky groups, in the same way that Mustafin does. The last one is done by means of asking to a hyperbolic group $\Gamma \subset \mathrm{PGL}_d(K)$ that it and their associated complex $\mathscr{B}_{\mathcal{L}_\Gamma}$ verify certain conditions. Previously, we start by defining the Bruhat-Tits building over a complete, discrete valuation field and study some properties. We construct the subcomplex $\mathscr{B}_{\mathcal{L}}$ associated to a compact $\mathcal{L}$ and we study it in relation with its minimal 1-skeleton, its points at infinity ($\mathcal{L}$) and the maps introduced above (the minimal star, the minimal differential and the flow). Even if $\mathscr{B}_{\mathcal{L}}$ is not a building, it has some nice properties that we present by means of its apartments, of the minimal edges, of the associated open sets and of the points of $\mathcal{L}$. Some technical details lead us to continue our study on dimension 2, in particular, the definition of harmonic cochains, as remarked above. With this tools, we can proof the main theorem of the chapter, which is the isomorphism

$$\mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}}, \mathbb{Z}) \cong \mathscr{M}(\mathcal{L}, \mathbb{Z})_0$$

when $\mathscr{B}_{\mathcal{L}}$ is a building of dimension 2. This allows to reduce the proof that the defining map of the analytic variety associated to the Mustafin uniformized variety is an analytic torus to the fact that the map

$$H_1(\Gamma\backslash\mathscr{B}_{\mathcal{L}_\Gamma}, \mathbb{Z}) \longrightarrow \mathrm{Hom}(\mathrm{C}^1_{\mathrm{har}}(\Gamma\backslash\mathscr{B}_{\mathcal{L}_\Gamma}, \mathbb{Z}), \mathbb{Z})$$

is injective.

Conventions

In this section I will give some definitions related to the notation employed along this work.

The notation I will define over objects of certain categories, like groups or vector spaces, works over the morphisms of the corresponding categories too.

Along this work all fields will be commutative. For topological spaces, compactness means quasi-compactness plus the Hausdorff property, and so locally compact spaces are also Hausdorff by definition.

If H is an abelian group, following the standard notation for the extension of scalars we will denote $H_K := H \otimes K$, where K is a field of characteristic 0, and the tensor product is taken over $\mathbb{Z}$. In addition, for us the dual group of an abelian group H will be

$$H^\vee = Hom_{\mathbb{Z}}(H, \mathbb{Z}).$$

Assume now V is a vector space over a field K. Its dual will be denoted V^* in order to distinguish it of the dual as abelian group. The annihilator of a vector subspace $W \subset V$ is

$$W^\perp := \{\omega \in V^* | \omega(v) = 0 \ \forall v \in W\}$$

If $\mathbb{R} \subseteq K$, a lattice in V is a discrete subgroup $\Lambda \subseteq V$ with rank equal to the dimension of V as $\mathbb{R}$-vector space, in particular, $\Lambda \cong \mathbb{Z}^{dim_{\mathbb{R}}(V)}$. If moreover, V has an inner product $\langle \, , \, \rangle$, the dual of a lattice (with respect to the inner product) is

$$\Lambda^\# = \{x \in V \mid \langle x, \lambda \rangle \in \mathbb{Z} \ \forall \ \lambda \in \Lambda\},$$

which is also a lattice.

We shall denote by log the natural logarithm.

By a complete extension $L|K$ over a complete base field K with absolute value $|\cdot|$ we will refer to a field L containing K, complete with respect to an absolute value $|\cdot|_L$ extending $|\cdot|$.

For any $x \in K$, $r \in \mathbb{R}_{\geq 0}$, we consider the ball in the completion $\mathbb{C}_K := \widehat{\overline{K}}$ of the algebraic closure of K, $B(x,r) := \{y \in \mathbb{C}_K \mid |y - x| \leq r\}$.

Now, let K be any field, let V be an n-dimensional K-vector space, and denote by V^* its dual, so $\mathbb{P}_V = \operatorname{Proj}(S^\bullet(V^*))$ is the projective space associated to V, whose K-rational points correspond to the 1-dimensional subspaces of V (thus, with the traditional notation we have $\mathbb{P}(V) = \mathbb{P}_V(K)$). We also will consider the dual projective space $\mathbb{P}_{V^*} = \operatorname{Proj}(S^\bullet(V))$, whose K-rational hyperplanes are in correspondence with the K-rational points of $\mathbb{P}_V$. Given such a point $z \in \mathbb{P}(V)$, we denote by H_z the corresponding hyperplane in $\mathbb{P}_{V^*}$. We will identify V with $S^1(V)$, and the field K with $S^0(V)$.

We will write $G := \operatorname{PGL}(V)$; it is naturally isomorphic to the group of automorphisms of $\mathbb{P}_V$ as K-algebraic variety. It acts also on $\mathbb{P}_{V^*}$ by the usual contragredient representation (if $\gamma \in G,\ \omega \in \mathbb{P}(V^*)$, then $\gamma \cdot \omega := \omega\gamma^{-1}$).

Chapter 1

Jacobians of graphs

By analogy with the classical case for complex algebraic curves one may consider the Jacobians of finite graphs. These appear to us when we consider the dual graph of the reduction of a curve over a p-adic field with totally degenerate reduction.

Moreover, graphs are more simple objects than curves, and also than metric graphs. It is because of this that we start studying them and their Jacobians in order to get familiar with distinct notions and notations which will appear with variations through this thesis.

First of all, we define a graph and some related notions which will appear henceforth and which will be also generalized.

Second we introduce discrete analytic tori and principally polarized discrete abelian varieties before compare different ways to compute them.

Then, we recall the construction of the Jacobian of a graph made in [BdlHN97] and we show that it is a principally polarized discrete abelian variety. We finish the chapter given two different constructions of the Jacobians and proving that they are equivalent. The last recalls tools provided by [Inf06, Ch. 3], while the previous one remakes the construction made using harmonic cochains in [AdS03], mainly in the section 4.2, for the 1-dimensional case, but with integral coefficients.

1.1 Introduction

Definition 1.1.1. *A graph (or undirected graph) G consists of a set $V = V(G)$ of vertices, a set $E = E(G)$, disjoint from V, of edges, and an incidence function*

$$\psi : E \longrightarrow V^{(2)}$$

that associates to each edge of G an unordered pair of (not necessarily distinct) vertices of G (what is also called an unweighted undirected multigraph). If the 2 vertices are the same we shall say the corresponding edge is a loop. If 2 edges have associated the same pair of vertices we shall say that they are parallel edges.

A directed graph is a pair (V, E) of disjoint sets (of vertices and edges) together with a map

$$s \times t : E \longrightarrow V \times V.$$

We call $s(e)$ the source vertex of e and $t(e)$ its target vertex.

An orientation of an undirected graph $G = (V, E)$ is a directed graph $G' = (V', E')$ such that $V' = V$, $E' = E$ and

$$s \times t : E \longrightarrow V \times V$$

verifies $\psi(e) = \{s(e), t(e)\}$ for any e. An oriented graph is a graph with an orientation.

A graph $G = (V, E)$ (in particular, an oriented graph) has associated naturally a directed graph $(V, \hat{E})$ where $\hat{E} = \hat{E}(G) = E \sqcup E$, which is the union of two copies of E, and two maps

$$s \times t : \hat{E} \longrightarrow V \times V \qquad \textit{and} \qquad o : \hat{E} \longrightarrow \hat{E},$$

the last written as $o(e) =: \overline{e}$ and called the opposite of e, satisfying for each $e \in E$ that $\overline{e} \neq e$, $\overline{\overline{e}} = e$ and $s(e) = t(\overline{e})$. In fact, the function o is given by mapping an edge of a copy of E to the same edge in the other copy of E, and both copies define oriented graphs with opposite orientations.

The genus of a graph is its genus as topological space, which will be denoted $g(G)$. The degree of a vertex v of G, d_v, is the number of edges incident to it.

For any graph G, we have a 2 to 1 map

$$E_t : \hat{E} \longrightarrow E.$$

An orientation in G is given by one preimage in $\hat{E}(G)$ of each edge of E for that map.

1.2 Discrete analytic tori

We need to know what kind of object are we looking for. Following the analogy with the classical case, we know that Jacobians of complex curves are principally polarized abelian varieties. Below we define a discrete analogue of them.

Definition 1.2.1. *A discrete analytic torus is a triple*

$$(H^{1,0}, H^{0,1}, \nu : H^{1,0} \longrightarrow H^{0,1}),$$

where $H^{1,0}$ and $H^{0,1}$ are free abelian groups of the same rank (finite), ν is an injective morphism. A polarization in a discrete analytic torus is a morphism $\psi : H^{1,0} \longrightarrow H^{0,1\vee}$ being injective and satisfying the conditions

$$\psi(\lambda)(\nu(\lambda)) > 0 \, \forall \; \lambda \in H^{1,0} \smallsetminus \{0\}$$
$$\psi(\lambda)(\nu(\lambda')) = \psi(\lambda')(\nu(\lambda)) \, \forall \; \lambda, \lambda' \in H^{1,0}.$$

A discrete abelian variety is a discrete analytic torus which admits a polarization. A polarized discrete abelian variety is a discrete analytic torus together with a polarization.

Definition 1.2.2. *A principally polarized discrete abelian variety (from now on,* ppdav*, for short) is a quadruple*

$$(H^{1,0}, H^{0,1}, \nu : H^{1,0} \longrightarrow H^{0,1}, Q : H^{0,1}_{\mathbb{R}} \times H^{0,1}_{\mathbb{R}} \longrightarrow \mathbb{R}),$$

where $H^{1,0}$ and $H^{0,1}$ are free abelian groups of the same rank (finite), ν is an injective morphism, and Q is an inner product such that $H^{1,0\#} = H^{0,1}$ and $H^{0,1\#} = H^{1,0}$ looking at $H^{1,0}$ inside of $H^{0,1} \subseteq H^{0,1}_{\mathbb{R}}$ by means of ν.

Remark 1.2.3. *Note that the fact that ν is an injective morphism of abelian groups of the same rank makes it to have finite cokernel. Further, the condition $H^{1,0\#} = H^{0,1}$ implies that Q restricted to $H^{1,0}$ takes values in $\mathbb{Z}$.*

The dimension of a discrete analytic torus or of a (principally polarized) discrete abelian variety is the rank of the free abelian groups of its definition.

Definition 1.2.4. *A morphism between two ppdav's $(H^{1,0}, H^{0,1}, \nu, Q)$ and $(H'^{1,0}, H'^{0,1}, \nu', Q')$ is a pair of group homomorphisms $f^{1,0} : H^{1,0} \longrightarrow H'^{1,0}$*

and $f^{0,1} : H^{0,1} \longrightarrow H'^{0,1}$*, such that the diagrams*

$$\begin{CD} H^{1,0} @>{\nu}>> H^{0,1} \\ @V{f^{1,0}}VV @VV{f^{0,1}}V \\ H'^{1,0} @>{\nu'}>> H'^{0,1} \end{CD} \qquad\qquad \begin{array}{ccc} H^{0,1}_{\mathbb{R}} \times H^{0,1}_{\mathbb{R}} & \overset{Q}{\searrow} & \\ {\scriptstyle f^{0,1}_{\mathbb{R}} \times f^{0,1}_{\mathbb{R}}}\downarrow & & \mathbb{R} \\ H'^{0,1}_{\mathbb{R}} \times H'^{0,1}_{\mathbb{R}} & \underset{Q'}{\nearrow} & \end{array}$$

commute.

Remark 1.2.5. *The last definition allows us speak of isomorphisms. Then one also may see that giving an isomorphism class of ppdav's is the same that giving a matrix* A *with integer coefficients defining an inner product in* $\mathbb{R}^n$ *(that is, symmetric and positive definite) up to multiplication by* $GL_n(\mathbb{Z})$*. From the ppdav we obtain* A *as a matrix representing* Q *in a basis of* $H^{1,0}$*. Reciprocally, one has* $H^{1,0} := \mathbb{Z}^n$*, the canonical lattice in* $\mathbb{R}^n$*,* $H^{0,1} := \mathbb{Z}^{n\#}$*, the dual lattice, and* Q *the inner product defined by* A*.*

Now, by means of the next equivalences, we shall see how the definition of a ppdav completes the ones given before.

Theorem 1.2.6. *Given* $H^{1,0}$*,* $H^{0,1}$*, free abelian groups of the same rank (finite), the following are equivalent.*

a. *There are* ν *and* Q *such that* $(H^{1,0}, H^{0,1}, \nu, Q)$ *is a ppdav.*

b. *There are an isomorphism and a bilinear map*

$$H^{1,0} \xrightarrow[\cong]{\psi} H^{0,1\vee}, \qquad H^{1,0} \times H^{0,1\vee} \xrightarrow{\langle\, , \,\rangle} \mathbb{Z}$$

respectively, such that

$$\begin{aligned} &\langle \lambda, \psi(\lambda') \rangle = \langle \lambda', \psi(\lambda) \rangle \;\forall\, \lambda, \lambda' \in H^{1,0} \quad \textit{and} \\ &\langle \lambda, \psi(\lambda) \rangle > 0 \,\forall\; \lambda \in H^{1,0} \smallsetminus \{0\} \end{aligned}$$

c. *There are an injective morphism and an isomorphism*

$$H^{1,0} \overset{\nu}{\hookrightarrow} H^{0,1}, \qquad H^{1,0} \xrightarrow[\cong]{\psi} H^{0,1\vee}$$

respectively, such that

- $\psi(\lambda)(\nu(\lambda)) > 0 \,\forall\, \lambda \in H^{1,0} \setminus \{0\}$
- *The diagram*

$$\begin{array}{ccc} H^{1,0} & \xrightarrow{\nu} & H^{0,1} \\ \downarrow \psi & & \downarrow \psi^{\vee} \\ H^{0,1^{\vee}} & \xrightarrow{\nu^{\vee}} & H^{1,0^{\vee}} \end{array}$$

commutes.

Proof. If we start with the ppdav and we want to prove b, we define ψ as follows:

$$\psi(\lambda) = Q(\nu(\lambda), \_) \,\forall\, \lambda \in H^{1,0}$$

We have $\psi(\lambda) \in H^{0,1^{\vee}}$ since $H^{1,0^{\#}} = H^{0,1}$. It is an isomorphism. In order to prove that, take an $\omega \in H^{0,1^{\vee}}$ and tensor it by $\mathbb{R}$, obtaining thus an element of $H_{\mathbb{R}}^{0,1^{*}}$. Since Q is an inner product on $H_{\mathbb{R}}^{0,1}$, there exists a unique $\lambda \in H_{\mathbb{R}}^{0,1}$ such that $Q(\lambda, \_) = \omega_{\mathbb{R}}$. Using $\omega_{\mathbb{R}|H^{0,1}} \in H^{0,1^{\vee}}$ and $H^{0,1^{\#}} = H^{1,0}$ we conclude $\lambda \in \nu(H^{1,0})$. This proves surjectivity, but also injectivity since ω determines $\omega_{\mathbb{R}}$.
Now we define the bilinear map by

$$\langle \lambda, \omega \rangle = Q(\nu(\lambda), \nu(\psi^{-1}(\omega))) \,\forall\, \lambda \in H^{1,0},\ \omega \in H^{0,1^{\vee}}$$

The map $\langle\, ,\, \rangle$ satisfies the required properties due to Q provides them. Further, it induces the morphism

$$\begin{array}{ccc} H^{1,0} & \longrightarrow & \left(H^{0,1^{\vee}}\right)^{\vee} \cong H^{0,1} \\ \lambda & \longmapsto & \langle \lambda, \, \rangle \end{array}$$

which coincides with ν. To show this, we take an arbitrary element $\omega \in H^{0,1^{\vee}}$ and we note that for any $\lambda \in H^{1,0}$

$$\langle \lambda, \omega \rangle = Q(\nu(\lambda), \nu(\psi^{-1}(\omega))) = \omega(\nu(\lambda))$$

by definition of ψ and symmetry of Q.

We use the last idea to prove the reciprocal. Thus, we define ν from $\langle\, ,\, \rangle$ by

$$H^{1,0} \longrightarrow \left(H^{0,1^{\vee}}\right)^{\vee} \cong H^{0,1}$$

This definition means $\langle \lambda, \omega \rangle = \omega(\nu(\lambda))$ for all $\lambda \in H^{1,0}$ and $\omega \in H^{0,1^\vee}$. Furthermore, the properties of $\langle \, , \, \rangle$ do easy to check that ν is injective.
In order to define Q we first note that ψ and $\langle \, , \, \rangle$ give a bilinear map $\widetilde{Q} : H^{1,0} \times H^{1,0} \longrightarrow \mathbb{Z}$ $(\widetilde{Q}(\lambda, \lambda') := \langle \lambda, \psi(\lambda') \rangle)$. Further, we may tensor $\widetilde{Q}$ and ν with $\mathbb{R}$ getting

$$H^{1,0}_{\mathbb{R}} \times H^{1,0}_{\mathbb{R}} \xrightarrow{\widetilde{Q}_{\mathbb{R}}} \mathbb{Z}, \qquad H^{1,0}_{\mathbb{R}} \xrightarrow{\nu_{\mathbb{R}}} H^{0,1}_{\mathbb{R}}$$

The map $\nu_{\mathbb{R}}$ is a monomorphism (since $\mathbb{R}$ is $\mathbb{Z}$-flat and ν injective) of vector spaces of the same dimension, and then an isomorphism. Thus we define Q as the composition

$$H^{0,1}_{\mathbb{R}} \times H^{0,1}_{\mathbb{R}} \xrightarrow{\nu_{\mathbb{R}}^{-1} \times \nu_{\mathbb{R}}^{-1}} H^{1,0}_{\mathbb{R}} \times H^{1,0}_{\mathbb{R}} \xrightarrow{\widetilde{Q}_{\mathbb{R}}} \mathbb{Z}$$

Then, by going through the vector spaces, $\nu_{\mathbb{R}}$, $\widetilde{Q}$ and $\widetilde{Q}_{\mathbb{R}}$, one may check that Q is an inner product taking into account the properties of $\langle \, , \, \rangle$, and also that, by construction (of Q and ν), Q satisfies

$$Q(\nu(\lambda), \mu) = \psi(\lambda)(\mu) \, \forall \; \lambda \in H^{1,0}, \; \mu \in H^{0,1}$$

This implies $H^{1,0} \subseteq H^{0,1\#}$ and $H^{0,1} \subseteq H^{1,0\#}$. In order to prove the equalities, tensor ψ with $\mathbb{R}$. Then consider the following commutative diagram:

$$\begin{array}{ccc}
H^{0,1}_{\mathbb{R}} & & \\
\uparrow {\scriptstyle \nu_{\mathbb{R}} \,|\, \cong} & & \\
H^{1,0}_{\mathbb{R}} & \xrightarrow[\cong]{\psi_{\mathbb{R}}} & H^{0,1^*}_{\mathbb{R}} \\
\uparrow & & \uparrow \\
H^{1,0} & \xrightarrow[\cong]{\psi} & H^{0,1^\vee}
\end{array}$$

Let $\lambda \in H^{0,1}_{\mathbb{R}}$. Then $Q(\lambda, \_) \in H^{0,1^*}_{\mathbb{R}}$. Saying $\lambda \in H^{0,1\#}$ means that $Q(\lambda, \_)$ restricted to $H^{0,1}$ has image in $\mathbb{Z}$. In this case, since ψ is an isomorphism, there is an element $\lambda' \in H^{1,0}$ with $Q(\nu(\lambda'), \_) = Q(\lambda, \_)$ on $H^{0,1}$. The commutativity of the diagram implies that these two maps coincide on $H^{0,1}_{\mathbb{R}}$. Then $\lambda = \nu(\lambda')$ in $H^{1,0}$ inside $H^{0,1}_{\mathbb{R}}$, thus $H^{0,1\#} = H^{1,0}$.

Now, dualizing the maps ψ in the diagram we have

$$\begin{array}{ccc} H_{\mathbb{R}}^{0,1} & \xrightarrow[\cong]{{\psi_{\mathbb{R}}}^*} & H_{\mathbb{R}}^{1,0*} \\ \uparrow & & \uparrow \\ H^{0,1} & \xrightarrow[\cong]{\psi^{\vee}} & H^{1,0\vee} \end{array}$$

Let $\mu \in H^{1,0\#}$ in $H_{\mathbb{R}}^{0,1}$. This means that $Q(\nu_{\mathbb{R}}(\_), \mu) = {\psi_{\mathbb{R}}}^*(\mu) \in H_{\mathbb{R}}^{1,0*}$ restricted to $H^{1,0}$ takes values in $\mathbb{Z}$, that is, it belongs to $H^{1,0\vee}$. As before, there is an element $\mu' \in H^{0,1}$ such that $\psi^{\vee}(\mu')$ coincides with the restriction of ${\psi_{\mathbb{R}}}^*(\mu)$ to $H^{1,0}$, therefore by the commutativity of the diagram we obtain $\mu = \mu' \in H^{0,1}$ and then $H^{1,0\#} = H^{0,1}$.

Now we will proof the equivalence between b and c. Assume b is satisfied. We have already defined ν from $\langle\,,\,\rangle$ proving a. The properties are verified straightforward from the ones given in b, for example

$$\begin{aligned} \left(\psi^{\vee}(\nu(\lambda))\right)(\lambda') = \psi(\lambda')(\nu(\lambda)) = \langle \lambda, \psi(\lambda')\rangle = \langle \lambda', \psi(\lambda)\rangle = \psi(\lambda)(\nu(\lambda')) = \\ = \left(\psi^{\vee}(\nu(\lambda'))\right)(\lambda) \quad \forall\ \lambda, \lambda' \in H^{1,0} \end{aligned}$$

means that the square is commutative.

Let's assume now c. We may define

$$\begin{array}{ccc} H^{1,0} \times H^{1,0} & \xrightarrow{\widetilde{Q}} & \mathbb{Z} \\ (\lambda, \lambda') & \longmapsto & \widetilde{Q}(\lambda, \lambda') := (\psi(\lambda))(\nu(\lambda')) \end{array}$$

$$\begin{array}{ccc} H^{1,0} \times H^{0,1\vee} & \xrightarrow{\langle\,,\,\rangle} & \mathbb{Z} \\ (\lambda, \omega) & \longmapsto & \langle \lambda, \omega\rangle := \widetilde{Q}(\lambda, \psi^{-1}(\omega)) \end{array}$$

The required properties are verified immediately from the ones that we have via these definitions. □

1.3 The Picard group and the discrete Jacobian of a graph

Closely related to the Jacobian of a graph is the Picard group of a graph. We define both below following [BdlHN97] and [BN07], and we compare them.

Let G be a finite connected graph with its set of directed edges $\hat{E}(G)$.

Let $C^0(G,\mathbb{R})$ be the vector space of all real functions on $V(G)$, and let $C^1(G,\mathbb{R})$ be the vector space of all functions $g : \hat{E}(G) \longrightarrow \mathbb{R}$ such that $g(\overline{e}) = -g(e)$ for all $e \in \hat{E}(G)$. These are Euclidean spaces for inner products defined by

$$\langle f_1, f_2 \rangle_0 = \sum_{v \in V(G)} f_1(v) f_2(v)$$

and

$$\langle g_1, g_2 \rangle_1 = \frac{1}{2} \sum_{e \in \hat{E}(G)} g_1(e) g_2(e) = \sum_{e \in E(G)} g_1(e) g_2(e)$$

for all $f_1, f_2 \in C^0(G,\mathbb{R}),\ g_1, g_2 \in C^1(G,\mathbb{R})$. We have an exterior differential $d : C^0(G,\mathbb{R}) \longrightarrow C^1(G,\mathbb{R})$ defined by

$$(df)(e) = f(t(e)) - f(s(e))$$

with its adjoint operator $d^* : C^1(G,\mathbb{R}) \longrightarrow C^0(G,\mathbb{R})$ given by

$$(d^*g)(v) = \sum_{\substack{e \in \hat{E}(G) \\ t(e)=v}} g(e) = \sum_{\substack{e \in E(G) \\ t(e)=v}} g(e) - \sum_{\substack{e \in E(G) \\ s(e)=v}} g(e),$$

and thus a "Laplacian operator"

$$\Delta = d^* d : C^0(G,\mathbb{R}) \longrightarrow C^0(G,\mathbb{R})$$

We may restrict all these maps to the subgroups of integer functions $C^0(G,\mathbb{Z})$, $C^1(G,\mathbb{Z})$. We identify the first with the free abelian group on $V(G)$, the group of divisors on G, $\mathrm{Div}(G)$ by means of $f \longleftrightarrow \sum_{v \in V(G)} f(v)v$. Inside $\mathrm{Div}(G)$ we have the subgroup of degree 0 divisors

$$\mathrm{Div}^0(G) = \Big\{ \sum_{v \in V(G)} n_v v \mid \sum_{v \in V(G)} n_v = 0 \Big\}$$

One sees that $\operatorname{Im} \Delta \subseteq \mathrm{Div}^0(G)$ and defines the Picard group of G by

$$\mathrm{Pic}^0(G) := \mathrm{Div}^0(G) / \operatorname{Im}(\Delta)$$

We note this coincides with the definition of Jacobian of a graph given in [BN07].

Next, we define the lattice of integral flows of the graph as

$$\Lambda^1(G) := C^1(G,\mathbb{Z}) \bigcap \mathrm{Ker}(d^*)$$

and the first cohomology groups as

$$H^1(G,\mathbb{R}) = \mathrm{Coker}\,(d : C^0(G,\mathbb{R}) \longrightarrow C^1(G,\mathbb{R}))$$
$$H^1(G,\mathbb{Z}) = \mathrm{Coker}\,(d : C^0(G,\mathbb{Z}) \longrightarrow C^1(G,\mathbb{Z}))$$

Further, we have

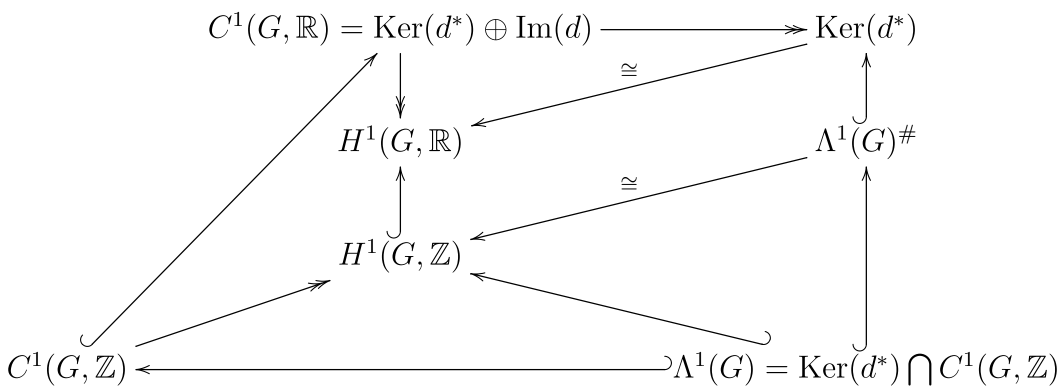
$$C^1(G,\mathbb{R}) = \mathrm{Ker}(d^*) \oplus \mathrm{Im}(d) \twoheadrightarrow \mathrm{Ker}(d^*)$$

If we restrict the projection of $C^1(G,\mathbb{R})$ onto $H^1(G,\mathbb{R})$ to $\mathrm{Ker}(d^*)$ we get an isomorphism, then we will identify $H^1(G,\mathbb{R})$ with $\mathrm{Ker}(d^*)$. By definition, we have $\Lambda^1(G) \subseteq \mathrm{Ker}(d^*)$. We consider its dual lattice inside of $\mathrm{Ker}(d^*)$ by $\langle\, ,\, \rangle_{1|\,\mathrm{Ker}(d^*)}$, $\Lambda^1(G)^{\#}$. From now on, we will denote the restriction $\langle\, ,\, \rangle_{1|\,\mathrm{Ker}(d^*)}$ by $(\, ,\,)$ looking at it as a bilinear form on $H^1(G,\mathbb{R})$. It is clear that $\Lambda^1(G) \subseteq \Lambda^1(G)^{\#}$. We also have

$$H^1(G,\mathbb{Z}) \hookrightarrow H^1(G,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \cong H^1(G,\mathbb{R})$$

and the isomorphism between $\mathrm{Ker}(d^*)$ and $H^1(G,\mathbb{R})$ restricts to an isomorphism between $\Lambda^1(G)^{\#}$ and $H^1(G,\mathbb{Z})$ ([BdlHN97, Prop. 3 (iii)]).

Gathering all this data, we obtain the next diagram:



Definition 1.3.1. *We define the (discrete) Jacobian torus of a graph G, $\mathrm{Jac}(G)$, as the quadruple*

$$(\Lambda^1(G), H^1(G,\mathbb{Z}),\ \Lambda^1(G) \longrightarrow H^1(G,\mathbb{Z})\,, (\, ,\,) : H^1(G,\mathbb{R}) \times H^1(G,\mathbb{R}) \longrightarrow \mathbb{R})$$

where the map is given by the composition $\Lambda^1(G) \hookrightarrow \Lambda^1(G)^{\#} \cong H^1(G,\mathbb{Z})$.

Once and for all, for us, the Jacobian of a graph will be its discrete Jacobian.

Proposition 1.3.2. *The Jacobian of a graph is a principally polarized discrete abelian variety.*

Proof. Since $\Lambda^1(G)$ and $\Lambda^1(G)^\#$ are lattices of the same vector space, they are free abelian groups of the same rank, and so $\Lambda^1(G)$ and $H^1(G, \mathbb{Z})$. The duality relations are shown in [BdlHN97, Prop. 3 (iii)]. □

Proposition 1.3.3. *The Picard group of a graph is given by the Jacobian, since*

$$\mathrm{Pic}^0(G) \cong H^1(G, \mathbb{Z})/H^1(G, \mathbb{Z})^\#$$

Proof. We have

$$H^1(G, \mathbb{Z})/H^1(G, \mathbb{Z})^\# \cong \Lambda^1(G)^\#/\Lambda^1(G) \cong \mathrm{Pic}^0(G)$$

where the first isomorphism comes from the one between $\mathrm{Ker}(d^*)$ and $H^1(G, \mathbb{R})$ and the second is given in [BdlHN97, Prop. 7 (iii)]. □

Remark 1.3.4. *Dually to the Jacobian we may define the (discrete) Albanese torus of G. One has $C_0(G, \mathbb{R})$, $C_1(G, \mathbb{R})$, $H_1(G, \mathbb{R})$, $H_1(G, \mathbb{Z})$ together with an inner product, then $H_1(G, \mathbb{Z})^\#$. In this case we have that $H_1(G, \mathbb{Z}) \subseteq H_1(G, \mathbb{Z})^\#$, and moreover*

$$\mathrm{Pic}^0(G) \cong H_1(G, \mathbb{Z})^\#/H_1(G, \mathbb{Z})$$

This is one of the goals of [KS08]. This is similar to the classical case, where the Jacobian variety and the Albanese variety are isomorphic in dimension 1.

1.4 Computing the Jacobian of a graph

Now, we will look at another way which will let us to compute the Jacobian of a graph, as we will prove.

The involved construction is studied in [dS01] and [AdS03], which are papers about a particular collection of p-adic varieties with totally degenerate reduction, certain quotients of Drinfeld's p-adic symmetric domains called p-adically uniformized varieties. A small variation of the one dimensional case is which gives the Jacobian.

Let $G = (V, E)$ be a finite connected oriented graph. Our goal is to compute its Jacobian by means of harmonic cochains as defined by de Shalit. In order to do that, we will reproduce in a simplified way and over the integers the construction done in [AdS03, §4.2]. Actually, that construction proceeds by considering the universal covering of G, is applied to it, and then invariants for the covering group (that is, the fundamental group of G) are taken. We skip all these steps and work directly over the graph, with the same result.

First of all, for each vertex v, we have to enumerate the target vertices of the (adjacent) edges which have v as source ($s(e) = v$) in one to one correspondence with them, even if there are coincidences among the vertices. We denote these enumerated vertices by $v_1, v_2, \ldots, v_{d_v}$, and the corresponding directed edges $e_1^v, e_2^v, \ldots, e_{d_v}^v \in \hat{E}(G)$ $(s(e_i^v) = v)$.

To start with the construction, consider for any vertex and any (oriented) edge the diagonal maps

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{-\Delta_v} & \mathbb{Z}^{d_v} \\ a_v & \longmapsto & (-a_v, \ldots, -a_v) \end{array} \qquad \begin{array}{ccc} \mathbb{Z} & \xrightarrow{-\Delta_e} & \mathbb{Z}^2 \\ b_e & \longmapsto & (-b_e, -b_e) \end{array}$$

and their corresponding products over all the vertices and (oriented) edges of the graph G:

$$\begin{array}{ccc} \prod_{v \in V(G)} \mathbb{Z} & \xrightarrow{\prod(-\Delta_v)} & \prod_{v \in V(G)} \mathbb{Z}^{d_v} \\ (a_v)_v & \longmapsto & (-a_v, \ldots, -a_v)_v \end{array} \qquad \begin{array}{ccc} \prod_{e \in E(G)} \mathbb{Z} & \xrightarrow{\prod(-\Delta_e)} & \prod_{e \in E(G)} \mathbb{Z}^2 \\ (b_e)_e & \longmapsto & (-b_e, -b_e)_e \end{array}$$

Note that since the products are finite, they are direct sums, and for example

$$\prod_{v \in V(G)} \mathbb{Z} \cong \mathrm{Div}(G).$$

Further, we are considering the factor $\mathbb{Z}^{d_v}$ with base $v, v_1, \ldots, v_{d_v - 1}$, that is, the vertex v itself, and the enumerated vertices removing the last.

Next, we define other two morphisms for each vertex and (oriented) edge,

$$\begin{array}{ccc} \mathbb{Z}^{d_v} & \longrightarrow & \mathbb{Z}^{d_v - 1} \\ (c_i^v)_{i=0 \div d_v - 1} & \longmapsto & (c_i^v - c_0^v)_{i=1 \div d_v - 1} \end{array} \qquad \begin{array}{ccc} \mathbb{Z}^2 & \longrightarrow & \mathbb{Z} \\ (a_1^e, a_2^e) & \longmapsto & a_2^e - a_1^e \end{array}$$

and we take the products, as above:

$$\prod_{v\in V(G)} \mathbb{Z}^{d_v} \xrightarrow{\partial_V^*} \prod_{v\in V(G)} \mathbb{Z}^{d_v-1}$$
$$((c_i^v)_{i=0\div d_v-1})_v \longmapsto ((c_i^v - c_0^v)_{i=1\div d_v-1})_v$$

$$\prod_{e\in E(G)} \mathbb{Z}^{2} \xrightarrow{\partial_E^*} \prod_{e\in E(G)} \mathbb{Z}$$
$$(a_1^e, a_2^e)_e \longmapsto (a_2^e - a_1^e)_e$$

Here we think of $\mathbb{Z}^{d_v-1}$ with base $e_1^v, e_2^v, \dots, e_{d_v-1}^v$.

To complete the structure we need, we define 3 morphisms more. The first is

$$\prod_{v\in V(G)} \mathbb{Z} \xrightarrow{d^0} \prod_{e\in E(G)} \mathbb{Z}$$
$$(a_v)_v \longmapsto (a_{t(e)} - a_{s(e)})_e$$

For the second morphism, first we have to inject the product over the vertices $v, v_1, \dots, v_{d_v-1}$, in the one adding the last vertex,

$$\prod_{v\in V(G)} \mathbb{Z}^{d_v} = \mathbb{Z}v \oplus \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_{d_v-1} \hookrightarrow \prod_{v\in V(G)} \mathbb{Z}^{d_v+1} = \mathbb{Z}v \oplus \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_{d_v}$$
$$((c_i^v)_{i=0\div d_v-1})_v \longmapsto ((c_i^v)_{i=0\div d_v})_v$$

where

$$c_{d_v}^v = d_v c_0^v - \sum_{i=1}^{i=d_v-1} c_i^v.$$

With this definition, we note the symmetry among the c_i^v for $i \neq 0$, since, if we change any of them by $c_{d_v}^v$, the relation is also satisfied. For $i \geq 1$ we may write $c_i^v = c_{v_i}^v$. With this notation we can define the next map:

$$\prod_{v\in V(G)} \mathbb{Z}^{d_v+1} \longrightarrow \prod_{e\in E(G)} \mathbb{Z}^{2}$$
$$((c_i^v)_{i=0\div d_v})_v \longmapsto \left(c_{s(e)}^{t(e)} - c_0^{s(e)}, c_0^{t(e)} - c_{t(e)}^{s(e)}\right)_e$$

We denote the composition of these two maps $\widetilde{d^0}$, which is the second morphism we were looking for.

For the third morphism we proceed in a similar way. We start taking into account the injection

$$\begin{array}{ccc}\prod_{v\in V(G)} \mathbb{Z}^{d_v-1} = \mathbb{Z}e_1^v \oplus \cdots \oplus \mathbb{Z}e_{d_v-1}^v & \hookrightarrow & \prod_{v\in V(G)} \mathbb{Z}^{d_v} = \mathbb{Z}e_1^v \oplus \cdots \oplus \mathbb{Z}e_{d_v}^v \\ ((b_i^v)_{i=1\div d_v-1})_v & \longmapsto & ((b_i^v)_{i=1\div d_v})_v\end{array}$$

where

$$b_{d_v}^v = -\sum_{i=1}^{d_v-1} b_i^v.$$

As before, we remark the symmetry among the b_i^v, in the relation. Consider an edge e, assume that $s(e) = v$ and $t(e) = v'$. There are i and j such that $v' = v_i$ and $v = v'_j$. We denote $b_i^v = b_e^v = b_e^{s(e)}$ and $b_j^{v'} = b_{\overline{e}}^{v'} = b_{\overline{e}}^{t(e)}$ (remember that $\overline{e}$ is the opposite edge of e). The composition of the last map with

$$\begin{array}{ccc}\prod_{v\in V(G)} \mathbb{Z}^{d_v} & \longrightarrow & \prod_{e\in E(G)} \mathbb{Z} \\ ((b_i^v)_{i=1\div d_v})_v & \longmapsto & (-b_{\overline{e}}^{t(e)} - b_e^{s(e)})_e\end{array}$$

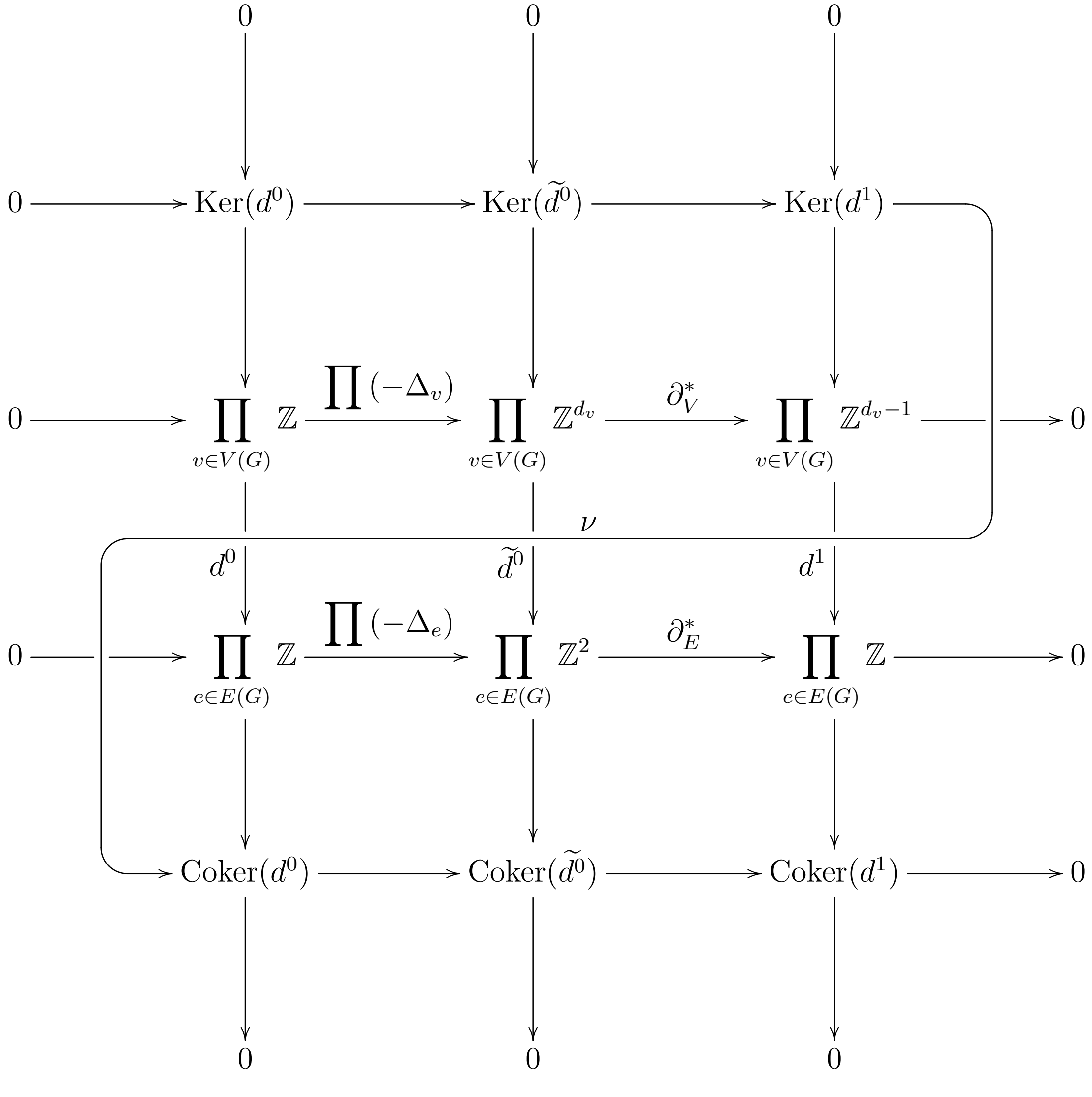
is, by definition, d^1. We get together all these morphisms in the next diagram:

$$\begin{array}{ccccccccc}0 & \longrightarrow & \prod_{v\in V(G)} \mathbb{Z} & \xrightarrow{\prod(-\Delta_v)} & \prod_{v\in V(G)} \mathbb{Z}^{d_v} & \xrightarrow{\partial_V^*} & \prod_{v\in V(G)} \mathbb{Z}^{d_v-1} & \longrightarrow & 0 \\ & & \downarrow d^0 & & \downarrow \widetilde{d}^0 & & \downarrow d^1 & & \\ 0 & \longrightarrow & \prod_{e\in E(G)} \mathbb{Z} & \xrightarrow{\prod(-\Delta_e)} & \prod_{e\in E(G)} \mathbb{Z}^2 & \xrightarrow{\partial_E^*} & \prod_{e\in E(G)} \mathbb{Z} & \longrightarrow & 0\end{array}$$

The exactness of the rows is clear and also the commutativity of the first square. We check the commutativity of the second square:

$$\begin{aligned} d^1\left(\partial_V^*\left(((c_i^v)_{i=0\div d_v-1})_v\right)\right)_e &= d^1\left(((c_i^v - c_0^v)_{i=1\div d_v-1})_v\right)_e = \\ = -c_{s(e)}^{t(e)} + c_0^{t(e)} - c_{t(e)}^{s(e)} + c_0^{s(e)} &= c_0^{t(e)} - c_{t(e)}^{s(e)} - c_{s(e)}^{t(e)} + c_0^{s(e)} = \\ = \partial_E^*\left(c_{s(e)}^{t(e)} - c_0^{s(e)}, c_0^{t(e)} - c_{t(e)}^{s(e)}\right)_e &= \partial_E^*\left(\widetilde{d}^0\left(\left((c_i^v)_{i=0\div d_v-1}\right)_v\right)\right)_e\end{aligned}$$

We may write the results of the differents maps in terms of e forgetting i's, that is, it does not matter if some vertex is v_{d_v}, because of the symmetry before remarked. As a consequence of the snake lemma we get a morphism ν, which we can see in the next diagram:



Definition 1.4.1. *We define the de Shalit Jacobian of the graph G as the triple*

$$(\mathrm{Ker}(d^1), \mathrm{Coker}(d^0), \nu : \mathrm{Ker}(d^1) \longrightarrow \mathrm{Coker}(d^0))$$

and we denote it by $\mathrm{Jac}^{dS}(G)$

Theorem 1.4.2. *There is a natural isomorphism between the de Shalit Jacobian and the discrete analytic torus determined by the Jacobian of a graph,*

$$\mathrm{Jac}^{dS}(G) \cong (\Lambda^1(G), H^1(G, \mathbb{Z}), \Lambda^1(G) \longrightarrow H^1(G, \mathbb{Z})).$$

In particular, the de Shalit Jacobian has a structure of ppdav isomorphic to $\mathrm{Jac}(G)$*, since we know that this discrete analytic torus admits a principal polarization.*

Proof.

Lemma 1.4.3. *There is an isomorphism* $\Lambda^1(G) \cong \mathrm{Ker}(d^1)$.

Proof. First, we remark that the lattice of integral flows $\Lambda^1(G)$ is nothing that the morphisms $g : \hat{E}(G) \longrightarrow \mathbb{Z}$ such that $g(\overline{e}) = -g(e)$, and

$$\sum_{\substack{e \in \hat{E}(G) \\ t(e)=v}} g(e) = 0$$

or, what is the same,

$$\sum_{\substack{e \in \hat{E}(G) \\ s(e)=v}} g(e) = 0.$$

Second, we define a map

$$b : \Lambda^1(G) \longrightarrow \prod_{v \in V(G)} \mathbb{Z}^{d_v - 1}$$

by

$$\Big(\big(b(g) \big)_i \Big)_v = g(e_i^v).$$

The image of this map is in the kernel of d^1:

$$d^1 \left(b(g) \right)_e = -b(g)_{\overline{e}}^{t(e)} - b(g)_e^{s(e)} = -g(\overline{e}) - g(e) = 0$$

Then we have $b : \Lambda^1(G) \longrightarrow \mathrm{Ker}(d^1)$.

Take now an element of

$$\mathrm{Ker}(d^1) \subseteq \prod_{v \in V(G)} \mathbb{Z}^{d_v - 1} \subseteq \prod_{v \in V(G)} \mathbb{Z}^{d_v}.$$

If we look at it inside $\prod_{v \in V(G)} \mathbb{Z}^{d_v}$, we may write it $((b_i^v)_{i=1 \div d_v})_v$. For any oriented edge e, we consider the vertex $s(e)$, then we have $e = e_i$ for some i, and we define $g(e) := b_i^{s(e)}$. Similarly to the way as we proceed in the last computation, we see that the condition of being in the kernel of d^1 implies $g(\overline{e}) = -g(e)$ ($g \in C^1(G, \mathbb{Z})$), and the immersion

$$\prod_{v \in V(G)} \mathbb{Z}^{d_v - 1} \subseteq \prod_{v \in V(G)} \mathbb{Z}^{d_v},$$

given by

$$b^v_{d_v} = -\sum_{i=1}^{d_v-1} b^v_i,$$

implies $g \in \mathrm{Ker}(d^*)$. It is clear that this construction is inverse to the map b, so this is an isomorphism. □

Remark 1.4.4. *The idea of this lemma is that these isomorphic groups are, in some way, groups of harmonic cochains as defined by de Shalit, but over the finite graph G. Indeed, if $\mathcal{T}_G \longrightarrow G$ is the universal covering of G with fundamental group Γ, $\mathrm{Ker}(d^1)$ coincides with $\mathrm{C}^1_{\mathrm{har}}(\mathcal{T}_G, \mathbb{Z})^\Gamma = H^0(\Gamma, \mathrm{C}^1_{\mathrm{har}}(\mathcal{T}_G, \mathbb{Z}))$ as it appears in [AdS03, §4.2] up to that we specify a different tree and that the group of values is $\mathbb{Z}$.*

We also have an isomorphism between $\mathrm{Coker}(d^0)$ and $H^1(G, \mathbb{Z})$. This is consequence of the 2 objects being cokernels of isomorphic morphisms, as we proof below.

Lemma 1.4.5. *There is an isomorphism of morphisms as described in the next commutative square:*

$$\begin{array}{ccc} C^0(G,\mathbb{Z}) & \xrightarrow{\ d\ } & C^1(G,\mathbb{Z}) \\ \cong \downarrow Ev_V & & \cong \downarrow Ev_E \\ \prod_{v\in V(G)} \mathbb{Z} & \xrightarrow{\ d^0\ } & \prod_{e\in E(G)} \mathbb{Z} \end{array}$$

Proof. The isomorphisms are defined as

$$\begin{array}{ccc} C^0(G,\mathbb{Z}) & \xrightarrow{\ Ev_V\ } & \prod_{v\in V(G)} \mathbb{Z} \\ f & \longmapsto & (f(v))_v \end{array} \qquad \begin{array}{ccc} C^1(G,\mathbb{Z}) & \xrightarrow{\ Ev_E\ } & \prod_{e\in E(G)} \mathbb{Z} \\ g & \longmapsto & (g(e))_e \end{array}$$

We show the commutativity:

$$Ev_E(d(f)) = (f(t(e)) - f(s(e)))_e = d^0((f(v))_v) = d^0(Ev_V(f))$$

□

Finally, we have to proof the commutativity of the square

$$\begin{array}{ccc}
\Lambda^1(G) & \longrightarrow & H^1(G,\mathbb{Z}) \\
\cong\Big\downarrow b & & \cong\Big\downarrow \overline{Ev_E} \\
\operatorname{Ker}(d^1) & \xrightarrow{\ \nu\ } & \operatorname{Coker}(d^0)
\end{array}$$

Take $g \in \Lambda^1(G)$. The image in $H^1(G,\mathbb{Z})$ is $\overline{g}$. Following

$$\overline{Ev_E}(\overline{g}) = \overline{(g(e))_e}$$

On the other hand, $b(g) = \big(g(e_i^v)_i\big)_v$. To finish we have to see its image by ν. To this end, we regard the definition of ν. We take as antiimage of $\big(g(e_i^v)_i\big)_v$ by the map $\partial_V^* : \prod_{v\in V(G)} \mathbb{Z}^{d_v} \longrightarrow \prod_{v\in V(G)} \mathbb{Z}^{d_v-1}$ the element defined through

$$c_0^v := 0$$
$$c_i^v := g(e_i^v),\ i = 1 \div d_v - 1$$

After this, we compute $\widetilde{d^0}$. By means of the injection of $\prod_{v\in V(G)} \mathbb{Z}^{d_v}$ in $\prod_{v\in V(G)} \mathbb{Z}^{d_v+1}$ we get

$$c_{d_v}^v = -\sum_{i=1}^{d_v-1} g(e_i^v) = g(e_{d_v}^v).$$

Therefore

$$\widetilde{d^0}\Big(\big((c_i^v)_{i=0\div d_v-1}\big)_v\Big)_e = (g(\overline{e}), -g(e))_e = (-g(e), -g(e))_e.$$

And this element has $(g(e))_e$ as antiimage by $\prod(-\Delta_e)$. Then

$$\nu(b(g)) = \nu(\big(g(e_i^v)_i\big)_v) = \overline{(g(e))_e},$$

so the square commutes. □

To finish this chapter, we give another way to compute the Jacobian which is implicit in [Inf06, Ch. 3].

Next we consider now the map

$$\begin{array}{ccc}
\mathbb{Z}^{E(G)} & \xrightarrow{\ d\ } & \mathbb{Z}^{V(G)} \\
e & \longmapsto & t(e) - s(e)
\end{array}$$

and denote its kernel $T_1^{-1}(G) = \mathrm{Ker}(d)$, the space of cycles of G. Likewise we take into account the map

$$\begin{array}{ccc} \mathbb{Z}^{V(G)} & \xrightarrow{\quad d' \quad} & \mathbb{Z}^{E(G)} \\ & & \\ v & \longmapsto & \displaystyle\sum_{\substack{e \in E(G) \\ t(e)=v}} e - \sum_{\substack{e \in E(G) \\ s(e)=v}} e \end{array}$$

to define $T_0^1(G) = \mathrm{Coker}(d')$, and we consider the composition

$$T_1^{-1}(G) \hookrightarrow \mathbb{Z}^{E(G)} \twoheadrightarrow T_0^1(G)$$

which will be called N. Together with this we dualize

$$\mathbb{Z}^{V(G)} \xrightarrow{\; d' \;} \mathbb{Z}^{E(G)} \longrightarrow T_0^1(G) \longrightarrow 0$$

to obtain the exact sequence

$$0 \longrightarrow T_0^1(G)^{\vee} \longrightarrow \left(\mathbb{Z}^{E(G)}\right)^{\vee} \xrightarrow{\; d'^{\vee} \;} \left(\mathbb{Z}^{V(G)}\right)^{\vee}$$

Taking the dual bases to $V(G)$ and to $E(G)$ and the correponding isomorphisms we get a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_1^{-1}(G) & \longrightarrow & \mathbb{Z}^{E(G)} & \xrightarrow{\; d \;} & \mathbb{Z}^{V(G)} \\ & & \cong \Big\downarrow \Longleftarrow & & \cong \Big\downarrow & & \cong \Big\downarrow \\ 0 & \longrightarrow & T_0^1(G)^{\vee} & \longrightarrow & \left(\mathbb{Z}^{E(G)}\right)^{\vee} & \xrightarrow{\; d'^{\vee} \;} & \left(\mathbb{Z}^{V(G)}\right)^{\vee} \end{array}$$

where the commutativity of the second square comes from the fact that the transpose of the matrix of d' in the given bases is the matrix of d.

Theorem 1.4.6. *The quadruple*

$$(T_1^{-1}(G), T_0^1(G), N : T_1^{-1}(G) \longrightarrow T_0^1(G), T_1^{-1}(G) \longrightarrow T_0^1(G)^{\vee})$$

satisfies the condition c *of Theorem 1.2.6, then it is a ppdav which we will call the Chow Jacobian of* G *and denote by* $\mathrm{Jac}^{Ch}(G)$*. Furthermore, we have*

$$\mathrm{Jac}(G) \cong \mathrm{Jac}^{Ch}(G)$$

Proof. The way of proving this theorem is the next. First we consider $\mathrm{Jac}(G)$, which we know it is a ppdav by Proposition 1.3.2. Second, we have the map

$$\begin{array}{ccc} \Lambda^1(G) & \longrightarrow & H^1(G,\mathbb{Z})^{\vee} \\ \lambda & \longmapsto & (\lambda, \_) \end{array}$$

which together with the morphism $\Lambda^1(G) \longrightarrow H^1(G,\mathbb{Z})$, applying Theorem 1.2.6, satisfies the condition c of itself. Finally, the only thing rest to do is proving the isomorphism between the Jacobians, that is, looking for two isomorphisms $T_1^{-1}(G) \cong \Lambda^1(G)$, $T_0^1(G) \cong H^1(G,\mathbb{Z})$ resulting in commutative diagrams:

$$\begin{array}{ccc} T_1^{-1}(G) & \xrightarrow{\ N\ } & T_0^1(G) \\ \cong\downarrow & & \downarrow\cong \\ \Lambda^1(G) & \longrightarrow & H^1(G,\mathbb{Z}) \end{array} \qquad \begin{array}{ccc} T_1^{-1}(G) & \longrightarrow & T_0^1(G)^{\vee} \\ \cong\downarrow & & \downarrow\cong \\ \Lambda^1(G) & \longrightarrow & H^1(G,\mathbb{Z})^{\vee} \end{array}$$

where the horizontal arrows are the ones already given. Thus we will get all the statements at the same time.

To define $T_1^{-1}(G) \longrightarrow \Lambda^1(G)$, let $\sum_{e\in E(G)} m_e e$ be a cycle in $T_1^{-1}(G) \subseteq \mathbb{Z}^{E(G)}$ and associate to it the map g defined by

$$g(e) := \begin{cases} m_e, \text{ if } e \in E(G) \\ -m_{\overline{e}}, \text{ if } \overline{e} \in E(G) \end{cases}$$

We can write in such a way any map of $C^1(G,\mathbb{Z})$. Note further that

$$\not{d}\Big(\sum_{e\in E(G)} m_e e\Big) = 0$$

means the same that $d^*(g) = 0$. Therefore, we can reverse the construction. Thus we obtain a well defined map, which is an isomorphism. Actually we get more, an isomorphism between $\mathbb{Z}^{E(G)}$ and $C^1(G,\mathbb{Z})$, which induces the commutativity of the next diagram with exact rows

$$\begin{array}{ccccccc} \mathbb{Z}^{V(G)} & \xrightarrow{\ \not{d}'\ } & \mathbb{Z}^{E(G)} & \longrightarrow & T_0^1 & \longrightarrow & 0 \\ \cong\downarrow & & \cong\downarrow & \Longrightarrow & \cong\vdots & & \\ C^0(G,\mathbb{Z}) & \xrightarrow{\ d\ } & C^1(G,\mathbb{Z}) & \longrightarrow & H^1(G,\mathbb{Z}) & \longrightarrow & 0 \end{array}$$

Then, the commutativity of the first square follows immediately, since it is the composition of the next commutative squares:

$$\begin{array}{ccccc} T_1^{-1}(G) & \hookrightarrow & \mathbb{Z}^{E(G)} & \twoheadrightarrow & T_0^1(G) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \Lambda^1(G) & \hookrightarrow & C^1(G,\mathbb{Z}) & \twoheadrightarrow & H^1(G,\mathbb{Z}) \end{array}$$

Finally, let us denote the dual element in $\left(\mathbb{Z}^{E(G)}\right)^{\vee}$ of an edge e by δ_e ($\delta_e(e) = 1$, $\delta_e(e') = 0$ for $e' \neq e$). We want to prove the commutativity of the second square. Let us take an element $\sum_{e \in E(G)} m_e e$ in T_1^{-1}. If first we follow the way of the left-down corner we get the associated map $g \in \Lambda^1(G)$ and $(g, \_) \in H^1(G,\mathbb{Z})^{\vee}$. Following the other way, we get $\sum_{e \in E(G)} m_e \delta_e \in {T_0^1}^{\vee}$.

Next, take any element $\eta \in H^1(G,\mathbb{Z})$. We have $\eta = \overline{h}$ for some $h \in C^1(G,\mathbb{Z})$, and

$$(g, \eta) = \langle g, h \rangle = \sum_{e \in E(G)} g(e)h(e) = \sum_{e \in E(G)} m_e h(e)$$

by construction of g. On the other hand we compute

$$\left(\sum_{e \in E(G)} m_e \delta_e\right)\left(\sum_{e \in E(G)} \overline{h(e)e}\right) = \sum_{e \in E(G)} m_e h(e)$$

The equality of the two last terms computed give the commutativity of the square and finishes thus the proof. □

Remark 1.4.7. *Note that neither* $\mathrm{Jac}^{dS}(G)$ *nor* $\mathrm{Jac}^{Ch}(G)$ *depend on the orientation* $E(G)$ *on the graph. Indeed, they can be obtained from* $\mathrm{Jac}(G)$ *by the corresponding isomorphisms, and the construction of the Jacobian does not use* $E(G)$ *but just* $\hat{E}(G)$*.*

Chapter 2

The Albanese torus of a finite metric graph

Along this chapter we extend the notions studied in chapter 1 to metric graphs and, then, we give a construction of the Albanese torus of a weighted graph and the Abel-Jacobi map by means of measures on the ends of its universal cover along sections 2.1 and 2.4.

Caporaso and Viviani studied in [CV10] the Torelli theorem for tropical curves by means of the identification of these ones with metric graphs and of the Albanese torus of these graphs. Baker and Faber developed some tools in [BF11] -where they also identify a tropical curve with a compact connected metric graph of finite total length- to understand better the corresponding Abel-Jacobi map, from their definition of the Jacobian of a weighted graph, which generalizes the definition of the Jacobian of a graph we showed in the chapter 1 from [BdlHN97]. In particular, they introduce a way to compute the Albanese torus of a weighted graph by means of 1-forms.

First, we show some structural results on metric and weighted graphs and, in particular, on trees. Then we study the ends of a tree as a topological space (with open compact sets which generalize the ones defined in [Das05, §2.3] and in [AdS02, §1.6] for certain discrete Bruhat-Tits trees). In the next section we define harmonic measures and harmonic cochains and we prove a canonical isomorphism between them, before introducing integrals on compact sets, which we will use on the ends of the trees. Finally, we relate the 1-forms in [BF11] with the harmonic cochains on a finite metric graph and we rise them to its universal cover, we proof the isomorphism between the Γ-invariant harmonic measures and the abelianized of Γ, where it is the fundamental group of the graph, and we conclude with a way to compute its Albanese torus and its Abel-Jacobi map by means of the integration on the ends of the universal cover tree.

2.1 Graphs, their models and the topology on the ends of a tree

First we will give the definitions of weighted and metric graphs and we will show some essential properties of them. Then, we will introduce the classical notions of rays and ends, in order to finish studying a natural topology on the set of ends of an infinite tree.

Definition 2.1.1. *A weighted graph* $\mathfrak{G}$ *is a non empty set* $V = V(\mathfrak{G})$ *called vertex set together with a directed edge set* $\hat{E} = \hat{E}(\mathfrak{G})$*, a weight function*

$$\ell : \hat{E} \longrightarrow \mathbb{R}_{>0},$$

an edge assignment map

$$s \times t : \hat{E} \longrightarrow V \times V$$

which makes correspond to each edge e *a pair* $(s(e), t(e))$*, where* $s(e)$ *is called the source of* e *and* $t(e)$ *the target of* e*, and a bijection*

$$o : \hat{E} \longrightarrow \hat{E}$$

verifying $\ell(o(e)) = \ell(e)$*,* $s \times t(o(e)) = (t(e), s(e))$*,* $o(e) \neq e$ *and* $o(o(e)) = e$*. The edge* $o(e)$ *is called the opposite of* e *and denoted by* $\bar{e}$ *(cf. [BF11] and [Ser80]).*

The valence of a vertex is the number of edges whose source is that vertex.

Definition 2.1.2. *The topological realization of a weighted graph* $\mathfrak{G}$ *is a topological space* $G := |\mathfrak{G}|$ *formed by vertices in correspondence with the vertices of* $\mathfrak{G}$ *and, for each* $e \in \hat{E}(\mathfrak{G})/\{e \sim o(e)\}$*, by an homeomorphic copy of the interval* $[0, \ell(e)]$*, glued according to the structure of the weighted graph.*

If it admits a distance, it is a metric space, so we call it a metric graph, for which the length of their edges is given by the weight of the edges of $\mathfrak{G}$ *(the same definitions and notations that we have for a weighted graph work for a metric graph).*

Remark 2.1.3. *This definition of metric graph includes the one stated in [BF11, §3]. In fact, they are the same under the assumptions of compactness, conectedness and finite valence everywhere. Thus, as we mentioned in the beginning of this chapter, tropical curves will be implicit objects through it.*

Remark 2.1.4. *Note that, even though all the edges have a length in the topological realization of a graph, it is not necessarily a metric space. Take,* *for example, the graph formed by two vertices v, v' and infinite edges e_n between them in correspondence with the non-zero natural numbers, each of them of weight $\frac{1}{n}$. Then, we would define*

$$d(v, v') = \inf_{n\in\mathbb{N}_{>0}} \{\ell(e_n)\} = \inf_{n\in\mathbb{N}_{>0}} \left\{\frac{1}{n}\right\} = 0$$

But a distance only can be zero between two points if they are the same, so this is not a metric space.

Definition 2.1.5. *Given a metric graph G, a model for it is any weighted graph $\mathfrak{G}$ such that G is obtained as its topological realization, that is $G \cong |\mathfrak{G}|$. A minimal model is one in which all the vertices have valence greater than 2.*

Definition 2.1.6. *Given two models $\mathfrak{G}, \mathfrak{G}'$ of a metric graph G and edges $e \in \hat{E}(\mathfrak{G}), e' \in \hat{E}(\mathfrak{G}')$ such that $|e'| \subset |e|$, we say that they have the same orientation or they preserve the orientation if it is the same in $\mathbb{R}$ after the homeomorphism $\rho_e : |e| \xrightarrow{\cong} [0, \ell(e)]$ (which preserves the orientation by definition, since $\rho_e(s(e)) < \rho_e(t(e))$).*

Remark 2.1.7. *For an edge $e \in \hat{E}(\mathfrak{G})$, the topological realizations of edges $|e|$ and $|o(e)|$ give the same set in $|\mathfrak{G}|$ but with opposite orientations.*

Definition 2.1.8. *A cycle in a weighted graph (resp. in a metric graph) is a subgraph whose topological realization (resp. itself) is homeomorphic to $\mathbb{S}^1$.*

A tree $\mathfrak{T}$ is a connected graph (weighted or metric respectively) without cycles. This is equivalent to say that given two vertices $v, v' \in V(\mathfrak{T})$, there exists a unique set of edges $P_{v,v'} \subset \hat{E}(\mathfrak{T})$ such that there is an homeomorphism $\rho_0 : |P_{v,v'}| \longrightarrow [0, r] \subset \mathbb{R}$ verifying $\rho_0(|v|) = 0$ and $\rho_0(|v'|) = r$.

We will denote the path $|P_{v,v'}|$ by $[v, v']$.

By definition, other topological notions (like connectedness...) will apply to a weighted graph if and only if they apply to its topological realization.

Let $\mathfrak{G} = (V, \hat{E})$ be a weighted graph such that $G = |\mathfrak{G}|$ is connected. As in the definitions of tree, a path between two vertices $v, v' \in V(\mathfrak{G})$ is a set of edges $P \subset \hat{E}(\mathfrak{G})$ such that there is an homeomorphism $\rho_0 : |P| \longrightarrow [0, r] \subset \mathbb{R}$ verifying $\rho_0(|v|) = 0$ and $\rho_0(|v'|) = r$. The difference is that for a graph we can have many paths between two vertices. The length of a path P is

$$\ell(P) := \sum_{e\in P} \ell(e)$$

Let us denote by $\mathfrak{P}_{v,v'}$ the set of paths between two vertices $v, v' \in V(\mathfrak{G})$.

Proposition 2.1.9. *Let $\mathfrak{G} = (V, \hat{E})$ be a weighted graph such that $G = |\mathfrak{G}|$ is connected. If for all pair of different vertices $v, v' \in V$ it satisfies*

$$\inf_{P \in \mathfrak{P}_{v,v'}} \ell(P) > 0,$$

then $G = |\mathfrak{G}|$ is a metric graph.

Proof. Given two different vertices $v, v' \in |V|$ we define

$$d(v, v') := \inf_{P \in \mathfrak{P}_{v,v'}} \ell(P), \text{ and otherwise } d(v, v) = 0.$$

Any other point, which is inside an edge e, can be thought as a vertex of valence two with the corresponding distances to the vertices of e, so the distance function extends to all G.

By definition $d(v, v') \geq 0$ for all $v, v' \in G$ and it is a symmetric map. By the hypothesis $d(v, v') = 0$ if and only if $v = v'$, so we only have to see the triangle inequality.

Since two paths between v and v', and between v' and v'' respectively extend to a path between v and v'' if they do not cut through another path, and otherwise allow to build a shorter path between v and v'', we get

$$d(v, v'') = \inf_{P \in \mathfrak{P}_{v,v''}} \ell(P) \leq \inf_{P \in \mathfrak{P}_{v,v'}} \ell(P) + \inf_{P \in \mathfrak{P}_{v',v''}} \ell(P) = d(v, v') + d(v', v'')$$

as we desired. □

Corollary 2.1.10. *The topological realization of a weighted graph $\mathfrak{G} = (V, \hat{E})$ such that for all $v, v' \in V$ there are a finite number of paths joining them is a metric graph. In particular, this is the case of trees and finite graphs.*

Lemma 2.1.11. *Let $\mathfrak{G}$ a weighted graph given by the sets $(V, \hat{E})$ of vertices and edges respectively and let $G := |\mathfrak{G}|$ be its topological realization. Let $p, p' \in G$ be two points connected for at least a path $P \subset G$ between them, so we have an homeomorphism $\rho : P \longrightarrow [0, \lambda] \subset \mathbb{R}$ such that $\rho(p) = 0$ and $\rho(p') = \lambda'$. Then $P \bigcap |V(\mathfrak{G})|$ is finite.*

Proof. Observe that if p is not the topological realization of a vertex of V, then is inside $|e|$ for an $e \in \hat{E}$ and such that one of the connected components of $|e \setminus \{p\}|$ is inside P and the other has empty intersection with P; therefore, we can extend P by $|e|$ and assume that $p = |v_0|$ for $v_0 \in V$, and identically for p'. Now we get, $|e| \bigcap P \neq \emptyset$ if and only if $|e| \subset P$.

For any edge $e \in \hat{E}$ such that $|e| \subset P$ take the image by ρ of the interior of its topological realization; it is an open interval $U_e := \rho(|e|) = (z_e - h_e, z_e +$

$h_e) \subset [0,\lambda]$. For any $|v| \in |V| \bigcap P$ different from p, p', let e^v the edge such that $t(e^v) = v$ and $|e^v| \subset P$ and e_v the one such that $s(e_v) = v$ and $|e_v| \subset P$. Take the open interval $U_v := (\rho(|v|) - h_{e^v}/2, \rho(|v|) + h_{e_v}/2) \subset [0,\lambda]$. For p take the open $U_p := [0, h_{e_p}/2) \subset [0,\lambda]$, and for p' the open $U_{p'} := (\lambda - h_{e^{p'}/2}, \lambda]$. Thus, we have

$$[0,\lambda] = U_p \cup \bigcup_{|e|\subset P} U_e \cup \bigcup_{|v|\in|V|\bigcap P} U_v \cup U_{p'}.$$

Since the interval is compact, we can make this covering with a finite number of these open sets given by $\hat{E}' \subset \hat{E}, V' \subset V$, both finite. But then we have $|V'| = P \bigcap |V(\mathfrak{G})|$ and we finish. □

In particular, given two vertices v, v' of a weighted tree $\mathfrak{T}$, we get that $[v, v'] \bigcap |V(\mathfrak{T})|$ is finite. As a consequence, we can extend the length of the edges of a tree given by the weight map to a metric on $|\mathfrak{T}|$

$$d : |\mathfrak{T}| \times |\mathfrak{T}| \longrightarrow \mathbb{R}_{\geq 0}$$

by $d(v, v') = \sum_{e\in P_{v,v'}} \ell(e)$ and by "linearity".

We will consider the free abelian group $\mathbb{Z}[\hat{E}(\mathfrak{G})]$ generated by the directed edges of $\mathfrak{G}$.

Given a weighted graph $\mathfrak{G}$, the star of a vertex v, $\mathrm{St}(v)$, is the set of edges of $\mathfrak{G}$ with source v.

Let $\mathfrak{G} = (V, \hat{E})$ be a weighted graph and $\mathfrak{H}$ be a finite weighted subgraph of $\mathfrak{G}$. We define the star of the subgraph as

$$\mathrm{St}(\mathfrak{H}) := \{e \in \hat{E} |\ s(e) \in \mathfrak{H}, e \notin \hat{E}(\mathfrak{H})\}$$

Note that this generalizes the definition of $\mathrm{St}(v)$ for a vertex v.

Definition 2.1.12. *A ray in a weighted tree $\mathfrak{T}$ is an infinite subtree whose topological realization is homeomorphic to $[0, +\infty)$ (so the condition of being infinite is equivalent by the lemma 2.1.11 to say that there are no $v, v' \in V(\mathfrak{T})$ such that the ray is inside $[v, v']$), or equivalently, it is the tree generated by an injective sequence of vertices, that is, an injective map $\mathbb{N} \longrightarrow V(\mathfrak{T})$ such that $[v_{n-1}, v_n] \cap [v_n, v_{n+1}] = \{v_n\}$ for all $n \geq 1$. Two rays are equivalent if they differ in a finite subgraph of the union, which is the same that their intersection being another ray. An end of $\mathfrak{T}$ is an equivalence class of rays.*

Let us denote its set of ends by $\mathcal{E}(\mathfrak{T})$.

Proposition 2.1.13. *Let $\mathfrak{T}$ and $\mathfrak{T}'$ be two weighted trees such that $|\mathfrak{T}| \cong |\mathfrak{T}'|$ (this is the same that having two models for a given metric tree). Then there is a natural bijection $\mathcal{E}(\mathfrak{T}) \cong \mathcal{E}(\mathfrak{T}')$.*

Proof. Let $[r] \in \mathcal{E}(\mathfrak{T})$ be an end. Let us denote the homeomorphism of the topological realizations by $T : |\mathfrak{T}| \longrightarrow |\mathfrak{T}'|$. Since $|r| \cong T(|r|) \subset |\mathfrak{T}'| \cong |\mathfrak{T}|$ is infinite, there are no $p, p' \in |\mathfrak{T}'|$ such that $T(|r|)$ is contained in the path between p and p', and then, there are no $v, v' \in V(\mathfrak{T}')$ such that $[v, v'] \supset T(|r|)$.

Write $\rho' : T(|r|) \xrightarrow{\cong} [0, +\infty)$ and let $p_0 = \rho'^{-1}(0) \in T(|r|)$ be the starting point of this half-line. If it is not a vertex $|v_0| \in |V(\mathfrak{T}')|$, it is inside $|e| \in |\hat{E}(\mathfrak{T})|$ such that $|t(e)| \in T(|r|)$ (because of the previous remark that $T(|r|) \nsubseteq [v, v']$). Define then $v_0 := t(e) \in V(\mathfrak{T}')$. Apply the same reasoning to the half-line $\rho'^{-1}([\rho(v_0) + 1, +\infty)$ to get $v_1 \in V(\mathfrak{T}')$, and so on, so we get a ray $(v_0, v_1, v_2, \dots)$ in $\mathfrak{T}'$ and an end in $\mathcal{E}(\mathfrak{T}')$.

Now we have defined the map $\mathcal{E}(\mathfrak{T}) \longrightarrow \mathcal{E}(\mathfrak{T}')$. Similarly we have a map in the opposite direction. Since the topological realization of the initial ray and of the ray resulted of the composition of both maps coincide on the topological realization of the tree, their intersection is infinite, so another ray, therefore the composition is the identity and we get the claimed bijection. □

Definition 2.1.14. *We define a ray in a metric tree T as the topological realization of a ray in a model $\mathfrak{T}$. We say that two of them are equivalent if their intersection is the realization of another ray in a model of T, and an end of T is a class of rays in it. Let us denote its set of ends by $\mathcal{E}(T) \cong \mathcal{E}(\mathfrak{T})$.*

By the previous proposition, this definitions are independent of the models chosen.

Definition 2.1.15. *Let e be an edge of $\mathcal{T}$. We define $\mathcal{B}(e)$ as the set of ends in $\mathcal{E}(\mathcal{T})$ classes of rays r such that $e \subset r$ and any homeomorphism $\rho_r : r \xrightarrow{\cong} [0, +\infty)$ preserves the orientation of e, that is, $\rho_r(s(e)) < \rho_r(t(e))$.*

Note that we can do this definition for e an edge of a model of $\mathcal{T}$ and $\mathcal{B}(|e|) = \mathcal{B}(e)$ due to the previous definitions.

Proposition 2.1.16. *These sets satisfy the next properties:*

- *Since $\mathcal{T}$ is a tree, $\mathcal{E}(\mathcal{T}) = \mathcal{B}(e) \sqcup \mathcal{B}(\bar{e})$ for each edge e.*
- *After considering a model of $\mathcal{T}$, for any vertex v its star gives rise to another partition*
$$\mathcal{E}(\mathcal{T}) = \bigsqcup_{e \in \mathrm{St}(v)} \mathcal{B}(e).$$

- *Let e, e' be edges of a model of $\mathcal{T}$. Then*

$$\mathcal{B}(e)\bigcap\mathcal{B}(e')\begin{cases} = \mathcal{B}(e) = \mathcal{B}(e'), & \textit{if there is another model of } \mathcal{T} \textit{ with} \\ & \textit{an edge } e'' \textit{ such that } |e|, |e'| \subset |e''| \\ & \textit{preserving orientation} \\ = \emptyset, \ \ \textit{if } s(e), s(e') \in [t(e), t(e')] \subset \mathcal{T} & \\ = \mathcal{B}(e) \subset \mathcal{B}(e'), & \textit{if } s(e) \in [t(e), s(e')] \textit{ and } s(e') \notin [t(e), t(e')] \\ = \mathcal{B}(e') \subset \mathcal{B}(e), & \textit{if } s(e') \in [t(e'), s(e)] \textit{ and } s(e) \notin [t(e'), t(e)] \\ \begin{cases} \neq \emptyset \textit{ and} \\ \subsetneq \mathcal{B}(e), \mathcal{B}(e') \end{cases}, & \begin{matrix} \textit{if } [s(e), s(e')] \textit{ is not an edge and} \\ s(e), s(e') \notin [t(e), t(e')] \end{matrix} \end{cases}$$

- *In the last of the previous cases, for any end $\varepsilon \in \mathcal{B}(e)\bigcap\mathcal{B}(e')$ there exists an edge e'' such that $\varepsilon \in \mathcal{B}(e'') \subset \mathcal{B}(e)\bigcap\mathcal{B}(e')$.*

Proof. Given any ray in $\mathcal{T}$, it is clear that we can make a unique equivalent ray starting either by an edge e or by its opposite, so the first assertion follows. We also can choose a unique equivalent ray starting by a given vertex v, which determines the edge of $\mathrm{St}(v)$, thus, we get the second claim too.

Next, we take two edges e, e'' in $\mathcal{T}$ such that $e \subset e''$ preserving the orientation. Then, the inclusion $\mathcal{B}(e'') \subset \mathcal{B}(e)$ is clear. But given a ray through e ("well oriented"), since e'' is an edge the ray always can be extended to an equivalent ray through e'', so $\mathcal{B}(e) = \mathcal{B}(e'')$.

Now assume $s(e), s(e') \in [t(e), t(e')]$. Observe that any homeomorphism $\rho : [t(e), t(e')] \xrightarrow{\cong} I \subset \mathbb{R}$, where I is a closed interval, reverses the orientation of e' with respect to e, so $\mathcal{B}(e)\bigcap\mathcal{B}(e') = \emptyset$ by definition of these subsets of ends.

Next, take $s(e) \in [t(e), s(e')]$, $s(e') \notin [t(e), t(e')]$ and an end in $\mathcal{B}(e)$, which can be given by a ray starting by $s(e)$ and through $t(e)$. Since $s(e) \in [t(e), s(e')]$, we can extend the ray to an equivalent one starting by $s(e')$, and $s(e') \notin [t(e), t(e')]$ implies that $t(e')$ belongs to the extended ray, therefore, the end is in $\mathcal{B}(e')$ too.

Finally, assume that $[s(e), s(e')]$ is not and edge and $s(e), s(e') \notin [t(e), t(e')]$. The first condition implies that there is a proper vertex $v \neq s(e), s(e')$ in $[s(e), s(e')]$, that is, a vertex in every model of $\mathcal{T}$, therefore having valence at least 3. Take the unique minimal path containing e and e', and so also v. observe that if $t(e) \notin [s(e), s(e')]$, then $[t(e), t(e')]\bigcap[s(e), s(e')] = \emptyset$, due to the second condition, but this would have as a consequence that there is a proper vertex in at least one of the edges, which cannot occur. Then $t(e), t(e') \in [s(e), s(e')]$. Next, if $[t(e'), t(e)] \subset [s(e), s(e')]$ preserving the orientation and with $t(e) \neq t(e')$, a proper vertex in $[s(e), s(e')]$ should be in e

or e' which is also imposible Therefore we get $[t(e), t(e')] \subset [s(e), s(e')]$ and $v \in [t(e), t(e')]$ necessarily. Since v has valence at least 3, there is a ray starting at v whose class is in $\mathcal{B}(e)$ and $\mathcal{B}(e')$, so the intersection is non empty. Since $[s(e), s(e')]$ reverses the orientation of one edge with respect the other, the intersection is neither $\mathcal{B}(e)$ nor $\mathcal{B}(e')$. Finally, and because of the same reason, any end in the intersection is the class of a ray starting by v and which does not pass through none of the edges e, e', so through a third edge e'' belonging to the star of v. Reciprocally, any ray in this way belongs to the intersection, so $\mathcal{B}(e'') \subset \mathcal{B}(e) \bigcap \mathcal{B}(e')$. □

Because of the last property, we can take the empty set with the sets $\mathcal{B}(e)$ as the basis for a topology in $\mathcal{E}(\mathcal{T})$, which, from now on, will be the considered topology there.

Given a tree $\mathcal{T}$ and an edge e in it, we denote by $\mathcal{T}_e^{t(e)}$ and $\mathcal{T}_e^{s(e)}$ to the connected components of $\mathcal{T} \setminus \mathring{e}$ containing $t(e)$ and $s(e)$ respectively. Observe that $\mathcal{T}_e^{s(e)} = \mathcal{T}_{\overline{e}}^{t(\overline{e})}$.

Remark 2.1.17. *We deduce a number of properties on the ends of the tree.*

- *The first property of the previous proposition implies that the sets $\mathcal{B}(e)$ are open and closed.*
- *Along the first paragraph of the proof we have noted that there is a bijection between the ends and the topological rays starting at a fixed vertex.*
- *A continuous action of a group Γ on $\mathcal{T}$ induces a continuous action of Γ on $\mathcal{E}(\mathcal{T})$.*
- *Note that given an edge e in $\mathcal{T}$, the connected component $\mathcal{T}_e^{t(e)}$ contains rays representing exactly the ends of $\mathcal{B}(e)$. Thus, we get*

$$\mathcal{B}(e) \cong \mathcal{E}(\mathcal{T}_e^{t(e)}) \text{ and } \mathcal{B}(\overline{e}) = \mathcal{E}(\mathcal{T}_e^{s(e)}).$$

  Therefore, all the properties of $\mathcal{E}(\mathcal{T})$ depending on hypotheses on $\mathcal{T}$ will apply to the open sets $\mathcal{B}(e)$ while $\mathcal{T}_e^{t(e)}$ satisfy the same hypotheses.

Proposition 2.1.18. *The topological space $\mathcal{E}(\mathcal{T})$ is Hausdorff.*

Proof. Take two distinct ends $\varepsilon, \varepsilon' \in \mathcal{E}(\mathcal{T})$.

Take a vertex v_0 in $\mathcal{T}$. If there is one edge $e \in \mathrm{St}(v_0)$ such that $\varepsilon, \varepsilon' \in \mathcal{B}(e)$ take the vertex $v_1 := t(e)$. Repeat this construction for v_1 to get v_2 and for v_i to get v_{i+1}. It has to be finite, since otherwise we would get a ray representing both ends and they would be equal.

So we can take a vertex v in $\mathcal{T}$ such that there are $e, e' \in \mathrm{St}(v)$ and $\varepsilon \in \mathcal{B}(e)$ and $\varepsilon' \in \mathcal{B}(e')$. Thus, we have disjoint open sets containing two distinct ends for any couple of them. □

Lemma 2.1.19. *Given three ends $\varepsilon, \varepsilon', \varepsilon'' \in \mathcal{E}(\mathcal{T})$, there exists a unique vertex $v \in \mathcal{T}$ such that there are three edges $e, e', e'' \in \mathrm{St}(v)$ verifying $\varepsilon \in \mathcal{B}(e)$, $\varepsilon' \in \mathcal{B}(e')$ and $\varepsilon'' \in \mathcal{B}(e'')$.*

Proof. Let us start proving the existence. As we proved in the previous proposition, we can take a vertex v_0 with $e_0, e_0' \in \mathrm{St}(v_0)$ such that $\varepsilon \in \mathcal{B}(e_0)$ and $\varepsilon' \in \mathcal{B}(e_0')$. If there is no $e'' \in \mathrm{St}(v_0)$ distinct of the other two whose open set contain ε'', this end is contained in $\mathcal{B}(e_0)$ or $\mathcal{B}(e_0')$. We can assume that $\varepsilon'' \in \mathcal{B}(e_0)$. Take $v_1 = t(e_0)$. Then $\varepsilon, \varepsilon''$ are contained in $\mathcal{B}(e_1), \mathcal{B}(e'')$ for $e_1, e'' \in \mathrm{St}(v_1) \setminus \{\overline{e_0}\}$ and $\varepsilon' \in \mathcal{B}(\overline{e_0})$. If $e_1 \neq e''$ we have find the vertex we are looking for; otherwise, we repeat this reasoning with $v_2 = t(e_1)$ and so on. Again as in the proof of the previous proposition, this process has to be finite, since we are defining rays representing ε and ε'' which are different, and in the finite step we get the vertex v.

For any other vertex $v' \neq v$, take the path $[v', v]$ joining them. At least two of the three edges e, e', e'' are not in these path, therefore the corresponding ends belong to the same open set of the edges of $\mathrm{St}(v')$. □

Proposition 2.1.20. *Let $\mathfrak{T} = (V, \hat{E})$ be a model for $\mathcal{T}$ and let $F \subset \hat{E}$ be a well oriented finite set of edges, meaning that it satisfies the following hypothesis:*

- *it cannot exist an edge e of $\mathcal{T}$ and edges $e', e'' \in F$ such that $|e'| \subset e$, preserving the orientation and $|e''| \subset e$ reversing the orientation.*

Take the source vertices of F, $\sigma := \sigma(F) := \{s(e) |\ e \in F\}$ and denote by $\mathfrak{T}_\sigma$ the subtree generated by σ. Then

1. *The open sets $\{\mathcal{B}(e)\}_{e \in F}$ are pairwise disjoint if and only if $F \bigcap \hat{E}(\mathfrak{T}_\sigma)$ is empty, which means $|F| \subset |\mathrm{St}(\mathfrak{T}_\sigma)|$.*

2. *The equality $\bigcup_{e \in F} \mathcal{B}(e) = \mathcal{E}(\mathcal{T})$ occurs if and only if $\mathrm{St}(\mathfrak{T}_\sigma) \subset F$.*

Proof. We will show the claims by induction on the cardinal of vertices $n = \#V(\mathfrak{T}_\sigma)$.

If $n = 1$, then $\mathfrak{T}_\sigma = \{v\} = \sigma(F)$, $F \subset \mathrm{St}(v)$, the sets $\mathcal{B}(e)$ with $e \in \mathrm{St}(v)$ are pairwise disjoint and $\bigsqcup_{e \in F} \mathcal{B}(e) = \mathcal{E}(\mathcal{T})$ if and only if $F = \mathrm{St}(v)$.

Next, assume $n > 1$ and let $v \in \sigma = \sigma(F)$ be a vertex with valence 1 in $\mathfrak{T}_\sigma$. Consider the non empty set $F_v := \{e \in F | \ s(e) = v\}$, proper in F since $n > 1$, and let e_v be the edge of $\mathfrak{T}_\sigma$ with target $t(e_v) = v$. Then, if $F' := (F \setminus F_v) \cup \{e_v\}$, we get the next remarkable properties:

- $\sigma' := (\sigma \setminus \{v\}) \cup \{s(e_v)\} = \sigma(F')$,
- and $\#V(\mathfrak{T}_{\sigma'}) = n - 1$, so we may apply the induction hypothesis on F'.
- $\hat{E}(\mathfrak{T}_{\sigma'}) = \hat{E}(\mathfrak{T}_\sigma) \setminus \{e_v, \overline{e_v}\}$, so

$$F' \cap \hat{E}(\mathfrak{T}_{\sigma'}) = (F \setminus F_v) \bigcap \big(\hat{E}(\mathfrak{T}_\sigma) \setminus \{e_v, \overline{e_v}\}\big).$$

- $\mathrm{St}(\mathfrak{T}_{\sigma'}) = (\mathrm{St}(\mathfrak{T}_\sigma) \setminus \mathrm{St}(v)) \cup \{e_v\}$.
- $\mathrm{St}(\mathfrak{T}_\sigma) = (\mathrm{St}(\mathfrak{T}_{\sigma'}) \setminus \{e_v\}) \cup (\mathrm{St}(v) \setminus \{\overline{e_v}\})$.

Suppose that $F \bigcap \hat{E}(\mathfrak{T}_\sigma) = \emptyset$. Therefore $F' \bigcap \hat{E}(\mathfrak{T}_{\sigma'}) = \emptyset$. Then, by induction hypothesis, the open sets $\{\mathcal{B}(e)\}_{e \in F'}$ are pairwise disjoint and, in particular, $\mathcal{B}(e_v) \cap \mathcal{B}(e) = \emptyset$ for all $e \in F \setminus F_v$. Recall now that

$$\mathcal{B}(e_v) = \bigsqcup_{e \in \mathrm{St}(v) \setminus \overline{e_v}} \mathcal{B}(e)$$

and that $F_v \subset \mathrm{St}(v)$. But, e_v and $\overline{e_v}$ are edges of $\mathfrak{T}_\sigma$, so $F \bigcap \hat{E}(\mathfrak{T}_\sigma) = \emptyset$ implies that $e_v, \overline{e_v} \notin F$, and therefore we get that the sets $\{\mathcal{B}(e)\}_{e \in F}$ are also pairwise disjoint.

Now assume that $F \bigcap \hat{E}(\mathfrak{T}_\sigma) \neq \emptyset$. Then, either $F' \bigcap \hat{E}(\mathfrak{T}_{\sigma'}) \neq \emptyset$, or $F' \bigcap \hat{E}(\mathfrak{T}_{\sigma'}) = \emptyset$ but

$$\emptyset \neq F \bigcap \hat{E}(\mathfrak{T}_\sigma) \subset \{e_v, \overline{e_v}\}.$$

In this last case, $F' \bigcap \hat{E}(\mathfrak{T}_{\sigma'}) = \emptyset$ and $F \bigcap \hat{E}(\mathfrak{T}_\sigma) \subset \{e_v, \overline{e_v}\}$, when $e_v \in F$, then $\mathcal{B}(e_v) \cap \mathcal{B}(e) \neq \emptyset$ for any $e \in F_v \neq \emptyset$. In the case $\overline{e_v} \in F$, the fact that $e_v \in F'$ and so that $\mathcal{B}(e_v) \cap \mathcal{B}(e) = \emptyset$ for any $e \in F \setminus F_v$ (by induction on F'), together with $\mathcal{B}(\overline{e_v}) = \mathcal{E}(\mathcal{T}) \setminus \mathcal{B}(e_v)$, implies that $\mathcal{B}(\overline{e_v}) \cap \mathcal{B}(e) \neq \emptyset$ for any $e \in F \setminus F_v$.

If $F' \bigcap \hat{E}(\mathfrak{T}_{\sigma'}) \neq \emptyset$, the sets $\{\mathcal{B}(e)\}_{e \in F'}$ are not pairwise disjoint, and the collection of sets $\{\mathcal{B}(e)\}_{e \in F}$ include the same except maybe $\mathcal{B}(e_v)$, besides the $\{\mathcal{B}(e)\}_{e \in F_v}$.

Therefore, if there are $e, e' \in F \setminus F_v$ such that $\mathcal{B}(e) \cap \mathcal{B}(e') \neq \emptyset$ we get the claim. Otherwise there is an $e_0 \in F \setminus F_v$ such that $\mathcal{B}(e_0) \cap \mathcal{B}(e_v) \neq \emptyset$

and $e_v \notin F$. By definition of e_v we have that $s(e_v) \in [t(e_v), s(e_0)]$. Then, taking in consideration the proposition 2.1.16 we get $s(e_0) \notin [t(e_v), t(e_0)]$ (and $\mathcal{B}(e_0) \cap \mathcal{B}(e_v) = \mathcal{B}(e_v) \subset \mathcal{B}(e_0)$), since otherwise we would have $s(e_0) \in [t(e_v), t(e_0)]$ and, as a consequence, $\mathcal{B}(e_0) \cap \mathcal{B}(e_v) = \emptyset$.

Take now an edge $e_1 \in F_v$. Assume first $e_1 \neq \overline{e_v}$. Then we obtain that $\mathcal{B}(e_1) \subset \mathcal{B}(e_v) \subset \mathcal{B}(e_0)$ and that the sets $\{\mathcal{B}(e)\}_{e\in F}$ are not pairwise disjoint as we wanted. To finish the proof of the the first equivalence, we just have to deal with the case $F_v = \{\overline{e_v}\}$. Since F is well oriented, there is some vertex of valence three in $\mathcal{T}$ between $s(e_0)$ and $t(e_v)$ (excluding them). Then $\mathcal{B}(e_0) \cap \mathcal{B}(\overline{e_v}) \neq \emptyset$ by proposition 2.1.16.

Recalling the properties we have noted above, we get that $\mathrm{St}(\mathfrak{T}_\sigma) \subset F$ implies $\mathrm{St}(\mathfrak{T}_{\sigma'}) \subset F'$, so, by hypothesis, $\bigcup_{e\in F'} \mathcal{B}(e) = \mathcal{E}(\mathcal{T})$. By definition, we know that each edge of F' is an edge of F except at most e_v, but we have that $\mathrm{St}(v) \setminus \{\overline{e_v}\} \subset \mathrm{St}(\mathfrak{T}_\sigma) \subset F$ and $\mathcal{B}(e_v) = \bigsqcup_{e\in \mathrm{St}(v)\setminus \overline{e_v}} \mathcal{B}(e)$, so

$$\mathcal{E}(\mathcal{T}) = \bigcup_{e\in F'} \mathcal{B}(e) \subset \bigcup_{e\in F} \mathcal{B}(e) = \mathcal{E}(\mathcal{T}).$$

Suppose that $\mathrm{St}(\mathfrak{T}_\sigma) \not\subset F$. This means that there is an edge $e \in \mathrm{St}(\mathfrak{T}_\sigma) \setminus F$, in particular with $s(e) \in \sigma = \sigma(F)$. We may assume that the vertex v we chose above in order to apply the induction method is different from $s(e)$. It is clear that $e \notin F'$, and by the assumption $e \in \mathrm{St}(\mathfrak{T}_{\sigma'})$, so $\mathrm{St}(\mathfrak{T}_{\sigma'}) \not\subset F'$ and $\bigcup_{e\in F'} \mathcal{B}(e) \neq \mathcal{E}(\mathcal{T})$.

Finally, as we have seen before, we have

$$\bigcup_{e\in F} \mathcal{B}(e) = \bigcup_{e\in F\setminus F_v} \mathcal{B}(e) \cup \bigcup_{e\in F_v} \mathcal{B}(e) \subset \bigcup_{e\in F'} \mathcal{B}(e) \cup \mathcal{B}(e_v) \subset \bigcup_{e\in F'} \mathcal{B}(e) \subsetneq \mathcal{E}(\mathcal{T}).$$

□

Corollary 2.1.21. *Let $\{e_i\}_{i\in I}$ be a finite set of directed edges in $\mathcal{T}$ such that the open sets $\mathcal{B}(e_i)$ for $i \in I$ are pairwise disjoint. Then*

$$\bigsqcup_{i\in I} \mathcal{B}(e_i) = \mathcal{E}(\mathcal{T}) \iff \{e_i\}_{i\in I} = \mathrm{St}(\mathfrak{T})$$

for the finite subtree $\mathfrak{T}$ with source vertices $\{s(e_i)\}_{i\in I}$, or $\{e_i\}_{i\in I} = \{e_1, e_2\}$ existing an edge e in $\mathcal{T}$ such that $e_1 \subset e$ and $e_2 \subset \overline{e}$.

Proposition 2.1.22. *Let $\mathcal{T}$ be a locally finite metric tree. Let E_0 be a set of edges of $\mathcal{T}$ such that $\mathcal{E}(\mathcal{T}) = \bigcup_{e\in E_0} \mathcal{B}(e)$. Then, there is a finite subset $E_1 \subset E_0$ such that $\mathcal{E}(\mathcal{T}) = \bigcup_{e\in E_1} \mathcal{B}(e)$.*

Proof. We assume that E_0 is an infinite set.

If there is an $e \in E_0$ such that $\overline{e} \in E_0$ we take $E_1 = \{e, \overline{e}\}$, so from now on we assume there is no such an edge e.

Consider now the subgraph $\mathcal{T} \setminus \bigcup_{e \in E_0} \mathring{e}$. It is a union of trees being its connected components, indexed by a set which we will denote I:

$$\mathcal{T} \setminus \mathring{E}_0 = \mathcal{T} \setminus \bigcup_{e \in E_0} \mathring{e} = \bigcup_{i \in I} \mathcal{T}_i.$$

If I is finite, then E_0 is also finite. Thus, assume that I is an infinite set and let us denote

$$E_i^+ = \{e \in E_0 |\ t(e) \in \mathcal{T}_i\},$$

$$E_i^- = \{e \in E_0 |\ s(e) \in \mathcal{T}_i\},$$

and

$$E_i = E_i^+ \cup E_i^- \neq \emptyset.$$

Observe that for $e \in E_0$, $e \in E_i^+$ means that $t(e) \in \mathcal{T}_i$ and $s(e) \notin \mathcal{T}_i$, which is equivalent to say that $\mathcal{T}_i \subset \mathcal{T}_e^{t(e)}$ (recall that $\mathcal{T}_e^{t(e)}$is the connected component of $\mathcal{T} \setminus \mathring{e}$ touching $t(e)$).

Next, let us divide the rest of the proof in three different cases.

First, assume that there exists $i \in I$ such that there are edges $e \neq e'$ and $\{e, e'\} \subset E_i^+$. Then, $e' \in \mathcal{T}_e^{t(e)}$ and $e \in \mathcal{T}_{e'}^{t(e')}$, and therefore $\mathcal{T}_e^{s(e)} \subset \mathcal{T}_{e'}^{t(e')}$. Thus,

$$\mathcal{E}(\mathcal{T}) = \mathcal{B}(e) \cup \mathcal{B}(\overline{e}) = \mathcal{B}(e) \cup \mathcal{E}(\mathcal{T}_e^{s(e)}) \subset \mathcal{B}(e) \cup \mathcal{E}(\mathcal{T}_{e'}^{t(e')}) = \mathcal{B}(e) \cup \mathcal{B}(e'),$$

so we take $E_1 = \{e, e'\}$.

In the second place, assume that there exists $i \in I$ such that $E_i^+ = \emptyset$.

If $\mathcal{E}(\mathcal{T}_i) \neq \emptyset$, there exists $\varepsilon \in \mathcal{E}(\mathcal{T}_i)$ and $e \in E_0$ such that $\varepsilon \in \mathcal{B}(e)$. Then $e \notin E_i$; otherwise, $e \in E_i^-$, so $\mathcal{T}_i \subset \mathcal{T}_e^{s(e)}$ and $\varepsilon \in \mathcal{E}(\mathcal{T}_i) \subset \mathcal{E}(\mathcal{T}_e^{s(e)}) = \mathcal{B}(\overline{e})$. Consider the unique path connecting e with $\mathcal{T}_i$. Since $e \notin E_i$, there is a unique edge $e' \neq e$ in this path, such that $e' \in E_i = E_i^-$, so $e \in \mathcal{T}_{e'}^{t(e')}$. The fact that $\varepsilon \in \mathcal{E}(\mathcal{T}_i) \cap \mathcal{B}(e)$ implies that $\mathcal{T}_i \subset \mathcal{T}_e^{t(e)}$ and $\mathcal{T}_{e'}^{s(e')} \subset \mathcal{T}_e^{t(e)}$. Thus,

$$\mathcal{E}(\mathcal{T}) = \mathcal{B}(e') \cup \mathcal{B}(\overline{e'}) = \mathcal{B}(e') \cup \mathcal{E}(\mathcal{T}_{e'}^{s(e')}) \subset \mathcal{B}(e') \cup \mathcal{E}(\mathcal{T}_e^{t(e)}) = \mathcal{B}(e') \cup \mathcal{B}(e),$$

so we take $E_1 = \{e, e'\}$.

If $\mathcal{E}(\mathcal{T}_i) = \emptyset$, the tree $\mathcal{T}_i$ is finite, since $\mathcal{T}$ is locally finite, so the set $E_1 := E_i = E_i^-$ is also finite. For any $j \neq i$ take the unique path connecting

$\mathcal{T}_j$ with $\mathcal{T}_i$. As above, it contains an edge $e_j \in E_i$ such that $\mathcal{T}_j \subset \mathcal{T}_{e_j}^{t(e_j)}$. Now, we get that

$$\mathcal{E}(\mathcal{T}) = \bigsqcup_{j \in I} \mathcal{E}(\mathcal{T}_j) \subset \bigsqcup_{e \in E_1} \mathcal{E}(\mathcal{T}_e^{t(e)}) = \bigsqcup_{e \in E_1} \mathcal{B}(e),$$

as we wanted.

Finally, assume that for all $i \in I$, $E_i^+ = \{e_i\}$. Fix an $i_0 \in I$. Let $i_1 \in I$ be the index such that $s(e_{i_0}) \in \mathcal{T}_{i_1}$. Note that $e_{i_0} \in E_{i_1}^-$, so we have $e_{i_1} \neq e_{i_0}$. Generally, let $i_{n+1} \in I$ be the index such that $s(e_{i_n}) \in \mathcal{T}_{i_{n+1}}$. Consider the ray defined by the sequence $\{t(e_{i_n})\}_{n \in \mathbb{N}}$, which can be seen as

$$\overline{e_{i_0}} \cup [s(e_{i_0}), t(e_{i_1})] \cup \overline{e_{i_1}} \cup [s(e_{i_1}), t(e_{i_2})] \cup \overline{e_{i_2}} \cup [s(e_{i_2}), t(e_{i_3})] \cup \overline{e_{i_3}} \cup \dots$$

(observe that $[s(e_{i_n}), t(e_{i_{n+1}})] \subset \mathcal{T}_{i_{n+1}}$, so the previous union is disjoint xcept at the vertices). We are going to show that this end cannot be in any $\mathcal{B}(e)$ for $e \in E_0$, so we will get a contradiction from which we will deduce that this case cannot occur.

Assume there exists $e \in E_0$ such that $\mathcal{B}(e)$ contains the previously built end. Therefore, the ray starting at e and defining this end cuts $\mathcal{T}_{i_n}$ and contains $\overline{e_{i_n}}$ for all $n \in \mathbb{N}$ big enough. Let us denote the edges in the intersection of this ray with E_0 by f_n consecutively starting from e, so $f_0 := e$ and for all n big enough $e_{i_n} = f_{m(n)}$, where $m : \mathbb{N}_{\geq N} \longrightarrow \mathbb{N}$ is an increasing map. Therefore, there exists an $m \in \mathbb{N}$ such that f_m has the same orientation of the ray (like f_0) and f_{m+1} has the opposite orientation (like any e_{i_n} for any n big enough). Then, if $\mathcal{T}_{i(m)}$ is the connected component between f_m and $f_m + 1$, both edges verify that $t(f_m), t(f_{m+1}) \in \mathcal{T}_{i(m)}$, and therefore, $f_m, f_{m+1} \in E_{i(m)}^+$, which contradicts the hypothesis that this set consists in a unique element.

Summarizing, the end given by the sequence $\{t(e_{i_n})\}_{n \in \mathbb{N}}$ cannot exist, so the third case, whose hypothesis is that each E_i^+ consists in a unique element, does not occur. □

Corollary 2.1.23. *If $\mathcal{T}$ is a locally finite tree, $\mathcal{E}(\mathcal{T})$ is compact, and in the same way, all the open subsets $\mathcal{B}(e)$ are also compact.*

Proof. The sets $\mathcal{B}(e)$ form a basis of open sets, so one only has to apply he definitions and the previous proposition to conclude. □

Remark 2.1.24. *Compact metric graphs are locally finite, and so their universal coverings. In particular, we may apply the previous corollary to the universal covering trees of tropical curves seen as metric graphs.*

2.2 Harmonic cochains on a graph and harmonic measures on a compact set

We give general definitions of harmonic cochains over any weighted graph and of harmonic measures over a suitable compact set, and we prove the isomorphism between the harmonic measures on the ends of a locally finite metric tree and the harmonic cochains on that tree.

In the next chapter, given a compact subset $\mathcal{L} \subset \mathbb{P}^1(K)$ we will build a tree $\mathcal{T}_K(\mathcal{L})$, whose ends correspond to $\mathcal{L}$, so we will be able to apply the result to this tree, proving the assertion by van der Put in [vdP92, Ex. 2.1.1].

2.2.1 Harmonic cochains on a graph

Recall that a harmonic cochain is a morphism $c : \mathbb{Z}[\hat{E}(\mathfrak{G}))] \to \mathbb{Z}$ verifying

- $c(\bar{e}) = -c(e)$ for any $e \in \hat{E}(\mathfrak{G})$, and
- $c\Big(\sum_{e \in \mathrm{St}(v)} e\Big) = 0$ for any vertex $v \in V(\mathfrak{G})$.

We denote the set of harmonic cochains of $\mathfrak{G}$ by $\mathrm{C}^1_{\mathrm{har}}(\mathfrak{G}, \mathbb{Z})$.

Observe that, if we subdivide an oriented edge e in two oriented edges e_1 and e_2, then the properties tell that any harmonic cochain verifies that $c(e_1) = c(e_2)$. Hence, given a (locally finite) metric graph and two arbitrary models for it, there is a canonical isomorphism between their harmonic cochains, so we can define them for the metric graph $G = |\mathfrak{G}|$, and we can write $\mathrm{C}^1_{\mathrm{har}}(G, \mathbb{Z}) := \mathrm{C}^1_{\mathrm{har}}(\mathfrak{G}, \mathbb{Z})$.

Lemma 2.2.1. *Let $\mathfrak{H}$ be a finite weighted subgraph of $\mathfrak{G}$. Then, any harmonic cochain c satisfies $c\Big(\sum_{e \in \mathrm{St}(\mathfrak{H})} e\Big) = 0$.*

Proof. First observe the following properties of stars:

$$\mathrm{St}(\mathfrak{H}) \sqcup \hat{E}(\mathfrak{H}) = \mathrm{St}(V(\mathfrak{H})) = \bigsqcup_{v \in V(\mathfrak{H})} \mathrm{St}(v)$$

Next note that an edge belongs to $\mathfrak{H}$ if and only if its opposite also do. Then, taking into consideration the first equality of stars and the previous remark, because of the first property of the harmonic cochains we get

$$c\Big(\sum_{e \in \mathrm{St}(\mathfrak{H})} e\Big) = \sum_{e \in \mathrm{St}(\mathfrak{H})} c(e) = \sum_{e \in \mathrm{St}(V(\mathfrak{H}))} c(e)$$

and because of the second equality of stars and the second property of harmonic cochains we finish as follows:

$$\sum_{e\in \mathrm{St}(V(\mathfrak{H}))} c(e) = \sum_{v\in V(\mathfrak{H})} \sum_{e\in \mathrm{St}(v)} c(e) = 0$$

□

2.2.2 Harmonic measures on a compact set

In order to get another point of view for the harmonic cochains we have to define the harmonic measures on a suitable compact space. Previously we will define distributions.

Remark 2.2.2. *Let X be a Hausdorff topological space and assume that it has a basis formed by open compact subsets (or similarly, every point has a neighbourhood basis formed by compact open subsets). Then, it is clear that X is locally compact, and it is also easy to check that it is totally disconnected. If X is a topological group locally compact and totally disconnected, then the opposite implication is also true (what is usually called a locally profinite group). In particular, these conditions are equivalent and satisfied for subspaces of finite dimensional projective spaces over local fields.*

Definition 2.2.3. *Let X be a Hausdorff topological space and assume that it has a basis formed by open compact subsets. Let $C_c^\infty(X,\mathbb{Z})$ be the space of all $\mathbb{Z}$-valued, compactly supported, locally constant functions on X. For any abelian group A we call*

$$\mathcal{D}(X,A) = \mathrm{Hom}(C_c^\infty(X,\mathbb{Z}), A)$$

the space of A-valued distributions on X.

Lemma 2.2.4. *With the previous hypotheses, any $f \in C_c^\infty(X,\mathbb{Z})$ is a finite linear combination of characteristic functions on open compact sets of X.*

Proof. Since f has compact support, there exists a compact $\mathcal{K} \subset X$ such that $X \setminus \mathcal{K} \subset f^1(\{0\})$ and

$$\mathcal{K} = \bigsqcup_{m\in\mathbb{Z}\setminus\{0\}} f^{-1}(\{m\}).$$

As the sets $f^{-1}(\{m\})$ are open, in this union only appear a finite set. But these sets are closed contained in the compact $\mathcal{K}$, so they are compact. Then we get a finite sum

$$f = \sum_{f^{-1}(\{m\})\subset\mathcal{K}} m\chi_{f^{-1}(\{m\})}.$$

□

Definition 2.2.5. *Let X be a compact space such that the compact open subsets form a basis for the topology. Given an abelian group A, an A-valued measure μ on X is a function on the compact open subsets of X such that applied to a finite disjoint union of compact open subsets is equal to the sum of the images of these subsets. The set of A-valued measures on X is denoted $\mathscr{M}(X, A)$.*

Proposition 2.2.6. *With the same hypotheses from previous definition we have an isomorphism of abelian groups*

$$\Upsilon : \mathcal{D}(X, A) \xrightarrow{\cong} \mathscr{M}(X, A),$$

given by $\Upsilon(f)(\mathcal{U}) = f(\chi_{\mathcal{U}})$, where $\chi_{\mathcal{U}} \in C_c^{\infty}(X, \mathbb{Z})$ is the characteristic function of $\mathcal{U}$.

Proof. First of all we have to see that this map is well defined. Indeed, if U is open and compact $\chi_U \in C_c^{\infty}(X, \mathbb{Z})$, and given disjoint open compact sets $\mathcal{U}, \mathcal{V} \subset X$ we have

$$\Upsilon(f)(\mathcal{U} \sqcup \mathcal{V}) = f(\chi_{\mathcal{U} \sqcup \mathcal{V}}) = f(\chi_{\mathcal{U}} + \chi_{\mathcal{V}}) = f(\chi_{\mathcal{U}}) + f(\chi_{\mathcal{V}}) = \Upsilon(f)(\mathcal{U}) + \Upsilon(f)(\mathcal{V}).$$

The injectivity is straightforward because of the lemma 2.2.4. In order to prove that it is exhaustive we only have to define an $f \in \mathcal{D}(X, A)$ by $f(\chi_{\mathcal{U}}) = \mu(\mathcal{U})$ and extending by linearity, again by the lemma named just before. Thus, we get clearly $\Upsilon(f) = \mu$. □

Definition 2.2.7. *Let $\mu \in \mathscr{M}(X, \mathbb{Z})$ be a $\mathbb{Z}$-valued measure on X. We say that μ is harmonic if the total volume $\mu(X)$ is 0. We denote the set of harmonic measures by $\mathscr{M}(X, \mathbb{Z})_0$.*

The isomorphic image in $\mathcal{D}(X, \mathbb{Z})$ is denoted by $\mathcal{D}(X, \mathbb{Z})_{har}$ and called the set of harmonic distributions.

Remark 2.2.8. *When X is compact we have an exact sequence*

$$0 \longrightarrow \mathbb{Z} \longrightarrow C_c^{\infty}(X, \mathbb{Z}) \longrightarrow C_c^{\infty}(X, \mathbb{Z})/\mathbb{Z} \longrightarrow 0.$$

so dualizing we get an exact sequence

$$0 \longrightarrow \mathrm{Hom}(C_c^{\infty}(X, \mathbb{Z})/\mathbb{Z}, \mathbb{Z}) \longrightarrow \mathrm{Hom}(C_c^{\infty}(X, \mathbb{Z}), \mathbb{Z}) \longrightarrow \mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$$

naturally isomorphic to

$$0 \longrightarrow \mathscr{M}(X, \mathbb{Z})_0 \longrightarrow \mathscr{M}(X, \mathbb{Z}) \longrightarrow \mathbb{Z}$$

where the last arrow maps a measure to its value on X.

2.2.3 Relating harmonic cochains with harmonic measures

Recall that if a tree $\mathcal{T}$ is locally finite, the set of ends $\mathcal{E}(\mathcal{T})$ is compact, so from now on along this chapter we are going to assume this hypothesis.

Theorem 2.2.9. *Any harmonic cochain c of a locally finite metric tree $\mathcal{T}$ determines a unique harmonic measure $\mu(c)$ in $\mathscr{M}(\mathcal{E}(\mathcal{T}),\mathbb{Z})_0$ by defining $\mu(c)(\mathcal{B}(e)) = c(e)$ for any directed edge e in $\mathcal{T}$. This induces an isomorphism between $\mathscr{M}(\mathcal{E}(\mathcal{T}),\mathbb{Z})_0$ and $\mathrm{C}^1_{\mathrm{har}}(\mathcal{T},\mathbb{Z})$.*

Proof. Essentially, all we have to check is that the map

$$\mathrm{C}^1_{\mathrm{har}}(\mathcal{T},\mathbb{Z}) \longrightarrow \mathscr{M}(\mathcal{E}(\mathcal{T}),\mathbb{Z})_0$$

given by the description above is well defined.

First, it is enough to characterize a harmonic measure over the sets $\mathcal{B}(e)$ since these are a basis for the topology of $\mathcal{E}(\mathcal{T})$.

Next, take a model $\mathfrak{T} = (V, \hat{E})$ for $\mathcal{T}$. We just have to see that for any open compact set $\mathcal{U} \subset \mathcal{E}(\mathcal{T})$ and for any partition $\mathcal{U} = \bigsqcup_{e\in I} \mathcal{B}(e)$ with $I \subset \hat{E}$ finite, the sum $\sum_{e\in I} c(e)$ is invariant. Let us take two finite partitions of $\mathcal{U}$:

$$\mathcal{U} = \bigsqcup_{e\in I} \mathcal{B}(e) = \bigsqcup_{e\in I'} \mathcal{B}(e)$$

Since $\mathcal{U}$ is open and compact so it is the complement $\mathcal{V} = \mathcal{E}(\mathcal{T}) \setminus \mathcal{U}$ and we can consider another finite partition $\mathcal{V} = \bigsqcup_{e\in\tilde{I}} \mathcal{B}(e)$, $\tilde{I} \subset \hat{E}$. Then we have

$$\mathcal{E}(\mathcal{T}) = \mathcal{U} \sqcup \mathcal{V} = \bigsqcup_{e\in I} \mathcal{B}(e) \sqcup \bigsqcup_{e\in\tilde{I}} \mathcal{B}(e) = \bigsqcup_{e\in I'} \mathcal{B}(e) \sqcup \bigsqcup_{e\in\tilde{I}} \mathcal{B}(e)$$

Therefore, by the previous corollary, we get $I \sqcup \tilde{I} = \mathrm{St}(\mathfrak{T})$ and $I' \sqcup \tilde{I} = \mathrm{St}(\mathfrak{T}')$ for certain finite subtrees of $\mathcal{T}$ (or any or both disjoint unions can be the degenerated case, which the reader can do as an easy exercise). Then we have

$$\sum_{e\in I} c(e) + \sum_{e\in\tilde{I}} c(e) = \sum_{e\in I\sqcup\tilde{I}} c(e) = \sum_{e\in\mathrm{St}(\mathfrak{T})} c(e) = 0$$

and

$$\sum_{e\in I'} c(e) + \sum_{e\in\tilde{I}} c(e) = \sum_{e\in I'\sqcup\tilde{I}} c(e) = \sum_{e\in\mathrm{St}(\mathfrak{T}')} c(e) = 0$$

after apply lemma 2.2.1, so we get

$$\sum_{e\in I} c(e) = \sum_{e\in I'} c(e)$$

as we wanted to prove.

Once we have the map well defined, it follows immediately from the definition that it is an isomorphism of abelian groups. Indeed, the kernel has to be zero and the same definition together with the fact that the sets $\mathcal{B}(e)$ are a basis for the topology of $\mathcal{E}(\mathcal{T})$ provide the exhaustivity. □

2.3 Harmonic integration on locally finite metric trees

In this section we introduce integration on compact sets from measures, then we relate them by means of the isomorphism of the measures with the distributions and of its definition as a dual space of some locally constant functions. Later, we integrate on the ends of a tree and we use this to define a way to integrate degree zero divisors on the tree.

Inspired by [Lon02, Prop. 5], we consider the next lemma.

Lemma 2.3.1. *Let X be a compact space (in particular Hausdorff) such that the compact open subsets form a basis for the topology. Let A be a complete topological abelian group such that a basic system of neighbourhoods of the zero consists of open subgroups. Let $f : X \longrightarrow A$ be a continuous function and let $\mu \in \mathscr{M}(X, \mathbb{Z})$ be a $\mathbb{Z}$-valued measure on X. Then, the limit*

$$\lim_{\overrightarrow{\mathcal{C}_\alpha}} \sum_{\substack{\mathcal{U}_n^\alpha \in \mathcal{C}_\alpha \\ t_n^\alpha \in \mathcal{U}_n^\alpha}} \mu(\mathcal{U}_n^\alpha) f(t_n^\alpha),$$

taken over the direct system of finite covers $\mathcal{C}_\alpha = \mathcal{C}_\alpha(X)$ of X by disjoint open compact subsets $\mathcal{U}_n^\alpha$, and where the t_n^α are arbitrary points in them, exists in A and is independent of the choice of the t_n^α's.

Proof. Let $N \subset A$ be an open supgroup neighbourhood of 0. The sets $f^{-1}(x + N)$ with $x \in A$ form an open covering of X, which can be refined to a covering by compact open subsets. We take a finite subcovering $\mathcal{C}_{\alpha(N)}$. Then, for any refined covering $\alpha \geq \alpha(N)$ and for any $\mathcal{U}_n^\alpha \in \mathcal{C}_\alpha$, $t, t' \in \mathcal{U}_n^\alpha$ implies that $f(t) - f(t') \in N$. This tell us that

$$\left\{ \sum_{\substack{\mathcal{U}_n^\alpha \in \mathcal{C}_\alpha \\ t_n^\alpha \in \mathcal{U}_n^\alpha}} \mu(\mathcal{U}_n^\alpha) f(t_n^\alpha) \right\}_\alpha$$

is a “Cauchy sequence” in α (quotation marks since there is not a unique sequence, but a direct system, so given α, α', there exists $\alpha'' > \alpha, \alpha'$), and it converges since A is complete. Further we get the independence of the choice of the t_n^α's. $\square$

Definition 2.3.2. *Let X be a compact space (in particular Hausdorff) such that the compact open subsets form a basis for the topology. Let A be a complete topological abelian group. Let $f : X \longrightarrow A$ be a continuous function and let $\mu \in \mathscr{M}(X, \mathbb{Z})$ be a $\mathbb{Z}$-valued measure on X. Then, the integral of f with respect to μ is defined as*

$$\int_X f d\mu := \int_X f(t) d\mu(t) := \lim_{\overrightarrow{\mathcal{C}_\alpha}} \sum_{\substack{\mathcal{U}_n^\alpha \in \mathcal{C}_\alpha \\ t_n^\alpha \in \mathcal{U}_n^\alpha}} \mu(\mathcal{U}_n^\alpha) f(t_n^\alpha) \in A$$

taken over the direct system of finite covers $\mathcal{C}_\alpha = \mathcal{C}_\alpha(X)$ of X by disjoint open compact subsets $\mathcal{U}_n^\alpha$, and where the t_n^α are arbitrary points in them.

Proposition 2.3.3. *For any measure $\mu \in \mathscr{M}(X, \mathbb{Z})$, we have*

1. *For any compact open subset U of X, and for any $a \in A$, denote by $\chi_{U,a}(t)$ the function mapping $x \in X$ to a if $x \in U$, and to 0 otherwise. Then $\int_X \chi_{U,a} d\mu = a\mu(U)$.*
2. *If $f, g : X \longrightarrow A$ are continuous functions on X and the corresponin integrals exist, then*

$$\int_X (f+g) d\mu = \left(\int_X f d\mu\right) + \left(\int_X g d\mu\right)$$

Remark 2.3.4. *The lemma previous to the definition tells the existence of the integral under a strong hypothesis on A. The last proposition, tells its existence if f is an A-linear combination of characteristic functions $\chi_{\mathcal{U}} : X \longrightarrow \mathbb{Z}$, which gives a reinterpretation of the proposition 2.2.6 for A-valued measures.*

$$\Upsilon^{-1} : \mathscr{M}(X, A) \xrightarrow{\cong} \mathcal{D}(X, A)$$

is given by

$$\Upsilon^{-1}(\mu)(\chi_{\mathcal{U}}) = \int\!\!\!\!\!\!\times_X \chi_{\mathcal{U}} d\mu.$$

and so,

$$\Upsilon^{-1}(\mu)(f) = \int\!\!\!\!\!\!\times_X f d\mu,$$

that is, for each measure μ we have a well defined map

$$\int\!\!\!\!\!\times_X d\mu : C_c^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} A \longrightarrow A.$$

The comentary below shows the trivial existence if f is constant and μ is harmonic.

Note that for any harmonic measure μ and for any constant function $f : X \longrightarrow A$ such that $f(x) = a$ for all $x \in X$, we have $\displaystyle\int_X f d\mu = 0$.

Let $\mathcal{T}$ be a metric tree, and let $\mathcal{A}$ be a finite set of points in $\mathcal{T}$. Consider the subtree of $\mathcal{T}$ generated by the points of $\mathcal{A}$, that is the minimal substree containing all these points. We denote it by $\mathcal{T}_\mathcal{A}$. It is clearly finite and there is a unique retraction map

$$\mathrm{r}_\mathcal{T}^\mathcal{A} : \mathcal{T} \longrightarrow \mathcal{T}_\mathcal{A}$$

which maps every point $p \in \mathcal{T} \setminus \mathcal{T}_\mathcal{A}$ to the nearest point in $\mathcal{T}_\mathcal{A}$. This has a sense, since $\mathcal{T}_\mathcal{A}$ is compact and path-connected, and $\mathcal{T}$ is a metric space, so there exists $\inf_{a\in\mathcal{T}_\mathcal{A}}\{d(p,a)\}$. Further, the a reaching this infimum is unique, since for any other $a' \in \mathcal{T}_\mathcal{A}$ the path connecting it with p pass through a and, therefore, has a greater length.

Since the topological realization of a ray $r = (v_0, v_1, \dots)$ is connected, there is an N such that for all $n \geq N$ $\mathrm{r}_\mathcal{T}^\mathcal{A}(v_n)$ stabilizes, so we can associate this point to the end defined by this ray, therefore we get a map

$$\mathrm{r}_\mathcal{T}^\mathcal{A} : \mathcal{E}(\mathcal{T}) \longrightarrow \mathcal{T}_\mathcal{A}.$$

Moreover, when $\mathcal{A} \subset \widetilde{\mathcal{A}}$ we have $\mathcal{T}_\mathcal{A} \subset \mathcal{T}_{\widetilde{\mathcal{A}}}$ and the restriction of the retraction map

$$\mathrm{r}_{\widetilde{\mathcal{A}}}^\mathcal{A} := \mathrm{r}_{\mathcal{T}|\mathcal{T}_{\widetilde{\mathcal{A}}}}^\mathcal{A} : \mathcal{T}_{\widetilde{\mathcal{A}}} \longrightarrow \mathcal{T}_\mathcal{A},$$

so $\mathrm{r}_\mathcal{T}^\mathcal{A} = \mathrm{r}_{\widetilde{\mathcal{A}}}^\mathcal{A} \circ \mathrm{r}_\mathcal{T}^{\widetilde{\mathcal{A}}}$.

Definition 2.3.5. *Let $\mathcal{T}$ be a metric tree, let $\mathcal{A}$ be a finite set of points in $\mathcal{T}$, and let $D := \sum_{p\in\mathcal{A}} m_p p \in \mathbb{Z}[\mathcal{T}]_0$ be a degree zero divisor. With these elements we build the map*

$$f_D : \mathcal{E}(\mathcal{T}) \longrightarrow \mathbb{R}$$

given by

$$f_D(\varepsilon) = -\frac{1}{2} \sum_{p\in\mathcal{A}} m_p d(p, \mathrm{r}_\mathcal{T}^\mathcal{A}(\varepsilon)).$$

This function is clearly locally constant and, therefore, continuous.

Remark 2.3.6. *For $D = 0$ we have $f_0 \equiv 0$.*

Next, let $D = p' - p$. Then we have

$$f_{p'-p}(\varepsilon) = \frac{1}{2}\left(d(p, \mathrm{r}^{\mathcal{A}}_{\mathcal{T}}(\varepsilon)) - d(p', \mathrm{r}^{\mathcal{A}}_{\mathcal{T}}(\varepsilon))\right)$$

and, if $[p, p']$ is an edge

$$f_{p'-p}(\varepsilon) = \begin{cases} \frac{1}{2}d(p,p') & \text{if } \varepsilon \in \mathcal{B}(p,p') \\ -\frac{1}{2}d(p,p') & \text{if } \varepsilon \in \mathcal{B}(p',p). \end{cases}$$

Lemma 2.3.7. *Let $\mathcal{T}$ be a metric tree, let $\mathcal{A}$ be a finite set of points in $\mathcal{T}$, and let $D := \sum_{p\in\mathcal{A}} m_p p \in \mathbb{Z}[\mathcal{T}]_0$ be a degree zero divisor. Then, for all $p' \in \mathcal{T}$*

$$\sum_{p\in\mathcal{A}} m_p d(p, \mathrm{r}^{\mathcal{A}}_{\mathcal{T}}(p')) = \sum_{p\in\mathcal{A}} m_p d(p, p').$$

Proof. We just have to note that for any $p \in \mathcal{T}_{\mathcal{A}}$, $\mathrm{r}^{\mathcal{A}}_{\mathcal{T}}(p') \in [p', p]$ and compute

$$\sum_{p\in\mathcal{A}} m_p d(p,p') = \sum_{p\in\mathcal{A}} m_p \left(d(p', \mathrm{r}^{\mathcal{A}}_{\mathcal{T}}(p')) + d(\mathrm{r}^{\mathcal{A}}_{\mathcal{T}}(p'), p)\right) =$$

$$= \sum_{p\in\mathcal{A}} m_p d(p, \mathrm{r}^{\mathcal{A}}_{\mathcal{T}}(p')) + d(p', \mathrm{r}^{\mathcal{A}}_{\mathcal{T}}(p')) \sum_{p\in\mathcal{A}} m_p = \sum_{p\in\mathcal{A}} m_p d(p, \mathrm{r}^{\mathcal{A}}_{\mathcal{T}}(p')).$$

□

Proposition 2.3.8. *Let $\mathcal{T}$ be a metric tree, let $\mathcal{A}, \mathcal{A}'$ be two finite sets of points in $\mathcal{T}$, and let $D := \sum_{p\in\mathcal{A}} m_p p$, $D' := \sum_{p\in\mathcal{A}'} m'_p p \in \mathbb{Z}[\mathcal{T}]_0$ be two degree zero divisors. Then*

$$f_{D+D'} = f_D + f_{D'}.$$

Proof. Let $\varepsilon \in \mathcal{E}(\mathcal{T})$. Let us denote $\widetilde{\mathcal{A}} := \mathcal{A} \cup \mathcal{A}'$. On one hand we have

$$-2f_{D+D'} = \sum_{p\in\widetilde{\mathcal{A}}} (m_p + m'_p) d(p, \mathrm{r}^{\widetilde{\mathcal{A}}}_{\mathcal{T}}(\varepsilon)) =$$

$$= \sum_{p\in\mathcal{A}} m_p d(p, \mathrm{r}^{\widetilde{\mathcal{A}}}_{\mathcal{T}}(\varepsilon)) + \sum_{p\in\mathcal{A}'} m'_p d(p, \mathrm{r}^{\widetilde{\mathcal{A}}}_{\mathcal{T}}(\varepsilon)).$$

On the other hand, since $\mathrm{r}^{\mathcal{A}}_{\mathcal{T}} = \mathrm{r}^{\mathcal{A}}_{\widetilde{\mathcal{A}}} \circ \mathrm{r}^{\widetilde{\mathcal{A}}}_{\mathcal{T}}$, by the previous lemma applied to $\mathrm{r}^{\widetilde{\mathcal{A}}}_{\mathcal{T}}(\varepsilon)$ we get

$$-2f_D(\varepsilon) = \sum_{p\in\mathcal{A}} m_p d(p, \mathrm{r}^{\mathcal{A}}_{\mathcal{T}}(\varepsilon)) = \sum_{p\in\mathcal{A}} m_p d(p, \mathrm{r}^{\widetilde{\mathcal{A}}}_{\mathcal{T}}(\varepsilon)).$$

We proceed identically for $\mathcal{A}', D'$ and we deduce the claimed equality. □

Corollary 2.3.9. *As a consequence, any map f_D is determined by the maps $f_{p'-p}$ where $[p,p']$ is an edge in $\mathcal{T}$.*

Definition 2.3.10. *Given a locally finite metric tree $\mathcal{T}$, a harmonic measure $\mu \in \mathscr{M}(\mathcal{E}(\mathcal{T}),\mathbb{Z})_0$ and a divisor $D := \sum_{p\in\mathcal{A}} m_p p \in \mathbb{Z}[\mathcal{T}]_0$ we define*

$$\int_D d\mu := \int_{\mathcal{E}(\mathcal{T})} f_D d\mu \in \mathbb{R}.$$

Lemma 2.3.11. *Let p, p' be points in $\mathcal{T}$ such that $[p,p']$ is an edge and $\mu \in \mathscr{M}(\mathcal{E}(\mathcal{T}),\mathbb{Z})_0$. Then we have*

$$\int_{p'-p} d\mu = d(p,p')\mu(\mathcal{B}(p,p')).$$

Proof. We use the definition and the computation of the function previously remarked:

$$\int_{p'-p} d\mu = \int_{\mathcal{E}(\mathcal{T})} f_{p'-p} d\mu = \mu(\mathcal{B}(p,p'))\frac{1}{2}d(p,p') - \mu(\mathcal{B}(p',p))\frac{1}{2}d(p,p') =$$

$$= \frac{1}{2}d(p,p')\left(\mu(\mathcal{B}(p,p')) - \mu(\mathcal{B}(p',p))\right) = d(p,p')\mu(\mathcal{B}(p,p'))$$

where the last equality is due to the harmonicity of μ and to the covering by an edge and its opposite. □

The latter lemma, the proposition 2.3.3 and the proposition 2.3.8 give us a well defined abelian groups morphism

$$\int_\bullet d : \mathbb{Z}[\mathcal{T}]_0 \longrightarrow \mathrm{Hom}(\mathscr{M}(\mathcal{E}(\mathcal{T}),\mathbb{Z})_0, \mathbb{R}).$$

Lemma 2.3.12. *Let Γ be a group acting continuously on $\mathcal{T}$. Then, the previous map commutes with the Γ-operation.*

Proof. We want to see that

$$\int_{\gamma\cdot D} d = \gamma \cdot \int_D d$$

for any $\gamma \in \Gamma$. That is to say that for any $\gamma \in \Gamma$ and $\mu \in \mathscr{M}(\mathcal{E}(\mathcal{T}),\mathbb{Z})_0$ we have

$$\int_{\mathcal{E}(\mathcal{T})} f_{\gamma D} d\mu = \int_{\gamma\cdot D} d\mu = \gamma \cdot \int_D d\mu = \int_D d(\gamma^{-1}\mu) = \int_{\mathcal{E}(\mathcal{T})} f_D d(\gamma^{-1}\mu)$$

Let us to compute the first integral:

$$\int_{\mathcal{E}(\mathcal{T})} f_{\gamma D} d\mu = \lim_{\overrightarrow{\mathcal{C}_\alpha}} \sum_{\substack{\mathcal{U}_n^\alpha \in \mathcal{C}_\alpha \\ t_n^\alpha \in \mathcal{U}_n^\alpha}} \mu(\mathcal{U}_n^\alpha) f_{\gamma D}(t_n^\alpha) = \lim_{\overrightarrow{\mathcal{C}_\alpha}} \sum_{\substack{\mathcal{U}_n^\alpha \in \mathcal{C}_\alpha \\ t_n^\alpha \in \mathcal{U}_n^\alpha}} \mu(\mathcal{U}_n^\alpha)(\gamma f_D)(t_n^\alpha) =$$

$$= \lim_{\overrightarrow{\mathcal{C}_\alpha}} \sum_{\substack{\mathcal{U}_n^\alpha \in \mathcal{C}_\alpha \\ t_n^\alpha \in \mathcal{U}_n^\alpha}} \mu(\mathcal{U}_n^\alpha) f_D(\gamma^{-1} t_n^\alpha) = \lim_{\overrightarrow{\mathcal{C}_\alpha}} \sum_{\substack{\mathcal{U}_n^\alpha \in \mathcal{C}_\alpha \\ t_n^\alpha \in \mathcal{U}_n^\alpha}} \mu(\gamma \mathcal{U}_n^\alpha) f_D(t_n^\alpha) =$$

$$= \lim_{\overrightarrow{\mathcal{C}_\alpha}} \sum_{\substack{\mathcal{U}_n^\alpha \in \mathcal{C}_\alpha \\ t_n^\alpha \in \mathcal{U}_n^\alpha}} (\gamma^{-1}\mu)(\mathcal{U}_n^\alpha) f_D(t_n^\alpha) = \int_{\mathcal{E}(\mathcal{T})} f_D d(\gamma^{-1}\mu)$$

Therefore we get the claimed compatibility of the action of Γ with the map.

□

Proposition 2.3.13. *A group Γ acting continuously on $\mathcal{T}$ induces an homomorphism of abelian groups*

$$\int d : \Gamma^{ab} \longrightarrow \mathrm{Hom}(\mathscr{M}(\mathcal{E}(\mathcal{T}), \mathbb{Z})_0^\Gamma, \mathbb{R}).$$

Proof. Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z}[\mathcal{T}]_0 \longrightarrow \mathbb{Z}[\mathcal{T}] \longrightarrow \mathbb{Z} \longrightarrow 0.$$

The action of Γ gives us a long homology sequence from which we extract the first connecting morphism

$$H_1(\Gamma, \mathbb{Z}) \longrightarrow H_0(\Gamma, \mathbb{Z}[\mathcal{T}]_0) = \mathbb{Z}[\mathcal{T}]_{0_\Gamma}$$

which maps γ to $\gamma p - p$ for any $p \in \mathcal{T}$, and the lemma gives us a morphism

$$\mathbb{Z}[\mathcal{T}]_{0_\Gamma} \longrightarrow \mathrm{Hom}(\mathscr{M}(\mathcal{E}(\mathcal{T}), \mathbb{Z})_0, \mathbb{R})_\Gamma = \mathrm{Hom}(\mathscr{M}(\mathcal{E}(\mathcal{T}), \mathbb{Z})_0^\Gamma, \mathbb{R}).$$

Thus, composing we get

$$\Gamma^{ab} \cong H_1(\Gamma, \mathbb{Z}) \longrightarrow \mathrm{Hom}(\mathscr{M}(\mathcal{E}(\mathcal{T}), \mathbb{Z})_0^\Gamma, \mathbb{R}).$$

□

2.4 The Albanese torus of a finite metric graph via integration

Along this section, first we recall the different discrete and analytic torus appeared in the chapter 1, in order to compare with the Albanese torus of a finite metric graph such as it is defined in [CV10]. Then, we recall the way for computing it implicit in [BF11] and we use this to rise the computation on the ends of the universal covering tree of the graph and to build the Abel-Jacobi map.

In addition, we proof the fundamental isomorphism between the Γ-invariant measures on the ends of the universal covering of a graph with the abelianized of Γ, where this group is the fundamental group of the given finite metric graph.

The Albanese torus of a finite metric graph

Let us start recalling some ideas shown along the previous chapter and taken from different papers as [BdlHN97], [KS00], [KS08], [CV10] or [BF11].

We have seen along the first chapter that the definition of the discrete Jacobian torus of a graph G consists essentially in a quotient $H^1(G,\mathbb{Z})/H^1(G,\mathbb{Z})^{\#}$ with the metric $(\ ,\ )$ on $H^1(G,\mathbb{R})$ induced by the inner product $\langle\ ,\ \rangle_1$ on $C^1(G,\mathbb{R})$. Dually, the discrete Albanese torus consists in a quotient $H_1(G,\mathbb{Z})^{\#}/H_1(G,\mathbb{Z})$ with the metric on $H^1(G,\mathbb{R})$ induced by the corresponding inner product on $C_1(G,\mathbb{R})$.

If we consider the undiscrete or analytic versions, the Jacobian of G is the flat torus $H^1(G,\mathbb{R})/H^1(G,\mathbb{Z})$ together with the same metric obtained as above, while the Albanese torus is $H_1(G,\mathbb{R})/H_1(G,\mathbb{Z})$ together with the corresponding flat metric.

Modifying the notation of first chapter, and denoting by $(\ ,\ )$ all the inner products, we call now

$$\mathrm{Jac}(G) = \big(H^1(G,\mathbb{R})/H^1(G,\mathbb{Z}), (\ ,\ )\big)$$

$$\mathrm{Jac}'(G) = \big(H^1(G,\mathbb{Z})/H^1(G,\mathbb{Z})^{\#}, (\ ,\ )\big)$$

$$\mathrm{Alb}(G) = (H_1(G,\mathbb{R})/H_1(G,\mathbb{Z}), (\ ,\ ))$$

$$\mathrm{Alb}'(G) = \big(H_1(G,\mathbb{Z})^{\#}/H_1(G,\mathbb{Z}), (\ ,\ )\big)$$

and we have $\mathrm{Alb}'(G) \hookrightarrow \mathrm{Alb}(G)$. Further, we have seen $\mathrm{Jac}'(G) \cong \mathrm{Pic}^0(G) \cong \mathrm{Alb}'(G)$.

Recall the definition of the Albanese torus of a finite metric graph (or more generally, of a tropical curve) (see, for example [CV10, Def. 4.1.4]). We only consider metric graphs G with all vertices of valence at least 2. By the introduction of section 2.1, for any edge e of G we have a length $\ell(e) \in \mathbb{R}_{>0}$.

We choose an orientation for each edge of G, and we consider the free abelian group $C_1(G,\mathbb{Z}) = \mathbb{Z}[E(G)]$ generated by these oriented edges of G and the map $\partial : \mathbb{Z}[E(G)] \to \mathbb{Z}[V(G)] = C_0(G,\mathbb{Z})$ given by $\partial(e) = t(e) - s(e)$, where $t(e)$ is the target of e and $s(e)$ is the source. Then $H_1(G,\mathbb{Z}) = \mathrm{Ker}(\partial)$.

For any metric graph we can define the paring

$$(\ , \ ) : C_1(G,\mathbb{Z}) \times C_1(G,\mathbb{Z}) \to \mathbb{R}$$

by $(e,e') = 0$ if $e' \neq e$ and $e' \neq \overline{e}$ (the opposite edge of e), $(e,e) = \ell(e)$ and $(e,\overline{e}) = -\ell(e)$, which induces a symmetric positive definite bilinear map on $C_1(G,\mathbb{R}) \cong C_1(G,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$.

If we are dealing with several metric graphs and we need to specify in which we are applying the pairing, we shall denote it by $(\ , \ )_G$.

From now on along this section we assume the graphs are finite unless we specify the contrary.

The previous inner product determines a flat metric on the homology group $H_1(G,\mathbb{R}) = \mathrm{Ker}(\partial_{\mathbb{R}})$.

Definition 2.4.1. *The Albanese torus of a finite metric graph G is the torus given by $H_1(G,\mathbb{R})/H_1(G,\mathbb{Z})$ together with the metric determined by $(\ , \ )$.*

$$\mathrm{Alb}(G) = (H_1(G,\mathbb{R})/H_1(G,\mathbb{Z}), (\ , \ ))$$

On the other hand, in [BF11] the authors consider a model $\mathfrak{G}$ for the graph G with one orientation for each edge, instead of taking each edge with its opposite, as we do here. Then, their definition of the 1-forms on $\mathfrak{G}$ is equivalent to the elements of the quotient of the real vector space with formal basis $\{de : \ e \in \hat{E}\}$ by the subspace generated by the elements $de + d\overline{e}$. Following, they take into consideration the harmonic 1-forms, which are those 1-forms $\omega = \sum \omega_e de$ such that for each vertex $v \in V$

$$\sum_{s(e)=v} \omega_e = 0 \left(\iff \sum_{t(e)=v} \omega_e = 0 \right),$$

and denote them by $\Omega(\mathfrak{G})$.

Let us consider the set $E = E(\mathfrak{G})$ of edges that they consider, formed by exactly an edge e for each couple $e, \overline{e}$ in $\hat{E} = \hat{E}(\mathfrak{G})$, so we have for an abelian group A (here, $\mathbb{Z}$ or $\mathbb{R}$)

$$A[E] \cong A[\hat{E}]/\{e + \overline{e}\}_{e \in \hat{E}}$$

and the A-cochains on the graph is the dual (over A) of the right-hand side, isomorphic to the dual of $A[E]$.

Consider the free abelian groups $C_1(\mathfrak{G}, A) := A[E]$ and $C_0(\mathfrak{G}, A) := A[V]$, and the differential map $\partial_1 : C_1(\mathfrak{G}, A) \longrightarrow C_0(\mathfrak{G}, A)$ given by $\partial_1(e) = t(e) - s(e)$. Then, on one hand, $H_1(\mathfrak{G}, A) = \mathrm{Ker}(\partial_1) \cong H_1(G, A)$ (that is the isomorphism between singular and simplicial homology), and on the other hand, by means of the identification of $C_1(\mathfrak{G}, A)$ and $C_0(\mathfrak{G}, A)$ with their duals made in [BF11, § 2.1] (which we will denote with hats instead of the notation used in that paper to avoid confusion), we have

$$\begin{array}{ccccccc} 0 \longrightarrow & H_1(\mathfrak{G}, A) & \longrightarrow & C_1(\mathfrak{G}, A) & \xrightarrow{\partial_1} & C_0(\mathfrak{G}, A) \\ & \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\ 0 \longrightarrow & \hat{H}_1(\mathfrak{G}, A) & \longrightarrow & \hat{C}_1(\mathfrak{G}, A) & \xrightarrow{d^*} & \hat{C}_0(\mathfrak{G}, A) \end{array}$$

as we have seen in theorem 1.4.6 for unweighted graphs. This implies our homology group coincides with the defined in the paper by Baker and Faber. They also introduce a pairing

$$\begin{array}{ccc} \Omega(\mathfrak{G}) \times C_1(\mathfrak{G}, \mathbb{R}) & \longrightarrow & \mathbb{R} \\ (\omega, \alpha) & \longmapsto & \int_\alpha \omega \end{array}$$

which extends by linearity the equality

$$\int_{e'} de = \begin{cases} \ell(e) & \text{if } e = e', \\ 0 & \text{if } e \neq e'. \end{cases}$$

and which restricts to a pairing

$$\Omega(\mathfrak{G}) \times H_1(\mathfrak{G}, \mathbb{R}) \longrightarrow \mathbb{R}.$$

Then, as commented in [BF11, Rem. 2.3 (2)], we get a natural construction of the Albanese torus.

Lemma 2.4.2. *Given a finite metric graph G, for any model $\mathfrak{G}$ the restricted pairing to the homology is perfect, and so we have*

$$\mathrm{Alb}(G) \cong \Omega(\mathfrak{G})^*/H_1(\mathfrak{G}, \mathbb{Z}).$$

Proof. This is the lemma 2.1 in [BF11]. □

We already see that this object does not depend on the chosen model. In fact, this is proven in [BF11, Lem. 2.9]. But let us see in another way. Consider the real harmonic cochains on any locally finite weighted graph $\mathfrak{G}$, which are maps $c : \mathbb{R}[\hat{E}(\mathfrak{G})] \longrightarrow \mathbb{R}$ verifying the same properties that the harmonic cochains. We denote them by $\mathrm{C}^1_{\mathrm{har}}(\mathfrak{G}, \mathbb{R})$. As the harmonic cochains (with integer values), these also satisfy

$$\mathrm{C}^1_{\mathrm{har}}(\mathfrak{G}, \mathbb{R}) \cong \mathrm{C}^1_{\mathrm{har}}(G, \mathbb{R}).$$

We also have a natural star map

$$\mathrm{St} : A[V] \longrightarrow A[\hat{E}] \longrightarrow A[E]$$

composing the projection with the map which associates the divisor $\sum_{e \in \mathrm{St}(v)} e$ to the vertex $v \in V$, understanding the star which appear in the summation as defined in previous sections. Thus, we can define the harmonic cochains as the kernel of the dual map, obtaining an exact sequence

$$0 \longrightarrow \mathrm{C}^1_{\mathrm{har}}(\mathfrak{G}, A) \longrightarrow \mathrm{Hom}_A(A[E], A) \longrightarrow \mathrm{Hom}_A(A[V], A).$$

which, when $A = \mathbb{R}$, is isomorphic to the exact sequence

$$0 \longrightarrow \Omega(\mathfrak{G}) \longrightarrow \mathbb{R}[E] \longrightarrow \mathbb{R}[V]$$

where a function $f \in \mathbb{R}[E]^*$ corresponds to the divisor $\sum_{e \in E(G)} f(e) de$ - writing the formal symbol de instead of e-, similarly for vertices, and the last map makes correspond the divisor $\sum_{v \in V(\mathfrak{G})} \left(\sum_{e \in \mathrm{St}(v)} m_e \right) v$ to $\sum_{e \in E(G)} m_e de$.

Thus we got $\Omega(\mathfrak{G}) \cong \mathrm{C}^1_{\mathrm{har}}(\mathfrak{G}, \mathbb{R}) \cong \mathrm{C}^1_{\mathrm{har}}(G, \mathbb{R})$ when G is finite.

Let us observe too, that under the assumption of finiteness of G, and so, of its vertices and its edges, and since $\mathbb{R}$ is $\mathbb{Z}$-flat, we have

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[E], \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}[E], \mathbb{R})$$

and

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[V], \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}[V], \mathbb{R}),$$

and in addition we get an exact sequence

$$0 \longrightarrow \mathrm{C}^1_{\mathrm{har}}(G, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \longrightarrow \mathbb{R}[E]^* \longrightarrow \mathbb{R}[V]^*$$

which results in a natural isomorphism $\mathrm{C}^1_{\mathrm{har}}(G, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathrm{C}^1_{\mathrm{har}}(G, \mathbb{R})$.

Remark 2.4.3. *In [BF11] the authors build the Jacobian of a finite weighted group and show that the Albanese torus of a finite metric group is isomorphic to the direct limit of the Jacobians of the models. We will not repeat that construction here, but we would like to mention that it generalizes naturally the construction of the Jacobian of an unweigthed graph in [BdlHN97], which we describe in the section1.3 of the first chapter.*

The Abel-Jacobi map

Next, we have yet a connected finite metric graph G and a model $\mathfrak{G}$. In fact, whichever model it is, it verifies

$$\mathrm{Alb}(G) \cong \Omega(\mathfrak{G})^*/H_1(\mathfrak{G}, \mathbb{Z}).$$

Definition 2.4.4. *Given two weighted graphs $\mathfrak{G}$ and $\mathfrak{G}'$, we say that $\mathfrak{G}'$ refines $\mathfrak{G}$ if there exist an injection $\nu_V : V(\mathfrak{G}) \longrightarrow V(\mathfrak{G}')$ and a surjection $\pi_E : E(\mathfrak{G}') \longrightarrow E(\mathfrak{G})$ such that for any edge e of $\mathfrak{G}$ there exist vertices $v_0 = \nu_V(s(e)), v_1, \ldots, v_n = \nu_V(t(e))$ and edges $e_1, \ldots, e_n \in E(\mathfrak{G}')$ satisfying $\pi_E^{-1}(e) = \{e_1, \ldots, e_n\}$, $\sum_{i=1}^n \ell(e_i) = \ell(e)$ and $s \times t(e_i) = (v_{i-1}, v_i)$ for each $i = 1, \ldots, n$.*

To construct the Abel-Jacobi map, fix a point $p \in G$. For any other point $q \in G$, consider another model $\mathfrak{G}_{p,q}$ of G containing p and q as vertices and refining $\mathfrak{G}$. Consider a path $P \subset \hat{E}(\mathfrak{G}_{p,q})$ from p to q in $\mathfrak{G}_{p,q}$, which we can see as a sum of the edges in P, that is a 1-chain $\mathbb{Z}[\hat{E}(\mathfrak{G}_{p,q})]$, and let α_P its image by the projection

$$\mathbb{Z}[\hat{E}(\mathfrak{G}_{p,q})] \longrightarrow \mathbb{Z}[E(\mathfrak{G}_{p,q})] = C_1(\mathfrak{G}_{p,q}, \mathbb{Z}).$$

Thus, our construction coincides with which precedes the proof of the theorem 2.11 in [BF11] α_P. With the notation in that reference

$$\alpha_P = \sum_{e \in E(\mathfrak{G})} \epsilon(P, e)e.$$

The pairing previously introduced induces a map

$$\int_\bullet : C_1(\mathfrak{G}_{p,q}, \mathbb{Z}) \longrightarrow \Omega(\mathfrak{G}_{p,q})^*,$$

so we get

$$\int_{\alpha_P} \in \Omega(\mathfrak{G}_{p,q})^*.$$

Lemma 2.4.5. *This construction is compatible with the refinement and so, it does not depend on the model $\mathfrak{G}_{p,q}$. Then, it gives an Abel-Jacobi map*

$$\Phi_p : G \longrightarrow \mathrm{Alb}(G)$$

defined by

$$\Phi_p(q) := \int_{\alpha_P}.$$

Proof. We constructed a map $\Phi_p : G \longrightarrow \Omega(\mathfrak{G}_{p,q})^*$. The difference of the images in two models goes to an integer cycle in a common refinement, which goes to zero in the Albanese torus (cf. paragraphs at the begining of section 4 of [BF11]). Therefore, we can compose with the isomorphisms

$$\frac{\Omega(\mathfrak{G}_{p,q})^*}{H_1(\mathfrak{G}_{p,q},\mathbb{Z})} \cong \frac{\Omega(\mathfrak{G})^*}{H_1(\mathfrak{G},\mathbb{Z})} \cong \mathrm{Alb}(G)$$

in a compatible way. □

A finite metric graph and its universal covering

Let G be a connected finite metric graph. It is well known that it has a universal covering space $\mathcal{T}_G$ which is a connected locally finite tree, being infinite if G is not a tree. It is clear that $\mathcal{E}(\mathcal{T}_G) = \emptyset$ if and only if G is a tree.

Let $\Gamma := \pi_1(G, v)$ for any v in G. Then Γ acts freely on $\mathcal{T}_G$ and $G \cong \Gamma\backslash\mathcal{T}_G$. Let us denote the universal covering projection by

$$\pi_G : \mathcal{T}_G \longrightarrow G.$$

Further, the action on $\mathcal{T}_G$ induces an action of Γ on $\mathcal{E}(\mathcal{T}_G)$.

Proposition 2.4.6. *Each non neutral element $\gamma \in \Gamma$ has exactly two distinct fixed points in $\mathcal{E}(\mathcal{T})$.*

Proof. Take any point $p \in \mathcal{T}_G$ and consider $\gamma^m p$ for $m \in \mathbb{Z}$. Since Γ acts freely on $\mathcal{T}_G$, these points for $m > 0$ have to define an end, and for $m < 0$ a different end, which are two fixed points. If there were three different ends fixed by γ, the vertex determined by them (lemma 2.1.19) would be fixed, so we would contradict the fact that the action is free. □

Let us take $\tilde{v}$ in $\mathcal{T}_G$ such that $\pi_G(\tilde{v}) = v$. We have maps

$$\begin{array}{ccccc} \Gamma & \twoheadrightarrow & \Gamma^{ab} = \pi_1(G,v)^{ab} & \xrightarrow[\cong]{\varpi} & H_1(G,\mathbb{Z}) \\ \gamma & \longmapsto & \overline{\gamma} & \longmapsto & \pi_G([\tilde{v},\gamma\tilde{v}]) \end{array}$$

Then, we have a symmetric, positive definite bilinear form

$$(\ ,\ )_{\pi_1(G,v)} : \Gamma^{ab} \times \Gamma^{ab} \longrightarrow \mathbb{R}$$

defined by $(\gamma, \gamma')_{\pi_1(G,v)} = (\varpi(\gamma), \varpi(\gamma'))$.

Proposition 2.4.7. *Any finite metric graph G satisfies $H_1(G,\mathbb{Z}) \cong \mathrm{C}^1_{\mathrm{har}}(G,\mathbb{Z})$.*

Proof. Take a model $\mathfrak{G}$ for G and recall that $H_1(G,\mathbb{Z}) \cong H_1(\mathfrak{G},\mathbb{Z})$ (it is the equivalence between singular and simplicial homologies). We want to prove $H_1(\mathfrak{G},\mathbb{Z}) \cong \mathrm{C}^1_{\text{har}}(\mathfrak{G},\mathbb{Z})$.

Given a cycle

$$z = \sum_{e\in\hat{E}(\mathfrak{G})} n_e \cdot e \in H_1(\mathfrak{G},\mathbb{Z}) \subset \mathbb{Z}[\hat{E}(\mathfrak{G})],$$

we associate to it a harmonic cochain $c(z)$ defined by $c(z)(e) := n_e$ and $c(z)(\bar{e}) := -n_e$ for any $e \in E(G)$. Indeed, we have

$$0 = \partial(z) = \sum_{e\in E(G)} n_e \cdot (t(e) - s(e)) = \sum_{v\in V(\mathfrak{G})} \left(\sum_{t(e)=v} n_e - \sum_{s(e)=v} n_e \right)$$

what implies that for any $v \in V(\mathfrak{G})$

$$\sum_{e\in \mathrm{St}(v)} c(e) = \sum_{s(e)=v} n_e - \sum_{t(e)=v} n_e = 0$$

Reciprocally, for each harmonic cochain c we get a cycle

$$z_c := \sum_{e\in E(G)} c(e)\cdot e.$$

This correspondence defines the bijection. $\square$

Since Γ acts on $\mathcal{T}_G$, it also acts on the harmonic cochains on the tree and since the action is free, we get

$$\mathrm{C}^1_{\text{har}}(\mathcal{T}_G,\mathbb{Z})^\Gamma \cong \mathrm{C}^1_{\text{har}}(\Gamma\backslash\mathcal{T}_G,\mathbb{Z}) = \mathrm{C}^1_{\text{har}}(G,\mathbb{Z}) \cong H_1(G,\mathbb{Z}) \cong \Gamma^{ab}$$

Corollary 2.4.8. *The map* $\mu : \Gamma^{ab} \to \mathscr{M}(\mathcal{E}(\mathcal{T}),\mathbb{Z})_0^\Gamma$ *defined by*

$$\mu_\gamma(\mathcal{B}(e)) := \mu(\gamma)(\mathcal{B}(e)) := \frac{(\pi_G(e), \varpi(\gamma))}{\ell(e)}.$$

over an edge e *is a natural isomorphism such that for any* $\gamma,\gamma' \in \Gamma$*, we have*

$$(\gamma,\gamma')_{\pi_1(G,v)} = \int_{\gamma p - p} d\mu_{\gamma'}$$

where p *is any point of* $\mathcal{T}$*.*

Proof. The previous results together with the section 2.2 give the composition of isomorphisms

$$\begin{array}{ccccccccc} \Gamma^{ab} & \xrightarrow[\cong]{} & H_1(G,\mathbb{Z}) & \xrightarrow[\cong]{} & \mathrm{C}^1_{\mathrm{har}}(G,\mathbb{Z}) & \xrightarrow[\cong]{} & \mathrm{C}^1_{\mathrm{har}}(\mathcal{T}_G,\mathbb{Z})^\Gamma & \xrightarrow[\cong]{} & \mathscr{M}(\mathcal{E}(\mathcal{T}_G),\mathbb{Z})_0^\Gamma \\ \gamma & \longmapsto & \varpi(\gamma) & \longmapsto & c(\varpi(\gamma)) & \longmapsto & & & \mu(c(\varpi(\gamma))) \end{array}$$

which assigns to $\gamma \in \Gamma^{ab}$ the harmonic cochain defined by

$$\mu(c(\varpi(\gamma)))(\mathcal{B}(e)) = c(\varpi(\gamma))(e) = \frac{(\pi_G(e), \varpi(\gamma))}{\ell(e)}$$

Since the set of points of valence greater than 2 in the path from p to γp is finite (by the lemma 2.1.11), then we get the equality

$$(\gamma, \gamma')_{\pi_1(G,v)} = \int_{\gamma p - p} d\mu_{\gamma'}$$

decomposing the path linearly, applying the lemma 2.3.11 and the additivity of the integral with respect to the path and taking into account the definition of the map μ. If $\gamma p - p = \sum_{i=1}^r p_i - p_{i-1}$, where $[p_{i-1}, p_i]$ are edges, then

$$\int_{\gamma p - p} d\mu_{\gamma'} = \sum_{i=1}^r \int_{p_i - p_{i-1}} d\mu_{\gamma'} = \sum_{i=1}^r d(p_i, p_{i-1}) \mu_{\gamma'}(\mathcal{B}(p_{i-1}, p_i)) =$$

$$= \sum_{i=1}^r d(p_i, p_{i-1}) \frac{(\pi_G([p_{i-1}, p_i]), \varpi(\gamma'))}{d(p_{i-1}, p_i)} = \sum_{i=1}^r (\pi_G([p_{i-1}, p_i]), \varpi(\gamma')) =$$

$$= (\varpi(\gamma), \varpi(\gamma')) = (\gamma, \gamma')_{\pi_1(G,v)}$$

□

The Albanese torus and the Abel-Jacobi map via integration on the universal covering

Note that we have by the proposition 2.3.13 a map

$$\int d : H_1(G,\mathbb{Z}) \cong \Gamma^{ab} \longrightarrow \mathrm{Hom}(\mathscr{M}(\mathcal{E}(\mathcal{T}),\mathbb{Z})_0^\Gamma, \mathbb{R}).$$

given by

$$\int d(\gamma)(\mu) = \int_{\gamma p - p} d\mu.$$

Theorem 2.4.9. *The Albanese torus of a (connected) finite metric graph G satisfies*

$$\mathrm{Alb}(G) \cong \frac{\mathrm{Hom}(\mathscr{M}(\mathcal{E}(\mathcal{T}), \mathbb{Z})_0^{\Gamma}, \mathbb{R})}{H_1(G, \mathbb{Z})}.$$

Proof. By the lemma 2.4.2 we have $\mathrm{Alb}(G) \cong \Omega(\mathfrak{G})^*/H_1(\mathfrak{G}, \mathbb{Z})$ where $\mathfrak{G}$ is a model for G. We also know $\Omega(\mathfrak{G}) \cong \mathrm{C}^1_{\mathrm{har}}(G, \mathbb{R}) \cong \mathrm{C}^1_{\mathrm{har}}(G, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ and $H_1(\mathfrak{G}, \mathbb{Z}) \cong H_1(G, \mathbb{Z})$ naturally. Moreover

$$\mathrm{Hom}(\mathscr{M}(\mathcal{E}(\mathcal{T}_G), \mathbb{Z})_0^{\Gamma}, \mathbb{R}) \cong \mathrm{Hom}(\mathrm{C}^1_{\mathrm{har}}(\mathcal{T}_G, \mathbb{Z})^{\Gamma}, \mathbb{R}) \cong \mathrm{Hom}(\mathrm{C}^1_{\mathrm{har}}(G, \mathbb{Z}), \mathbb{R}) \cong$$

$$\cong \mathrm{Hom}_{\mathbb{R}}(\mathrm{C}^1_{\mathrm{har}}(G, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}, \mathbb{R}) \cong \mathrm{Hom}_{\mathbb{R}}(\mathrm{C}^1_{\mathrm{har}}(G, \mathbb{R}), \mathbb{R}) = \mathrm{C}^1_{\mathrm{har}}(G, \mathbb{R})^*.$$

Therefore, to end we only have to see that the next square is commutative:

$$\begin{array}{ccccc}
H_1(\mathfrak{G}, \mathbb{Z}) & \hookrightarrow & H_1(\mathfrak{G}, \mathbb{R}) & \xrightarrow[\cong]{\int_\bullet} & \Omega(\mathfrak{G})^* \\
 & & & & \downarrow \cong \\
\cong \downarrow & & & & \mathrm{C}^1_{\mathrm{har}}(G, \mathbb{R})^* \\
 & & & & \downarrow \cong \\
H_1(G, \mathbb{Z}) & & \xrightarrow{\int d} & & \mathrm{Hom}(\mathscr{M}(\mathcal{E}(\mathcal{T}_G), \mathbb{Z})_0^{\Gamma}, \mathbb{R}).
\end{array}$$

Take $\alpha = \sum_{e \in E(\mathfrak{G})} m_e e \in H_1(\mathfrak{G}, \mathbb{Z})$. We also will denote by α its topological realization, which is the cycle that corresponds to it in $H_1(G, \mathbb{Z})$. Recall that $\Gamma^{ab} \cong H_1(G, \mathbb{Z})$, thus, there is $\gamma \in \Gamma$ such that $\alpha = \varpi(\gamma)$, that is the projection of a path from a vertex $\tilde{v} \in \mathcal{T}_G$ to $\gamma \cdot \tilde{v}$. Then, m_e counts how many times appear edges representing e in that path in $\mathcal{T}_G$ with orientation. If $\tilde{e}$ is one of those edges in $\pi_G^{-1}(e)$, that is the value $\sum_{\gamma \in \Gamma} \epsilon([\tilde{v}, \gamma\tilde{v}], \gamma\tilde{e})$ with the notation introduced before the proof of theorem 2.11 in [BF11], applied at $\mathcal{T}_G$. That is

$$m_e = \sum_{\gamma \in \Gamma} \epsilon([\tilde{v}, \gamma\tilde{v}], \gamma\tilde{e}) = \frac{(\varpi(\gamma), e)}{\ell(e)}.$$

Take also a harmonic measure $\mu \in \mathscr{M}(\mathcal{E}(\mathcal{T}_G), \mathbb{Z})_0^{\Gamma}$. The corresponding harmonic 1-form is

$$\sum_{e \in E(\mathfrak{G})} \mu(\mathcal{B}(e))de \in \Omega(\mathfrak{G}).$$

By the isomorphism $\mathscr{M}(\mathcal{E}(\mathcal{T}_G), \mathbb{Z})_0^{\Gamma} \cong \Gamma^{ab}$, we have $\mu = \mu_{\gamma'}$ for $\gamma' \in \Gamma^{ab}$ and, as above,

$$\mu(\mathcal{B}(e)) = \frac{(e, \varpi(\gamma'))}{\ell(e)}$$

by the corollary 2.4.8. This result also allows us to conclude the sought commutativity, as follows:

$$\int_\alpha (\mu) = \int_{\sum_{e\in E(\mathfrak{G})} m_e e} \left(\sum_{e\in E(\mathfrak{G})} \mu(\mathcal{B}(e))de \right) = \sum_{e\in E(\mathfrak{G})} m_e \mu(\mathcal{B}(e))\ell(e) =$$

$$= \left(\sum_{e\in E(\mathfrak{G})} m_e e, \sum_{e\in E(\mathfrak{G})} \mu(\mathcal{B}(e))e \right) = (\varpi(\gamma), \varpi(\gamma')) = \int_{\gamma p - p} d\mu.$$

□

Theorem 2.4.10. *Given any point $p \in G$, the Abel-Jacobi map with base point p is given by*

$$\begin{array}{ccc} G & \xrightarrow{\quad i_p \quad} & \dfrac{\mathrm{Hom}(\mathscr{M}(\mathcal{E}(\mathcal{T}), \mathbb{Z})_0^\Gamma, \mathbb{R})}{\int d\,(H_1(G, \mathbb{Z}))} \cong \mathrm{Alb}(G) \\ q & \longmapsto & \displaystyle\int_{\tilde{p}}^{\tilde{q}} d : \mu \longmapsto \int_{\tilde{q}-\tilde{p}} d\mu \end{array}$$

where $\pi_G(\tilde{p}) = p$ and $\pi_G(\tilde{q}) = q$.

Proof. First, observe that the map is well defined. Indeed, assume first we define it fixing a point $\tilde{p} \in \mathcal{T}_G$. Two representants of $q \in G$ in $\mathcal{T}_G$ have the form $\tilde{q}$ and $\gamma\tilde{q}$ for some $\gamma \in \Gamma$, and so we have

$$\int_{\gamma\tilde{q}-\tilde{p}} d - \int_{\tilde{q}-\tilde{p}} d = \int_{\gamma\tilde{q}-\tilde{q}} d \in \int d(H_1(G, \mathbb{Z}))$$

That is, $i_{\tilde{p}}$ does not depend on the chosen representant of q. Next, changing $\tilde{p}$ by $\gamma\tilde{p}$ and taking a fixed representant $\tilde{q}$ of q, we get $i_{\tilde{p}}(q) = i_{\gamma\tilde{p}}(q)$ by an identical reasoning, therefore i_p is well defined.

Now we want to see that for any $q \in G$, $\Phi_p(q) = i_p(q)$. By the lemma 2.4.5, we may take a model $\mathfrak{G}$ of G containing p and q as vertices. Thus, for any path P in $\mathfrak{G}$ from p to q we have

$$\int_{\alpha_P} = \Phi_p(q)$$

with the notation introduced above the mentioned lemma. In particular, if we take representants $\tilde{p}$ and $\tilde{q}$ of p and q respectively, we take as P the projection of the path from $\tilde{p}$ to $\tilde{q}$. Then, we want to see that

$$\int_{\alpha_P} = \int_{\tilde{q}-\tilde{p}} d.$$

As in the proof of the corollary 2.4.8, we subdivide the path $[\tilde{p}, \tilde{q}]$ in subpaths $[p_i, p_{i+1}]$, $i = 0, \ldots, r$, which are projected to the topological realizations in G of edges in $\mathfrak{G}$, where $p_0 = \tilde{p}$ and $p_r = \tilde{q}$. Let $e_i \in \hat{E}(\mathfrak{G})$ be the edge whose topological realization is $\pi_G([p_{i-1}, p_i])$, with the same orientation. Note that the projection of $\sum_{i=1}^r e_i$ in $\mathbb{Z}[E(\mathfrak{G})]$ is $\alpha_P = \sum_{e \in E(\mathfrak{G})} \epsilon(P, e)e$ by construction.

Now, take a harmonic measure $\mu \in \mathscr{M}(\mathcal{E}(\mathcal{T}), \mathbb{Z})_0^\Gamma$. We evaluate and proceed like in the proof of the previous theorem

$$i_p(q)(\mu) = \int_{\tilde{p}}^{\tilde{q}} d\mu = \sum_{i=1}^r \int_{p_{i-1}}^{p_i} d\mu = \sum_{i=1}^r \ell(e_i)\mu(\mathcal{B}(e_i)),$$

$$\Phi_p(q)(\mu) = \int_{\alpha_P} (\mu) = \sum_{e \in E(\mathfrak{G})} \epsilon(P, e)\mu(\mathcal{B}(e))\ell(e),$$

and, since $\mu(\mathcal{B}(\overline{e}_i)) = -\mu(\mathcal{B}(e_i))$ and $\epsilon(P, e)$ counts how many times appears e in the set $\{e_1, \ldots, e_r\}$ with orientation,

$$\sum_{i=1}^r \ell(e_i)\mu(\mathcal{B}(e_i)) = \sum_{e \in E(\mathfrak{G})} \epsilon(P, e)\mu(\mathcal{B}(e))\ell(e),$$

obtaining $i_p(q)(\mu) = \Phi_p(q)(\mu)$ for all μ, as we wanted to prove. □

Chapter 3

The Abel-Jacobi map for Mumford curves via integration

Mumford built in 1972 some algebraic curves associated to certain subgroups of the linear group $\mathrm{PGL}_2(K)$, when K is a complete field with respect to a discrete absolute value, analogous to a construction of Schottky over the complex numbers. He restricted to the case of discrete absolute value and used the geometry given by formal schemes.

This was generalized to every non-archimedean absolute value by Gerritzen and van der Put in [GvdP80] in 1980. They named such curves Mumford curves. Shortly after Mumford's paper, Drinfeld and Manin in [MD73] showed that the Jacobian of a Mumford curve is isomorphic to an analytic torus (in the rigid-analytic geometry) and that it can be built with some theta functions in the case K is a finite extension of the p-adic numbers. This construction was also done in the general case by Gerritzen and van der Put in [GvdP80]. Both took advantage of rigid analytic geometry, introduced by Tate some years ago.

More recently, Dasgupta showed in his thesis ([Das04]) an equivalent construction of the Jacobian to the ones cited above, but restricted to the local case, by means of multiplicative integrals, defined previously by Darmon in [Dar01] and generalized by Longhi in [Lon02].

Before that, in 1990 Berkovich introduced an alternative analytic theory to the one of Tate in his seminal book [Ber90]. The biggest difference over a variety consists in introducing more points instead of removing Zariski open sets. This does not impede getting equivalent categories of "good" enough analytic varieties which can be seen as generic fibres of formal schemes, thanks to works of Raynaud, Bosch and Lütkebohmert. Concurrently, tropical geometry was developed and found in big relation with Berkovich analytic geometry.

In this chapter, we give a new construction of the Jacobian of a Mumford curve over any complete non-archimedean field, departing from Berkovich geometry, and giving so a new and enlightening point of view.

It should be also recognized a great parallelism of this work with part of the paper by van der Put [vdP92]. Some of the results are directly related to results by Baker and Rabinoff appeared in [BR15] in slightly different language.

In order to get the asserted goal, we make the basic constructions given by Berkovich theory in sections 3.1 and 3.2, from which, later, in the section 3.6, we build our Mumford curve. They are the Berkovich projective line together $({\mathbb{P}^1_K}^*)^{an}$ with its skeleton $\mathcal{T}_K$, which coincides with the Bruhat-Tits building of $\mathrm{PGL}_2(K)$, the locally finite subtree $\mathcal{T}_K(\mathcal{L})$ associated to a compact set $\mathcal{L}$ and the retraction map

$$\mathrm{r}_{\mathcal{L}} : ({\mathbb{P}^1_K}^*)^{an} \longrightarrow \overline{\mathcal{T}_K(\mathcal{L})}.$$

Through the sections 3.4 and 3.5 we develope the theory of multiplicative integrals and analytic functions that we need -completed later in the sections 3.7 and 3.8. Essentially, we define these integrals, we build the ones in which we are interested and we relate them to analytic functions through the Poisson formula and the map

$$\tilde{\mu} : \mathcal{O}(\Omega_{\mathcal{L}})^* :\longrightarrow \mathcal{M}(\mathcal{L}, \mathbb{Z})_0$$

Later we study the automorphic forms for a Schottky group $\Gamma \subset \mathrm{PGL}_2(K)$, and the last part of this work gather all previous topics to build the desired Abel-Jacobi map.

Through this chapter K will be a complete field with respect to a non-trivial non-archimedean absolute value$|\cdot| := |\cdot|_K$. The ring of integers of K will be denoted by $\mathcal{O}_K = \{x \in K \mid |x| \leq 1\}$, its maximal ideal by $\mathfrak{m}_K = \{x \in K \mid |x| < 1\}$, and its residue field by $k := \mathcal{O}_K/\mathfrak{m}_K$.

If the absolute value $|\cdot|$ is discretely valued, we will assume $-\log|x| \in \mathbb{Z}$ for any $x \in K^*$, so it is the discrete valuation v_K associated to $|\cdot|$. Otherwise, we also define the valuation of x by $v_K(x) := -\log|x|$.

Taking the 2-dimensional vector space $V = KX_0 \oplus KX_1 \cong K^2$, we see the dual projective line ${\mathbb{P}^1_K}^*$ over K as the projective spectrum of the polynomial ring $K[X_0, X_1]$, that is $\mathbb{P}_{V^*} = \mathrm{Proj}(S^\bullet V)$. Its K-rational points correspond to $(K^2)^* \setminus \{(0,0)\}$ modulo homothety. We denote the class of (x_0, x_1) by $[x_0 : x_1]$.

The infinite point in the dual projective line will be $\infty = [0 : 1]$ and we embed K in $\mathbb{P}^{1*}(K)$ by means of $i^*(z) = [1 : -z]$. Therefore, an $f \in K[X_0, X_1]$

defines a function $L \longrightarrow L$ for any extension $L|K$, that by abuse of notation we also denote f, by $f(z) := f(1, -z)$.

On the other hand we inject K in $\mathbb{P}^1(K)$ by $i(z) = [z : 1]$, taking as infinity of the projective line $[1 : 0]$.

Given a point $p = [a : b] \in \mathbb{P}^{1^*}(K)$, we will denote its corresponding point $p^* = [-b : a] \in \mathbb{P}^1(K)$ (or if $p \in \mathbb{P}^1(K)$, then $p^* \in \mathbb{P}^{1^*}(K)$). Note that this implies $i^*(z) = i(z)^*$ for all $z \in K$ and $(\gamma \cdot p)^* = \gamma \cdot p^*$ for all $p \in \mathbb{P}^1(K)$ (or $p \in \mathbb{P}^{1^*}(K)$), $\gamma \in PGL_2(K)$. Furthermore, $\infty^* = \infty$.

More generally, given a set of points $S \subset \mathbb{P}^1(K)$ (resp. $S' \subset \mathbb{P}^{1^*}(K)$) we denote $S^* := \{p^* | p \in S\} \subset \mathbb{P}^{1^*}(K)$ (resp. $S'^* := \{p^* | p \in S'\} \subset \mathbb{P}^1(K)$).

3.1 Trees and Skeletons

The main objective of this section is the construction of a metric tree associated to an arbitrary compact set $\mathcal{L} \subset \mathbb{P}^1(K)$, study its structure and define the open sets associated to its edges. This subtree generalizes to a non-discrete setting the one defined by Mumford in [Mum72a] and gives an alternative and more complete construction to the one given in [GvdP80, Ch. 1]. In order to do it, we recall some well known notions coming from Berkovich analytic geometry and Bruhat-Tits theory. This first part is mainly extracted from [Bak08], but it is also greatly indebted to [Wer04], where some ideas we recall here and further on are shown.

Consider the Berkovich analytic projective line $(\mathbb{P}^{1}_{K}{}^*)^{an}$ defined over K, which is the set of all the multiplicative seminorms on the polynomial ring $K[X_0, X_1]$ extending $|\ |$ on K modulo an equivalence relation which is specified below; that is, the maps

$$\alpha : K[X_0, X_1] \to \mathbb{R}_{\geq 0}$$

such that

1. $\alpha_{|K} = |\ |$.
2. $\alpha(X_0 K + X_1 K) \neq \{0\}$.
3. $\alpha(f \cdot g) = \alpha(f) \cdot \alpha(g)$
4. $\alpha(f + g) \leq \max\{\alpha(f), \alpha(g)\}$

with $\alpha \sim \beta$ if there exists a constant $C \in \mathbb{R}_{>0}$ such that $\alpha(f) = C^d \beta(f)$ for all $f \in K[X_0, X_1]$ homogeneous of degree d and for all $d \geq 0$.

We associate to an $x \in {\mathbb{P}^1_K}^*(K)$, $x \neq \infty = [0:1]$ and an $r \in \mathbb{R}_{\geq 0}$ an element $\alpha(x,r) \in ({\mathbb{P}^1_K}^*)^{an}$ by defining

$$\alpha(x,r)(f) = \sup\{|f(y)| \ : \ y \in B(x,r)\} \text{ for } f \in K[X_0, X_1]$$

and $\alpha(\infty, 0)(f) := |f(0,1)|$.

We will call these seminorms the ones associated to the balls (or to K-rational points if $r = 0$).

Remark 3.1.1. *Let $f = \lambda X_0 + \mu X_1 \in V = KX_0 \oplus KX_1$. Then,*

$$\alpha(x,r)(f) = \max\{|\lambda - \mu x|, |\mu| r\}.$$

Indeed, for $y \in B(x,r)$ we have

$$|\lambda - \mu y| = |\lambda - \mu x + \mu(x-y)| \leq \max\{|\lambda - \mu x|, |\mu||x-y|\}$$

whose maximum is reached at $|y - x| = r$, and so

$$\alpha(x,r)(f) = \sup\{|\lambda \cdot 1 + \mu \cdot (-y)| : \ y \in B(x,r)\} \leq \max\{|\lambda - \mu x|, |\mu| r\}.$$

In addition we have trivially $|\lambda - \mu x| \leq \alpha(x,r)(f)$, and so, the only case which concerns us is $|\lambda - \mu x| < |\mu| r$, which is equivalent to

$$\left|\frac{\lambda}{\mu} - x\right| < r$$

Then, for any y such that

$$\left|\frac{\lambda}{\mu} - x\right| < |y - x| < r$$

we have $|\lambda - \mu x| < |\mu||x-y|$ and so $|\lambda - \mu y| = |\lambda - \mu x + \mu(x-y)| = |\mu||x-y|$. Finally, we have a sequence $(y_n)_n$ inside $B(x,r)$ such that $\lim_n |x - y_n| = r$, and therefore

$$\alpha(x,r)(f) = \sup\{|\lambda - \mu y|, \ y \in B(x,r)\} = \sup\{|\mu||x-y|, \ y \in B(x,r)\} = |\mu| r.$$

In particular, when $q \in K$ and $f = qX_0 + 1X_1$, identifying them we get $\alpha(x,r)(q) = \max\{|q-x|, r\}$.

Definition 3.1.2. *We call maximal skeleton of $({\mathbb{P}^1_K}^*)^{an}$ and denote $\mathcal{T}_K$ the set of points associated to balls with $r > 0$, and the compactified skeleton $\overline{\mathcal{T}_K}$ is the skeleton together with the (points associated to) rational points ${\mathbb{P}^1}^*(K)$. It is well known that this set is a topological space, and together with a natural metric, which we will recall in the following, forms an $\mathbb{R}$-tree ([BPR13]).*

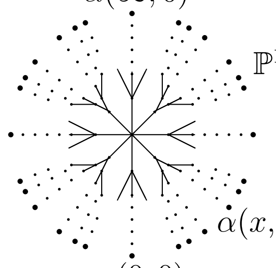


Figure 3.1: The Berkovich projective line $(\mathbb{P}^{1}_{K}{}^{*})^{an}$.

Remark 3.1.3. *If K is algebraically closed, then it is well-known (look at [Ber90]) that the points in $(\mathbb{P}^{1}_{K}{}^{*})^{an}$ can be divided in four types, the type we being associated to K-rational points, types II and III associated to (closed) balls with center some $x \in \mathbb{P}^{1*}(K)$, and with radius $r \in |K^*|$ or $r \in \mathbb{R}_{>0} \backslash |K^*|$ respectively, and a fourth type associated to sequences of nesting balls with empty intersection. Then the topological space $(\mathbb{P}^{1}_{K}{}^{*})^{an}$ has the structure of an $\mathbb{R}$-tree. The maximal skeleton $\mathcal{T}_K$ of $(\mathbb{P}^{1}_{K}{}^{*})^{an}$ is the set of points of type II and III, which is an $\mathbb{R}$-subtree, and $\overline{\mathcal{T}_K}$ is the set of points of type I, II and III.*

Recall that in [BPR13] is defined a skeleton in $(\mathbb{P}^{1}_{K}{}^{*})^{an}$ and corollary 5.56. asserts that $\mathcal{T}_K$ is the inductive limit of all their skeleta. Note also that $(\mathbb{P}^{1}_{K}{}^{*})^{an}$ is homeomorphic to the inverse limit of the set of all skeleta with respect to the natural retraction maps ([BPR13, Thm. 5.57.]).

Given any two distinct points x_0 and $x_1 \in \mathbb{P}^{1*}(K) \backslash \{\infty\}$, if $R = |x_0 - x_1|$, we will denote by $x_0 \vee x_1 := \alpha(x_0, R) = \alpha(x_1, R)$. For any two classes of seminorms $\alpha_0 = \alpha(x_0, r_0)$ and $\alpha_1 = \alpha(x_1, r_1) \in \mathcal{T}_K$, either the corresponding balls verify $B(x_0, r_0) \cap B(x_1, r_1) \neq \emptyset$, in which case $\alpha(x_i, r_i) = \alpha(y, r_i)$ for all the points $y \in B(x_0, r_0) \cap B(x_1, r_1)$ and $i = 0,\ 1$, and we denote $\alpha_0 \vee \alpha_1 := \alpha(y, \max(r_0, r_1))$, or they verify $B(x_0, r_0) \cap B(x_1, r_1) = \emptyset$ and we denote $\alpha_0 \vee \alpha_1 := x_0 \vee x_1$.

Let us consider two points $\alpha = \alpha(x, r), \alpha' = \alpha(x, r')$ of the $\mathbb{R}$-tree $\overline{\mathcal{T}_K}$, with $0 \leq r \leq r'$ and $x \neq \infty$. We denote the (oriented) path from α to α' as $P(\alpha, \alpha')$, being as a set of points $\{\alpha(x, s) | r \leq s \leq r'\} \cong [r, r'] \subset \mathbb{R}_{\geq 0}$. The (oriented) path $P(\alpha', \alpha)$ from α' to α is the same set of points oriented with the opposite direction. Finally, the (oriented) path $P(\alpha(x, r), \alpha(\infty, 0))$ from $\alpha(x, r)$ to $\alpha(\infty, 0)$ is the set of points

$$\{\alpha(x, s) | s \geq r\} \bigcup \{\alpha(\infty, 0)\} \cong [r, \infty] \subset \mathbb{R}_{\geq 0} \bigcup \{\infty\}$$

with the orientation given by the isomorphism (as above), and we define similarly the opposite path $P(\alpha(\infty, 0), \alpha(x, r))$ reversing the orientation. Given two arbitrary points $\alpha, \alpha' \in \overline{\mathcal{T}_K} \setminus \{\alpha(\infty, 0)\}$, the (oriented) path $P(\alpha, \alpha')$ from α to α' is the path $P(\alpha, \alpha \vee \alpha')$ followed by the path $P(\alpha \vee \alpha', \alpha')$.

Recall that given any two distinct points x_0 and $x_1 \in \mathbb{P}^{1*}(K)$, there is a unique line in $\mathcal{T}_K$ going from x_0 to x_1, the open path $\mathring{P}(\alpha(x_0, 0), \alpha(x_1, 0))$ -the interior of the path $P(\alpha(x_0, 0), \alpha(x_1, 0))$. This line is homeomorphic as a metric tree to $\mathbb{R}$, and we denote it by $\mathbb{A}_{\{x_0,x_1\}}$: it is called an apartment of the skeleton $\mathcal{T}_K$. Its closure is, by definition, $\overline{\mathbb{A}_{\{x_0,x_1\}}} = \mathbb{A}_{\{x_0,x_1\}} \cup \{x_0, x_1\}$.

Given two points $\alpha_0 = \alpha(x_0, r_0)$ and $\alpha_1 = \alpha(x_0, r_1) \in \mathbb{A}_{\{x_0,\infty\}}$, we define

$$d(\alpha_0, \alpha_1) = \left| \log \frac{r_1}{r_0} \right|,$$

and in general we define

$$d(\alpha_0, \alpha_1) := d(\alpha_0, \alpha_0 \vee \alpha_1) + d(\alpha_0 \vee \alpha_1, \alpha_1).$$

Then d determines a well defined metric on $\mathcal{T}_K$.

A seminorm on V is $\alpha : V = X_0K + X_1K \longrightarrow \mathbb{R}_{\geq 0}$ satisfying (2) and (4) as above in the definition of multiplicative seminorm on $K[X_0, X_1]$, and $\alpha(\lambda v) = |\lambda|\alpha(v)$ for $\lambda \in K$, $v \in V$. We say that a seminorm α on V is diagonalizable if there exists a basis v_0, v_1 of V such that

$$\alpha(v) = \max\{|\omega_0(v)|\alpha(v_0), |\omega_1(v)|\alpha(v_1)\}$$

for all $v \in V$, where ω_0, ω_1 is the dual basis of v_0, v_1. We denote that seminorm as $\alpha_{(v_0,v_1),(\rho_0,\rho_1)}$ with $\rho_0 := \alpha(v_0)$ and $\rho_1 := \alpha(v_1)$.

Remark 3.1.4 (The action of $\mathrm{PGL}_2(K)$ on $(\mathbb{P}^{1\,*}_K)^{an}$)**.** *The left action of $\mathrm{PGL}_2(K)$ on V induces a left action on $K[X_0, X_1] \cong S^\bullet V$. Then, it also induces a left action on $(\mathbb{P}^{1\,*}_K)^{an}$ by defining $(\gamma \cdot \alpha)(f) := \alpha(\gamma^{-1} \cdot f)$.*

For any $\gamma \in \mathrm{PGL}_2(K)$ we get $\gamma \cdot \alpha(x, 0) = \alpha(\gamma \cdot x, 0)$, making the injection $\mathbb{P}^{1*}(K) \longrightarrow (\mathbb{P}^{1\,*}_K)^{an}$ defined by $x \mapsto \alpha(x, 0)$ equivariant. We also have

$$\gamma \cdot \alpha_{(v_0,v_1),(\rho_0,\rho_1)} = \alpha_{(\gamma \cdot v_0, \gamma \cdot v_1),(\rho_0,\rho_1)}.$$

Next we are going to identify the $\mathbb{R}$-tree $\mathcal{T}_K$ with the Bruhat-Tits building of $\mathrm{PGL}_2(K)$, which is the set of diagonalizable norms on K^2 up to homothety.

Proposition 3.1.5. *The seminorm $\alpha(x, r)$ restricted to V is the seminorm $\alpha := \alpha_{(v_0,v_1),(\rho_0,\rho_1)}$, diagonalizable with respect to the basis $v_0 = (1, 0)$, $v_1 = (x, 1)$ and such that $\rho_0 = 1$ and $\rho_1 = r$ when $x \neq \infty$, meanwhile $\alpha(\infty, 0) = \alpha_{((1,0),(0,1)),(0,1)}$.*

Proof. The identification works by means of restricting any seminorm in $\overline{\mathcal{T}_K}$ to $KX_0 + KX_1$, by means of its identification with K^2. When the seminorm is $\alpha(x,r)$ for $x \in K \subset \mathbb{P}^{1*}(K)$ and $r \geq 0$, and we apply it to a vector $v = (a,b) = (a-bx)v_0 + bv_1$, we have

$$\alpha(x,r)(v) = \max\{|a-bx|, |b|r\} = \alpha(v)$$

Observe that $\omega_0(a,b) = a - bx$ and also that the seminorm on K^2 associated to a rational point x has x^* as its kernel, that is to say, the set of vectors $w \in K^2$ with $|\omega_0(w)| = 0$ is the subspace generated by $(x,1)$.

In the case of $\alpha(\infty,0)$ we have

$$\alpha(\infty,0)(v) = |b| = \max\{|a|0, |b|1\} = \alpha_{((1,0),(0,1)),(0,1)}(v)$$

□

In the following result we will specify how the correspondence between classes of seminorms with form $\alpha(x,r)$ and diagonalizable seminorms on V works.

Proposition 3.1.6. *Let v_0, v_1 be a basis of V, $\omega_0, \omega_1 \in V^*$ be its dual basis, $y_0 = [\omega_0], y_1 = [\omega_1] \in \mathbb{P}^{1*}(K)$ and $\rho_0, \rho_1 \in \mathbb{R}_{\geq 0}$. We suppose that $y_0, y_1 \neq \infty$ (look at proposition above for the case in which one point is ∞), and then we may take $\omega_i = (1, -y_i)$ for $i = 1, 2$ (by means of i^*).*

With these hypotheses we get:

$$\text{If } \rho_1 < \rho_0, \qquad [\alpha_{(v_0,v_1),(\rho_0,\rho_1)}] = [\alpha_{(v_0,v_1),(1,\frac{\rho_1}{\rho_0})}]$$

and

$$\alpha_{(v_0,v_1),(1,\frac{\rho_1}{\rho_0})} = \alpha\left(y_0, \frac{\rho_1}{\rho_0}|y_0 - y_1|\right).$$

$$\text{If } \rho_0 < \rho_1, \qquad [\alpha_{(v_0,v_1),(\rho_0,\rho_1)}] = [\alpha_{(v_0,v_1),(\frac{\rho_0}{\rho_1},1)}]$$

and

$$\alpha_{(v_0,v_1),(\frac{\rho_0}{\rho_1},1)} = \alpha\left(y_1, \frac{\rho_0}{\rho_1}|y_0 - y_1|\right).$$

$$\text{If } \rho_1 = \rho_0, \qquad [\alpha_{(v_0,v_1),(\rho_0,\rho_1)}] = [\alpha_{(v_0,v_1),(1,1)}]$$

and

$$\alpha_{(v_0,v_1),(1,1)} = \alpha\left(y_0, |y_0 - y_1|\right) = \alpha\left(y_1, |y_0 - y_1|\right).$$

Reciprocally, and for $r \leq |y_0 - y_1|$

$$\alpha(y_0, r) = \alpha_{(v_0,v_1),\left(1,\frac{r}{|y_0-y_1|}\right)}$$

$$\alpha(y_1, r) = \alpha_{(v_0,v_1),\left(\frac{r}{|y_0-y_1|},1\right)}.$$

Proof. Assume, just for simplicity, that $\rho_0, \rho_1 \neq 0$, meaning that α is a norm. Define $\alpha := \alpha_{(v_0,v_1),(\rho_0,\rho_1)}$.

Next, we start at the end. By definition $\alpha \in \mathbb{A}_{\{y_0,y_1\}}$, so $\alpha \in P(y_0, y_0 \vee y_1)$ or $\alpha \in P(y_0 \vee y_1, y_1)$; for some $r \leq |y_0 - y_1|$, in the first case we would get $\alpha = \alpha(y_0, r)$ and in the second we would $\alpha = \alpha(y_1, r)$ up to homothety. Without loss of generality we suppose the first case. Let us take an arbitrary vector $v = (a, b) \in V$. We have

$$\alpha(y_0, r)(v) = \max\{|a - by_0|, |b|r\},$$

$$\alpha(a, b) = \max\{|a - by_0|\rho_0, |a - by_1|\rho_1\} \sim \max\{|a - by_0|, |a - by_1|\frac{\rho_1}{\rho_0}\}$$

We note that if $|a - by_0| < |b|r$, we have $[\alpha](v) = [\alpha(y_0, r)](v)$ if and only if

$$|b|r = |a - by_1|\frac{\rho_1}{\rho_0},$$

or also

$$\frac{\rho_1}{\rho_0} = \frac{|b|r}{|a - by_1|}.$$

But since we have $|b||y_0 - y_1| \geq |b|r > |a - by_0|$, then we get $|a - by_1| = |a - by_0 + b(y_0 - y_1)| = \max\{|a - by_0|, |b||y_0 - y_1|\} = |b||y_0 - y_1|$, so

$$\frac{\rho_1}{\rho_0} = \frac{r}{|y_0 - y_1|}$$

Therefore we obtain

$$[\alpha] = \left[\alpha\left(y_0, \frac{\rho_1}{\rho_0}|y_0 - y_1|\right)\right]$$

after assuming $r \leq |y_0 - y_1|$, that is $\rho_1 \leq \rho_0$. In the same way, when $\rho_1 \geq \rho_0$ we get

$$[\alpha] = \left[\alpha\left(y_1, \frac{\rho_0}{\rho_1}|y_0 - y_1|\right)\right].$$

We see the extreme cases too, that is, when $\rho_1 = 0$ then $[\alpha] = [\alpha(y_0, 0)]$, and when $\rho_0 = 0$, $[\alpha] = [\alpha(y_1, 0)]$. □

We keep together the last two results in the next:

Corollary 3.1.7. *The maximal and the compactified skeletons $\mathcal{T}_K$ and $\overline{\mathcal{T}_K}$ can be canonically identified with the set of classes modulo homothety of non-trivial diagonalizable norms and seminorms on K^2 respectively. These are the Bruhat-Tits building of* $\mathrm{PGL}_2(K)$ *and its compactification.*

Proof. The classes of seminorms associated to balls correspond to the classes of diagonalizable norms and seminorms on K^2 by the two previous results. □

And now we are going to show that d is invariant with respect to the action of $\mathrm{PGL}_2(K)$.

Consider any apartment $\mathbb{A}_{\{x_0,x_1\}}$ for $x_0, x_1 \in \mathbb{P}^{1*}(K)$ and choose representatives $\omega_0, \omega_1 \in V^*$ respectively. Let $v_0, v_1 \in V$ be the dual basis of ω_0, ω_1. For any two elements in this apartment $\alpha := \alpha_{(v_0,v_1),(\rho_0,\rho_1)}$, $\alpha' := \alpha_{(v_0,v_1),(\rho'_0,\rho'_1)}$ we define a distance in this apartment as:

$$d_{x_0,x_1}(\alpha,\alpha') := \left|\log\left(\frac{\rho_1\rho'_0}{\rho_0\rho'_1}\right)\right| = \left|\log\left(\frac{\rho_1}{\rho_0}\right) - \log\left(\frac{\rho'_1}{\rho'_0}\right)\right|$$

Note that the homeomorphism (up to orientation) $\mathbb{A}_{\{x_0,x_1\}} \longrightarrow \mathbb{R}$ is given by

$$\alpha \mapsto \log\left(\frac{\rho_1}{\rho_0}\right),$$

so d_{x_0,x_1} is the transported distance from the natural one in $\mathbb{R}$.

Proposition 3.1.8. *The two definitions of distance coincide, that is, for any $x_0, x_1 \in \mathbb{P}^{1*}(K)$ we have*

$$d_{|\mathbb{A}_{\{x_0,x_1\}}} = d_{x_0,x_1}$$

Proof. For any $\alpha := \alpha_{(v_0,v_1),(\rho_0,\rho_1)}, \alpha' := \alpha_{(v_0,v_1),(\rho'_0,\rho'_1)} \in \mathbb{A}_{\{x_0,x_1\}}$ we want to see $d(\alpha,\alpha') = d_{x_0,x_1}(\alpha,\alpha')$.

First, we can assume and we do that there exists an $x \in \mathbb{P}^{1*}(K)$ such that $\alpha, \alpha' \in \mathbb{A}_{\{x,\infty\}}$. Otherwise $d(\alpha,\alpha') = d(\alpha, \alpha\vee\alpha') + d(\alpha\vee\alpha',\alpha')$ and by definition d_{x_0,x_1} satisfies the same equality.

Moreover, it is enough to prove that if $\alpha,\alpha' \in \mathbb{A}_{\{x_0,x_1\}} \bigcap \mathbb{A}_{\{y_0,y_1\}}$ then $d_{x_0,x_1}(\alpha,\alpha') = d_{y_0,y_1}(\alpha,\alpha')$, since for the particular case $y_0 = x,\ y_1 = \infty$ we have $d_{x,\infty} = d$.

We may reduce to the case $y_0 = x_0$ by applying the result in two steps. Let us denote $x_2 := y_1 \in \mathbb{P}^{1*}(K)$ and let it be represented by

$$\omega_2 = \lambda\omega_0 + \mu\omega_1 \in V^*,\ \mu \neq 0.$$

Then

$$u_0 = v_0 - \frac{\lambda}{\mu}v_1, u_2 = \frac{v_1}{\mu} \in V$$

is the dual basis of ω_0, ω_2. Now we have that

$$\alpha := \alpha_{(v_0,v_1),(\rho_0,\rho_1)} = \alpha_{(u_0,u_2),(\eta_0,\eta_2)} \text{ with } \eta_0 = \max\left\{\rho_0, \left|\frac{\lambda}{\mu}\right|\rho_1\right\},\ \eta_2 = \left|\frac{1}{\mu}\right|\rho_1$$

and

$$\alpha' := \alpha_{(v_0,v_1),(\rho_0',\rho_1')} = \alpha_{(u_0,u_2),(\eta_0',\eta_2')} \text{ with } \eta_0' = \max\left\{\rho_0', \left|\frac{\lambda}{\mu}\right| \rho_1'\right\},\ \eta_2' = \left|\frac{1}{\mu}\right| \rho_1'$$

Note that $\rho_1 = |\mu|\eta_2$ implies that $\eta_0 = \max\{\rho_0, |\lambda|\eta_2\}$. Furthermore

$$\rho_0 = \alpha(v_0) = \max\{\eta_0, |\lambda|\eta_2\},$$

since $v_0 = u_0 + \lambda u_2$. Then $\eta_0 = \rho_0$. Identically we get $\eta_0' = \rho_0'$.

Therefore

$$d_{x_0,x_2}(\alpha, \alpha') = \left|\log \frac{\eta_2\eta_0'}{\eta_0\eta_2'}\right| = \left|\log \frac{\rho_1\rho_0'}{\rho_0\rho_1'}\right| = d_{x_0,x_1}(\alpha, \alpha')$$

□

Corollary 3.1.9. *The distance is* $\mathrm{PGL}_2(K)$*-invariant, that is to say,*

$$d(\alpha, \alpha') = d(\gamma \cdot \alpha, \gamma \cdot \alpha')$$

for any $\gamma \in \mathrm{PGL}_2(K)$.

Proof. First we recall that $\gamma \cdot \alpha_{(v_0,v_1),(\rho_0,\rho_1)} = \alpha_{(\gamma\cdot v_0,\gamma\cdot v_1),(\rho_0,\rho_1)}$. Let us to take now any apartment $\mathbb{A}_{\{x_0,x_1\}}$ which contains α, α' as above. Then $d(\alpha, \alpha') = d_{x_0,x_1}(\alpha, \alpha') = d_{\gamma\cdot x_0,\gamma\cdot x_1}(\gamma\cdot\alpha, \gamma\cdot\alpha') = d(\gamma\cdot\alpha, \gamma\cdot\alpha')$, where the second equality is due to the remark 3.1.4. □

Let x_0, x_1 and x_2 be three distinct points in $\mathbb{P}^{1*}(K)$. Then there exists a unique point $t(x_0, x_1, x_2) \in \mathcal{T}_K$ which is contained in the three apartments they form. If $x_2 = \infty$, then $t(x_0, x_1, \infty) = \alpha(x_0, R) = x_0 \vee x_1$, where $|x_1 - x_0| = R$. If none of them is equal to ∞, it corresponds to the smallest ball containing all three points.

Observe that the points $t(x_0, x_1, x_2)$ are always of type II, so they have the form $\alpha(x_0, r)$ with $r \in |K^*|$.

Definition 3.1.10. *Let* $\mathcal{L}$ *be a subset of* $\mathbb{P}^1(K)$ *which contain at least two points. Denote by*

$$\mathcal{T}_K(\mathcal{L}) := \bigcup_{\{x_0^*,x_1^*\}\subset\mathcal{L}} \mathbb{A}_{\{x_0,x_1\}} = \bigcup_{\{x_0,x_1\}\subset\mathcal{L}^*} \mathbb{A}_{\{x_0,x_1\}}$$

the tree associated to $\mathcal{L}$ *(which is the subspace of* $\mathcal{T}_K$ *generated by the lines between two points corresponding of points in* $\mathcal{L}$*). Note that*

$$\overline{\mathcal{T}_K(\mathcal{L})} := \mathcal{T}_K(\mathcal{L}) \cup \mathcal{L}^*$$

with the natural topology.

It is clear that for any extension of fields $L|K$ the tree associated to $\mathcal{L}$ is always the same: $\mathcal{T}_L(\mathcal{L}) = \mathcal{T}_K(\mathcal{L})$, $\overline{\mathcal{T}_L(\mathcal{L})} = \overline{\mathcal{T}_K(\mathcal{L})}$.

We will show in the sequel that $\mathcal{T}_K(\mathcal{L})$ is a locally finite metric tree if $\mathcal{L}$ is compact.

Lemma 3.1.11. *The points of the form $t(x_0, x_1, x_2)$ for three distinct points $x_0, x_1, x_2 \in \mathcal{L}^*$ are the points in $\mathcal{T}_K(\mathcal{L})$ with valence greater than 2.*

Suppose that $\infty, x_0 \in \mathcal{L}^*$ and consider a point $\alpha := \alpha(x_0, r) \in \mathcal{T}_K(\mathcal{L})$ of the form $t(x_0, x_1, \infty)$ for some $x_1 \in \mathcal{L}^*$. Then

$$\{y \in \mathcal{L}^* \setminus \{x_0, \infty\} \mid \alpha = t(x_0, y, \infty)\} = \{y \in \mathcal{L}^* \mid |y - x_0| = r\}.$$

Moreover, there is a bijection between the set of directions from $\alpha(x_0, r)$ except the ones which connect with ∞ and x_0, and the image of the map

$$\psi : \{y \in \mathcal{L}^* \mid |y - x_0| = r\} \to k^*$$

given by:

$$\psi(y) = \frac{y - x_0}{x_1 - x_0} \pmod{\mathfrak{m}_K}.$$

Proof. The unique claim that needs a proof is the bijection. From the equality shown, we see that a direction can be identified with a set of points $E_y \subset \{y \in \mathcal{L}^* \mid |y - x_0| = r\}$ such that $|y' - y''| < r$ for all $y', y'' \in E_y$. Thus, the only thing we have to prove is that $\psi(y) = \psi(y')$ if and only if $|y - y'| < r$.

To start with this equivalence we note that $\psi(y) = \psi(y')$ means that there exists $z \in \mathfrak{m}_K$, or equivalently $|z| < 1$, such that

$$\frac{y - x_0}{x_1 - x_0} = \frac{y' - x_0}{x_1 - x_0} + z$$

We may write this equality as $y - y' = z(x_1 - x_0)$ and taking absolute value $|y - y'| = |z|r < r$. Finally, the other option, $|z| = 1$, is that for which $\psi(y) \neq \psi(y')$. □

Proposition 3.1.12. *If $\mathcal{L}$ is compact, then $\mathcal{T}_K(\mathcal{L})$ is a locally finite metric tree, that is to say, any vertex has a finite number of directions arriving to it and any finite lenght path contains only a finite number of vertices of valence greater than 2.*

Proof. We suppose $\mathcal{L}$ has at least three points and $\infty \in \mathcal{L}^*$ without loss of generality.

Note that $|\mathcal{L}| = |\mathcal{L}^*|$ and that $\mathcal{L}$ is compact if and only if $\mathcal{L}^*$ is compact.

In order to prove the first claim consider a vertex $\alpha(x_0, r) \in \mathcal{T}_K(\mathcal{L})$ that we may assume of the form $t(x_0, x_1, \infty)$ for some x_0 and $x_1 \in \mathcal{L}^*$. Since $\mathcal{L}^*$ is compact and $\{y \in K \mid |y - x_0| = r\}$ is closed, their intersection $\{y \in \mathcal{L}^* \mid |y - x_0| = r\}$ is compact. Now, given any $t \in k^*$, the set $\psi^{-1}(\{t\})$ is an open subset (the previous proof shows it is an open ball). Then, if the point had infinite directions arriving to it, the image of ψ would be infinite so the compact set $\{y \in \mathcal{L}^* \mid |y - x_0| = r\}$ would be covered by an infinite number of disjoint open subsets and we would get a contradiction.

To get the second claim we can reduce us to show it for a path with the form $P(\alpha(x, r), \alpha(x, r'))$ with $0 < r \leq r'$. We have to show that the set

$$S_{r,r'} := \{s \in [r, r'] |\ \exists y \in \mathcal{L}^* : |y - x| = s\}$$

is finite. Consider the set

$$\{y \in \mathcal{L}^* |\ r \leq |y - x| \leq r'\} = \mathcal{L}^* \bigcap \Big(B(x, r') \setminus \mathring{B}(x, r) \Big) =$$

$$= \bigcup_{s \in S_{r,r'}} \{y \in \mathcal{L}^* |\ |y - x| = s\}$$

Since it is a closed in $\mathcal{L}^*$, then it is compact. Further, the subsets

$$\mathcal{L}^*_{x,s} := \{y \in \mathcal{L}^* |\ |y - x| = s\} = \bigcup_{y \in \mathcal{L}^*_{x,s}} \Big(\mathcal{L}^* \cap \mathring{B}(y, s) \Big)$$

are open, so we can get a finite covering by them, and this implies necessarily that $S_{r,r'}$ is finite. □

Definition 3.1.13. *With the hypotheses of definition 3.1.10 we say that $\mathcal{T}_K(\mathcal{L})$ is perfect if for any $\alpha \in \mathcal{T}_K(\mathcal{L})$ and for any $r \in \mathbb{R}_{>0}$ there exists $\alpha' \in \mathcal{T}_K(\mathcal{L})$ with valence greater than 2 and such that $d(\alpha, \alpha') > r$.*

One can show that this definition is compatible with the one of perfect set, so $\mathcal{T}_K(\mathcal{L})$ is perfect if and only if $\mathcal{L}$ is perfect (all the points in $\mathcal{L}$ are accumulation points), clearly equivalent to $\mathcal{L}^*$ being perfect. For example, if $\mathcal{L}$ is a finite set, then $\mathcal{T}_K(\mathcal{L})$ is not perfect, since it has just a finite number of vertices of valence greater than 2.

Definition 3.1.14. *We will call a topological (oriented) edge $\mathfrak{e} := \mathfrak{e}_{\alpha,\beta}$ (of $\mathcal{T}_K(\mathcal{L})$) a non trivial path $P(\alpha, \beta) \subset \mathcal{T}_K(\mathcal{L})$, such that all its interior points have valence two in $\mathcal{T}_K(\mathcal{L})$. We will call the length of $\mathfrak{e}$ the distance $d(\alpha, \beta)$, and we will denote it by $l(\mathfrak{e})$.*

From now on, let $\mathcal{L} \subset \mathbb{P}^1(K)$ be a compact subset with at least two points.

Proposition 3.1.15. *Since $\mathcal{L}$ is compact, there is a bijection between $\mathcal{L}$ and $\mathcal{E}(\mathcal{T}_K(\mathcal{L}))$.*

Proof. Let us handle with $\mathcal{L}^*$ instead of $\mathcal{L}$ for simplicity. From the definition of $\mathcal{T}_K(\mathcal{L})$ we build a map $\varepsilon : \mathcal{L}^* \longrightarrow \mathcal{E}(\mathcal{T}_K(\mathcal{L}))$. Indeed, to any point $x_0 \in \mathcal{L}^*$ we may choose any ray in any apartment $\mathbb{A}_{\{x_1,x_0\}}$ getting close to x_0, which gives a well defined end. Further, it is clearly injective.

Now, to prove exhaustivity, we take an end given by a ray $\{\alpha(x_n, r_n)\}$. After taking a different model of $\mathcal{T}_K(\mathcal{L})$ and an equivalent ray, we may assume and we do $r_n \neq r_{n+1}$ for all n.

There are two options. Either $r_{n+1} > r_n$ for all n or there exists an N such that $r_{n+1} < r_n$ for all $n \geq N$.

Indeed, assume $r_{n-1} > r_n < r_{n+1}$. Then we have

$$\alpha(x_n, r_{n-1}) \in P\big(\alpha(x_n, r_n), \alpha(x_{n-1}, r_{n-1})\big),$$

since, if $|x_{n-1} - x_n| < r_{n-1}$ we have $\alpha(x_{n-1}, r_{n-1}) = \alpha(x_n, r_{n-1})$ and otherwise $P\big(\alpha(x_n, r_n), \alpha(x_{n-1}, r_{n-1})\big)$ is the path $P\big(\alpha(x_n, r_n), \alpha(x_n, |x_n - x_{n-1}|)\big)$ followed by the path $P\big(\alpha(x_n, |x_n - x_{n-1}|), \alpha(x_{n-1}, r_{n-1})\big)$. And in the same way $\alpha(x_n, r_{n+1}) \in P(\alpha(x_n, r_n), \alpha(x_{n+1}, r_{n+1}))$. Therefore

$$P\big(\alpha(x_{n-1}, r_{n-1}), \alpha(x_n, r_n)\big) \bigcap P\big(\alpha(x_{n+1}, r_{n+1}), \alpha(x_n, r_n)\big) \supset$$

$$\supset P\big(\alpha(x_n, r_n), \alpha(x_n, \min\{r_{n-1}, r_{n+1}\})\big) \neq \{\alpha(x_n, r_n)\}$$

against the definition of ray.

Therefore, by taking an equivalent ray, we may assume that $\{r_n\}_n$ is increasing or decreasing from the beginning.

Moreover, since $\mathcal{L}^*$ is compact, the sequence $\{x_n\}_n$ has a convergent partial subsequence $\{x_{n_m}\}_m$, which defines an equivalent ray, thus we assume $\lim_{n\to\infty} x_n = x \in \mathcal{L}^*$. When $x \neq \infty$, we also can suppose without loss of generality, and we do, that $x = 0$.

Then, I claim that the preimage of this end is ∞ when $\{r_n\}_n$ is an increasing sequence and it is $0 (= x)$ when the sequence is decreasing.

If $\{r_n\}_n$ is increasing, choose $r_1 \geq 0$. The limit previously computed gives an N such that for all $n \geq N$ we have $|x_n| < r_1 < r_n$, so

$$\alpha(x_n, r_n) = \alpha(0, r_n) \in \mathbb{A}_{\{0,\infty\}},$$

therefore the given end is the image of ∞.

Thus, to finish we assume the sequence $\{r_n\}_n$ is decreasing.

Now assume that there exists an $r > 0$ such that $r_n > r$ for all n. Since $\lim_{n\to\infty} x_n = 0$ we have that there would exist an N such that $|x_n| < r$ for all $n \geq N$. This would imply that $\alpha(x_n, r_n) = \alpha(0, r_n)$ and we would have all the ray contained in $P\big(\alpha(x_0, r_0), \alpha(0, r)\big)$ a contradiction with the definition of ray, which has to be infinite. Therefore $\lim_{n\to\infty} r_n = 0$.

Next, take an $\alpha(x_n, r_n)$ such that $|x_n| > r_n$. Since we assume the sequence of the radii is decreasing, by the same resoning

$$\alpha(x', r') \in P\big(\alpha(x_n, r_n), \alpha(x_{n+1}, r_{n+1})\big) \text{ implies } r_{n+1} < r' < r_n.$$

This gives $r_n \geq |x_n - x_{n+1}|$. Otherwise ($r_n < |x_n - x_{n+1}|$) we would have that the path $P\big(\alpha(x_n, r_n), \alpha(x_{n+1}, r_{n+1})\big)$ is $P\big(\alpha(x_n, r_n), \alpha(x_n, |x_n - x_{n+1}|)\big)$ followed by $P\big(\alpha(x_n, |x_n - x_{n+1}|), \alpha(x_{n+1}, r_{n+1})\big)$, getting a contradiction through the first part. Therefore we obtain

$$|x_n| > r_n \geq |x_n - x_{n+1}| \Longrightarrow |x_n| = |x_{n+1}|,$$

and by the same reasoning $|x_n| = |x_m|$ for all $m \geq n$, which contradices that $\lim_{n\to\infty} x_n = 0$.

Then, we got $|x_n| \leq r_n$ for all n and so $\alpha(x_n, r_n) = \alpha(0, r_n) \in \mathbb{A}_{\{0,\infty\}}$, so the end is the one given by the image of 0, as we desired.

Finally, note that if we had $\lim_{n\to\infty} x_n = \infty \in \mathcal{L}^*$, we could assume $\{|x_n|\}_n$ is strictly increasing, and further, by similar reasonings as above, we would obtained $\{r_n\}_n$ being a strictly increasing sequence and

$$|x_{n+1}| = |x_n - x_{n+1}| > r_n \Longrightarrow r_{n+1} > |x_n - x_{n+1}| = |x_{n+1}|,$$

$$r_n \geq |x_{n+1}| \Longrightarrow r_{n+1} > r_n \geq |x_{n+1}|,$$

so we would get the end of the ray in $\mathbb{A}_{\{0,\infty\}}$, as the image of ∞.

□

Next, we particularize the definition 2.1.15.

Definition 3.1.16. *Let $\mathfrak{e}$ be the topological edge of $\mathcal{T}_K(\mathcal{L})$ induced by the path $P(\alpha, \beta)$ (in particular, $\alpha \neq \beta$). We may define a subset of $\mathcal{L}$ associated to it as*

$$\mathcal{B}(\mathfrak{e}) := \mathcal{B}(\alpha, \beta) := \{x \in \mathcal{L} |\ \alpha \notin P(x^*, \beta)\} = \{x \in \mathcal{L} |\ \beta \in P(x^*, \alpha)\}.$$

Corollary 3.1.17. *The bijection $\mathcal{L} \cong \mathcal{E}(\mathcal{T}_K(\mathcal{L}))$ is an homeomorphism. In particular, the topology defined in the previous chapter coincide with the given topology in $\mathcal{L}$.*

Proof. Note that if $\alpha = \alpha(x, r)$, $\alpha' = \alpha(x, s)$ and $x \in K$, either $r < s$ and so $\mathcal{B}(\alpha, \alpha') = \mathcal{L} \setminus \mathring{B}(x, s)$, or $r > s$ and then $\mathcal{B}(\alpha, \alpha') = \mathcal{L} \bigcap B(x, s)$.

And now we have finished, since the balls and their complementaries are a basis of open sets in $\mathcal{L}$ and the sets associated to edges are a basis of open sets in $\mathcal{E}(\mathcal{T}_K(\mathcal{L}))$. □

Thus, we have the properties of proposition 2.1.16, and, as a consequence, these sets are an open basis of the topology of $\mathcal{L}$ in the strong sense, meaning that any compact open set of $\mathcal{L}$ is a finite disjoint union of them.

3.2 The retraction map

We build the retraction map $\mathrm{r}_{\mathcal{L}} : ({\mathbb{P}^1_K}^*)^{an} \longrightarrow \overline{\mathcal{T}_K(\mathcal{L})}$ generalizing the reduction map constructed by Werner in [Wer04] to the trees introduced in the previous section, which, on the other hand, gives the complete description over all the Berkovich analytic points of the reduction map named in [Das05, 2.3.]. Further, we do not restrict to a local field.

Through this section $L|K$ will be an arbitrary extension of valued complete fields.

Given any compact subset with at least two points $\mathcal{L} \subset \mathbb{P}^1(K)$ we define $\Omega_{\mathcal{L}}(L) := {\mathbb{P}^1}^*(L) \setminus \mathcal{L}^*$. We also define the diameter of $\mathcal{L}^*$ as

$$d_{\mathcal{L}^*} = \begin{cases} \inf\{r \geq 0 |\ \mathcal{L}^* \subset B(x,r) \text{ for some } x \in \mathcal{L}^*\} \text{ if } \infty \notin \mathcal{L}^* \\ +\infty \text{ if } \infty \in \mathcal{L}^* \end{cases}$$

Note that we may fix $x \in \mathcal{L}^*$ and the definition is independent of the chosen point x. Observe also that we can do the same definition for $\mathcal{L}$ obtaining $d_{\mathcal{L}} = d_{\mathcal{L}^*}$, so we can speak about the diameter of $\mathcal{L}$ and denote by $d_{\mathcal{L}}$ the diameter of $d_{\mathcal{L}^*}$.

Definition 3.2.1. *Let $\mathcal{L} \subset \mathbb{P}^1(K)$ be as just above. We define the retraction map $\mathrm{r}_{\mathcal{L}} : {\mathbb{P}^1}^*(L) \to \overline{\mathcal{T}_K(\mathcal{L})}$ to be*

$$\mathrm{r}_{\mathcal{L}}(x) = \begin{cases} x, \text{ if } x \in \mathcal{L}^* \\ \alpha(x, \inf\{s \geq 0 \mid B(x,s) \cap \mathcal{L}^* \neq \emptyset\}), \text{ if } B(x, d_{\mathcal{L}}) \cap \mathcal{L}^* \neq \emptyset \text{ and } x \notin \mathcal{L}^* \\ \alpha(y, d_{\mathcal{L}}) \text{ for any } y \in \mathcal{L}^*, \text{ if } B(x, d_{\mathcal{L}}) \cap \mathcal{L}^* = \emptyset \end{cases}$$

for $x \neq \infty$, and

$$\mathrm{r}_{\mathcal{L}}(\infty) = \begin{cases} \alpha(y, d_{\mathcal{L}}) \text{ for any } y \in \mathcal{L}^*, \text{ if } \infty \notin \mathcal{L}^* \\ \alpha(\infty, 0), \text{ if } \infty \in \mathcal{L}^* \end{cases}$$

We also define $\mathrm{r}_{\mathcal{L}} : \Omega_{\mathcal{L}}(L) \to \mathcal{T}_K(\mathcal{L})$ as the restriction.

Remark 3.2.2. *The retraction map leaves fixed the points of $\mathcal{L}^*$. On the other hand, if $x \notin \mathcal{L}^*$, the point $\mathrm{r}_{\mathcal{L}}(x)$ is the only point of the path $P(\alpha(x,0), \mathrm{r}_{\mathcal{L}}(x)) \subset \overline{\mathcal{T}_K}$ which is in $\mathcal{T}_K(\mathcal{L})$.*

Now we want to extend this map to $r_{\mathcal{L}} : \overline{\mathcal{T}_L} \longrightarrow \overline{\mathcal{T}_K(\mathcal{L})}$. First, if $\alpha \in \overline{\mathcal{T}_K(\mathcal{L})}$, then $r_{\mathcal{L}}(\alpha) = \alpha$.

Next, consider $\alpha \in \mathcal{T}_L \setminus \overline{\mathcal{T}_K(\mathcal{L})}$. Then $\alpha = \alpha(x, r)$ for some $x \in L \setminus \mathcal{L}^*$ and some $r > 0$.

If $B(x, r) \bigcap \mathcal{L}^* = \emptyset$, we define $r_{\mathcal{L}}(\alpha) := r_{\mathcal{L}}(x)$. We only need to show that $r_{\mathcal{L}}(\alpha)$ does not depend on the chosen x. When $B(x, d_{\mathcal{L}}) \cap \mathcal{L}^* \neq \emptyset$, $r_{\mathcal{L}}(x) = \alpha(x, s)$ and $s > r$ since $\alpha(x, r) \notin \overline{\mathcal{T}_K(\mathcal{L})}$. Hence, if $\alpha(x, r) = \alpha(y, r)$, then $\alpha(x, s) = \alpha(y, s)$. Otherwise, it is clear.

In the other case, $B(x, r) \bigcap \mathcal{L}^* \neq \emptyset$, we have $\infty \notin \mathcal{L}^*$ and $\mathcal{L}^* \subset B(x, r)$ (so $r > d_{\mathcal{L}}$). Then we define $r_{\mathcal{L}}(\alpha) := r_{\mathcal{L}}(\infty)$.

Proposition 3.2.3. *The retraction map is a retraction. As a consequence, if $\Gamma \subset \mathrm{PGL}_2(K)$ acts on $\mathcal{L}$, it is Γ-equivariant.*

Proof. It follows from the previous remark and construction that the map is a retraction in the strict sense. The consequence is due to the fact that the projective linear group acts continuously on $\mathcal{T}_K$ and Γ leaves $\mathcal{T}_K(\mathcal{L})$ invariant. □

Next, let us recall that $\mathbb{C}_K$ embeds isometrically into a spherically complete nonarchimedean field $\mathbb{K}$, since it admits a maximally complete extension by [Kru32, Thm. 24], and this condition is equivalent to spherical completeness by [Kap42, Thm. 4]. We know by [Ber90, §1.4] that $(\mathbb{P}^{1\,*}_{\mathbb{K}})^{an}$ has no type IV points so we get

$$r_{\mathcal{L}} : (\mathbb{P}^{1\,*}_{\mathbb{K}})^{an} = \overline{\mathcal{T}_{\mathbb{K}}} \longrightarrow \overline{\mathcal{T}_K(\mathcal{L})}$$

Note that from the beginning of the formalization of the retraction map, each time that we define it taking an infimum ($r_{\mathcal{L}}(\alpha) = \alpha(x, \inf\{...\})$) we get this element is inside the tree $\mathcal{T}_K(\mathcal{L})$ since $\mathcal{L}$ is compact.

The following lemma is clear from the properties of the retraction map.

Lemma 3.2.4. *If we have two subsets $\mathcal{L}' \subset \mathcal{L} \subset \mathbb{P}^1(K)$ as above, then*

$$r_{\mathcal{L}'}(\alpha) = r_{\mathcal{L}'}(r_{\mathcal{L}}(\alpha)) \text{ for any } \alpha \in \overline{\mathcal{T}_K}.$$

Lemma 3.2.5. *For any two points y_0, y_1 in $\mathbb{P}^{1*}(K)$, with respective representatives in $(K^2)^*$ given by ω_0, ω_1 and having dual basis v_0, v_1, and for any $\alpha \in \overline{\mathcal{T}_L}$, the point $r_{\{[v_0],[v_1]\}}(\alpha)$ is the seminorm η diagonalized by v_0 and v_1 up to equivalence, with $\eta(v_i) = \alpha(v_i)$ for $i = 0$ and 1, that is $[r_{\{[v_0],[v_1]\}}(\alpha)] = [\alpha_{(v_0,v_1),(\alpha(v_0),\alpha(v_1))}]$.*

Proof. If $\alpha \in \mathbb{A}_{\{y_0,y_1\}}$ there is nothing to prove. From now on we assume this is not the case.

If one of the two points, let us assume y_1, is ∞, then, writing $\alpha = \alpha(x,r)$,

$$\mathrm{r}_{\{\infty,[v_1]\}}(\alpha) = \alpha(x, |y_0 - x|) = \alpha(y_0, |y_0 - x|) = \alpha_{(v_0,v_1),(1,|x-y_0|)}$$

Now we compute

$$\alpha(x,r)(v_0) = \alpha(x,r)(1,0) = \max\{1,0\} = 1,$$

$$\alpha(x,r)(v_1) = \alpha(x,r)(y_0,1) = \max\{|y_0 - x|, r\} = |x - y_0|,$$

since $|x - y_0| > r$ due to $\alpha \notin \mathbb{A}_{\{y_0,y_1\}}$.

Next, suppose $y_0, y_1 \neq \infty$, and then we can take $\omega_i = (1, -y_i)$ for $i = 0,1$, so

$$v_0 = \left(\frac{y_1}{y_1 - y_0}, \frac{1}{y_1 - y_0}\right) \text{ and } v_1 = \left(\frac{y_0}{y_0 - y_1}, \frac{1}{y_0 - y_1}\right).$$

Furthermore, either $\{y_0, y_1\} \subset B(x,r)$ or $B(x,r) \bigcap \{y_0, y_1\} = \emptyset$.

In the first case

$$\mathrm{r}_{\{[v_0],[v_1]\}}(\alpha) = \mathrm{r}_{\{[v_0],[v_1]\}}(\infty) = \alpha(y_0, |y_0 - y_1|\}) = \alpha_{(v_0,v_1),(1,1)}$$

We just need to show that $\alpha(v_0) = \alpha(v_1)$. We have

$$\alpha(x,r)(v_0) = \alpha(x,r)\left(\frac{y_1}{y_1 - y_0}, \frac{1}{y_1 - y_0}\right) = \max\left\{\left|\frac{y_1 - x}{y_1 - y_0}\right|, \left|\frac{r}{y_1 - y_0}\right|\right\}$$

and, identically,

$$\alpha(x,r)(v_1) = \max\left\{\left|\frac{y_0 - x}{y_0 - y_1}\right|, \left|\frac{r}{y_0 - y_1}\right|\right\}.$$

Since the condition $\{y_0, y_1\} \subset B(x,r)$ tells us that $r \geq |y_0 - x|, |y_1 - x|$ we get the required equality $\alpha(v_0) = \alpha(v_1)$.

In the second case, being satisfied $B(x,r) \bigcap \{y_0, y_1\} = \emptyset$, we have

$$\mathrm{r}_{\{[v_0],[v_1]\}}(\alpha) = \mathrm{r}_{\{[v_0],[v_1]\}}(x) =$$

$$= \begin{cases} \alpha(x, \min\{|x - y_0|, |x - y_1|\}), \text{ if } B(x, |y_0 - y_1|) \cap \{y_0, y_1\} \neq \emptyset \\ \alpha(y_0, |y_0 - y_1|), \text{ if } B(x, |y_0 - y_1|) \cap \{y_0, y_1\} = \emptyset \end{cases}$$

after noticing that the diameter is $d_{\{[v_0],[v_1]\}} = |y_0 - y_1|$. Now, the fact that the intersection $B(x, |y_0 - y_1|) \cap \{y_0, y_1\}$ is empty is equivalent to the inequality $|y_0 - x| = |y_1 - x| > |y_0 - y_1|$ and $\alpha(y_0, |y_0 - y_1|) = \alpha_{(v_0,v_1),(1,1)}$. All the rest of the proof for this situation works exactly equal as above taking into account that the condition $B(x,r)\bigcap\{y_0,y_1\}=\emptyset$ implies $|y_0-x|(=|y_1-x|)>r$.

Finally, when $B(x,|y_0-y_1|)\cap\{y_0,y_1\}\neq\emptyset$ we have $|y_i-x|\leq|y_0-y_1|$ for $i=0,1$ and at least for one i, $|y_i-x|=|y_0-y_1|$; assume this equality for y_1. Then, on one hand we get

$$\alpha(x,\min\{|x-y_0|,|x-y_1|\})=\alpha(x,|x-y_0|)=\alpha(y_0,|x-y_0|)=\alpha_{(v_0,v_1),\left(1,\frac{|x-y_0|}{|y_0-y_1|}\right)}$$

On the other hand we have

$$\alpha(x,r)(v_0)=\max\left\{\left|\frac{y_1-x}{y_1-y_0}\right|,\left|\frac{r}{y_1-y_0}\right|\right\}=\left|\frac{y_1-x}{y_1-y_0}\right|$$

and

$$\alpha(x,r)(v_1)=\max\left\{\left|\frac{y_0-x}{y_0-y_1}\right|,\left|\frac{r}{y_0-y_1}\right|\right\}=\left|\frac{y_0-x}{y_0-y_1}\right|$$

since $B(x,r)\bigcap\{y_0,y_1\}=\emptyset$. Therefore, maintaining and employing the assumption $|y_1-x|=|y_0-y_1|\geq|y_0-x|$, we obtain

$$\alpha_{(v_0,v_1),(\alpha(v_0),\alpha(v_1))}=\alpha_{(v_0,v_1),\left(\frac{|y_1-x|}{|y_1-y_0|},\frac{|y_0-x|}{|y_0-y_1|}\right)}=\alpha_{(v_0,v_1),\left(1,\left|\frac{x-y_0}{y_0-y_1}\right|\right)},$$

and so the claimed equality. Note that if we had assumed $|y_0-x|=|y_0-y_1|$ we would have got

$$\alpha(x,\min\{|x-y_0|,|x-y_1|\})=\alpha(y_1,|x-y_1|)=\alpha_{(v_0,v_1),\left(\frac{|x-y_1|}{|y_0-y_1|},1\right)}=$$

$$=\alpha_{(v_0,v_1),(\alpha(v_0),\alpha(v_1))}$$

too. □

With the notation of the previous lemma note that $y_0^*=[v_1]$ and $y_1^*=[v_0]$.

Lemma 3.2.6. *Let $\mathcal{L}\subset\mathbb{P}^1(K)$ be a compact subset with at least two points. For any two seminorms $\alpha,\alpha'\in\overline{\mathcal{T}_L}$ such that $\alpha_{|K[X_0,X_1]}=\alpha'_{|K[X_0,X_1]}$, then $\mathrm{r}_{\mathcal{L}}(\alpha)=\mathrm{r}_{\mathcal{L}}(\alpha')$.*

Proof. If $\mathcal{L}^*=\{y_0,y_1\}$ the claim is true due to the last lemma. Otherwise, we always can find two points $y_0,y_1\in\mathcal{L}^*$ such that their retractions $\mathrm{r}_{\mathcal{L}}(\alpha),\mathrm{r}_{\mathcal{L}}(\alpha')\in\mathbb{A}_{\{y_0,y_1\}}$. Then, using this hypothesis for the outer equalities together with lemmas 3.2.4 and 3.2.5 for the interior equalities, we get

$$\mathrm{r}_{\mathcal{L}}(\alpha)=\mathrm{r}_{\{y_0^*,y_1^*\}}(\mathrm{r}_{\mathcal{L}}(\alpha))=\mathrm{r}_{\{y_0^*,y_1^*\}}(\alpha)=$$

$$=\mathrm{r}_{\{y_0^*,y_1^*\}}(\alpha')=\mathrm{r}_{\{y_0^*,y_1^*\}}(\mathrm{r}_{\mathcal{L}}(\alpha'))=\mathrm{r}_{\mathcal{L}}(\alpha')$$

□

Finally, recall that we have a retraction map $\mathrm{r}_{\mathcal{L}} : \overline{\mathcal{T}_L} \longrightarrow \overline{\mathcal{T}_K(\mathcal{L})}$ with two important particular cases:

$$\mathrm{r}_{\mathcal{L}} : \overline{\mathcal{T}_K} \longrightarrow \overline{\mathcal{T}_K(\mathcal{L})}$$

and

$$\mathrm{r}_{\mathcal{L}} : (\mathbb{P}^{1\,*}_{\mathbb{K}})^{an} = \overline{\mathcal{T}_{\mathbb{K}}} \longrightarrow \overline{\mathcal{T}_K(\mathcal{L})}$$

Now we want extend the first retraction map to $\mathrm{r}_{\mathcal{L}} : (\mathbb{P}^{1\,*}_{K})^{an} \longrightarrow \overline{\mathcal{T}_K(\mathcal{L})}$. Note first that $(\mathbb{P}^{1\,*}_{K})^{an} \cong (\mathbb{P}^{1\,*}_{\mathbb{C}_K})^{an}/\operatorname{Gal}(\mathbb{C}_K|K)$ by [Ber90, Cor. 1.3.6], so we may assume for a while $K = \mathbb{C}_K$ in order to define the extension.

Then, by remark 3.1.3 we only have to do this for the points of type IV. Let us take such a seminorm point $\alpha \in (\mathbb{P}^{1\,*}_{K})^{an}$. It is a limit of ball seminorms $\{\alpha(x_i, r_i)\}_{i\in\mathbb{N}}$ such that

$$r_{i+1} \leq r_i, \qquad B(x_{i+1}, r_{i+1}) \subset B(x_i, r_i)$$

$$r := \lim_{i\to\infty} r_i > 0 \text{ and } \bigcap_{i\in\mathbb{N}} B(x_i, r_i) = \emptyset$$

We consider the balls of the same center and radio with points in the spherical completion $\mathbb{K}$, that is $B_{\mathbb{K}}(x_i, r_i) := \{y \in \mathbb{K} \mid |y - x_i| \leq r_i\}$. Denote the associated seminorms in $(\mathbb{P}^{1\,*}_{\mathbb{K}})^{an}$ by $\alpha_{\mathbb{K}}(x_i, r_i)$.

Therefore, on one hand we have $\alpha_{\mathbb{K}}(x_i, r_i)_{|K[X_0,X_1]} = \alpha(x_i, r_i)$ and on the other hand we obtain $\bigcap_{i\in\mathbb{N}} B_{\mathbb{K}}(x_i, r_i) \neq \emptyset$, so it is a ball $B_{\mathbb{K}}(\hat{x}, r)$ which has an associated seminorm $\alpha_{\mathbb{K}}(\hat{x}, r) \in (\mathbb{P}^{1\,*}_{\mathbb{K}})^{an}$. Thus we get

$$\alpha = \lim_{i\to\infty} \alpha(x_i, r_i) = \lim_{i\to\infty} \alpha_{\mathbb{K}}(x_i, r_i)_{|K[X_0,X_1]} = \alpha_{\mathbb{K}}(\hat{x}, r)_{|K[X_0,X_1]}$$

Finally, we may take $\mathrm{r}_{\mathcal{L}}(\alpha) := \mathrm{r}_{\mathcal{L}}(\alpha_{\mathbb{K}}(\hat{x}, r))$ which is well defined by the last lemma above.

Remark 3.2.7. *This construction of* $\mathrm{r}_{\mathcal{L}} : (\mathbb{P}^{1\,*}_{K})^{an} \longrightarrow \overline{\mathcal{T}_K(\mathcal{L})}$ *and the lemma 3.2.5 allows us to note that when* $\overline{\mathcal{T}_K(\mathcal{L})} = \overline{\mathcal{T}_K}$*, this definition coincides with the given by Werner in [Wer04].*

Remark 3.2.8. *The retraction map we have built restricts to another retraction map on* $\Omega^{an}_{\mathcal{L}} := (\mathbb{P}^{1\,*}_{K})^{an} \setminus \mathcal{L}^*$*, making correspond to the square of inclusions*

$$\begin{array}{ccc} \overline{\mathcal{T}_K(\mathcal{L})} & \hookrightarrow & (\mathbb{P}^{1\,*}_{K})^{an} \\ \uparrow & & \uparrow \\ \mathcal{T}_K(\mathcal{L}) & \hookrightarrow & \Omega^{an}_{\mathcal{L}} \end{array}$$

the square of retractions

$$\begin{array}{ccc} ({\mathbb{P}^1_K}^*)^{an} & \xrightarrow{\mathrm{r}_{\mathcal{L}}} & \overline{\mathcal{T}_K(\mathcal{L})} \\ \uparrow & & \uparrow \\ \Omega^{an}_{\mathcal{L}} & \xrightarrow{\mathrm{r}_{\mathcal{L}}} & \mathcal{T}_K(\mathcal{L}). \end{array}$$

3.3 The discrete cross ratio

In this section we show with wide generality some results relating the cross ratio of 4 points in $\mathbb{P}^1(\mathbb{C}_K)$ with the tree they generate.

Recall that, given four points $a_1, a_2, z_1, z_2 \in {\mathbb{P}^1_K}^*(\mathbb{C}_K)$, the cross ratio is defined as

$$\left(\begin{array}{c} a_1 : z_1 \\ a_2 : z_2 \end{array} \right) = \frac{(a_1 - z_1)(a_2 - z_2)}{(a_1 - z_2)(a_2 - z_1)}$$

Note that formally

$$\left(\begin{array}{c} a_1 : z_1 \\ a_2 : z_2 \end{array} \right) = \left(\begin{array}{c} z_1 : a_1 \\ z_2 : a_2 \end{array} \right) = \left(\begin{array}{c} a_2 : z_2 \\ a_1 : z_1 \end{array} \right)$$

and given a fifth point $z_3 \in {\mathbb{P}^1_K}^*(\mathbb{C}_K)$,

$$\left(\begin{array}{c} a_1 : z_1 \\ a_2 : z_2 \end{array} \right) \left(\begin{array}{c} a_1 : z_2 \\ a_2 : z_3 \end{array} \right) = \left(\begin{array}{c} a_1 : z_1 \\ a_2 : z_3 \end{array} \right)$$

The next result is known, at least the particular cases and when K is local ([MD73], [BDG04]), but we prefer to expose a general and new proof using our results.

Proposition 3.3.1. *Let $a_1, a_2, z_1, z_2 \in {\mathbb{P}^1_K}^*(\mathbb{C}_K)$ be points such that $a_1 \neq a_2$ and $z_1 \neq z_2$. Then*

$$v_K \left(\left(\begin{array}{c} a_1 : z_1 \\ a_2 : z_2 \end{array} \right) \right) = \left(\mathbb{A}_{\{a_1, a_2\}}, \mathbb{A}_{\{z_1, z_2\}} \right)_{\mathcal{T}_{\mathbb{C}_K}}.$$

Proof. To begin, recall the definition of the first term,

$$v_K \left(\left(\begin{array}{c} a_1 : z_1 \\ a_2 : z_2 \end{array} \right) \right) = -\log \left| \frac{(a_1 - z_1)(a_2 - z_2)}{(a_1 - z_2)(a_2 - z_1)} \right|.$$

If $a_i = z_j$ for some i, j it is clear that the valuation of the cross ratio and the intersection pairing of the apartments are identically $\pm\infty$ with the sign depending on the combination. Next we will considerate the case in which the four points are distinct.

Let us suppose first that one of the four points is ∞. By the absolute symmetry among them, we can put $z_2 = \infty$. Then, on one hand we have

$$v_K\left(\left(\begin{array}{c} a_1 : z_1 \\ a_2 : z_2 \end{array}\right)\right) = -\log\left|\frac{a_1 - z_1}{a_2 - z_1}\right|$$

On the other hand we will compute the intersection of $\mathbb{A}_{\{a_1,a_2\}}$ with $\mathbb{A}_{\{z_1,\infty\}}$. Note that we may write $\alpha(z_1, r)$ with $r \in \mathbb{R}_{>0}$ for the points of the second apartment. Let us assume without loss of generality that $|a_1 - z_1| < |a_2 - z_1|$, so we see that the intersection between the apartments goes from the point $\alpha(z_1, |z_1 - a_1|)$ to the point $\alpha(z_1, |z_1 - a_2|)$ and the distance between them, which is the length of the intersection, and it is the product of the pairing (with positive sign because the assumption), is

$$\left|\log\frac{|a_2 - z_1|}{|a_1 - z_1|}\right| = -\log\left|\frac{a_1 - z_1}{a_2 - z_1}\right| = v_K\left(\left(\begin{array}{c} a_1 : z_1 \\ a_2 : z_2 \end{array}\right)\right)$$

as we wanted to see.

To finish the proof we have to deal with the case in which none of the four points is ∞. Let us consider the compact set $\mathcal{L}' := \{a_1, a_2\}$ and the radii

$$\begin{aligned} r_1 &:= \min(|z_1 - a_1|, |z_1 - a_2|), \text{ and} \\ r_2 &:= \min(|z_2 - a_1|, |z_2 - a_2|). \end{aligned}$$

Once more, we can do the assumption $r_1 \leq r_2$ without loss of generality. We will consider three cases:

We suppose first $|a_1 - a_2| \geq r_2 \geq r_1$.

On one hand it can occur that there is an $i \in \{1, 2\}$ such that $r_1 = |z_1 - a_i|$ and $r_2 = |z_2 - a_i|$. Then, the starting and ending points of the intersection between $\mathbb{A}_{\{a_1,a_2\}}$ and $\mathbb{A}_{\{z_1,z_2\}}$ are $\alpha(a_i, r_1)$ and $\alpha(a_i, r_2)$ respectively (so the intersection pairing is the distance with positive sign), or the intersection is empty or just a point if $r_1 = r_2$. Anyway,

$$\left(\mathbb{A}_{\{a_1,a_2\}}, \mathbb{A}_{\{z_1,z_2\}}\right)_{\mathcal{T}_{\mathbb{C}_K}} = d(\alpha(a_i, r_1), \alpha(a_i, r_2)) = \left|\log\frac{r_2}{r_1}\right| = -\log\frac{r_1}{r_2}$$

If $i = 1$, $r_1 \leq |z_1 - a_2| \leq \max\{r_1, |a_1 - a_2|\}$ so $|z_1 - a_2| = |a_1 - a_2|$, and $r_2 \leq |z_2 - a_2| \leq \max\{r_2, |a_1 - a_2|\}$ so $|z_2 - a_2| = |a_1 - a_2|$. If $i = 2$, the same computation gives a similar result. In any case we always get

$$v_K\left(\left(\begin{array}{c} a_1 : z_1 \\ a_2 : z_2 \end{array}\right)\right) = -\log\left|\frac{(a_1 - z_1)(a_2 - z_2)}{(a_1 - z_2)(a_2 - z_1)}\right| = -\log\frac{r_1|a_1 - a_2|}{r_2|a_1 - a_2|} = -\log\frac{r_1}{r_2}.$$

On the other hand, writing $\{i,j\}=\{1,2\}$ we have $r_1=|a_i-z_1|$ and $r_2=|a_j-z_2|$. We may assume $i=1$ and $j=2$. The starting and ending points of the intersection are $\alpha(a_1,r_1)$ and $\alpha(a_2,r_2)$. So we have

$$\left(\mathbb{A}_{\{a_1,a_2\}},\mathbb{A}_{\{z_1,z_2\}}\right)_{\mathcal{T}_{\mathbb{C}_K}}=d(\alpha(a_1,r_1),\alpha(a_2,r_2))=$$

$$=d(\alpha(a_1,r_1),\alpha(a_1,|a_1-a_2|))+d(\alpha(a_2,r_2),\alpha(a_2,|a_1-a_2|))=-\log\frac{r_1r_2}{|a_1-a_2|^2}$$

(Note that if we assumed $i=2$ and $j=1$, the intersection pairing would be minus the distance.)

Further, $r_2\geq|a_1-z_2|\geq\max\{|a_1-a_2|,r_2\}\geq|a_1-a_2|$ so $|a_1-z_2|=|a_1-a_2|$ and identically $|a_2-z_1|=|a_1-a_2|$. Therefore

$$v_K\left(\left(\begin{array}{c}a_1:z_1\\a_2:z_2\end{array}\right)\right)=-\log\left|\frac{(a_1-z_1)(a_2-z_2)}{(a_1-z_2)(a_2-z_1)}\right|=-\log\frac{r_1r_2}{|a_1-a_2|^2}.$$

In second place we suppose $r_2>|a_1-a_2|\geq r_1$. We can assume $r_1=|z_1-a_1|$. Let us observe that $r_2=|z_2-a_1|=|z_2-a_2|$. The starting and ending points of the intersection are $\alpha(a_1,r_1)$ and $\alpha(a_1,|a_1-a_2|)$, so

$$\left(\mathbb{A}_{\{a_1,a_2\}},\mathbb{A}_{\{z_1,z_2\}}\right)_{\mathcal{T}_{\mathbb{C}_K}}=d(\alpha(a_1,r_1),\alpha(a_1,|a_1-a_2|))=-\log\frac{r_1}{|a_1-a_2|}$$

(Note that if we assumed $r_1=|z_1-a_2|$, the distance would appear with a minus, and so we would get the inverse value.)

Since we have $|z_1-a_2|=|a_1-a_2|$ by an argument as above, we get

$$v_K\left(\left(\begin{array}{c}a_1:z_1\\a_2:z_2\end{array}\right)\right)=-\log\left|\frac{(a_1-z_1)(a_2-z_2)}{(a_1-z_2)(a_2-z_1)}\right|=$$

$$=-\log\frac{r_1r_2}{r_2|a_1-a_2|}=-\log\frac{r_1}{|a_1-a_2|}.$$

Finally, the third case is $r_1\geq r_2>|z_1-z_2|$. In this case the intersection of the apartments is empty so the intersection pairing of the apartments is zero, and since $|z_1-a_1|=|z_1-a_2|$ and $|z_2-a_1|=|z_2-a_2|$, the valuation of the cross ratio vanishes as well. □

Corollary 3.3.2. *Let $\mathcal{L}\subset\mathbb{P}^1_K(K)$ be a compact set with at least two points. If a_1,a_2,z_1,z_2 are in $\mathcal{L}^*$ or even in ${\mathbb{P}^1_K}^*(K)$, the pairing can be done in $\mathcal{T}_K$,*

$$v_K\left(\left(\begin{array}{c}a_1:z_1\\a_2:z_2\end{array}\right)\right)=\left(\mathbb{A}_{\{a_1,a_2\}},\mathbb{A}_{\{z_1,z_2\}}\right)_{\mathcal{T}_K}$$

while if $a_1,a_2\in\mathcal{L}^*$ and $z_1,z_2\in\Omega_{\mathcal{L}}(\mathbb{C}_K)$, we may restrict to $\mathcal{T}_K(\mathcal{L})$:

$$v_K\left(\left(\begin{array}{c}a_1:z_1\\a_2:z_2\end{array}\right)\right)=\left(\mathbb{A}_{\{a_1,a_2\}},P\big(\mathrm{r}_{\mathcal{L}}(z_1),\mathrm{r}_{\mathcal{L}}(z_2)\big)\right)_{\mathcal{T}_K(\mathcal{L})}.$$

3.4 Multiplicative Integrals

The following definition was introduced by Longhi in [Lon02] as a generalization of the one given by Darmon in [Dar01].

Definition 3.4.1. *Let X be a compact space such that the compact open subsets form a basis for the topology, and let G be a complete topological abelian group (written multiplicatively). Let $f : X \longrightarrow G$ be a continuous function and let $\mu \in \mathscr{M}(X, \mathbb{Z})$ be a $\mathbb{Z}$-valued measure on X. The multiplicative integral of f with respect to μ is defined as*

$$\times\!\!\!\!\!\!\int_X f d\mu := \times\!\!\!\!\!\!\int_X f(t) d\mu(t) := \lim_{\overrightarrow{\mathcal{C}_\alpha}} \prod_{\substack{\mathcal{U}_n^\alpha \in \mathcal{C}_\alpha \\ t_n^\alpha \in \mathcal{U}_n^\alpha}} f(t_n^\alpha)^{\mu(\mathcal{U}_n^\alpha)}$$

where the limit is taken over the direct system of finite covers $\mathcal{C}_\alpha = \mathcal{C}_\alpha(X)$ of X by disjoint open compact subsets $\mathcal{U}_n^\alpha$, and the t_n^α are arbitrary points in them.

Proposition 3.4.2. *If G has a basic system of neighbourhoods of the identity consisting of open subgroups the integral is well defined, since the limit exists and it does not depend on the choice of the t_n^α's.*

Proof. Look at [Lon02, Prop. 5] or at our proof of lemma 2.3.1. □

Proposition 3.4.3. *For any measure $\mu \in \mathscr{M}(X, \mathbb{Z})$, we have*

1. *For any compact open subset U of X, and for any $\gamma \in G$, denote by $\chi_{U,\gamma}(t)$ the function mapping $x \in X$ to γ if $x \in U$, and to 1 otherwise. Then $\times\!\!\!\!\!\!\int_X \chi_{U,\gamma} d\mu = \gamma^{\mu(U)}$.*
2. *If $f, g : X \longrightarrow G$ are continuous functions on X such that the corresponding integrals exist, then*

$$\times\!\!\!\!\!\!\int_X (f \cdot g) d\mu = \left(\times\!\!\!\!\!\!\int_X f d\mu \right) \left(\times\!\!\!\!\!\!\int_X g d\mu \right)$$

Note that for any harmonic measure $\mu \in \mathscr{M}(X, \mathbb{Z})_0$ and for any constant function $f : X \longrightarrow G$ such that $f(x) = \lambda$ for all $x \in X$, we have $\times\!\!\!\!\!\!\int_X f d\mu = 1$.

Now, let $\mathcal{L}$ be a compact subset of $\mathbb{P}^1(K)$ with at least two points and let $L|K$ be an arbitrary complete extension of fields. We get from them the set $\mathcal{L}^* \subset \mathbb{P}^{1^*}(K)$, the space $\Omega_{\mathcal{L}}(L)$ and the tree $\mathcal{T}_K(\mathcal{L})$. With these objects we give the next definitions and lemmas.

Definition 3.4.4. *Let $\mathcal{P}$ be a finite set of points in $\Omega_{\mathcal{L}}(L)$, and consider $D := \sum_{p\in\mathcal{P}} m_p p$ a divisor of degree zero. We denote by f_D the element of* $\mathrm{Maps}(\mathcal{L}, L^*)/L^*$ *which is defined up to scalars as follows: if we choose representatives $v_p \in (L^2)^*$ for any $p \in \mathcal{P}$ and $v_q \in K^2$ for q, then*

$$f_D(q) := \prod_{p\in\mathcal{P}} v_p(v_q)^{m_p}$$

does not depend on v_q. Any other election of the vectors v_p change f_D to λf_D for some $\lambda \in L^*$.

Similarly, let $\mathcal{A}$ be a finite set of points in $\overline{\mathcal{T}_L}$, and consider the degree zero divisor $D := \sum_{[\alpha]\in\mathcal{A}} m_{[\alpha]}[\alpha]$, then we denote by $|f|_D$ the element of $\mathrm{Maps}(\mathcal{L}, \mathbb{R}_{>0})/\mathbb{R}^*_{>0}$ *being defined up to scalars by*

$$|f|_D(q) = \prod_{\alpha\in\mathcal{A}} \alpha(q)^{m_{[\alpha]}}$$

(remind that the points $[\alpha]$ are classes modulo homothety of diagonalizable seminorms α).

We note that we will be flexible when using these notations, not making difference between the map and the class of the map.

We note also that any representant of f_D can be seen as a map which extends to a meromorphic function on $\mathbb{P}^1$ with divisor D.

Remark 3.4.5. *Given divisors D, D' with the suitable support we have the equalities $f_{D+D'} = f_D f_{D'}$ and $f_{-D} = f_D^{-1}$, or also $|f|_{D+D'} = |f|_D |f|_{D'}$ and $|f|_{-D} = |f|_D^{-1}$.*

In particular, for any points $p, p', p'' \in \Omega_{\mathcal{L}}$ and $\alpha, \alpha', \alpha'' \in \mathcal{T}_L$ we have

$$f_{p'-p} = f_{p'-p''} f_{p''-p} \text{ and } |f|_{\alpha'-\alpha} = |f|_{\alpha'-\alpha''} |f|_{\alpha''-\alpha}.$$

Remark 3.4.6. *We can see the degree zero divisor 0 as the divisor $0p$ for any $p \in \Omega_{\mathcal{L}}(L)$. Therefore, as $m_p = 0$, we get $f_0 \equiv 1$ and $|f|_0 \equiv 1$.*

As a particular case, if we consider the divisor $D := \alpha(x, s) - \alpha(x, r)$ in $\mathcal{T}_K(\mathcal{L})$, where $s > r$, then we have

$$|f|_D(q) = \begin{cases} \frac{s}{r} & \text{if } q \in B(x^*, r) \\ \frac{s}{|q-x^*|} & \text{if } q \in B(x^*, s) \setminus B(x^*, r) \\ 1 & \text{if } q \notin B(x^*, s) \end{cases}$$

for any $q \in \mathcal{L}$.

Observe that, if the path from $\alpha(x, r)$ to $\alpha(x, s)$ is a topological edge, then $\mathcal{L}^* \cap (B(x, s - \epsilon) \setminus B(x, r))$ is empty for any $s - r > \epsilon > 0$ (and so the corresponding intersection with $\mathcal{L}$), and then $|f|_D(q) = 1$ or $\frac{s}{r}$ for any $q \in \mathcal{L}$. Moreover, by the remark 3.4.5, any $|f|_D$ is determined by the divisors of this type.

Proposition 3.4.7. *For any degree zero divisor $D \in \mathbb{Z}[\mathcal{T}_K(\mathcal{L})]_0$ we have the equality of maps*

$$-\log |f|_D(x) = f_D(\varepsilon(x^*))$$

where f_D is the map on the ends of the tree $\mathcal{T}_K(\mathcal{L})$ in definition 2.3.5 (there is not ambiguity since the other map f_D given in definition 3.4.4 has no sense for divisors D on the tree), $x \in \mathcal{L}$ and $\varepsilon(x^*)$ is the corresponding point seen as an end of $\mathcal{T}_K(\mathcal{L})$.

Proof. Let $\alpha' := \alpha(x, s)$ and $\alpha(x, r)$ with $r < s$ and assume that $P(\alpha, \alpha')$ is a topological edge. We have $B(x^*, r) \cap \mathcal{L}^* = \mathcal{B}(\alpha', \alpha)$ and $\mathcal{L}^* \setminus B(x^*, s) = \mathcal{B}(\alpha, \alpha')$. Since $|f|_D$ is well defined up to scalars, after multiply by $\left(\frac{s}{r}\right)^{-\frac{1}{2}}$, we also have

$$|f|_D(q) = \begin{cases} \left(\frac{s}{r}\right)^{\frac{1}{2}} & \text{if } q \in \mathcal{B}(\alpha', \alpha) \\ \left(\frac{s}{r}\right)^{-\frac{1}{2}} & \text{if } q \in \mathcal{B}(\alpha, \alpha') \end{cases}$$

Then we get

$$-\log |f|_D(q) = \begin{cases} -\frac{1}{2}d(\alpha, \alpha') & \text{if } q \in \mathcal{B}(\alpha', \alpha) \\ \frac{1}{2}d(\alpha, \alpha') & \text{if } q \in \mathcal{B}(\alpha, \alpha') \end{cases}$$

By the remark 3.4.5, this becomes true for any divisor $D = \alpha' - \alpha$ such that $P(\alpha, \alpha')$ is a topological edge.

Then, when $P(\alpha, \alpha')$ is an edge we get

$$-\log |f|_{\alpha'-\alpha}(q) = f_{\alpha'-\alpha}(\varepsilon(q^*)).$$

Therefore, remark 3.4.5 and corollary 2.3.9 imply that for any degree zero divisor $D \in \mathbb{Z}[\mathcal{T}_K(\mathcal{L})]_0$ we get the equality

$$-\log |f|_D(q) = f_D(\varepsilon(q^*)),$$

as we desired. □

Lemma 3.4.8. *Let $\mathcal{A}$ be a finite set of points in $\overline{\mathcal{T}_K}$, let $D := \sum_{\alpha \in \mathcal{A}} m_\alpha \alpha$ be a degree zero divisor and consider its retraction $\mathrm{r}_\mathcal{L}(D) := \sum_{\alpha \in \mathcal{A}} m_\alpha \, \mathrm{r}_\mathcal{L}(\alpha)$. Then $|f|_D = |f|_{\mathrm{r}_\mathcal{L}(D)}$ in $\mathrm{Maps}(\mathcal{L}, \mathbb{R}_{>0})/\mathbb{R}^*_{>0}$.*

Proof. First of all, observe that in the case $\mathcal{L}^* = \{y_0, y_1\}$ this is a consequence of lemma 3.2.5.

Now we do the general case. Fix $x \in \mathcal{L}^*$ and consider any point $y \in \mathcal{L}^*$, $x \neq y$. Take $\mathcal{L}' := \{y^*, x^*\} \subset \mathcal{L}$. Using the previous case twice (and taking some representatives) we get that

$$|f|_{D_{|\mathcal{L}'}} = |f|_{\mathrm{r}_{\mathcal{L}'}(D)_{|\mathcal{L}'}} = |f|_{\mathrm{r}_{\mathcal{L}'}(\mathrm{r}_{\mathcal{L}}(D))_{|\mathcal{L}'}} = |f|_{\mathrm{r}_{\mathcal{L}}(D)_{|\mathcal{L}'}}$$

by applying lemma 3.2.4. Since this equality is satisfied for all $\mathcal{L}'^*$ with x fixed, it is satisfied for $\mathcal{L}^*$ too (if we looked to the maps representing these classes modulo homothety, it would appear some scalar at the end of the equality which would not depend on $\mathcal{L}'^*$ or on y due to the fixed x). □

Definition 3.4.9. *Given any degree 0 divisor $D = \sum_{i\in I} m_i p_i$ with support in $\Omega_{\mathcal{L}}(L)$ (i.e. $m_i \in \mathbb{Z}$, $p_i \in \Omega_{\mathcal{L}}(L)$, being I a finite set and with $\sum_{i\in I} m_i = 0$) we choose v_i in $(L^2)^*$ representatives of the $p_i \in \mathbb{P}^{1^*}(L)$ and consider the map up to scalars $f_D \in \mathrm{Maps}(\mathcal{L}, L^*)/K^*$ given by a representant $\prod_{i\in I} v_i(x)^{m_i}$ (which depend on the v_i's). Let $\mu \in \mathscr{M}(\mathcal{L}, \mathbb{Z})_0$ be a $\mathbb{Z}$-valued harmonic measure on $\mathcal{L}$.*

We define

$$\times\!\!\!\!\!\!\int_{\mathcal{L},D} d\mu := \times\!\!\!\!\!\!\int_{\mathcal{L}} f_D d\mu \in L^*,$$

which is well defined since the integral does not depend on f_D but only on D (and L^* satisfies the hypothesis of proposition 3.4.2). Indeed, although the representant of f_D depend on the elections of the representatives in $(L^2)^*$ of the points in $\mathbb{P}^{1^*}(L)$, the multiplicative integral does not, since the measure is harmonic.

In general, when some $\mathcal{L}$ was fixed previously -as along this section-, we will omit its corresponding set, writing

$$\times\!\!\!\!\!\!\int_{D} d\mu := \times\!\!\!\!\!\!\int_{\mathcal{L},D} d\mu,$$

meanwhile we will specify the other sets over which we will integrate.

Note also that when $D = 0$, we have $\times\!\!\!\!\!\!\int_{0} d\mu = 1$, since $f_0 \equiv 1$.

Therefore, this definition gives us a morphism of groups

$$\begin{array}{ccc} \mathbb{Z}[\Omega_{\mathcal{L}}(L)]_0 & \xrightarrow{\times\!\!\!\!\!\int_{\bullet} d} & \mathrm{Hom}(\mathscr{M}(\mathcal{L}, \mathbb{Z})_0, L^*) \\ D & \longmapsto & \times\!\!\!\!\!\!\int_{D} d : \mu \longmapsto \times\!\!\!\!\!\!\int_{D} d\mu \end{array}$$

Lemma 3.4.10. *Let $\Gamma \subset \mathrm{PGL}_2(K)$ be a subgroup acting on $\mathcal{L}$ and so on $\mathcal{L}^*$. Then, the map $\times\!\!\!\!\!\!\int_{\bullet} d$ is Γ-equivariant.*

Proof. We want to see that $\times\!\!\!\!\!\!\int_{\gamma\cdot D} d = \gamma \cdot \times\!\!\!\!\!\!\int_{D} d$ for any $\gamma \in \Gamma$. That is to say that for any $\gamma \in \Gamma$ and $\mu \in \mathscr{M}(\mathcal{L}, \mathbb{Z})_0$ we have

$$\times\!\!\!\!\!\!\int_{\mathcal{L}} f_{\gamma D} d\mu = \times\!\!\!\!\!\!\int_{\gamma\cdot D} d\mu = \gamma \cdot \times\!\!\!\!\!\!\int_{D} d\mu = \times\!\!\!\!\!\!\int_{D} d(\gamma^{-1}\mu) = \times\!\!\!\!\!\!\int_{\mathcal{L}} f_D d(\gamma^{-1}\mu)$$

Let us to compute the first integral:

$$\times\!\!\!\!\!\!\int_{\mathcal{L}} f_{\gamma D} d\mu = \varinjlim_{\mathcal{C}_\alpha(\mathcal{L})} \prod_{\substack{\mathcal{U}_n^\alpha \in \mathcal{C}_\alpha(\mathcal{L}) \\ t_n^\alpha \in \mathcal{U}_n^\alpha}} f_{\gamma D}(t_n^\alpha)^{\mu(\mathcal{U}_n^\alpha)} = \varinjlim_{\mathcal{C}_\alpha(\mathcal{L})} \prod_{\substack{\mathcal{U}_n^\alpha \in \mathcal{C}_\alpha(\mathcal{L}) \\ t_n^\alpha \in \mathcal{U}_n^\alpha}} (\gamma f_D)(t_n^\alpha)^{\mu(\mathcal{U}_n^\alpha)} =$$

$$= \varinjlim_{\mathcal{C}_\alpha(\mathcal{L})} \prod_{\substack{\mathcal{U}_n^\alpha \in \mathcal{C}_\alpha(\mathcal{L}) \\ t_n^\alpha \in \mathcal{U}_n^\alpha}} f_D(\gamma^{-1} t_n^\alpha)^{\mu(\mathcal{U}_n^\alpha)} = \varinjlim_{\mathcal{C}_\alpha(\mathcal{L})} \prod_{\substack{\mathcal{U}_n^\alpha \in \mathcal{C}_\alpha(\mathcal{L}) \\ t_n^\alpha \in \mathcal{U}_n^\alpha}} f_D(t_n^\alpha)^{\mu(\gamma \mathcal{U}_n^\alpha)} =$$

$$= \varinjlim_{\mathcal{C}_\alpha(\mathcal{L})} \prod_{\substack{\mathcal{U}_n^\alpha \in \mathcal{C}_\alpha(\mathcal{L}) \\ t_n^\alpha \in \mathcal{U}_n^\alpha}} f_D(t_n^\alpha)^{(\gamma^{-1}\mu)(\mathcal{U}_n^\alpha)} = \times\!\!\!\!\!\!\int_{\mathcal{L}} f_D d(\gamma^{-1}\mu)$$

Therefore we get the claimed compatibility of the action of Γ with the map. □

Definition 3.4.11. *Given any degree 0 divisor $D = \sum_{i\in I} m_i \alpha_i$ with support in $\mathcal{T}_K(\mathcal{L})$ (i.e. $m_i \in \mathbb{Z}$, $\alpha_i \in \mathcal{T}_K(\mathcal{L})$, with I a finite set and $\sum_{i\in I} m_i = 0$) consider the map up to scalars $|f|_D \in \mathrm{Maps}(\mathcal{L}, \mathbb{R}_{>0})/\mathbb{R}^*_{>0}$ given by a representant $\prod_{i\in I} \alpha_i(x)^{m_i}$. Let $\mu \in \mathscr{M}(\mathcal{L}, \mathbb{Z})_0$ be a $\mathbb{Z}$-valued harmonic measure on $\mathcal{L}^*$.*

We define

$$\left| \times\!\!\!\!\!\!\int_{\mathcal{L}} \right|_D d\mu := \times\!\!\!\!\!\!\int_{\mathcal{L}} |f|_D d\mu \in \mathbb{R}_{>0}$$

when the corresponding limit exist, since, as above, its value only depends on D, but not on the representant of $|f|_D$, because of the harmonicity of the measure.

We will follow the same rule that above with respect to $\mathcal{L}$, omiting it when it is a given fixed set and specifying only in case of need:

$$\left| \times\!\!\!\!\!\!\int \right|_D d\mu := \left| \times\!\!\!\!\!\!\int_{\mathcal{L}} \right|_D d\mu.$$

Remark 3.4.12. *We cannot claim yet the existence of the integral since we cannot apply proposition 3.4.2, but we are going to prove this after the next remark.*

Remark 3.4.13. *From this definition and the proposition 3.4.7 we get*

$$-\log\left|\times\!\!\!\!\!\!\int\right|_D d\mu = \int_D d\mu$$

through the homeomorphisms $\mathcal{L} \cong \mathcal{L}^* \cong \mathcal{E}(\mathcal{T}_K(\mathcal{L}))$ inducing the isomorphism between measures $\mathscr{M}(\mathcal{L},\mathbb{Z})_0 \cong \mathscr{M}(\mathcal{E}(\mathcal{T}_K(\mathcal{L})),\mathbb{Z})_0$, which we identify denoting μ in both sides.

Remark 3.4.14. *As we anticipated in the remark 3.4.12, the latter has as a consequence the existence of*

$$\left|\times\!\!\!\!\!\!\int\right|_D d\mu$$

since

$$\int_D d\mu$$

exists, as we proved through the section 2.3, and the logarithm is a bijection.

Lemma 3.4.15. *Let $\mathcal{P}$ be a finite set of points in $\Omega_{\mathcal{L}}(L)$, and consider a degree zero divisor $D := \sum_{p\in\mathcal{P}} m_p p$. Denote by $\alpha_D := \sum_{p\in\mathcal{P}} m_p\alpha_p$, where α_p is the seminorm associated to p. Then $|f_D| = |f|_{\alpha_D}$ in $\mathrm{Maps}(\mathcal{L},\mathbb{R}_{>0})/\mathbb{R}^*_{>0}$.*

Proof. Take $q \in \mathcal{L}$ and representatives as in the definition 3.4.4. For the sake of simplicity we will assume all the points p and q are non infinite (then we can choose $v_q = (q,1)$ and $v_p = (1,-p)$).

$$|f_D|(q) = |f_D(q)| = |\prod_{p\in\mathcal{P}} v_p(v_q)^{m_p}| = \prod_{p\in\mathcal{P}} |q-p|^{m_p} = |\prod_{p\in\mathcal{P}} \alpha_p(q)| = |f|_{\alpha_D}(q)$$

by having into account for the fourth equality the remark 3.1.1. □

Proposition 3.4.16. *For any degree 0 divisor $D = \sum_{i\in I} m_i p_i$ with support in $\Omega_{\mathcal{L}}(L)$, consider the divisor $\mathrm{r}_{\mathcal{L}}(D) := \sum_{i\in I} m_i\,\mathrm{r}_{\mathcal{L}}(p_i)$ on $\mathcal{T}_K(\mathcal{L})$. Then, for any $\mathbb{Z}$-valued harmonic measure $\mu \in \mathscr{M}(\mathcal{L},\mathbb{Z})_0$ on $\mathcal{L}$, we have*

$$\left|\times\!\!\!\!\!\!\int_D d\mu\right| = \left|\times\!\!\!\!\!\!\int\right|_{\mathrm{r}_{\mathcal{L}}(D)} d\mu.$$

Proof. Applying the lemmas 3.4.15 and 3.4.8 we obtain

$$\left|⨍_D d\mu\right| = ⨍_{\mathcal{L}} |f_D| d\mu = ⨍_{\mathcal{L}} |f|_{\alpha_D} d\mu = ⨍_{\mathcal{L}} |f|_{\mathrm{r}_{\mathcal{L}}(D)} d\mu = \left|⨍\right|_{\mathrm{r}_{\mathcal{L}}(D)} d\mu$$

□

Lemma 3.4.17. *Given $x \in \mathcal{L}^*$, for any two points $\alpha(x,r), \alpha(x,s) \in \mathcal{T}_K(\mathcal{L})$, with $s > r$, such that the path $P(\alpha(x,r), \alpha(x,s))$ is a topological edge, then*

$$\left|⨍\right|_{\alpha(x,s)-\alpha(x,r)} d\mu = \left(\frac{s}{r}\right)^{\mu(B(x^*,r)\cap\mathcal{L})}$$

Proof. We have

$$|f|_D(q) = \begin{cases} \frac{s}{r} & \text{if } q \in B(x^*,r) \\ 1 & \text{if } q \notin B(x^*,r) \end{cases}$$

and these are the only two possibilities. Hence $|f|_D(q) = \chi_{U,\frac{s}{r}}$ for the open set $U = B(x^*,r)$ in the notation of Proposition 3.4.3.

Now, if we denote by $D = \alpha(x,s) - \alpha(x,r)$, and by applying Proposition 3.4.3, we get

$$\left|⨍\right|_D d\mu = ⨍_{\mathcal{L}} |f|_D d\mu = ⨍_{\mathcal{L}} \chi_{U,\frac{s}{r}} d\mu = \left(\frac{s}{r}\right)^{\mu(B(x^*,r)\cap\mathcal{L})}.$$

□

Proposition 3.4.18. *For any $\alpha, \alpha' \in \mathcal{T}_K(\mathcal{L})$ such that $P(\alpha,\alpha')$ is a topological edge, then*

$$v_K\left(⨍_{\alpha'-\alpha} d\mu\right) = -\log\left|⨍\right|_{\alpha'-\alpha} d\mu = d(\alpha,\alpha')\mu(\mathcal{B}(\alpha,\alpha')).$$

Proof. Proposition 3.4.16 gives us

$$v_K\left(⨍_{\alpha'-\alpha} d\mu\right) = -\log\left|⨍_{\alpha'-\alpha} d\mu\right| = -\log\left|⨍\right|_{\alpha'-\alpha} d\mu$$

and applying the remark 3.4.13 together with the lemma 2.3.11 we obtain

$$-\log\left|⨍\right|_{\alpha'-\alpha} d\mu = \int_{\alpha'-\alpha} d\mu = d(\alpha,\alpha')\mu(\mathcal{B}(\alpha,\alpha')).$$

□

Remark 3.4.19. *We may show this result in a more expressive way writing the topological edge as $\mathfrak{e}$ and defining its boundary $\partial\mathfrak{e}$ as the difference of its target minus its source -as usual in homology theory.*

Recall that by the theorem 2.2.9 we have $\mathscr{M}(\mathcal{L},\mathbb{Z})_0 \cong \mathrm{C}^1_{\mathrm{har}}(\mathcal{T}_K(\mathcal{L}),\mathbb{Z})$ in such a way that to each harmonic measure μ corresponds a harmonic cochain c_μ such that $c_\mu(\mathfrak{e}) = \mu(\mathcal{B}(\mathfrak{e}))$.

Thus, by abuse of notation we can write $\mu(\mathfrak{e}) = \mu(\mathcal{B}(\mathfrak{e}))$, and we will do.

Therefore, we may write the proposition as

$$v_K\left(\times\!\!\!\!\!\!\int_{\partial\mathfrak{e}}\right) = l(\mathfrak{e})\mu(\mathfrak{e}).$$

3.5 The Poisson Formula

In this section we will show in our context the Poisson formula made by Longhi in [Lon02, Thm. 6]. To show this, we recall and study in detail a map introduced by van der Put in [vdP92, Thm. 2.1], which assigns a harmonic measure to any invertible analytic function, and to which we will give later uses.

Let $\mathcal{L} \subset \mathbb{P}^1(K)$ be a compact set with at least two points and consider the abelian group of harmonic measures $\mathscr{M}(\mathcal{L},\mathbb{Z})_0$. For any two different points $a, b \in \mathcal{L}$ we define the harmonic measure $\mu_{a,b}$ by

$$\mu_{a,b}(\mathcal{U}) := \begin{cases} 1 \text{ if } a \in \mathcal{U},\ b \notin \mathcal{U} \\ -1 \text{ if } b \in \mathcal{U},\ a \notin \mathcal{U} \\ 0, \text{ otherwise} \end{cases}$$

In particular, on the open compact subsets $\mathcal{B}(e) \subset \mathcal{L}$, which determine the measure because of being a basis, we note that

$$\mu_{a,b}(e) := \begin{cases} 1 \text{ if } e \in P(b^*, a^*) \\ -1 \text{ if } e \in P(a^*, b^*) \\ 0, \text{ otherwise} \end{cases}$$

For any $a, b \in \mathcal{L}$ we take representatives $\tilde{a}, \tilde{b} \in K^2$ and for any complete extension $L|K$ we define the function $\omega_{\tilde{a}-\tilde{b}} : \Omega_{\mathcal{L}}(L) \longrightarrow L^*$ as

$$\omega_{\tilde{a}-\tilde{b}}(z) := \frac{\tilde{a}(z)}{\tilde{b}(z)} = \frac{z(\tilde{a})}{z(\tilde{b})}.$$

Note that identifying z with $(1, -z)$ or $(0, 1)$ if it is ∞, this is an analytic function on $\Omega_{\mathcal{L}}(L)$ depending on a, b up to a constant.

Let us write for any $p, q \in \mathcal{L}^*$, $u_{p,q}(z) := \omega_{\tilde{p}^* - \tilde{q}*}$ for suitable representants, so we can put

$$u_{p,q}(z) := \frac{z-p}{z-q}$$

where we consider the usual convention when some of the two points are ∞ ([GvdP80, Ch. 2(2.2)]), that is

$$u_{p,q}(z) := \begin{cases} 1 \text{ if } p = q = \infty \\ z - p \text{ if } p \neq \infty = q \\ \dfrac{1}{z-q} \text{ if } p = \infty \neq q \end{cases}$$

On the other hand, let us recall part of the definition 3.4.9. For any degree 0 divisor $D = \sum_{i \in I} m_i p_i$ with support in $\Omega_{\mathcal{L}}(L)$ we could build as above a map up to scalars $f_D \in \text{Maps}(\mathcal{L}, L^*)/L^*$. Let us fix an element $b_0 \in \mathcal{L}$. Along this section we will choose a representant of f_D satisfying $f_D(b_0) = 1$, so f_D will be a well defined function.

We write the usual notation $\mathcal{O}(\Omega_{\mathcal{L}})$ for the analytic functions on the analytic space $\Omega_{\mathcal{L}} := (\mathbb{P}^{1\,*}_K)^{an} \setminus \mathcal{L}^*$, and we write $\mathcal{O}(\Omega_{\mathcal{L}})^*$ for the ones which vanish nowhere. Then we have $\omega_{\tilde{a}-\tilde{b}} \in \mathcal{O}(\Omega_{\mathcal{L}})^*$.

Let $\mathfrak{e}$ be a topological edge of $\mathcal{T}_K(\mathcal{L})$ induced by a path $P(\alpha(x,r), \alpha(x,s))$ with $x \in \mathcal{L}^*$ and $r \leq s$. Then we define the (closed) annulus associated to $\mathfrak{e}$ as $R(\mathfrak{e}) := R_x(r,s) := B(x,s) \setminus \mathring{B}(x,r)$, and the open annulus associated to $\mathfrak{e}$ as $\mathring{R}(\mathfrak{e}) := \mathring{R}_x(r,s) := \mathring{B}(x,s) \setminus B(x,r)$.

We recall the following result from [Thu05, Lem. 2.2.1.].

Lemma 3.5.1. *Given $x \in \mathcal{L}^*$, and any two points $\alpha(x,r), \alpha(x,s) \in \mathcal{T}_K(\mathcal{L})$, with $r < s$, such that the path $P(\alpha(x,r), \alpha(x,s))$ is a topological edge (i.e. $\mathring{R}_x(r,s) \cap \mathcal{L}^* = \emptyset$), for any $\omega \in \mathcal{O}(\Omega_{\mathcal{L}})^*$ there exists $k \in \mathbb{Z}$ such that for any interior path $P' = P(\alpha(x,r'), \alpha(x,s')) \subset P(\alpha(x,r), \alpha(x,s))$ $(r \leq r' \leq s' \leq s)$ satisfying $R_x(r',s') \cap \mathcal{L}^* = \emptyset$, the function $|\omega(z)(z-x)^{-k}|$ is constant on $R_x(r',s')$.*

Proof. For any $0 < r' \leq s'$ let us consider $R_x(r',s')^{an}$, the Berkovich analytic annulus associated to $R_x(r',s')$. Now we can assume without any problem that $x = 0$. Then, we have the isomorphism

$$\mathcal{O}(R_0(r',s')^{an}) \cong K\langle r'T^{-1}, s'^{-1}T\rangle$$

where

$$K\langle r'T^{-1}, s'^{-1}T\rangle =$$

$$= \left\{ \sum_{n=-\infty}^{\infty} a_n T^n : \ |a_n| r'^n \to 0 \text{ as } n \to -\infty, \ |a_n| s'^n \to 0 \text{ as } n \to \infty \right\}$$

We will prove first the case $r' = s' = 1$. We have $\omega \in \mathcal{O}(\Omega_{\mathcal{L}})^*$ and then the restriction of ω is a unit in $K\langle T^{-1}, T\rangle$. Such an element can be expressed as $\omega = c \cdot \omega_1$ for $c \in K^*$ such that $\|\omega\|_{R_0(1,1)} = |c|$ and $\omega_1 \in \mathcal{O}_K\langle T^{-1}, T\rangle^*$. Therefore, the reduction of ω_1 to $k[T^{-1}, T]^*$ is also invertible so it has the form bT^n for $b \in k$, $n \in \mathbb{Z}$, and so we deduce that we can write $\omega_1 = \tilde{b}T^n + \omega_2 = \tilde{b}T^n(1 + \omega_2')$ with $\tilde{b} \in \mathcal{O}_K^*$, $\omega_2 \in \mathfrak{m}_K\langle T^{-1}, T\rangle$, $\omega_2' = \tilde{b}^{-1}T^{-n}\omega_2$ and $\|\omega_2'\|_{R_0(1,1)} = \|\omega_2\|_{R_0(1,1)} < 1$, so that

$$|\omega(z)z^{-n}| = |c\tilde{b}||1 + \omega_2'(z)| = |c\tilde{b}|.$$

Observe that writing $\omega = \sum_{n\in\mathbb{Z}} a_n T^n$ the supremum norm can be expressed by $\|\omega\|_{R_0(1,1)} := \max\{|a_m|\}$ and this is reached at just one m, which is n.

From now on we consider the case $r' < s'$. Now ω is a unit $\sum_{n\in\mathbb{Z}} a_n T^n$ in $K\langle r'T^{-1}, s'^{-1}T\rangle$, so for any $r'' \in [r', s']$, the image of ω by the restriction homomorphism $K\langle r'T^{-1}s'^{-1}, T\rangle \longrightarrow K\langle r''T^{-1}, r''^{-1}T\rangle$ is also a unit. Next note that after a non archimedean extension $K'|K$ we have $r'' \in |K'^*|$ so there is an isomorphism $K'\langle r''T^{-1}, r''^{-1}T\rangle \cong K'\langle T^{-1}, T\rangle$. □

Definition 3.5.2. *We say that a sequence of functions $(\omega_n)_n$ in $\mathcal{O}(\Omega_{\mathcal{L}})^*$ converge uniformly to a function $\omega \in \mathcal{O}(\Omega_{\mathcal{L}})^*$ if for each edge e of $\mathcal{T}_K(\mathcal{L})$ and for all $\epsilon > 0$ there exists an $n_0 = n(e, \epsilon)$ such that for any $N \geq n_0$ we have $\|\omega - \omega_N\|_{R(|e|)} < \epsilon$ (recall that $|e|$ means the topological realization of e).*

We will write $\lim_{N\to\infty} \omega_N = \omega$.

Theorem 3.5.3. *There exists a morphism $\tilde{\mu} : \mathcal{O}(\Omega_{\mathcal{L}})^* \longrightarrow \mathscr{M}(\mathcal{L}, \mathbb{Z})_0$ with kernel K^* and such that commutes with limits in the following sense: if $\lim_{N\to\infty} \omega_N = \omega$, then $\tilde{\mu}(\omega) = \lim_{N\to\infty} \tilde{\mu}(\omega_N)$.*

Proof. Let us consider $\omega \in \mathcal{O}(\Omega_{\mathcal{L}})^*$. We have to define $\tilde{\mu}(\omega)$ over each (directed) edge e of $\mathcal{T}_K(\mathcal{L})$. By the proposition 2.1.16 we may assume that $|e|$ or $|\overline{e}|$ is contained in a topological edge given by $P(\alpha(x, r), \alpha(x, s))$ with $r < s$ and $x \in \mathcal{L}^* \cap K$. Depending on if this happens with e or $\overline{e}$, we define $\tilde{\mu}(\omega)(e) := k$ or $\tilde{\mu}(\omega)(e) := -k$ respectively, where k is the integer obtained in the above lemma. Henceforth we will work on this edge to prove its properties.

First, $\tilde{\mu}(\omega)$ is a harmonic measure because of the definition and the residue theorem ([FvdP04, Thm. 2.3.3 (2)]).

From the way we have defined the map $\tilde{\mu}$ it is clear that it is a morphism and that K^* is inside its kernel. From the definition of $\tilde{\mu}$, the fact that $\Omega_{\mathcal{L}}$ is connected implies that if $\tilde{\mu}(\omega) = 0$, then the absolute value of ω is a constant, and since bounded analytic functions on $\Omega_{\mathcal{L}}$ are constant ([GvdP80, Ch. 4 Cor. (2.5)]), we get $\mathrm{Ker}(\tilde{\mu}) = K^*$.

And now let us see the commutativity with limits in the sense we told. We want to check the equality $\tilde{\mu}(\omega)(e) = \lim_{N\to\infty} \tilde{\mu}(\omega_N)(e)$ for any edge e that we can take as above.

We know by hypothesis that for any $\epsilon > 0$ there exists an $n_0 = n(e,\epsilon)$ such that for any $N \geq n_0$ we have $\|\omega - \omega_N\|_{R_x(r,s)} < \epsilon$. Note that if we just take $\epsilon = \inf_{z\in R_x(r,s)} \{|\omega(z)|\}$, which is strictly positive since $R_x(r,s)$ is compact, then for any $z \in R_x(r,s)$ we get $|\omega(z) - \omega_N(z)| < |\omega(z)|$ and so $|\omega_N(z)| = |\omega(z)|$, therefore $\tilde{\mu}(\omega_N)(e) = \tilde{\mu}(\omega)(e)$. $\square$

Proposition 3.5.4. *The morphism* $\tilde{\mu} : \mathcal{O}(\Omega_{\mathcal{L}})^* \longrightarrow \mathscr{M}(\mathcal{L},\mathbb{Z})_0$ *satisfies the following properties:*

1. *For any two different points* $a, b \in \mathcal{L}$,

$$\tilde{\mu}\left(\omega_{\tilde{a}-\tilde{b}}\right) = \mu_{b,a}$$

independently of the chosen representants of a *and* b*. In particular, for any* $p, q \in \mathcal{L}^*$ *we have* $\tilde{\mu}\left(u_{p,q}\right) = \mu_{q^*,p^*}$.

2. *It is natural with the meaning that if* $\mathcal{L} \subset \mathcal{L}'$ *are both compacts, it commutes with restriction maps:*

$$\begin{array}{ccc} \mathcal{O}(\Omega_{\mathcal{L}})^* & \xrightarrow{\tilde{\mu}} & \mathscr{M}(\mathcal{L},\mathbb{Z})_0 \\ \downarrow & & \downarrow \\ \mathcal{O}(\Omega_{\mathcal{L}'})^* & \xrightarrow{\tilde{\mu}} & \mathscr{M}(\mathcal{L}',\mathbb{Z})_0 \end{array}$$

In particular it does not depend on $\mathcal{L}$*, since given any compacts* $\mathcal{L}_1,\ \mathcal{L}_2$*, the definition coincides in* $\mathcal{L}_1 \cap \mathcal{L}_2$.

3. *It commutes with the action of* $\mathrm{PGL}_2(K)$*, that is, for each* $\gamma \in \mathrm{PGL}_2(K)$ *the diagram*

$$\begin{array}{ccc} \mathcal{O}(\Omega_{\mathcal{L}})^* & \xrightarrow{\tilde{\mu}} & \mathscr{M}(\mathcal{L},\mathbb{Z})_0 \\ \downarrow{\scriptstyle\gamma_*} & & \downarrow{\scriptstyle\gamma_*} \\ \mathcal{O}(\Omega_{\gamma\mathcal{L}})^* & \xrightarrow{\tilde{\mu}} & \mathscr{M}(\gamma\mathcal{L},\mathbb{Z})_0 \end{array}$$

is commutative, where $\gamma_*(\omega) = \gamma \cdot \omega$ *and* $\gamma_*(\mu) = \gamma \cdot \mu$*. (Note that* $\Omega_{\gamma\mathcal{L}} = \gamma\Omega_{\mathcal{L}}$*.)*

Proof. First, we want to see $\tilde{\mu}\left(\omega_{\tilde{a}-\tilde{b}}\right)(e) = \mu_{b,a}(e)$.

If $a, b \in \mathcal{B}(e) = \mathcal{L} \setminus \mathring{B}(x^*, s)$, for $z \in R_x(r, s)$ we have

$$|\omega_{\tilde{a}-\tilde{b}}(z)| = \left|\frac{z(\tilde{a})}{z(\tilde{b})}\right| = \frac{|x - a^*|}{|x - b^*|}$$

(taking into account the above convention if a or b are ∞), which is a constant, so $\tilde{\mu}(\omega_{\tilde{a}-\tilde{b}})(e) = 0 = \mu_{b,a}(e)$.

If $a, b \in \mathcal{B}(\overline{e}) = \mathcal{L} \cap B(x^*, r)$, $z \in R_x(r, s)$ verifies

$$|z(\tilde{a})| = |z - a^*| = |z - x| = |z - b^*| = |z(\tilde{b})|,$$

so that we also get a constant ($|\omega_{\tilde{a}-\tilde{b}|R_x(r,s)}| \equiv 1$) and the equality as above.

Finally, assuming $a \in \mathcal{B}(e) = \mathcal{L} \setminus \mathring{B}(x^*, s),\ b \in \mathcal{B}(\overline{e}) = \mathcal{L} \cap B(x^*, r)$, then

$$|\omega_{\tilde{a}-\tilde{b}}(z)| = \left|\frac{z(\tilde{a})}{z(\tilde{b})}\right| = \frac{|x - a^*|}{|z - b^*|} = \frac{|x - a^*|}{|x - z|}\frac{|x - z|}{|z - b^*|} = |x - a^*| \cdot |z(x^*)|^{-1},$$

therefore $\tilde{\mu}(\omega_{\tilde{a}-\tilde{b}})(e) = -1 = \mu_{b,a}(e)$ (once more, one should consider the case in which a is ∞, but we would get a similar result).

Second, the naturality is a direct consequence of the definition of the $\tilde{\mu}$ through the above lemma.

The third property is equivalent to say $\gamma \cdot \tilde{\mu}(\omega)(e) = \tilde{\mu}(\gamma \cdot \omega)(e)$ for all $\omega \in \mathcal{O}(\Omega_{\mathcal{L}})^*$ and $e \in \mathcal{T}_K(\gamma \cdot \mathcal{L})$, and the left side of the equality is $\tilde{\mu}(\omega)(\gamma^{-1} \cdot e)$. Then, this also follows from the definition by means of the lemma and from the isomorphism $\gamma^* : \mathcal{O}(R(|e|)) \xrightarrow{\cong} \mathcal{O}(R(|\gamma^{-1}e|))$, by which $\gamma^*(\omega) = \gamma^{-1} \cdot \omega$. □

As Longhi remarks ([Lon02]), we may compute a multiplicative integral on $\mathcal{L}$ by means of fixing a vertex $v_0 \in \mathcal{T}_K(\mathcal{L})$ and defining $l_{v_0}(e)$ as the number of intermediate vertices between v_0 and e in a previously fixed model for $\mathcal{T}_K(\mathcal{L})$. Then we have

$$\oint_{\mathcal{L}} f d\mu = \lim_{n\to\infty} \prod_{\substack{l_{v_0}(e)=n \\ t_e \in \mathcal{B}(e)}} f(t_e)^{\mu(e)}$$

Theorem 3.5.5 (Poisson Formula)**.** *Let $u \in \mathcal{O}(\Omega_{\mathcal{L}})^*$ and $z_0 \in \Omega_{\mathcal{L}}$. Then, for any $z \in \Omega_{\mathcal{L}}$ the next identity is satisfied:*

$$\frac{u(z)}{u(z_0)} = \oint_{z-z_0} d\tilde{\mu}(u)$$

Proof. We follow the proof of [Lon02, Thm. 6].

The partial products

$$\prod_{\substack{l_{v_0}(e)=N \\ t_e\in\mathcal{B}(e)}} f_{z-z_0}\left(t_e\right)^{\tilde{\mu}(u)(e)}$$

converge uniformly on $\Omega_{\mathcal{L}}$ so the integral built with them is a nowhere vanishing analytic function of z. Since by the previous theorem the kernel of $\tilde{\mu}$ is K^*, in order to prove the identity it is enough to see that $\tilde{\mu}(u(z)) = \tilde{\mu}\left(\times\!\!\!\!\!\!\int_{\mathcal{L}} f_{z-z_0}(t)d\tilde{\mu}(u)(t)\right)$. Further, note that

$$f_{z-z_0}(t_e) = f_{z-z_0}(t_e)/f_{z-z_0}(b_0) = \frac{\tilde{z}(\tilde{t_e})}{\tilde{z}_0(\tilde{t_e})}\frac{\tilde{z}_0(\tilde{b_0})}{\tilde{z}(\tilde{b_0})} = \frac{\tilde{z}(\tilde{t_e})}{\tilde{z}(\tilde{b_0})}\frac{\tilde{z}_0(\tilde{b_0})}{\tilde{z}_0(\tilde{t_e})} = c\cdot\omega_{\tilde{t_e}-\tilde{b_0}}(z),$$

$$c \in K(\tilde{z}_0)^*$$

Therefore we have $\tilde{\mu}(f_{z-z_0}(t_e)) = \mu_{b_0,t_e}$ also by the previous theorem. Then, by the commutativity of $\tilde{\mu}$ and limits we obtain

$$\tilde{\mu}\left(\times\!\!\!\!\!\!\int_{\mathcal{L}^*} f_{z-z_0}(t)d\tilde{\mu}(u)(t)\right) = \tilde{\mu}\Big(\lim_{N\to\infty}\prod_{\substack{l_{v_0}(e)=N \\ t_e\in\mathcal{B}(e)}} f_{z-z_0}\left(t_e\right)^{\tilde{\mu}(u)(e)}\Big) =$$

$$= \lim_{N\to\infty}\sum_{\substack{l_{v_0}(e)=N \\ t_e\in\mathcal{B}(e)}} \tilde{\mu}(u)(e)\tilde{\mu}\left(f_{z-z_0}(t_e)\right) = \lim_{N\to\infty}\sum_{\substack{l_{v_0}(e)=N \\ t_e\in\mathcal{B}(e)}} \tilde{\mu}(u)(e)\mu_{b_0,t_e}$$

Let us evaluate on an edge e' of $\mathcal{T}_K(\mathcal{L})$. We may assume e' points away from b_0, so $b_0 \in \mathcal{B}(\overline{e'})$. We have $e' \subset P(b_0^*, t_e^*)$ if and only if $t_e \in \mathcal{B}(e')$, so we get

$$\tilde{\mu}\left(\times\!\!\!\!\!\!\int_{\mathcal{L}} f_{z-z_0}(t_e)d\tilde{\mu}(u)(t)\right)(e') = \lim_{N\to\infty}\sum_{\substack{l_{v_0}(e)=N \\ t_e\in\mathcal{B}(e)}} \tilde{\mu}(u)(e)\mu_{b_0,t_e}(e') =$$

$$= \lim_{N\to\infty}\sum_{\substack{l_{v_0}(e)=N \\ t_e\in\mathcal{B}(e)\cap\mathcal{B}(e')}} -\tilde{\mu}(u)(e) = \tilde{\mu}(u)(e')$$

where the last equality is due to harmonicity applied to the sum independent of $N \geq l_{v_0}(e')$. □

Corollary 3.5.6 (Extended Poisson Formula). *Take* $u \in \mathcal{O}(\Omega_{\mathcal{L}})^*$. *Then, given any degree 0 divisor* $D = \sum m_p p$ *of* $\Omega_{\mathcal{L}}$, *we have*

$$\prod_{p \in \mathrm{Supp}(D)} u(p)^{m_p} = \oint_D d\tilde{\mu}(u)$$

Corollary 3.5.7. *The morphism* $\tilde{\mu} : \mathcal{O}(\Omega_{\mathcal{L}})^* \longrightarrow \mathcal{M}(\mathcal{L}, \mathbb{Z})_0$ *is surjective and for each* $z_0 \in \Omega_{\mathcal{L}}$ *it has a section* $\mathcal{I}_{z_0} : \mathcal{M}(\mathcal{L}, \mathbb{Z})_0 \longrightarrow \mathcal{O}(\Omega_{\mathcal{L}})^*$. *As a consequence we get a (non-unique, non-canonical) isomorphism*

$$\mathcal{O}(\Omega_{\mathcal{L}})^* \cong K^* \times \mathcal{M}(\mathcal{L}, \mathbb{Z})_0.$$

Proof. Let us take a harmonic measure $\mu \in \mathcal{M}(\mathcal{L}, \mathbb{Z})_0$. Let $z_0 \in \Omega_{\mathcal{L}}$ be any point. Then, as along the proof of the Poisson formula, we see that the function

$$\mathcal{I}_{\mu, z_0}(z) := \oint_{z - z_0} d\mu$$

is analytic on $\Omega_{\mathcal{L}}$, and once more, the same steps with μ instead of $\tilde{\mu}(u)$ prove that $\tilde{\mu}(\mathcal{I}_{\mu, z_0}) = \mu$. Then, we define the section by $\mathcal{I}_{z_0}(\mu) := \mathcal{I}_{\mu, z_0}$ and we check that it is a morphism of groups:

$$\mathcal{I}_{z_0}(\mu + \mu')(z) = \oint_{z - z_0} d(\mu + \mu') = \oint_{z - z_0} d\mu \oint_{z - z_0} d\mu' = \left(\mathcal{I}_{z_0}(\mu)\mathcal{I}_{z_0}(\mu')\right)(z)$$

Finally, by theorem 3.5.3 we got the short exact sequence

$$0 \longrightarrow K^* \longrightarrow \mathcal{O}(\Omega_{\mathcal{L}})^* \longrightarrow \mathcal{M}(\mathcal{L}, \mathbb{Z})_0 \longrightarrow 0$$

which, with the section morphism, gives the asserted isomorphism by elementary homological algebra. □

3.6 Schottky groups and their limit sets

Along this section we recall Schottky groups and their main properties, and we build the Mumford curve, for which we want to give its Jacobian via the isomorphism with the Albanese variety, and its associated graph. The main novelty is the "Berkovich analytification" of some results in [GvdP80].

Given any $\gamma \in \mathrm{PGL}_2(K)$, we say that γ is hyperbolic if the (two) eigenvalues of γ have two distinct absolute values. In particular, hyperbolic elements have infinite order. We have

$$\gamma = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = \left[\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\right]$$

with $\lambda_i \in \overline{K}$ and

$$\left|\frac{(a+d)^2}{ad-bc}\right| = \left|\frac{(\lambda_1+\lambda_2)^2}{\lambda_1\lambda_2}\right|.$$

If there was $|\lambda_1| = |\lambda_2|$, we would have

$$\left|\frac{(\lambda_1+\lambda_2)^2}{\lambda_1\lambda_2}\right| \leq \frac{|\lambda_1|^2}{|\lambda_1|^2} = 1,$$

while in our case we may assume wlog $\max\{|\lambda_1|, |\lambda_2|\} = |\lambda_1|$ so we have $|\lambda_1 + \lambda_2| = |\lambda_1|$ and so

$$\left|\frac{(\lambda_1+\lambda_2)^2}{\lambda_1\lambda_2}\right| = \frac{|\lambda_1|^2}{|\lambda_1\lambda_2|} = \left|\frac{\lambda_1}{\lambda_2}\right| > 1.$$

so γ is hyperbolic if and only if that value is strictly greater than 1. We have even more: $\gamma \in \mathrm{PGL}_2(K)$ is hyperbolic if and only if it is conjugated to an element of $\mathrm{PGL}_2(\mathcal{O}_K)$ represented by a matrix

$$\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$$

with $q \in K$, $|q| < 1$ (look at [GvdP80, Ch. 1 Lem. I.1.4]). The idea is reduce the characteristic polynomial to k and apply the Hensel lemma, which we dispose of since our base field is complete non-Archimedean. In particular, the eigenvalues of any representant of γ are in K. From this we get that if γ is hyperbolic,

$$\{x \in \mathbb{P}^1(\mathbb{C}_K) |\ \gamma x = x\} \subset \mathbb{P}^1(K).$$

Given any subgroup $\Gamma \subset \mathrm{PGL}_2(K)$, we denote by $\mathcal{L}_\Gamma$ the closure in $\mathbb{P}^1(\mathbb{C}_K)$ of the set of fixed points for some element of Γ distinct of the identity and we call it the limit set of Γ (there is no risk of confussion with any other object appearing through this work).

$$\mathcal{L}_\Gamma := \overline{\{x \in \mathbb{P}^1(\mathbb{C}_K) |\ \exists\, \gamma \in \Gamma \setminus \{\mathbb{1}_\Gamma\} :\ \gamma x = x\}} \subset \mathbb{P}^1(\mathbb{C}_K)$$

But, from the previous remark we have $\mathcal{L}_\Gamma \subset \mathbb{P}^1(K)$. If $\gamma x = x$, then $\gamma'\gamma\gamma'^{-1}(\gamma' x) = \gamma' x$, therefore $\mathcal{L}_\Gamma$ is Γ-invariant, since the action of $\mathrm{PGL}_2(K)$ on $\mathbb{P}^1(\mathbb{C}_K)$ is continue.

Observe that $\mathcal{L}_\Gamma^*$ is the limit set of Γ in the dual projective line for the contragredient action, and that this set verifies the same properties that we have just mentioned.

Definition 3.6.1. *A Schottky group is a finite generated subgroup* $\Gamma \subset \mathrm{PGL}_2(K)$ *such that all its elements* $\gamma \neq \mathbb{1}_\Gamma$ *are hyperbolic and* $\overline{\Gamma p}$ *is compact for all* $p \in \mathbb{P}^1(\mathbb{C}_K)$.

Let $\gamma \in \mathrm{PGL}_2(K)$ be represented by

$$\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}.$$

with $|q| < 1$. Then, $\mathcal{L}_{\langle\gamma\rangle} = \{[1:0],[0:1]\} = \{\infty, 0\}$ is its set of fixed points, and for any $p \neq 0, \infty$,

$$\overline{\{\gamma^n p\}_{n\in\mathbb{Z}}} = \{0, \infty\} \bigsqcup \{\gamma^n p\}_{n\in\mathbb{Z}}$$

is compact. More specifically, we have

$$p_\gamma^+ := \lim_{n\to+\infty} \gamma^n p = 0 \quad \text{ and } \quad p_\gamma^- := \lim_{n\to-\infty} \gamma^n p = \infty.$$

Thus, $\Gamma = \langle\gamma\rangle$ is a Schottky group.

More generally, for any $\gamma' \in \mathrm{PGL}_2(K)$ hyperbolic, we have $\gamma' = \delta\gamma\delta^{-1}$ with γ as above. Thus, $\mathcal{L}_{\langle\gamma'\rangle} = \delta\mathcal{L}_{\langle\gamma\rangle}$ and $\overline{\langle\gamma'\rangle p} = \delta\overline{\langle\gamma\rangle\delta^{-1}p}$, and so, $\langle\gamma'\rangle$ is also a Schottky group.

Clearly, for $|q| \neq 1$ and $\delta \in \mathrm{PGL}_2(K)$, these are all the Schottky groups generated by one element.

Note one further thing. Consider an element $\alpha = \alpha(x,r)$ of the tree $\mathcal{T}_K$. Then one has

$$\gamma \cdot \alpha(x,r) = \alpha(qx, |q|r) \neq \alpha(x,r)$$

Thus, any hyperbolic element acts freely on $\mathcal{T}_K$.

Lemma 3.6.2. *For any $\gamma \in \Gamma$, $\mathcal{L}_\Gamma = \overline{\Gamma \cdot \mathcal{L}_{\langle\gamma\rangle}}$. In particular, it is compact.*

Proof. Since $\mathcal{L}_\Gamma$ is Γ-invariant and closed, we have $\overline{\Gamma \cdot \mathcal{L}_{\langle\gamma\rangle}} \subset \mathcal{L}_\Gamma$. As we can take closures, for the opposite inclusion it is enough to see that any fixed point p' for some element γ' of Γ is in $\overline{\Gamma \cdot \mathcal{L}_{\langle\gamma\rangle}}$. Indeed, we may assume that $p' \notin \mathcal{L}_{\langle\gamma\rangle}$. Then, one of the two points of $\mathcal{L}_{\langle\gamma\rangle}$ is not fixed by γ', let us write p_γ^ϵ. Then, $p' = \lim_{n\to\infty} \gamma' p_\gamma^\epsilon \in \overline{\Gamma \cdot \mathcal{L}_{\langle\gamma\rangle}}$, after taking the inverse of γ' if necessary. Thus, we conclude. □

Corollary 3.6.3. *If $\mathcal{L}_\Gamma$ has at least three points, it is perfect (it does not have isolated points), and in particular, an infinite compact set.*

Proof. The same proof as above applies here. □

Remark 3.6.4. *We could refine even deeper the results, as shown in [GvdP80, Ch. 1 (1.6)]) or also in the preprint [SX16, §5 and §6], but we already have all we need here.*

Since $\mathcal{L}_\Gamma$ is compact, we can consider the tree $\mathcal{T}_K(\mathcal{L}_\Gamma)$.

Proposition 3.6.5. *A Schottky group* Γ *acts freely on* $\mathcal{T}_K(\mathcal{L}_\Gamma)$ *(with the induced left action by* $\mathrm{PGL}_2(K)$ *on* $\mathcal{T}_K$*), and the quotient* $G_\Gamma := \Gamma\backslash\mathcal{T}_K(\mathcal{L}_\Gamma)$ *is a finite metric graph. Moreover, if* $\mathcal{L}' \subset \mathbb{P}^1(K)$ *is the union of* $\mathcal{L}_\Gamma$ *and a finite set of orbits of points by the action of* Γ*, then there exists a finite connected graph* $G_{\mathcal{L}'}$ *such that*

$$G_\Gamma \subset G_{\mathcal{L}'} \subset \Gamma\backslash\mathcal{T}_K(\mathcal{L}') \text{ and } (\Gamma\backslash\mathcal{T}_K(\mathcal{L}')) \setminus G_{\mathcal{L}'} = \bigsqcup_{\mathcal{R}_{\mathcal{L}'}} (0, +\infty)$$

where $\mathcal{R}_{\mathcal{L}'} = \Gamma\backslash(\mathcal{L}' \setminus \mathcal{L}_\Gamma)$ *is a finite set.*

Proof. The fact that Γ acts freely on $\mathcal{T}_K(\mathcal{L}_\Gamma)$ is a consequence of all its non-neutral elements are hyperbolic.

For the rest of the proof, we are inspired by the proof given in [GvdP80, Ch. 1 Lem. (3.2)]. Let B_Γ be a finite set of generators of Γ and their inverses containing the identity $\mathbb{1}_\Gamma$ too. Take $w \in \mathcal{T}_K(\mathcal{L}_\Gamma)$ and a finite subtree $\mathfrak{T}_w \subset \mathcal{T}_K(\mathcal{L}_\Gamma)$ containing $B_\Gamma \cdot w$. Then,

$$\mathfrak{T} = \bigcup_{\gamma\in\Gamma} \gamma \cdot \mathfrak{T}_w$$

is a subtree of $\mathcal{T}_K(\mathcal{L}_\Gamma)$. The only thing we have to verify is that it is connected, that is, given $\gamma, \gamma' \in \Gamma$ and $p \in \gamma \cdot \mathfrak{T}_w$, $p' \in \gamma' \cdot \mathfrak{T}_w$ there exists a path in $\mathfrak{T}$ between p and p'. Through operating by γ' on the path, we may suppose $\gamma' = \mathbb{1}_\Gamma$. Also, by an induction process it is enough to show this when $\gamma \in B_\Gamma$. So, with these hypotheses, we have p' and γw connected by a path in $\mathfrak{T}_w$, and γw and p connected by a path in $\gamma \cdot \mathfrak{T}_w$.

Now we will show $V(\mathfrak{T}) = V(\mathcal{T}_K(\mathcal{L}_\Gamma))$, from what we will get the finiteness of the quotient.

Let v be any vertex of $\mathcal{T}_K(\mathcal{L}_\Gamma)$ and consider a ray through v starting at w, whose end corresponds to a point of the limit set $z \in \mathcal{L}_\Gamma$ by proposition 3.1.15. Since $\mathcal{L}_\Gamma$ is the closure of the set of fixed points, we can take the ray corresponding to a fixed point z for some $\gamma \in \Gamma$. After considering the inverse of γ if necessary, for any $z_0 \in \mathbb{P}^1_K \setminus \mathcal{L}_\Gamma$, we have that $\lim_{n\to\infty} \gamma^n z_0 = z$. Then the fragments $P(\gamma^n w, \gamma^{n+1} w)$ belong to $\mathfrak{T}$ and the end of the ray starting at w which is contained in their union corresponds to z, so $v \in \mathfrak{T}$.

Thus we get that $V(G_\Gamma)$ is finite. But we know that $\mathcal{T}_K(\mathcal{L}_\Gamma)$ is locally finite, so we conclude the finiteness of the quotient.

For the second part, recall that $\mathcal{T}_K(\mathcal{L}_\Gamma) \subset \mathcal{T}_K(\mathcal{L}')$ and that we have the retraction map

$$\mathrm{r}_{\mathcal{L}_\Gamma} : \Omega_{\mathcal{L}_\Gamma} \longrightarrow \mathcal{T}_K(\mathcal{L}_\Gamma).$$

Choose a $p \in \Omega_{\mathcal{L}_\Gamma}$ such that $\Gamma \cdot p$ is one of the orbits added to $\mathcal{L}_\Gamma$ to form $\mathcal{L}'$. Take the open path $L_p := \mathring{P}(\mathrm{r}_{\mathcal{L}_\Gamma}(p), p)$ and then observe that $L_p \cap \mathcal{T}_K(\mathcal{L}_\Gamma) = \emptyset$. Now it is clear that

$$\Gamma \backslash \mathcal{T}_K(\mathcal{L}') = G_\Gamma \bigsqcup \left(\bigcup_{\pi_\Gamma(p) \in \mathcal{R}_{\mathcal{L}'}} \pi_\Gamma(L_p) \right)$$

but the $\pi_\Gamma(L_p)$ have not to be disjoint. Nevertheless, note that for any $\gamma \in \Gamma \setminus \{\mathbb{1}_\Gamma\}$ the intersection $L_{\gamma p} \cap L_p$ is empty, since otherwise, $\mathrm{r}_{\mathcal{L}_\Gamma}(p)$ would be a fixed vertex for γ, which contradicts the first claim of the result. Take now another $q \in \Omega_{\mathcal{L}_\Gamma}$ such that $\pi_\Gamma(q) \in \mathcal{R}_{\mathcal{L}'}$ and $\pi_\Gamma(q) \neq \pi_\Gamma(p)$. It may happen that for some $\gamma \in \Gamma$ (by the previous consideration, for at most one γ) we have $L_p \cap L_{\gamma q} \neq \emptyset$. In that case, in which $\mathrm{r}_{\mathcal{L}_\Gamma}(p) = \mathrm{r}_{\mathcal{L}_\Gamma}(\gamma q)$, let v_{pq} be the vertex of valence 3 in the tree $L_p \cup L_{\gamma q}$. Next, let v_p be one vertex of L_p such that all the possible v_{pq} with $\pi_\Gamma(q) \in \mathcal{R}_{\mathcal{L}'}$ are in the path $P(\mathrm{r}_{\mathcal{L}_\Gamma}(p), v_p)$. Finally take

$$G_{\mathcal{L}'} := \Gamma \backslash \left(\mathcal{T}_K(\mathcal{L}_\Gamma) \bigcup_{\pi_\Gamma(p) \in \mathcal{R}_{\mathcal{L}'}} \Gamma \cdot P(\mathrm{r}_{\mathcal{L}_\Gamma}(p), v_p) \right)$$

and the claim is immediate. □

Corollary 3.6.6. *If Γ is a Schottky group and $G_\Gamma := \Gamma \backslash \mathcal{T}_K(\mathcal{L}_\Gamma)$, then $\Gamma \cong \pi_1(G_\Gamma, v)$ for any vertex v of the quotient graph, so it is a free group, in particular, if it is generated by more than one element, it is non abelian, and $\pi_\Gamma : \mathcal{T}_K(\mathcal{L}_\Gamma) \longrightarrow G_\Gamma$ is the universal cover of the graph.*

We denote the rank of a Schottky group Γ by $g(\Gamma)$.

Theorem 3.6.7. *Let Γ be a Schottky group and consider $\mathcal{L} := \mathcal{L}_\Gamma$ and $\Omega := \Omega_{\mathcal{L}} = (\mathbb{P}^{1*})^{an} \setminus \mathcal{L}^*$. Then Γ acts on Ω and $C_\Gamma := \Gamma \backslash \Omega$ is a proper analytic space and so it is isomorphic to the analytification of a smooth projective algebraic curve of genus $g(\Gamma)$.*

Proof. You can see the proof with more detail in [GvdP80, Ch. 2 and 3]. Here, we will sketch it.

We will suppose that G_Γ has a model without loops. This is possible after a finite extension of the base field, if necessary. The general case can be done by means of Galois descent.

We consider the projection $\pi_\Gamma : \mathcal{T}_K(\mathcal{L}) \longrightarrow G_\Gamma$ and a metric graph model for $\mathcal{T}_K(\mathcal{L})$ given by a pair of sets $(V, \hat{E})$. The collection of vertices V is formed by points of the form $t(x_0, x_1, x_2)$ for $x_0, x_1, x_2 \in \mathbb{P}^1(K)$ such that it includes all the points of valency greater than 2, it is Γ-invariant and the metric graph model for G_Γ given by $\pi_\Gamma(V)$ has no loops. Recall that the set of open edges for the model of $\mathcal{T}_K(\mathcal{L})$ is the set of connected components of $\mathcal{T}_K(\mathcal{L}) \setminus V$, and the edges are obtained from the open ones adjoining the adherent vertices in two different ways, giving the two orientations for each edge. We will denote this set by $\hat{E}$.

Consider now the restriction of the retraction map, $\mathrm{r}_\mathcal{L} : \Omega_\mathcal{L} \longrightarrow \mathcal{T}_K(\mathcal{L})$. To each $e \in \hat{E}$, we associate $U(e) := \mathrm{r}_\mathcal{L}^{-1}(e)$, and, similarly, to a vertex $v \in V$ we associate $U(v) := \mathrm{r}_\mathcal{L}^{-1}(v)$. Then, the sets $U(e)$ and $U(v)$ are strictly affinoid and from them we get back Ω by gluing $U(e)$ with $U(e')$ through $U(v)$ when the edges e, e' have v as a common vertex.

Since the retraction map $\mathrm{r}_\mathcal{L}$ is Γ-equivariant, given two edges $e, e' \in \hat{E}$ such that $\pi_\Gamma(e) = \pi_\Gamma(e')$ so there exists $\gamma \in \Gamma$ such that $\gamma \cdot e = e'$, then $\gamma \cdot U(e) = U(e')$, and similarly for vertices. Therefore, gluing as before but taking into account these identifications, or what is the same, gluing according to the graph G_Γ we get the analytic space C_Γ, which is reduced and separated.

To prove that C_Γ is proper we are going to show that it is compact and its boundary (over K) is empty ([Tem15, Def. 4.2.13. (ii)]).

The compactness is because we can express C_Γ as a finite union of affinoids: the preimages of the stars of the vertices of G_Γ, which is a finite set.

To show that the boundary is empty, take any $x \in C_\Gamma$. We want to show there exists $x \in U$ affinoid such that $x \notin \partial U$. Consider the image of x by the induced retraction map in the quotients,

$$\mathrm{r}_{\mathcal{L},\Gamma} : C_\Gamma \longrightarrow G_\Gamma.$$

Now, $\mathrm{r}_{\mathcal{L},\Gamma}(x)$ is an interior point of a $\mathrm{St}(v)$ for some vertex v in the fixed model of G_Γ (if $\mathrm{r}_{\mathcal{L},\Gamma}(x)$ is a vertex we take $v = \mathrm{r}_{\mathcal{L},\Gamma}(x)$; otherwise v is any vertex of the edge to which $\mathrm{r}_{\mathcal{L},\Gamma}(x)$ belongs). Then, $\mathrm{r}_{\mathcal{L},\Gamma}^{-1}(\mathrm{St}(v))$ is the affinoid we are looking for.

Consider the following commutative diagram:

$$\begin{array}{ccc} \Omega & \xrightarrow{\mathrm{r}_\mathcal{L}} & \mathcal{T}_K(\mathcal{L}) \\ \downarrow{\scriptstyle \pi_\Gamma} & & \downarrow{\scriptstyle \pi_\Gamma} \\ C_\Gamma & \xrightarrow{\mathrm{r}_{\mathcal{L},\Gamma}} & G_\Gamma \end{array}$$

Choose a vertex $\tilde{v}$ in $\mathcal{T}_K(\mathcal{L})$ such that $\pi_\Gamma(\tilde{v}) = v$. Then π_Γ gives an isomorphism

$$\pi_\Gamma : \mathrm{St}(\tilde{v}) \xrightarrow{\sim} \mathrm{St}(v),$$

since there are no loops in G_Γ and the action of Γ in $\mathcal{T}_K(\mathcal{L})$ is free. It is clear that

$$\mathrm{r}_{\mathcal{L}}^{-1}(\mathrm{St}(\tilde{v})) = \bigcup_{\tilde{v}=s(e)} U(e)$$

and hence, by construction of C_Γ, π_Γ also induces an isomorphism

$$\pi_\Gamma : \mathrm{r}_{\mathcal{L}}^{-1}(\mathrm{St}(\tilde{v})) \xrightarrow{\sim} \mathrm{r}_{\mathcal{L},\Gamma}^{-1}(\mathrm{St}(v))$$

Now recall that $\partial U(e) = \{s(e), t(e)\} \subset U(e)$, since $U(e)$ is an annulus, therefore

$$\partial(\mathrm{r}_{\mathcal{L}}^{-1}(\mathrm{St}(\tilde{v}))) = \{t(e) |\ s(e) = \tilde{v}\}.$$

So we get $\partial(\mathrm{r}_{\mathcal{L},\Gamma}^{-1}(\mathrm{St}(v))) = \{\pi_\Gamma(t(e)) |\ s(e) = \tilde{v}\} \not\ni x$ as we wished. □

Remark 3.6.8. *We have used that the retraction* $\mathrm{r}_{\mathcal{L}_\Gamma} : \Omega_{\mathcal{L}_\Gamma} \longrightarrow \mathcal{T}_K(\mathcal{L}_\Gamma)$ *stated in the remark 3.2.8 descends to another retraction* $\mathrm{r}_{\mathcal{L},\Gamma}$ *giving place to a commutative square*

$$\begin{array}{ccc} \Omega_{\mathcal{L}_\Gamma} & \xrightarrow{\mathrm{r}_{\mathcal{L}_\Gamma}} & \mathcal{T}_K(\mathcal{L}_\Gamma) \\ \downarrow \pi_\Gamma & & \downarrow \pi_\Gamma \\ C_\Gamma & \xrightarrow{\mathrm{r}_{\mathcal{L}_\Gamma,\Gamma}} & G_\Gamma. \end{array}$$

Corollary 3.6.9. *If there exists a model of* G_Γ *which is without loops, then the map* $\Omega_{\mathcal{L}}(K) \longrightarrow C_\Gamma(K)$ *is surjective.*

Proof. Choose such a model. By the previous proof we have

$$C_\Gamma(K) = \bigcup_{e \in E(G_\Gamma)} U(e)(K), \qquad \Omega_{\mathcal{L}}(K) = \bigcup_{\tilde{e} \in E} U(\tilde{e})(K)$$

with the same notation. We may assume $\pi_\Gamma(\tilde{e}) = e$ so we conclude $U(\tilde{e}) = U(e)$. □

3.7 A peculiar symmetry

In this section we study some properties of the action of Γ on $\mathcal{T}_K$, a relation among the harmonic measures, and a symmetry among multiplicative integrals which can be useful to generalize the well known symmetry between theta functions.

Let $\Gamma \subset \mathrm{PGL}_2(K)$ be a Schottky group, and let $\mathcal{L} := \mathcal{L}_\Gamma \subset \mathbb{P}^1(K)$ be its limit set. We are going to show a new result which will led to a proof of the symmetry of the bilinear pairing defining the Albanese variety of the Mumford curve C_Γ.

We assume that $\Omega_{\mathcal{L}}(K) \neq \emptyset$ and contains at least the closures of two Γ-orbits of points. This is possible after a finite extension of K, meanwhile $\mathcal{L}$ remains invariant.

Let us define for any $p \in \Omega_{\mathcal{L}}(K)$ the compact set $\mathcal{L}_p := \mathcal{L} \cup \overline{\Gamma \cdot p^*} \subset \mathbb{P}^1(K)$ and for any $\gamma, \delta \in \Gamma$ the analytic function

$$u_{\gamma,\delta,p}(z) := u_{\gamma\delta p,\gamma p}(z) = \frac{z - \gamma\delta p}{z - \gamma p} \in \mathcal{O}(\Omega_{\mathcal{L}_p})$$

Consider now a point $q \in \Omega_{\mathcal{L}_p}(K)$, which is the same that a point $q \in \Omega_{\mathcal{L}}(K)$ verifying $\overline{\Gamma \cdot p} \cap \overline{\Gamma \cdot q} = \emptyset$. Then, for any $\rho \in \Gamma$, applying the invariance of the cross ratio we obtain

$$\frac{u_{\gamma,\delta,p}(\rho q)}{u_{\gamma,\delta,p}(q)} = \frac{u_{\gamma^{-1},\rho,q}(\delta p)}{u_{\gamma^{-1},\rho,q}(p)}$$

Recall from proposition 3.5.4 the equality of measures

$$\tilde{\mu}(u_{\gamma,\delta,p}) = \tilde{\mu}(u_{\gamma\delta p,\gamma p}) = \mu_{\gamma p^*,\gamma\delta p^*},$$

and then

$$\frac{u_{\gamma,\delta,p}(\rho q)}{u_{\gamma,\delta,p}(q)} = \mathop{\times\!\!\!\!\!\!\int}_{\mathcal{L}_p} f_{\rho q - q}(t) d\mu_{\gamma p^*,\gamma\delta p^*}$$

Therefore, putting together the two last ideas we have

$$\mathop{\times\!\!\!\!\!\!\int}_{\mathcal{L}_p} f_{\rho q - q}(t) d\mu_{\gamma p^*,\gamma\delta p^*} = \frac{u_{\gamma,\delta,p}(\rho q)}{u_{\gamma,\delta,p}(q)} =$$

$$= \frac{u_{\gamma^{-1},\rho,q}(\delta p)}{u_{\gamma^{-1},\rho,q}(p)} = \mathop{\times\!\!\!\!\!\!\int}_{\mathcal{L}_q} f_{\delta p - p}(t) d\mu_{\gamma^{-1} q^*,\gamma^{-1}\rho q^*}$$

For any $\delta \in \Gamma$, using corollary 2.4.8 applied to the tree $\mathcal{T}_K(\mathcal{L})$, one defines a harmonic measure $\mu_\delta \in \mathscr{M}(\mathcal{L}, \mathbb{Z})_0$, while we have just defined a harmonic measure $\mu_{\gamma p^*,\gamma\delta p*} \in \mathscr{M}(\mathcal{L}_p, \mathbb{Z})_0$ for each $\gamma \in \Gamma$. Note that $\mathcal{L} \subset \mathcal{L}_p$ and $\mathcal{T}_K(\mathcal{L}) \subset \mathcal{T}_K(\mathcal{L}_p)$. We consider compatible models for these trees, meaning that the model of $\mathcal{T}_K(\mathcal{L}_p)$ restricts to the model of $\mathcal{T}_K(\mathcal{L})$.

Proposition 3.7.1. *With the above notations, for any edge e of $\mathcal{T}_K(\mathcal{L}_p)$ and $\mathcal{T}_K(\mathcal{L})$ we have*

$$\sum_{\gamma\in\Gamma} \mu_{\gamma p^*,\gamma\delta p^*}(e) = -\mu_\delta(e)$$

and for any edge of $\mathcal{T}_K(\mathcal{L}_p)$ which is not inside $\mathcal{T}_K(\mathcal{L})$, then

$$\sum_{\gamma\in\Gamma} \mu_{\gamma p^*,\gamma\delta p^*}(e) = 0.$$

In order to prove the proposition, we observe first that

$$\sum_{\gamma\in\Gamma}\mu_{\gamma p^*,\gamma\delta p^*}(e)=\sum_{\gamma\in\Gamma}\mu_{p^*,\delta p^*}(\gamma^{-1}e)=$$

$$=\sum_{\gamma\in\Gamma}\mu_{p^*,\delta p^*}(\gamma e)=\sum_{\{\gamma\in\Gamma|\ \gamma e\in|P(p,\delta p)|\}}\mu_{p^*,\delta p^*}(\gamma e)$$

(where the bars for $|P(p,\delta p)|$ mean that we are considering just the underlying sets, without orientation) and we proceed by steps. The first step, which is the main one, lies essentially on the following lemma.

Lemma 3.7.2. *For any* $\delta\in\Gamma$ *and* $p\in\Omega_{\mathcal{L}}(K)$ *we have*

$$|\mathbb{A}_{\{p,\delta p\}}|\cap|\mathbb{A}_{\{\delta^2 p,\delta^3 p\}}|=\emptyset$$

and

$$\mathbb{A}_{\{p,\delta p\}}\cap\mathbb{A}_{\{\delta^{-1}p,\delta^2 p\}}=\mathbb{A}_{\{p,\delta p\}}\cap\mathbb{A}_{\delta}\subset\mathbb{A}_{\{p,\delta p\}}\cap\mathcal{T}_K(\mathcal{L}).$$

Proof. Since δ is hyperbolic it has the form

$$\delta=\delta'\begin{pmatrix} q & 0\\ 0 & 1\end{pmatrix}\delta'^{-1}$$

with $|q|<1$. Consider $p':=\delta^{-1}p\in\Omega_{\mathcal{L}}$. Then, if we prove the equalities of the lemma for

$$\begin{pmatrix} q & 0\\ 0 & 1\end{pmatrix}$$

and p' instead of δ and p, allowing δ act on the apartments we will get the claims. Therefore, we may assume

$$\delta=\begin{pmatrix} q & 0\\ 0 & 1\end{pmatrix}$$

with $|q|<1$. In particular, we have $\mathbb{A}_\delta=\mathbb{A}_{\{\infty,0\}}$.

Now, we want to show $|\mathbb{A}_{\{p,qp\}}|\cap|\mathbb{A}_{\{q^2p,q^3p\}}|=\emptyset$. Let us observe that $|q^3p|<|q^2p|<|qp|<p$, so

$$\mathbb{A}_{\{p,qp\}}\cap\mathbb{A}_{\{\infty,0\}}=P\left(\alpha(0,|p|),\alpha(0,|qp|)\right)$$
$$\mathbb{A}_{\{q^2p,q^3p\}}\cap\mathbb{A}_{\{\infty,0\}}=P\left(\alpha(0,|q^2p|),\alpha(0,|q^3p|)\right)$$

Therefore, if the intersection $|\mathbb{A}_{\{p,\delta p\}}|\cap|\mathbb{A}_{\{\delta^2p,\delta^3p\}}|$ was non empty it should occur in $\mathbb{A}_{\{\infty,0\}}$ since the total space is a tree, but it is clear that

$$P\left(\alpha(0,|p|),\alpha(0,|qp|)\right)\cap P\left(\alpha(0,|q^2p|),\alpha(0,|q^3p|)\right)=\emptyset,$$

and so we get the first claim.

In order to obtain the second claim we will prove

$$\left(\mathbb{A}_{\{p,qp\}}, \mathbb{A}_{\{q^{-1}p,q^2p\}}\right)_{\mathcal{T}_{\mathbb{C}_K}} = \left(\mathbb{A}_{\{p,qp\}}, \mathbb{A}_{\{\infty,0\}}\right)_{\mathcal{T}_{\mathbb{C}_K}}.$$

Applying the proposition 3.3.1 we see it is enough to check that

$$v_K\left(\left(\begin{array}{c} p : q^{-1}p \\ qp : q^2p \end{array}\right)\right) = v_K\left(\left(\begin{array}{c} p : \infty \\ qp : 0 \end{array}\right)\right),$$

so we compute:

$$v_K\left(\left(\begin{array}{c} p : q^{-1}p \\ qp : q^2p \end{array}\right)\right) = -\log\frac{|p-q^{-1}p||qp-q^2p|}{|p-q^2p||qp-q^{-1}p|} = -\log\frac{|q^{-1}p||qp|}{|p||q^{-1}p|} =$$

$$= -\log\frac{|qp|}{|p|} = v_K\left(\left(\begin{array}{c} p : \infty \\ qp : 0 \end{array}\right)\right)$$

□

Next, and under the hypotheses of the previous lemma, it allows us to subdivide the apartment $\mathbb{A}_{p,\delta p}$ in three paths:

$$\mathbb{A}_{\{p,\delta p\}} = S_{p,\delta p} \cup I_{p,\delta p} \cup T_{p,\delta p},$$

where

$$S_{p,\delta p} = P(p, t(p,\delta p, \delta^{-1}p))$$
$$I_{p,\delta p} = P(t(p,\delta p,\delta^{-1}p), t(p,\delta p,\delta^2 p))$$
$$T_{p,\delta p} = P(t(p,\delta p,\delta^2 p), \delta p)$$

Since the first part of the lemma tells that $|\mathbb{A}_{\{\delta^{-1}p,p\}}| \cap |\mathbb{A}_{\{\delta p,\delta^2 p\}}| = \emptyset$, this implies that $|S_{p,\delta p}| \cap |T_{p,\delta p}| = \emptyset$, the intersections of the interior of the paths are empty and the paths are well defined subpaths of $\mathbb{A}_{\{p,\delta p\}}$ with the same orientation.

The second part of the lemma implies that $I_{p,\delta p} \subset \mathcal{T}_K(\mathcal{L})$. With this tools, we proceed to get the next step:

Lemma 3.7.3. *Let e be an edge of $\mathcal{T}_K(\mathcal{L}_p)$ and consider the sets*

$$\Gamma_S^e := \{\gamma \in \Gamma |\ \gamma e \in |S_{p,\delta p}|\}$$
$$\Gamma_I^e := \{\gamma \in \Gamma |\ \gamma e \in |I_{p,\delta p}|\}$$
$$\Gamma_T^e := \{\gamma \in \Gamma |\ \gamma e \in |T_{p,\delta p}|\}$$

so that we have the decomposition

$$\{\gamma \in \Gamma |\ \gamma e \in |P(p,\delta p)|\} = \Gamma_S^e \sqcup \Gamma_I^e \sqcup \Gamma_T^e.$$

Then:

1. *There is a bijection $\Gamma_S^e \longleftrightarrow \Gamma_T^e$ which reverses the orientation of the edge in $\mathbb{A}_{\{p,\delta p\}}$, that is, if γ' corresponds to a γ such that γe is in $S_{p,\delta p}$ with the same orientation, the edge $\gamma' e$ is in $T_{p,\delta p}$ with the opposite orientation.*

2. *If e is not inside $\mathcal{T}_K(\mathcal{L})$, then $\Gamma_I^e = \emptyset$.*

Proof. 1. The bijection is defined by

$$\begin{aligned} \Gamma_S^e &\longrightarrow \Gamma_T^e \\ \gamma &\mapsto \ \delta\gamma \end{aligned}$$

Thus, if the directed edge γe is in

$$S_{p,\delta p} = P(p, t(p, \delta p, \delta^{-1} p)) = P(p, \delta^{-1} p) \cap P(p, \delta p),$$

the directed edge $\delta\gamma e$ is in

$$\delta P(p, \delta^{-1} p) \cap \delta P(p, \delta p) = P(\delta p, p) \cap P(\delta p, \delta^2 p) = T_{p,\delta p}.$$

In general, the orientation of γe with respect to $S_{p,\delta p}$ and $P(p, \delta p)$ is the same as the orientation of $\delta\gamma e$ with respect to $T_{p,\delta p}$ and $P(p, \delta p)$ so the opposite to the orientation of γe. Clearly, the inverse map is $\gamma \mapsto \delta^{-1}\gamma$.

2. The result is clear from the remark previous to the lemma. If e is not inside $\mathcal{T}_K(\mathcal{L})$, there is no γe inside $\mathcal{T}_K(\mathcal{L})$ for $\gamma \in \Gamma$, but

$$\Gamma_I^e = \{\gamma \in \Gamma | \ \gamma e \in |I_{p,\delta p}| \subset |\mathcal{T}_K(\mathcal{L})|\}$$

so $\Gamma_I^e = \emptyset$.

□

Proof of proposition 3.7.1. Let us see first the second claim. If e is not in $\mathcal{T}_K(\mathcal{L})$ we have

$$\sum_{\gamma \in \Gamma} \mu_{\gamma p^*, \gamma\delta p^*}(e) = \sum_{\{\gamma \in \Gamma | \ \gamma e \in |P(p,\delta p)|\}} \mu_{p^*,\delta p^*}(\gamma e) =$$

$$= \sum_{\gamma \in \Gamma_S^e} \mu_{p^*,\delta p^*}(\gamma e) + \sum_{\gamma \in \Gamma_I^e} \mu_{p^*,\delta p^*}(\gamma e) + \sum_{\gamma \in \Gamma_T^e} \mu_{p^*,\delta p^*}(\gamma e)$$

Because of the second part of the previous lemma the second summation is zero and because of the first part and the definition of $\mu_{p^*,\delta p*}$ the sum of the other two summations vanishes, so $\sum_{\gamma\in\Gamma}\mu_{\gamma p^*,\gamma\delta p^*}(e) = 0$ as we wanted to see.

We assume now that e is in $\mathcal{T}_K(\mathcal{L})$. We have the same equalities as before and also the cancellation of the two extreme summations so

$$\sum_{\gamma\in\Gamma}\mu_{\gamma p^*,\gamma\delta p^*}(e) = \sum_{\gamma\in\Gamma_I^e}\mu_{p^*,\delta p^*}(\gamma e)$$

and we want to prove this is equal to

$$-\mu_\delta(e) = -\frac{(\pi_\Gamma(e),\varpi(\delta))_{G_\Gamma}}{\ell(e)} = -\frac{(\pi_\Gamma(e),\pi_\Gamma(P(\alpha,\delta\alpha)))_{G_\Gamma}}{\ell(e)}$$

where $\pi_\Gamma : \mathcal{T}_K(\mathcal{L}) \longrightarrow G_\Gamma = \Gamma\backslash\mathcal{T}_K(\mathcal{L})$ is the covering projection and α is any vertex in $\mathbb{A}_\delta$. We take $\alpha = t(p,\delta p,\delta^{-1}p)$, so we have $\delta\alpha = t(p,\delta p,\delta^2 p)$ and

$$\mu_\delta(e) = \frac{(\pi_\Gamma(e),\pi(P(\alpha,\delta\alpha)))_{G_\Gamma}}{\ell(e)} = \sum_{\substack{|\gamma e|\subset|P(\alpha,\delta\alpha)|\\ \gamma\in\Gamma}}\frac{(\gamma e,P(\alpha,\delta\alpha))_{\mathcal{T}}}{\ell(e)} =$$

$$= \sum_{\gamma\in\Gamma_I^e}\frac{(\gamma e,P(\alpha,\delta\alpha))_{\mathcal{T}}}{\ell(e)} = -\sum_{\gamma\in\Gamma_I^e}\mu_{p^*,\delta p^*}(\gamma e) = -\sum_{\gamma\in\Gamma}\mu_{\gamma p^*,\gamma\delta p^*}(e)$$

where for the third equality we use the definition of α and the fact that the action of Γ on $\mathcal{T}$ is free, and for the fourth equality we use the definition of $\mu_{p^*,\delta p*}$. □

Corollary 3.7.4. *With the above notations we have*

$$\prod_{\gamma\in\Gamma}\,⨍_{\mathcal{L}_p} f^{-1}_{\rho q-q}(t)\,d\mu_{\gamma p^*,\gamma\delta p^*} = ⨍_{\mathcal{L}} f_{\rho q-q}(t)\,d\mu_\delta$$

Proof. It is direct from the proposition, taking into account that the inverse of the function $f_{\rho q-q}$ appears due to the negative sign in the equality

$$\sum_{\gamma\in\Gamma}\mu_{\gamma p^*,\gamma\delta p^*}(e) = -\mu_\delta(e).$$

□

Corollary 3.7.5. *Let $\Gamma \subset \mathrm{PGL}_2(K)$ be a Schottky group, and consider its limit set $\mathcal{L} := \mathcal{L}_\Gamma \subset \mathbb{P}^1(K)$. For any $\rho,\delta\in\Gamma$ and for any $p,q\in\Omega_{\mathcal{L}}(K)$ such that $\overline{\Gamma\cdot p}\cap\overline{\Gamma\cdot q} = \emptyset$ we get*

$$⨍_{\rho q-q} d\mu_\delta = ⨍_{\delta p-p} d\mu_\rho$$

Proof. Taking into account the last observation previous to the proposition and the corollary above we get

$$\oint_{\rho q-q} d\mu_\delta = \oint_{\mathcal{L}} f_{\rho q-q}(t) d\mu_\delta = \prod_{\gamma\in\Gamma} \oint_{\mathcal{L}_p} f^{-1}_{\rho q-q}(t) d\mu_{\gamma p^*,\gamma\delta p^*} =$$

$$= \prod_{\gamma\in\Gamma} \oint_{\mathcal{L}_q} f^{-1}_{\delta p-p}(t) d\mu_{\gamma^{-1}q^*,\gamma^{-1}\rho q^*} = \oint_{\mathcal{L}} f_{\delta p-p}(t) d\mu_\rho = \oint_{\delta p-p} d\mu_\rho$$

□

3.8 Automorphic Forms

The main goal of this section is to prove the theorem 3.8.16 using the analytic theory developed along this paper and some results from [BPR13], like the propositions 2.5, 2.10 and the slope formula theorem (5.15), instead of using [GvdP80, Ch. 2 (3.2)], whose proof requires the development of other analytic tools.

Let G be a metric graph.

Definition 3.8.1. *We call a tropical function on G a continuous function $f : G \longrightarrow \mathbb{R}$ such that there exists a model $\mathfrak{G}$ of G satisfying for each edge $e \in \hat{E}(\mathfrak{G})$ that the restriction*

$$f_{\big| |e|} : |e| \longrightarrow \mathbb{R}$$

is linear with integral slope, where by linear we mean that for every isometric embedding $[a, b] \longrightarrow |e|$, the composition $[a, b] \longrightarrow |e| \longrightarrow \mathbb{R}$ is linear.

Note that this is equivalent to say that for each model of G the function f is piecewise linear (with integral slopes) on each edge.

Suppose now that G is locally finite. Given a tropical function f on G and a model $\mathfrak{G}$ of G such that f verifies the "edge-linearity" condition stated on previous definition, we can associate to it a cochain D_f on the edges of $\mathfrak{G}$ defined by taking $D_f(e)$ to be the slope of f on e.

We call f a harmonic function if D_f is a harmonic cochain.

Remark 3.8.2. *If f is harmonic, $f_{\big| |e|}$ is linear for any edge of any model of G.*

Next, let Γ be a group with a left action on a metric graph G.

Definition 3.8.3. *A tropical function f on G is called an automorphic form for Γ if*

$$\forall\, \gamma \in \Gamma\ \exists\ c_f(\gamma) \in \mathbb{R} :\ f(z) = c_f(\gamma) + f(\gamma z)\ \forall z \in G$$

Lemma 3.8.4. *Let G be a locally finite metric graph on which acts a group Γ. Let f be an automorphic form for Γ. Then there exists a model $\mathfrak{G}$ of G, on which acts Γ, such that f is linear on its edges, the cochain D_f is Γ-invariant and so induces a cochain $\overline{D_f}$ on $\Gamma\backslash G$.*

Proof. Since f is tropical there exists a model of G such that f is linear on its edges. Now, the minimal Γ-invariant model refining the previous satisfies the claims of the lemma immediately, and D_f is Γ-invariant because f is automorphic for Γ. □

Lemma 3.8.5. *Let G be a locally finite metric graph on which acts a group Γ. Assume there exists a finite connected graph $G' \subset G/\Gamma$ such that*

$$(\Gamma\backslash G) \setminus G' = \bigsqcup_{i\in I} L_i \text{ where } I \text{ is finite and } L_i \cong (0,\infty)\ \forall i \in I$$

such that its closure inside $\Gamma\backslash G$ is $\overline{L_i} \cong [0,\infty)$ (we are choosing an orientation on L_i).

Then, any harmonic function on G being an automorphic form for Γ verifies:

1. *For any $i \in I$, the restricted cochain is constant: $\overline{D_f}_{|L_i} \equiv m_i \in \mathbb{Z}$.*
2. $\sum_{i\in I} m_i = 0$.

Proof. We take a suitable model of G -since f is harmonic, it only has to be Γ-invariant-. Since D_f is harmonic, so it is $\overline{D_f}$. Now, given two adjacent edges e, e' of L_i with the same orientation, due to the hypothesis on G and G' harmonicity implies $\overline{D_f}(e) + \overline{D_f}(\overline{e'}) = 0$, so $\overline{D_f}(e) = \overline{D_f}(e')$, and this extends obviously to any edge of L_i, so the first claim rests proved.

The second claim is a direct consequence of the lemma 2.2.1. □

From now on, let Γ be a fixed Schottky group, $\mathcal{L} = \mathcal{L}_\Gamma \subset \mathbb{P}^1(K)$ the set of fixed points of Γ, and $\Omega_{\mathcal{L}}$ as defined above. Let $L|K$ be a field extension.

Definition 3.8.6. *We will say that a $\mathbb{C}_K$-valued meromorphic function $f \neq 0$ on $\Omega_{\mathcal{L}}$ is an automorphic form for Γ (or Γ-automorphic form) with automorphy factor $c_f : \Gamma \longrightarrow \mathbb{C}_K^*$ if*

$$f(z) = c_f(\alpha) f(\alpha z)\ \forall z \in \Omega_{\mathcal{L}} \forall \alpha \in \Gamma.$$

We will call it L-automorphic if c_f takes values in L^*.

Let us denote the set of L-automorphic forms on $\Omega_{\mathcal{L}}$ by $\mathcal{A}_\Gamma(L)$.

Remark 3.8.7. *By definition, c_f is a group morphism.*

Proposition 3.8.8. *Given a point $z_0 \in \Omega_{\mathcal{L}}(K)$ and a Γ-invariant harmonic measure $\mu \in \mathcal{M}(\mathcal{L}, \mathbb{Z})_0^\Gamma$ the function on $\Omega_{\mathcal{L}}$*

$$\mathcal{I}_{\mu,z_0}(z) := \times\!\!\!\!\!\!\int_{z-z_0} d\mu$$

is an analytic and automorphic form for Γ with automorphy factor independent of z_0.

Proof. We already know it is analytic, as shown in the proof of theorem 3.5.5 and remarked in its corollary 3.5.7.

In order to see that it is automorphic for Γ let us show first that the integral

$$\times\!\!\!\!\!\!\int_{p-\gamma p} d\mu$$

does not depend on $p \in \Omega_{\mathcal{L}}$. Indeed, given $p, q \in \Omega_{\mathcal{L}}$ we have

$$\frac{\times\!\!\!\!\!\!\int_{p-\gamma p} d\mu}{\times\!\!\!\!\!\!\int_{q-\gamma q} d\mu} = \frac{\times\!\!\!\!\!\!\int_{p-q} d\mu}{\times\!\!\!\!\!\!\int_{\gamma p-\gamma q} d\mu} = 1$$

due to the Γ-equivariance of the integration and to the Γ-invariance of μ.

Therefore,

$$\frac{\mathcal{I}_{\mu,z_0}(z)}{\mathcal{I}_{\mu,z_0}(\gamma z)} = \frac{\times\!\!\!\!\!\!\int_{z-z_0} d\mu}{\times\!\!\!\!\!\!\int_{\gamma z-z_0} d\mu} = \times\!\!\!\!\!\!\int_{z-\gamma z} d\mu \in K^*$$

is its automorphy factor. □

Proposition 3.8.9. *For any $c \in \operatorname{Hom}(\Gamma^{ab}, L^*)$ there exists an L-automorphic form f such that $c = c_f$.*

Proof. Let us consider the group $\mathcal{M}(\Omega_{\mathcal{L}})^*$ of non-zero meromorphic functions on $\Omega_{\mathcal{L}}$ and its quotient Q by the constants, so we have the short exact sequence

$$0 \longrightarrow L^* \longrightarrow \mathcal{M}(\Omega_{\mathcal{L}})^* \longrightarrow Q \longrightarrow 0$$

After taking invariants under Γ we find the exact sequence

$$\mathcal{M}(C_\Gamma) \longrightarrow Q^\Gamma \longrightarrow \mathrm{Hom}(\Gamma^{ab}, L^*) \longrightarrow H^1(\Gamma, \mathcal{M}(\Omega_\mathcal{L})^*)$$

We end the proof recalling that $H^1(\Gamma, \mathcal{M}(\Omega_\mathcal{L})^*) = 0$ by [vdP92, Cor. 5.3] -since C_Γ is algebraic-, and noting that Q^Γ coincides with the group of L-automorphic forms modulo the constants. □

We may express this telling that the morphism

$$\mathcal{A}_\Gamma(L) \longrightarrow \mathrm{Hom}(\Gamma^{ab}, L^*)$$

is surjective.

Let us formalize the notion of infinite divisor as in [MD73, §2].

Definition 3.8.10. *We call a function $\boldsymbol{D} : \Omega_\mathcal{L}(\mathbb{C}_K) \longrightarrow \mathbb{Z}$ an infinite L-divisor on $\Omega_\mathcal{L}$ verifying the following properties:*

- $\boldsymbol{D}(z_1) = \boldsymbol{D}(z_2)$ *if* $z_1 = \Gamma z_2$.
- *The set* $\mathrm{Supp}(D) := \{z \in \Omega_\mathcal{L} |\ \boldsymbol{D}(z) \neq 0\}$ *has no limit points in* $\Omega_\mathcal{L}$ *and there is a finite extension* $L'|L$ *such that* $\mathrm{Supp}(D) \subset \Omega_\mathcal{L}(L')$.

We write such a divisor in the usual form

$$D = \sum_{n_z = \boldsymbol{D}(z) \neq 0} n_z z.$$

We will say that such an infinite divisor has finite representant $\tilde{D}$ if this is a finite divisor (that is it has finite support) such that

$$D = \sum_{\gamma \in \Gamma} \gamma \tilde{D} =: \Gamma \tilde{D}$$

We consider now the zeroes and poles of the automorphic forms. Note that if z is a zero (resp. pole) of order n of $f \in \mathcal{A}_\Gamma$, for each $\gamma \in \Gamma$, γz is a zero (resp. pole) of order n of f too.

Proposition 3.8.11. *Let f be a meromorphic function and e an edge of a model of $\mathcal{T}_K(\mathcal{L})$. Then, the set of zeroes and poles of f restricted to $U(e)$ is finite.*

Proof. First, a meromorphic function is the quotient of analytic functions so we may assume that f is analytic and we only have to show that it has a finite number of zeroes. But this is proved in [FvdP04, Prop. 3.3.6] as a consequence of the fact that the affinoid $U(e)$ is a disjoint union of closed discs, the Mittag-Leffler decomposition theorem and the Weierstrass preparation theorem. □

Corollary 3.8.12. *The set of zeros and poles of an automorphic form f on $\Omega_{\mathcal{L}}$ for Γ is a finite union of orbits of points of $\Omega_{\mathcal{L}}$.*

Proof. Consider a model for $\mathcal{T}_K(\mathcal{L})$ and denote the set of its edges $\hat{E}$. Consider also a set of edges $\hat{E}_\Gamma \subset \hat{E}$ in bijection by π_Γ with the edges on the induced model on G_Γ. Since the quotient graph is finite, so it is the set $\hat{E}_\Gamma$, and since this is a set of representatives of the graph G_Γ,

$$\bigcup_{\gamma\in\Gamma} \gamma \cdot \hat{E}_\Gamma = \hat{E}$$

Therefore, the affinoids $\gamma U(\hat{E}_\Gamma)$ with $\gamma \in \Gamma$ cover all $\Omega_{\mathcal{L}}$, where

$$U(\hat{E}_\Gamma) := \bigcup_{e\in\hat{E}_\Gamma} U(e).$$

Now, because of the previous proposition, the set $S_\Gamma(f)$ of zeroes and poles of f on $U(\hat{E}_\Gamma)$ is finite. And since this set is Γ-invariant and the orbit of $U(\hat{E}_\Gamma)$ covers $\Omega_{\mathcal{L}}$, the orbit of $S_\Gamma(f)$ is the set of zeroes and poles of f and it is a finite union of orbits of points. □

Let us denote $S(f)$ the set of zeroes and poles of an automorphic form f on $\Omega_{\mathcal{L}}$, and $\mathcal{L}_f := \mathcal{L}_\Gamma \cup S(f)^*$. The set $\mathcal{L}_f$ is compact, due to the previous proposition and the fact that Γ is a Schottky group.

Note that f has neither zeroes nor poles on $\Omega_{\mathcal{L}_f}$, so $f \in \mathcal{O}(\Omega_{\mathcal{L}_f})^*$.

Theorem 3.8.13. *Let f be an automorphic form for Γ on $\Omega_{\mathcal{L}}$. Then*

$$F = -\log|f|_{|\mathcal{T}_K(\mathcal{L}_f)}$$

is a harmonic and automorphic form for Γ on $\mathcal{T}_K(\mathcal{L}_f)$.

Proof. The first thing we have to check is that F is tropical, that is, given a model of $\mathcal{T}_K(\mathcal{L}_f)$ and an edge e of this model, the restriction of F on $|e|$ is piecewise linear on it.

Since we are going to apply lemma 3.5.1, we recall the notation used in it. We may suppose that the topological realization of the edge e is the path $|e| = P(\alpha(x,r), \alpha(x,s))$ with $x \in \mathcal{L}_f$, $r < s$ and such that its associated annulus satisfies $R_x(r,s) \cap \mathcal{L}_f = \emptyset$. We also do not loss generality assuming $x = 0$. Now we consider an isometric embedding

$$\exp : [r_0, s_0] \longrightarrow P(\alpha(0, \exp(r_0)), \alpha(0, \exp(s_0)))$$

where $r = \exp(r_0),\ s = \exp(s_0)$.

By the cited lemma, we know that $|f(z)| = r|z^k|$ for some $r \in \mathbb{R}_{>0}, k \in \mathbb{Z}$ on that path, and $z = \exp(w)$ for $w \in [r_0, s_0]$. Therefore

$$F(\exp(w)) = -\log|f(z)| = -k\log|z| - \log(r) = -kw - \log(r),$$

so we get the hoped linearity with integral slope k, and so F becomes tropical.

In the previous computation we got $D_F(e) = -k$. Recall also the map

$$\tilde{\mu} : \mathcal{O}(\Omega_{\mathcal{L}_f})^* \longrightarrow \mathscr{M}(\mathcal{L}_f, \mathbb{Z})_0$$

by which $\tilde{\mu}(f)(e) = k$. Therefore $D_F = -\tilde{\mu}(f)$, so this is a harmonic cochain and F is harmonic.

Finally we will show that F is automorphic for Γ on $\mathcal{T}_K(\mathcal{L}_f) \subset \Omega_{\mathcal{L}_f} \subset \Omega_{\mathcal{L}}$. Since f is automorphic on $\Omega_{\mathcal{L}}$ we have that for all $z \in \Omega_{\mathcal{L}}$ and $\gamma \in \Gamma$, $f(z) = c_f(\gamma)f(\gamma z)$ with $c_f(\gamma) \in \mathbb{C}_K^*$. Let us restrict to the case when $z \in \mathcal{T}_K(\mathcal{L}_f)$:

$$F(z) = -\log|f(z)| = -\log|c_f(\gamma)f(\gamma z)| = -\log|c_f(\gamma)| - \log|f(\gamma z)| =$$

$$= v_K(c_f(\gamma)) + F(\gamma z) \text{ with } v_K(c_f(\gamma)) \in \mathbb{R}$$

□

We maintain the same hypothesis of the theorem. Consider now the quotient $\Gamma\backslash\mathcal{T}_K(\mathcal{L}_f)$. By the proposition 3.6.5, its quotient is the disjoint union of a finite connected graphs with a finite union of "ends" which correspond to the classes of zeroes and poles of f modulo Γ -that is $\Gamma\backslash S(f)$- by the definition of $\mathcal{L}_f$. For any $x \in S(f)$ denote L_x the corresponding end oriented from the interior to the exterior, like in lemma 3.8.5. With the previous theorem, the next completes the slope formula (cf. [BPR13, 5.15]).

Proposition 3.8.14. *With the previous notation we get*

$$\overline{D_{F|L_x}} \equiv o_x(f)$$

Proof. In order to know the value of $\overline{D_{F|L_x}}$ we have to evaluate D_f on any edge e of L_x. We can assume its topological realization is of the form $P(\alpha(x,r), \alpha(x,s))$ with $r < s$. Note that, by the chosen orientation, we have $\overline{D_{F|L_x}} = D_f(\overline{e}) = -D_F(e)$. Finally, by what we have seen on the proof of the previous theorem or in lemma 3.5.1, we get $D_F(e) = -o_x(f)$, so $\overline{D_{F|L_x}} = o_x(f)$. □

Next, we want to build a finite degree zero divisor associated to an automorphic form on $\Omega_{\mathcal{L}}$. In order to get this, we have to refine the proof of corollary 3.8.12.

First, we note that there is a "semi-open" (connected) tree (open at some edges, closed at others) in $\mathcal{T}_K(\mathcal{L})$ in bijection with $G_\Gamma = \Gamma \backslash \mathcal{T}_K(\mathcal{L})$ by the projection map π_Γ.

To see this, take a maximal tree T_Γ of G_Γ and a set E_Γ^c of adjacent closed edges of $\mathcal{T}_K(\mathcal{L})$ such that its topological realization $|E_\Gamma^c|$ is a tree in bijection with T_Γ by π_Γ. Next take a set of open edges E_Γ^o of $\mathcal{T}_K(\mathcal{L})$ corresponding to the open edges which form $G_\Gamma \setminus T_\Gamma$, each one of them adjacent to some edge of E_Γ^c. Then we have that $|E_\Gamma^c \cup E_\Gamma^o|$ is a subtree of $\mathcal{T}_K(\mathcal{L})$ in bijection with $\pi_\Gamma(E_\Gamma^c \cup E_\Gamma^o) = G_\Gamma$, as the one we claimed the existence.

Now take

$$U(G_\Gamma) := \left(\bigcup_{e \in E_\Gamma^c} U(e) \right) \bigcup \left(\bigcup_{\mathring{e} \in E_\Gamma^o} U(\mathring{e}) \right)$$

By construction, for $\gamma' \in \Gamma \setminus \{\mathbb{1}_\Gamma\}$, we have

$$U(G_\Gamma) \bigcap (\gamma' \cdot U(G_\Gamma)) = \emptyset \text{ and } \bigcup_{\gamma \in \Gamma} \gamma \cdot U(G_\Gamma) = \Omega_{\mathcal{L}}.$$

Consider also the set $S_\Gamma(f) = S(f) \bigcap U(G_\Gamma)$ (note that in the proof of corollary 3.8.12 we used the same notation but with a slightly different meaning, since $U(\hat{E}_\Gamma) \neq U(G_\Gamma)$) and the finite divisor

$$D_f^\Gamma := \sum_{p \in S_\Gamma(f)} o_p(f) p$$

By the previous remark on unions and intersections on the orbit of $U(G_\Gamma)$ and the structure of $S(f)$, we get that the divisor of f satisfies

$$\sum_{p \in S(f)} o_p(f) p = \sum_{\gamma \in \Gamma} \gamma \cdot D_f^\Gamma$$

Proposition 3.8.15. *An automorphic form has associated an infinite divisor with finite representant of degree zero.*

Proof. Because of the previous considerations, the only we have to proof is that D_f^Γ has degree zero, that is

$$\sum_{p \in S_\Gamma(f)} o_p(f) = 0$$

Next note that there is a bijection between $S_\Gamma(f)$ and $\Gamma \backslash S(f)$. Further, by the previous theorem we have

$$\sum_{p \in S_\Gamma(f)} o_p(f) = \sum_{p \in S_\Gamma(f)} \overline{D_{F|L_p}} = \sum_{\pi_\Gamma(p) \in S(f)/\Gamma} \overline{D_{F|L_p}}$$

Finally, applying the lemma 3.8.5 to the quotient $\mathcal{T}_K(\mathcal{L}_f)$, which has as ends the sets L_p with $\pi_\Gamma(p) \in \Gamma\backslash S(f)$ by the proposition 3.6.5, we get that this sum is zero, as we wanted to prove. □

In order to go in depth, let us take into consideration a special kind of automorphic forms: theta functions.

For any $p, p' \in \Omega_{\mathcal{L}}(\mathbb{C}_K)$, the infinite product

$$\theta(p - p'; z) := \prod_{\gamma \in \Gamma} \frac{z - \gamma p}{z - \gamma p'}$$

defines a meromorphic function on $\Omega_{\mathcal{L}}$, clasically called theta function.

Its construction and the properties we report are done in [GvdP80, Ch. 2]. It is an L-automorphic form for Γ, where $L|K$ is any complete extension of fields such that $p, p' \in \Omega_{\mathcal{L}}(L)$. If p and p' are in the same Γ-orbit, the theta function is analytic. If $\Gamma p \neq \Gamma p'$, then $\theta(p - p'; z)$ has simple zeroes at the points of Γp and simple poles at the points of $\Gamma p'$ and no other zeroes or poles. The previous considerations show us that $\theta(p - p'; z)$ has associated an infinite divisor on $\Omega_{\mathcal{L}}$, which is $\Gamma(p - p') = \Gamma p - \Gamma p'$. Further, for any $\delta \in \Gamma$ and $p \in \Omega_{\mathcal{L}}$, the theta function $\theta(p - \delta p; z)$ does not depend on p.

Next we prove a simpler version of [GvdP80, Ch. 2 (3.2)].

Theorem 3.8.16. *Let f be an automorphic form on $\Omega_{\mathcal{L}}$. There is a finite divisor $\sum_{i=1}^{r}(p_i - q_i)$ which represent the infinite divisor associated to f and such that*

$$f(z) = \widetilde{f}(z) \cdot \theta(p_1 - q_1; z) \cdots \theta(p_r - q_r; z)$$

with $\widetilde{f}$ analytic function without zeroes on $\Omega_{\mathcal{L}}$. Further, if L is a field such that $p_i, q_i \in \Omega_{\mathcal{L}}(L)$, then f is L-automorphic.

Proof. First, with the notation of the previous proposition take

$$D_f^\Gamma = \sum_{i=1}^{r} (p_i - q_i)$$

Second, consider the automorphic form

$$\theta_{D_f^\Gamma}(z) := \theta(p_1 - q_1; z) \cdots \theta(p_r - q_r; z)$$

By definition, the zeroes and poles of it are the same as the ones of f, so $\widetilde{f}(z) := f(z)/\theta_{D_f^\Gamma}(z)$ is an analytic function.

The second claim is immediate. □

Therefore we have an infinite divisor on $\Omega_{\mathcal{L}}$ for any automorphic form. Indeed, the associated infinite divisor to the form of the theorem is

$$\Gamma \cdot \sum_{i=1}^{r} (p_i - q_i)$$

As a consequence we get a well defined degree zero divisor on the curve $\Gamma\backslash\Omega_{\mathcal{L}}(L) = C_\Gamma(L)$.

Finally let us take into consideration $\delta \in \Gamma$ and the analytic function $\theta(p - \delta p; z) \in \mathcal{O}(\Omega_{\mathcal{L}})^*$ for any $p \in \Omega_{\mathcal{L}}(K)$ (as above we assume $\Omega_{\mathcal{L}}(K) \neq \emptyset$, if necessary after a finite extension of the base field).

Theorem 3.8.17. *The image of $\theta(p - \delta p; z)$ by the morphism*

$$\tilde{\mu} : \mathcal{O}(\Omega_{\mathcal{L}})^* \longrightarrow \mathscr{M}(\mathcal{L}, \mathbb{Z})_0$$

is μ_δ. Moreover, it maps any (analytic) automorphic form to a Γ-invariant measure.

Proof. In the same way that we did before, we define $\mathcal{L}_p := \mathcal{L} \cup \overline{\Gamma \cdot p^*} \subset \mathbb{P}^1(K)$. We recall the analytic functions defined through section 3.5.

$$u_{\gamma p, \gamma\delta p}(z) = \frac{z - \gamma p}{z - \gamma\delta p} \in \mathcal{O}(\Omega_{\mathcal{L}_p})^*$$

so

$$\theta(p - \delta p; z) = \prod_{\gamma \in \Gamma} u_{\gamma p, \gamma\delta p}(z) \text{ on } \Omega_{\mathcal{L}_p}.$$

Now, theorem 3.5.3 gives us the map

$$\tilde{\mu} : \mathcal{O}(\Omega_{\mathcal{L}_p})^* \longrightarrow \mathscr{M}(\mathcal{L}_p, \mathbb{Z})_0$$

by which we map the previous functions:

$$\tilde{\mu}(\theta(p - \delta p; z)) = \tilde{\mu}\left(\prod_{\gamma \in \Gamma} u_{\gamma p, \gamma\delta p}(z)\right) = \sum_{\gamma \in \Gamma} \tilde{\mu}(u_{\gamma p, \gamma\delta p}(z)) = \sum_{\gamma \in \Gamma} \mu_{\gamma\delta p^*, \gamma p^*}$$

where the second equality is due to the fact that $\tilde{\mu}$ commutes with limits. Thus, applying results of previous sections, this measure coincides with $-\mu_{\delta^{-1}} = \mu_\delta$ when it is restricted to $\mathcal{L}$, so the image of $\theta(p - \delta p; z)$ by $\tilde{\mu}$ as an analytic function on $\mathcal{L}$ is μ_δ.

For the second claim, let us take an analytic K-automorphic form $f \in \mathcal{O}(\Omega_{\mathcal{L}})^*$. To be automorphic means that for any $\gamma \in \Gamma$, $\gamma \cdot f = c_\gamma f$ for some $c_\gamma \in K^*$.

Therefore, applying the Γ-equivariance of $\tilde{\mu}$ -recall the third part of proposition 3.5.4 and the Γ-invariance of $\mathcal{L}$- we get

$$\gamma \cdot \tilde{\mu}(f) = \tilde{\mu}(\gamma \cdot f) = \tilde{\mu}(c_\gamma f) = \tilde{\mu}(c_\gamma) + \tilde{\mu}(f) = \tilde{\mu}(f)$$

Finally, since we can apply this reasoning for any field K, this is true for all automorphic forms. □

Corollary 3.8.18. *If $f \in \mathcal{O}(\Omega_\mathcal{L})^*$ is an automorphic form, there exists a $\delta \in \Gamma$ such that $\tilde{\mu}(f) = \mu_\delta$.*

Proof. By the previous theorem we have $\tilde{\mu}(f) \in \mathscr{M}(\mathcal{L}, \mathbb{Z})_0^\Gamma$ and by the isomorphism $\Gamma^{ab} \cong \mathscr{M}(\mathcal{L}, \mathbb{Z})_0^\Gamma$ (corollary 2.4.8) there exists a $\delta \in \Gamma$ such that $\tilde{\mu}(f) = \mu_\delta$. □

We give a new proof of the complete result cited above [GvdP80, Ch. 2 (3.2)].

Corollary 3.8.19. *All analytic automorphic forms are products of the theta functions of the form $\theta(p - \delta p; z)$ by constants.*

Proof. This is due to the first claim of the theorem, to the previous corollary and to the fact that the kernel of $\tilde{\mu}$ are the constants. □

We finish this section extending the corollary 3.5.7.

Corollary 3.8.20. *We have a commutative rectangle of short exact sequences with sections for each $z_0 \in \Omega_\mathcal{L}$*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K^* & \longrightarrow & \mathcal{O}(\Omega_\mathcal{L})^* & \underset{\mathcal{I}_{z_0}}{\overset{\tilde{\mu}}{\rightleftarrows}} & \mathcal{M}(\mathcal{L}, \mathbb{Z})_0 & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & K^* & \longrightarrow & \mathcal{A}_\Gamma \cap \mathcal{O}(\Omega_\mathcal{L})^* & \underset{\mathcal{I}_{z_0}}{\overset{\tilde{\mu}}{\rightleftarrows}} & \mathcal{M}(\mathcal{L}, \mathbb{Z})_0^\Gamma & \longrightarrow & 0
\end{array}$$

and with (non-canonical) isomorphisms $\mathcal{O}(\Omega_\mathcal{L})^* \cong K^* \times \mathcal{M}(\mathcal{L}, \mathbb{Z})_0$ and

$$\mathcal{A}_\Gamma \cap \mathcal{O}(\Omega_\mathcal{L})^* \cong K^* \times \mathcal{M}(\mathcal{L}, \mathbb{Z})_0^\Gamma \cong K^* \times \Gamma^{ab}.$$

Proof. We had already built the first exact sequence with its section and the corresponding isomorphism by the corollary 3.5.7. The map $\tilde{\mu}$ restricts to analytic automorphic forms and Γ-invariant measures by the theorem 3.8.17. The same occurs to the section due to the proposition 3.8.8, so we get the exhaustivity and the isomorphism (using the corollary 2.4.8 for the last part). □

3.9 The Albanese variety and the Abel-Jacobi map

Using the results of the previous sections, we show that the Albanese variety and the Abel-Jacobi map of a Mumford curve can be described in terms of multiplicative integrals. The main theorem generalizes the result of Dasgupta [Das05, Thm. 2.5] to any field complete with respect a non-archimedean absolute value. We give, however, a distinct and independent proof.

3.9.1 The abelian variety T/Λ

Let $\Gamma \subset \mathrm{PGL}_2(K)$ be a Schottky group, let $\mathcal{L} := \mathcal{L}_\Gamma \subset \mathbb{P}^1(K)$ be its limit set and let $\Omega_{\mathcal{L}}$ be the functor which associates to any complete extension of fields $L|K$ the set of points $\Omega_{\mathcal{L}}(L)$.

Now we are going to do the following steps aimed at building an abelian variety associated to Γ in a natural way.

Take into consideration the short exact sequence

$$0 \longrightarrow \mathbb{Z}[\Omega_{\mathcal{L}}]_0 \longrightarrow \mathbb{Z}[\Omega_{\mathcal{L}}] \longrightarrow \mathbb{Z} \longrightarrow 0$$

where the first arrow is the injection of divisors of degree zero and the second arrow is the degree map. Since Γ acts on $\Omega_{\mathcal{L}}$, we can take the associated long homology sequence, and in particular, the morphism

$$\begin{array}{ccc} \Gamma^{ab} = H_1(\Gamma, \mathbb{Z}) & \longrightarrow & H_0(\Gamma, \mathbb{Z}[\Omega_{\mathcal{L}}]_0) = \mathbb{Z}[\Omega_{\mathcal{L}}]_{0\Gamma} \\ \gamma & \longmapsto & \gamma p - p \end{array}$$

independent of the chosen $p \in \Omega_{\mathcal{L}}$.

Since the map $⨍_{\bullet} d : \mathbb{Z}[\Omega_{\mathcal{L}}]_0 \longrightarrow \mathrm{Hom}(\mathscr{M}(\mathcal{L}, \mathbb{Z})_0, \mathbb{G}_{m,K})$ is Γ-equivariant we may take Γ-coinvariants, so we obtain

$$⨍_{\bullet} d : \mathbb{Z}[\Omega_{\mathcal{L}}]_{0\Gamma} \longrightarrow \mathrm{Hom}(\mathscr{M}(\mathcal{L}, \mathbb{Z})_0, \mathbb{G}_{m,K})_\Gamma = \mathrm{Hom}(\mathscr{M}(\mathcal{L}, \mathbb{Z})_0^\Gamma, \mathbb{G}_{m,K})$$

and after composing with the connecting map above we get

$$\begin{array}{ccccc} \Gamma^{ab} & \longrightarrow & \mathbb{Z}[\Omega_{\mathcal{L}}]_{0\Gamma} & \longrightarrow & \mathrm{Hom}(\mathscr{M}(\mathcal{L}, \mathbb{Z})_0^\Gamma, \mathbb{G}_{m,K}) \\ \gamma & & \longrightarrow & & ⨍_{\gamma p - p} d : \mu \mapsto ⨍_{\gamma p - p} d\mu \end{array}$$

Note that if $\mathcal{L} \neq \mathbb{P}^1(K)$, then we may take $p \in \Omega_{\mathcal{L}}(K)$. This occurs unless K is local and Γ is cocompact, in which case, since we may take p in any complete extension $L|K$, we also have $\times\!\!\!\!\!\!\int_{\gamma p - p} d\mu \in K^*$.

By corollary 3.6.6 Γ is the fundamental group of $\Gamma\backslash\mathcal{T}_K(\mathcal{L})$, therefore, by corollary 2.4.8 we get a pairing

$$\begin{array}{ccc} \Gamma^{ab} \times \Gamma^{ab} & \xrightarrow{\times\!\!\!\!\!\!\int_{\mathcal{L}}(\ ,\ )} & K^* \\ (\gamma, \gamma') & \longmapsto & \times\!\!\!\!\!\!\int_{\mathcal{L}} (\gamma, \gamma') := \times\!\!\!\!\!\!\int_{\gamma p - p} d\mu_{\gamma'} \end{array}$$

such that, by the corollary 2.4.8 and the proposition 3.4.16,

$$v_K \left(\times\!\!\!\!\!\!\int_{\mathcal{L}} (\gamma, \gamma') \right) = (\gamma, \gamma')_\Gamma$$

for all $\gamma, \gamma' \in \Gamma$. This equality implies that the pairing is positive definite. Further, using corollary 3.7.5 we get

$$\times\!\!\!\!\!\!\int_{\mathcal{L}} (\gamma, \gamma') = \times\!\!\!\!\!\!\int_{\gamma p - p} d\mu_{\gamma'} = \times\!\!\!\!\!\!\int_{\gamma' p - p} d\mu_{\gamma} = \times\!\!\!\!\!\!\int_{\mathcal{L}} (\gamma', \gamma)$$

so the pairing is symmetric too.

Summarizing, we have a morphism

$$\begin{array}{ccc} H_1(\Gamma, \mathbb{Z}) & \xrightarrow{\times\!\!\!\!\!\!\int_{\bullet} d} & \mathrm{Hom}(\mathscr{M}(\mathcal{L}, \mathbb{Z})_0^\Gamma, \mathbb{G}_{m,K}) := T \\ \gamma & \longmapsto & \times\!\!\!\!\!\!\int_{\gamma p - p} d : \mu \mapsto \times\!\!\!\!\!\!\int_{\gamma p - p} d\mu \end{array}$$

which is an isomorphism between $H_1(\Gamma, \mathbb{Z}) \cong \Gamma^{ab}$ and its image Λ, so that it is a free group of rank $g = \mathrm{rank}(\Gamma)$.

Note that, as a consequence of having

$$\times\!\!\!\!\!\!\int_{\gamma p - p} d\mu \in K^*$$

for any $\gamma \in \Gamma$, we get

$$\Lambda \subset T(K) = \mathrm{Hom}(\mathscr{M}(\mathcal{L}, \mathbb{Z})_0^\Gamma, K^*) \cong \mathrm{Hom}(\Gamma^{ab}, K^*) \cong (K^*)^g$$

Let us consider now the valuation map applied to this:

$$\begin{array}{ccc} (K^*)^g & \xrightarrow{v_K} & \mathbb{R}^g \\ (a_1, \ldots, a_g) & \longmapsto & (v_K(a_1), \ldots, v_K(a_g)) \end{array}$$

Lemma 3.9.1. *The subgroup $v_K(\Lambda) \subset \mathbb{R}^g$ is a lattice.*

Proof. Observe the way in which the isomorphism $T(K) \cong (K^*)^g$ works:

$$\begin{array}{ccc} T(K) & \longrightarrow & (K^*)^g \\ \displaystyle\int\!\!\!\!\times & \longmapsto & \left(\displaystyle\int\!\!\!\!\times (\mu_{\gamma_1}), \ldots, \int\!\!\!\!\times (\mu_{\gamma_g})\right) \end{array}$$

where $\gamma_1, \ldots, \gamma_g$ is a fixed basis of the free group Γ. In particular, Λ seen inside of $(K^*)^g$ is the multiplicative subgroup

$$\left\{\left(\int\!\!\!\!\times_{\mathcal{L}} (\gamma, \gamma_1), \ldots, \int\!\!\!\!\times_{\mathcal{L}} (\gamma, \gamma_g)\right)\right\}_{\gamma \in \Gamma}.$$

After applying the valuation map to this we get

$$\left(v_K\left(\int\!\!\!\!\times_{\mathcal{L}} (\gamma, \gamma_1)\right), \ldots, v_K\left(\int\!\!\!\!\times_{\mathcal{L}} (\gamma, \gamma_g)\right)\right) = ((\gamma, \gamma_1)_\Gamma, \ldots, (\gamma, \gamma_g)_\Gamma)$$

that is the image of the map

$$\begin{array}{ccc} \Gamma^{ab} & \longrightarrow & \mathrm{Hom}(\Gamma^{ab}, \mathbb{R}) \cong \mathbb{R}^g \\ \gamma & \longmapsto & v_K\left(\displaystyle\int\!\!\!\!\times_{\mathcal{L}} (\gamma, \cdot)\right) \end{array}$$

As Γ is generated by $\gamma_1, \ldots, \gamma_g$, $v_K(\Lambda) \subset \mathbb{R}^g$ is the subgroup generated by

$$((\gamma_1, \gamma_1)_\Gamma, \ldots, (\gamma_1, \gamma_g)_\Gamma), \ldots, ((\gamma_g, \gamma_1)_\Gamma, \ldots, (\gamma_g, \gamma_g)_\Gamma)$$

which, due to the fact that $(\, , \,)_\Gamma$ is positive definite, is isomorphic to $\mathbb{Z}^g$ so it is a discrete subgroup, and it has maximal rank. Therefore it is a lattice. □

Theorem 3.9.2. *The quotient*

$$T^{an}/\Lambda = \mathrm{Hom}(\mathscr{M}(\mathcal{L}, \mathbb{Z})_0^\Gamma, \mathbb{G}_{m,K})^{an}/\Lambda$$

is an abelian variety.

Proof. By [FvdP04, 6.4, p. 171] we obtain that this quotient is a proper analytic torus.

Note that by means of the identification $\mathscr{M}(\mathcal{L},\mathbb{Z})_0^\Gamma \cong \Gamma^{ab}$, the torus is defined by the map $\gamma \mapsto \times\!\!\!\!\!\!\int_{\mathcal{L}} (\gamma,\cdot)$.

This torus has principal polarization

$$\begin{array}{ccc} \Gamma^{ab}\cong\Lambda & \xrightarrow{\quad\mu^*\quad} & X(T)=\mathrm{Hom}_{K-\mathcal{G}rp}(T,\mathbb{G}_{m,K})\cong\Gamma^{ab} \\ \gamma & \longmapsto & \mu^*(\gamma): \times\!\!\!\!\!\!\int \longmapsto \times\!\!\!\!\!\!\int(\mu_\gamma) \end{array}$$

since

$$\mu^*(\gamma')\left(\times\!\!\!\!\!\!\int_{\gamma p-p} d\right) = \times\!\!\!\!\!\!\int_{\gamma p-p} d\mu_{\gamma'} = \times\!\!\!\!\!\!\int_{\mathcal{L}}(\gamma,\gamma')$$

and this form is symmetric and positive definite. Thus, we conclude that T^{an}/Λ is an abelian variety ([FvdP04, Thm. 6.6.1]). $\square$

Remark 3.9.3. *This statement rests on two main steps: one was proving the symmetry. The other one can be explained by the fact that composing with the valuation gives a real analytic torus (that is, an isogeny) which is the Albanese torus of $G_\Gamma = \Gamma\backslash\mathcal{T}_K(\mathcal{L}_\Gamma)$ by the theorem 2.4.9, since there is a commutative triangle*

$$\begin{array}{ccc} \Gamma^{ab} & \xrightarrow{\;\times\!\!\!\!\!\!\int_\bullet d\;} & \mathrm{Hom}(\mathscr{M}(\mathcal{L}_\Gamma,\mathbb{Z})_0^\Gamma,\mathbb{G}_{m,K}) \\ & \searrow^{\int_\bullet d} & \Big\downarrow v_K \\ & & \mathrm{Hom}(\mathscr{M}(\mathcal{E}(\mathcal{T}_K(\mathcal{L}_\Gamma)),\mathbb{Z})_0^\Gamma,\mathbb{R}). \end{array}$$

Indeed, we take into account the isomorphism $\mathcal{L}\cong\mathcal{E}(\mathcal{T}_K(\mathcal{L}_\Gamma))$ and the next equalities given by the proposition 3.4.16 and the remark 3.4.13 allow conclude the proof of the commutativity:

$$v_K\left(\times\!\!\!\!\!\!\int_{\gamma p-p} d\mu\right) = -\log\left|\times\!\!\!\!\!\!\int_{\gamma p-p} d\mu\right| =$$

$$= -\log\left|\times\!\!\!\!\!\!\int\right|_{\gamma\,\mathrm{r}_{\mathcal{L}}(p)-\mathrm{r}_{\mathcal{L}}(p)} d\mu = \int_{\gamma\,\mathrm{r}_{\mathcal{L}}(p)-\mathrm{r}_{\mathcal{L}}(p)} d\mu$$

3.9.2 The isomorphism with the Albanese variety and the Abel-Jacobi map

An isomorphism between abelian varieties

Our next goal is to get an isomorphism of abelian varieties

$$\operatorname{Alb}(C_\Gamma) \longrightarrow T/\Lambda$$

In order to show this we are going to use the well known isomorphism $\operatorname{Alb}(C_\Gamma) \cong \operatorname{Div}_0(C_\Gamma)/\operatorname{Prin}(C_\Gamma)$. First we will build for any extension of complete fields $L|K$ a map

$$\operatorname{Div}_0(C_\Gamma)(L) \longrightarrow (T/\Lambda)(L)$$

Then, let us fix any extension of complete fields $L|K$. Take a divisor $D \in \operatorname{Div}_0(C_\Gamma)(L)$. It can be written as

$$D = \sum_{p \in C_\Gamma(\mathbb{C}_L)} n_p p \qquad \text{verifying} \qquad D^\sigma = D \ \forall \sigma \in \operatorname{Gal}(\mathbb{C}_L|L)$$

and there exists a finite extension $L'|L$ such that $\operatorname{Supp}(D) \subset C_\Gamma(L')$ so that $D \in \operatorname{Div}_0(C_\Gamma(L'))$. Now, there is a finite field extension $\tilde{L}|L'$ such that G_Γ has no loops (in fact, this is true for almost any extension up to a finite number), so by corollary 3.6.9, the map $\Omega_{\mathcal{L}}(\tilde{L}) \longrightarrow C_\Gamma(\tilde{L})$ is surjective and thus, the maps

$$\Gamma\backslash\Omega_{\mathcal{L}}(\tilde{L}) \longrightarrow C_\Gamma(\tilde{L}) \qquad \text{and} \qquad \Gamma\backslash\mathbb{Z}[\Omega_{\mathcal{L}}(\tilde{L})]_0 \longrightarrow \operatorname{Div}_0(C_\Gamma(\tilde{L}))$$

are isomorphisms. Thus, we got a finite extension $\tilde{L}|L$ such that there is a divisor $\tilde{D} \in \mathbb{Z}[\Omega_{\mathcal{L}}(\tilde{L})]_0$ satisfying $\pi_\Gamma(\tilde{D}) = D$, that is

$$\forall \sigma \in \operatorname{Gal}(\tilde{L}|L) \ \exists \ \gamma_\sigma \in \Gamma \text{ such that } \tilde{D}^\sigma = \gamma_\sigma \tilde{D}.$$

The continuous arrows of the diagram

$$\begin{array}{ccc}
\mathbb{Z}[\Omega_{\mathcal{L}}(\tilde{L})]_{0\Gamma} & \xrightarrow{\ \fint_\bullet d\ } & \operatorname{Hom}(\mathscr{M}(\mathcal{L},\mathbb{Z})_0^\Gamma, \tilde{L}^*) = T(\tilde{L}) \\
\downarrow \ \tilde{D} & \longmapsto & \fint_{\tilde{D}} d : \mu \longmapsto \fint_{\tilde{D}} d\mu \qquad \searrow \\
\operatorname{Div}_0(C_\Gamma(\tilde{L})) & \dashrightarrow & T(\tilde{L})/\Lambda \\
\pi_\Gamma(\tilde{D}) = D & \longmapsto & \fint_D d
\end{array}$$

factorize by the dots arrow, since $\Gamma\backslash\mathbb{Z}[\Omega_{\mathcal{L}}(\tilde{L})]_0 \cong H_1(\Gamma,\mathbb{Z})\backslash\mathbb{Z}[\Omega_{\mathcal{L}}(\tilde{L})]_{0_\Gamma}$.

We can finish the construction of the map we told above thanks to the following result.

Lemma 3.9.4. *Given a finite extension $\tilde{L}|L$ and any $\tilde{D} \in \mathbb{Z}[\Omega_{\mathcal{L}}(\tilde{L})]_0$ satisfying*

$$\forall\sigma \in \mathrm{Gal}(\tilde{L}|L)\ \exists\ \gamma_\sigma \in \Gamma \text{ such that } \tilde{D}^\sigma = \gamma_\sigma \tilde{D},$$

we have

$$\left(⨍_{\tilde{D}} d\right)^\sigma \equiv ⨍_{\tilde{D}} d \quad (mod\ \Lambda)$$

Proof. We just have to note how it is defined the integral, as a limit of products of the function f_D. This is integrated over $\mathcal{L}$, set of K-rational points, so invariant by σ. Therefore, for any $\mu \in \mathscr{M}(\mathcal{L},\mathbb{Z})_0$ we have

$$\left(⨍_{\tilde{D}} d\mu\right)^\sigma \left(⨍_{\tilde{D}} d\mu\right)^{-1} = \left(⨍_{\mathcal{L}} f_{\tilde{D}} d\mu\right)^\sigma \left(⨍_{\tilde{D}} d\mu\right)^{-1} =$$

$$= ⨍_{\mathcal{L}} f_{\tilde{D}^\sigma} d\mu \left(⨍_{\tilde{D}} d\mu\right)^{-1} = ⨍_{\mathcal{L}} f_{\gamma_\sigma\tilde{D}} d\mu \left(⨍_{\mathcal{L}} f_{\tilde{D}} d\mu\right)^{-1} =$$

$$= ⨍_{\gamma_\sigma\tilde{D}-\tilde{D}} d\mu$$

independent of μ. Finally $⨍_{\gamma_\sigma\tilde{D}-\tilde{D}} d \in \Lambda$. □

Corollary 3.9.5. *Under the same hypothesis we get*

$$⨍_{\tilde{D}} d \in (T/\Lambda)(L)$$

Proof. It is immediate. □

Therefore, for $D \in \mathrm{Div}_0(C_\Gamma)(L)$ we have built a well defined element

$$⨍_{D} d \in (T/\Lambda)(L),$$

so we get the map

$$\mathrm{Div}_0(C_\Gamma)(L) \longrightarrow (T/\Lambda)(L)$$

Next we want to show its exhaustivity and compute its kernel. The next result is crucial to move forward:

Lemma 3.9.6. *Let $\tilde{D}$ be a degree zero divisor on $\Omega_{\mathcal{L}}$ which can be represented as $\sum_{i=1}^{r}(p_i - q_i)$ and let us define the automorphic form*

$$\theta_{\tilde{D}}(z) := \theta(p_1 - q_1; z) \cdots \theta(p_r - q_r; z)$$

Then its factor of automorphy is given by

$$c_{\theta_{\tilde{D}}}(\gamma) = \times\!\!\!\!\!\!\!\int_{\tilde{D}} d\mu_\gamma \quad \forall\, \gamma \in \Gamma$$

Proof. On one hand we have

$$c_{\theta_{\tilde{D}}}(\gamma) = \frac{\theta_{\tilde{D}}(z)}{\theta_{\tilde{D}}(\gamma z)} = \frac{\theta(p_1 - q_1; z) \cdots \theta(p_r - q_r; z)}{\theta(p_1 - q_1; \gamma z) \cdots \theta(p_r - q_r; \gamma z)} =$$

$$= \frac{\theta(z - \gamma z; p_1) \cdots \theta(z - \gamma z; p_r)}{\theta(z - \gamma z; q_1) \cdots \theta(z - \gamma z; q_r)}$$

where the last equality is due to the straightforward symmetry of theta functions. On the other hand, applying the theorem 3.8.17 and the extended Poisson formula (corollary 3.5.6) we have

$$\times\!\!\!\!\!\!\!\int_{\tilde{D}} d\mu_\gamma = \times\!\!\!\!\!\!\!\int_{\tilde{D}} d\tilde{\mu}(\theta(z_0 - \gamma z_0; \cdot)) = \prod_{i=1}^{r} \frac{\theta(z_0 - \gamma z_0; p_i)}{\theta(z_0 - \gamma z_0; q_i)}$$

Since the right sides of two last chains of equalities are independent of z and z_0 respectively, they are equal. □

Lemma 3.9.7. *If $h \in \mathcal{O}(\Omega_{\mathcal{L}})^*$ is an (analytic) automorphic form, its factor of automorphy c_h belongs to Λ.*

Proof. First, recall by corollary 3.8.18 that $\tilde{\mu}(h) = \mu_\delta$ for some $\delta \in \Gamma$. Next, let us compute its automorphic form on a $\gamma \in \Gamma$ by means of applying the Poisson formula:

$$c_h(\gamma) = \frac{h(z)}{h(\gamma z)} = \times\!\!\!\!\!\!\!\int_{z-\gamma z} d\tilde{\mu}(h) = \times\!\!\!\!\!\!\!\int_{z-\gamma z} d\mu_\delta = \times\!\!\!\!\!\!\!\int_{z-\delta z} d\mu_\gamma = \times\!\!\!\!\!\!\!\int_{z-\delta z} d(\mu_\gamma)$$

Finally, $\times\!\!\!\!\!\!\!\int_{z-\delta z} d$ belongs to Λ by definition. □

Proposition 3.9.8. *Given an automorphic form $h \in \mathcal{A}_\Gamma$ with factor of automorphy c_h, there is a finite divisor $\tilde{D}_h$ on $\Omega_{\mathcal{L}}$ such that the infinite divisor of h on $\Omega_{\mathcal{L}}$ is $D_h = \Gamma \cdot \tilde{D}_h$ and*

$$c_h \equiv c_{\theta_{\tilde{D}_h}} = \times\!\!\!\!\!\!\!\int_{\tilde{D}_h} d \ (mod\ \Lambda)$$

Proof. We take $\tilde{D}_h$ a finite divisor as in theorem 3.8.16, such that $D_h = \Gamma \cdot \tilde{D}_h$ and $h(z) = h'(z)\theta_{\tilde{D}_h}(z)$ with $h'(z)$ analytic. Then, by the previous lemmas we have

$$c_h = c_{h'} c_{\theta_{\tilde{D}_h}} \equiv c_{\theta_{\tilde{D}_h}} = \displaystyle{\times}\!\!\!\!\!\!\int_{\tilde{D}_h} d \pmod{\Lambda}$$

□

Corollary 3.9.9. *The map* $\mathrm{Div}_0(C_\Gamma)(L) \longrightarrow (T/\Lambda)(L)$ *factorize by the principal divisors of* C_Γ *and the resulting map*

$$\mathrm{Div}_0(C_\Gamma)(L)/\,\mathrm{Prin}(C_\Gamma)(L) \longrightarrow (T/\Lambda)(L)$$

is injective.

Proof. First we will show that the map factorize by the principal divisors.

A divisor of $Div_0(C_\Gamma)(L)$ is principal when it is the divisor of a meromorphic function on C_Γ, that is a Γ-invariant meromorphic function on $\Omega_{\mathcal{L}}$. Let D_h and h be such a divisor and such a function respectively. Since h is Γ-invariant, its factor of automorphy is constant equal to 1. Therefore, by the proposition we get

$$\displaystyle{\times}\!\!\!\!\!\!\int_{\tilde{D}_h} d \equiv 1 \pmod{\Lambda}$$

with $D_h = \Gamma \tilde{D}_h$, and so we obtain the factorization by the principal divisors.

Next we want to prove the injectivity of this factorized map. Take now a $D \in \mathrm{Div}_0(C_\Gamma)(L)$ such that

$$\displaystyle{\times}\!\!\!\!\!\!\int_{\tilde{D}} d \in \Lambda \text{ so there exists a } \delta \in \Gamma \text{ satisfying } \displaystyle{\times}\!\!\!\!\!\!\int_{\tilde{D}} d = \displaystyle{\times}\!\!\!\!\!\!\int_{\delta p - p} d$$

where $D = \Gamma \tilde{D}$ with $\tilde{D}$ divisor on $\Omega_{\mathcal{L}}$ and $p \in \Omega_{\mathcal{L}}$. Now, as above, we can build the automorphic form $\theta_{\tilde{D}}$, which has associated infinite divisor D. Further, let us consider the analytic function $\theta(\delta p - p; z)$, and write $c_{\tilde{D}}$ and c_δ for the factors of automorphy of the two last automorphic forms. Observe that

$$c_{\tilde{D}}(\gamma) = \displaystyle{\times}\!\!\!\!\!\!\int_{\tilde{D}} d\mu_\gamma = \displaystyle{\times}\!\!\!\!\!\!\int_{\delta p - p} d\mu_\gamma = c_\delta(\gamma).$$

Therefore, D is the divisor associated to the function $\theta_{\tilde{D}}(z)/\theta(\delta p - p; z)$, which is Γ-invariant, so it is principal and thus the injectivity is done.

□

Proposition 3.9.10. *There is an isomorphism*

$$\left(\mathrm{Div}_0(C_\Gamma)/\,\mathrm{Prin}(C_\Gamma)\right)(L) \longrightarrow (T/\Lambda)(L)$$

Proof. Let us check first that this map is well defined.

Consider a divisor D in $\left(\mathrm{Div}_0(C_\Gamma)/\operatorname{Prin}(C_\Gamma)\right)(L)$. Then, there is a Galois extension $\tilde{L}|L$ and a divisor $\tilde{D} \in \mathrm{Div}_0(C_\Gamma)(\tilde{L})$ such that

$$\tilde{D}^\sigma - \tilde{D} \in Prin(C_\Gamma)(\tilde{L}) \text{ for all } \sigma \in \mathrm{Gal}(\tilde{L}|L).$$

This implies that

$$\oint_{\tilde{D}^\sigma - \tilde{D}} d = 0_{T/\Lambda} \in (T/\Lambda)(\tilde{L})$$

and so, as in the proof of the lemma 3.9.4 we get the next equalities in $(T/\Lambda)(\tilde{L})$:

$$\left(\oint_{\tilde{D}} d\right)^\sigma = \oint_{\tilde{D}^\sigma} d = \oint_{\tilde{D}} d \quad \forall \sigma \in \mathrm{Gal}(\tilde{L}|L)$$

Therefore $\oint_D d \in (T/\Lambda)(L)$ and we get the morphism

$$\left(\mathrm{Div}_0(C_\Gamma)/\operatorname{Prin}(C_\Gamma)\right)(L) \longrightarrow (T/\Lambda)(L)$$

which is injective by the previous corollary.

Next we have to prove its exhaustivity. An element $\Xi \in (T/\Lambda)(L)$ can be seen in $T(\tilde{L})/\Lambda$, satisfying $\Xi^\sigma = \Xi$ for each $\sigma \in \mathrm{Gal}(\tilde{L}|L)$, where $\tilde{L}|L$ is a Galois extension. This element is the class of a $\xi \in T(\tilde{L}) \cong \mathrm{Hom}(\Gamma^{ab}, \tilde{L}^*)$ such that

$$\xi^\sigma \equiv \xi \pmod{\Lambda} \quad \text{for each} \quad \sigma \in \mathrm{Gal}(\tilde{L}|L),$$

which in turn is the factor of automorphy c_h of an automorphic form $h \in \mathcal{A}_\Gamma$, by the proposition 3.8.9. Now, by the proposition 3.9.8 we have

$$\oint_{\tilde{D}_h} d \equiv c_h = \xi \pmod{\Lambda} \qquad \text{and so} \qquad \oint_{D_h} d = \Xi$$

with $D_h \in \mathrm{Div}_0(C_\Gamma)(\tilde{L})$. By the hypothesis

$$\left(\oint_{D_h} d\right)^\sigma = \oint_{D_h} d \qquad \text{so} \qquad \oint_{D_h^\sigma - D_h} d = 0_{T/\Lambda}$$

what, due to the injectivity of the map, gives that $D_h^\sigma - D_h \in \operatorname{Prin}(C_\Gamma)(\tilde{L})$. But this for each $\sigma \in \mathrm{Gal}(\tilde{L}|L)$ implies that $D_h \in \left(\mathrm{Div}_0(C_\Gamma)/\operatorname{Prin}(C_\Gamma)\right)(L)$. □

Now we are ready to prove the main theorem, which generalizes to any complete field with respect to a non-trivial non-archimedean valuation [Das05, Thm. 2.5]:

Theorem 3.9.11. *There is an isomorphism over K of abelian varieties*

$$\mathrm{Alb}(C_\Gamma) \longrightarrow T/\Lambda$$

Proof. First, as we told above, we recall the isomorphism

$$\mathrm{Alb}(C_\Gamma) \cong \mathrm{Div}_0(C_\Gamma)/\,\mathrm{Prin}(C_\Gamma)$$

Second, we have built an analytic morphism of abelian varieties

$$\mathrm{Div}_0(C_\Gamma)/\,\mathrm{Prin}(C_\Gamma) \longrightarrow T/\Lambda$$

Since they are proper, by GAGA it is an algebraic morphism, and it also respects the group operations, so it is a morphism of abelian varieties. Further, it induces an isomorphism in the corresponding L-points for any extension of complete fields $L|K$, and this implies that it is an isomorphism. □

The Abel-Jacobi map

Corollary 3.9.12. *The abelian variety T^{an}/Λ is the Albanese variety of the curve C_Γ and the Abel Jacobi map is given, after having fixed some point $z_0 \in C_\Gamma$, by*

$$\begin{array}{ccc} C_\Gamma & \xrightarrow{\;i_{z_0}\;} & \mathrm{Hom}(\mathscr{M}(\mathcal{L},\mathbb{Z})_0^\Gamma, \mathbb{G}_{m,K})/\Lambda \\ z & \longmapsto & \displaystyle \times\!\!\!\!\!\!\int_{z-z_0} d \end{array}$$

Remark 3.9.13. *Next, we put together the remarks 3.6.8 and 3.9.3. This Abel-Jacobi map descends to the one of the associated graph by means of the retraction. That is, we have a commutative diagram*

$$\begin{array}{ccccccc} \Omega_{\mathcal{L}_\Gamma} & \twoheadrightarrow & C_\Gamma & \xrightarrow{\;i_{z_0}\;} & \dfrac{\mathrm{Hom}(\mathscr{M}(\mathcal{L},\mathbb{Z})_0^\Gamma, \mathbb{G}_{m,K})}{\Lambda} & \cong & \mathrm{Alb}(C_\Gamma) \\ \downarrow{\scriptstyle \mathrm{r}_{\mathcal{L}_\Gamma}} & & \downarrow{\scriptstyle \mathrm{r}_{\mathcal{L}_\Gamma,\Gamma}} & & \downarrow{\scriptstyle v_K} & & \\ \mathcal{T}_K(\mathcal{L}_\Gamma) & \twoheadrightarrow & G_\Gamma & \xrightarrow{\;i_p\;} & \dfrac{\mathrm{Hom}(\mathscr{M}(\mathcal{E}(\mathcal{T}_K(\mathcal{L}_\Gamma)),\mathbb{Z})_0^\Gamma, \mathbb{R})}{\displaystyle\int d\left(\Gamma^{ab}\right)} & \cong & \mathrm{Alb}(G_\Gamma). \end{array}$$

where $p = \mathrm{r}_{\mathcal{L}_\Gamma,\Gamma}(z_0)$. Indeed, for any $z \in C_\Gamma$ let $\widetilde{z} \in \Omega_{\mathcal{L}_\Gamma}$ be a representant, so that $\widetilde{p} := \mathrm{r}_{\mathcal{L}_\Gamma}(\widetilde{z}_0) \in \mathcal{T}_K(\mathcal{L}_\Gamma)$ is a representant for $p \in G_\Gamma$ and more generally,

$\mathrm{r}_{\mathcal{L}_\Gamma}(\widetilde{z})$ *is a representant for* $\mathrm{r}_{\mathcal{L}_\Gamma,\Gamma}(z)$*. Next, we compute as in the last remark named:*

$$v_K\left(\fint_{\widetilde{z}-\widetilde{z}_0} d\mu\right) = -\log\left|\fint_{\widetilde{z}-\widetilde{z}_0} d\mu\right| =$$

$$= -\log\left|\fint\right|_{\mathrm{r}_{\mathcal{L}_\Gamma}(\widetilde{z})-\widetilde{p}} d\mu = \int_{\mathrm{r}_{\mathcal{L}}(\widetilde{z})-\widetilde{p}} d\mu = i_p(\mathrm{r}_{\mathcal{L}_\Gamma,\Gamma}(z)).$$

This summarizes several results from [BR15] that we have recovered for Berkovich analytic Mumford curves, as the theorem 2.9 (which is, essentially, our corollary 2.4.8), the proposition 6.1 (it is what we have just proved in this remark) and the corollary 6.6, which says that there is a canonical isomorphism between the skeleton of the Albanese torus $\mathrm{Alb}(C_\Gamma)^{an}$ *of* C_Γ *and the Albanese torus* $\mathrm{Alb}(G_\Gamma)$ *of the skeleton of* C_Γ*, and that* v_K *coincides with the retraction to the skeleton, which, moreover, is the tropicalization map (cf. [Gub10, § 4]).*

Chapter 4

The conjectural construction of the Albanese variety of a non-Archimedean uniformized variety

Mustafin generalized "Mumford's construction of nonarchimedean uniformization for curves over a discrete valued field K to the multidimensional case" in [Mus78]. In order to do it, he introduced the Bruhat-Tits building $\mathscr{B}(G)$ associated to a $d+1$-dimensional vector space V, then, given a certain subgroup $\Gamma \subset \mathrm{PGL}(V)$ that he called normal hyperbolic and that genaralizes Schottky groups, he considered a subbuilding $\mathscr{B}_{\mathcal{L}_\Gamma} \subset \mathscr{B}(G)$ obtained from the dual set $\mathcal{L}_\Gamma$ of the set of limit points of Γ . With these objects, Mustafin built a formal scheme $\Omega_{\mathcal{L}_\Gamma}$ which arises from projective d-space over K by removing the dual hyperplanes of the points in $\mathcal{L}_\Gamma$ as a rigid analytic variety, and that uniformizes the object of his research, that he obtain as a quotient $\Gamma\backslash\Omega_{\mathcal{L}_\Gamma} =: X_\Gamma$ and that inherits a rigid analytic structure.

These uniformized varieties are algebraizable in some cases, like when they are abelian varieties (in which case the abelian variety is a quotient of a torus $\mathbb{G}^d_{m,K}$ by a lattice), or also when the base field is local and the group Γ is discrete, cocompact and without torsion. In the last case, $\mathcal{L}_\Gamma = \mathbb{P}(V)$, $\mathscr{B}_{\mathcal{L}_\Gamma} = \mathscr{B}(G)$ and $\Omega_{\mathcal{L}_\Gamma}$ is the rigid analytic space called p-adic symmetric space introduced by Drinfeld in [Dri74] generalizing the 1-dimensional p-adic upper half plane as a p-adic analogue of the real symmetric spaces.

Drinfeld remarked the importance of the cohomology of the p-adic symmetric spaces, which computed Schneider and Stuhler in [SS91], where they also computed the cohomology of their quotient varieties by the groups Γ. Later, in [dS01], de Shalit went deeper in the study of the rigid de Rham cohomology of the p-adic symmetric spaces, giving a new description in terms of "a certain space of harmonic cochains on the Bruhat-Tits building" and answering "a few questions left open in the original approach" by Schneider and Stuhler. This author continued this study together with Alon in [AdS02] (and in [AdS03]), where they related the different descriptions given of the rigid de Rham cohomology of $\Omega_{\mathbb{P}(V)}$. In particular, they translated a description given in [SS91] to the language of harmonic measures on the space of K-points of a certain flag variety which, combinatorially, can be seen as part of the spherical building boundary of the Bruhat-Tits building of $\mathrm{PGL}(V)$ (which is the Bruhat-Tits building of $\mathrm{GL}(V)$), and they described the isomorphism between these space of harmonic measures and the space of harmonic cochains on the building.

Finally, Raskind and Xarles defined in [RX07a] the notion of projective varieties with totally degenerate reduction, which applies to abelian varieties, to the quotients of the p-adic symmetric spaces by torsion free, discrete, cocompact subgroups $\Gamma \subset \mathrm{PGL}(V)$ and, more generally, to any nonarchimedean uniformized variety $X_\Gamma = \Gamma\backslash\Omega_{\mathcal{L}_\Gamma}$ being algebraizable as a projective variety. Then, in [RX07b], they associated to those varieties certain rigid analytic tori that they called "p-adic intermediate Jacobians", and a kind of Abel-Jacobi maps to them. As their complex analogues introduced by Griffith, the "extreme" p-adic intermediate Jacobians are the Picard variety and the Albanese variety.

The original motivation for this work is to give a more analytic construction of such tori, but early we had decided to focus on the Albanese varieties, since a big field for research is open only with their study. In this chapter we give a conjectural construction of the Albanese varieties in the paragraph 4.5.3 and in the following we study a way to prove that it is in fact a rigid analytic torus when the uniformized variety is a surface. The key step is the proof of the isomorphism between the harmonic measures on $\mathcal{L}_\Gamma$ with the harmonic cochains on $\mathscr{B}_{\mathcal{L}_\Gamma}$, which generalizes the isomorphisms proved by Schneider and Stuhler, and de Shalit and Alon when Γ is discrete, cocompact and torsion free ($\mathcal{L}_\Gamma = \mathbb{P}(V)$) and K is local.

4.1 The Bruhat-Tits building (over a discrete valuation field)

In this section, we introduce the Bruhat-Tits building following mainly the combinatorial approach by Mustafin and de Shalit in [Mus78] and [dS01] respectively. We also introduce some special subcomplexes $\mathscr{B}_{\mathcal{L}}$, that for certain compact sets $\mathcal{L} \subset \mathbb{P}(V)$ are the subbuildings in which we are interested. In particular, we start to study their minimal 1-skeleton.

Let K be a complete field with respect to a non-trivial discrete valuation v_K, let $\mathcal{O}_K$ be its valuation ring, let $\mathfrak{m}_K = (\pi_K)$ be its maximal ideal and $k = \mathcal{O}_K/\mathfrak{m}_K$ its residual field.

Let V be a $(d+1)$-dimensional K-vector space, and denote by V^* its dual, so $\mathbb{P}_V = \mathrm{Proj}(S^\bullet(V^*))$ is the projective space associated to V, whose K-rational points correspond to the 1-dimensional subspaces of V (so, with the traditional notation we have $\mathbb{P}(V) = \mathbb{P}_V(K)$). We will write $G := \mathrm{PGL}(V)$, the group of automorphisms of $\mathbb{P}_V$ as K-algebraic variety.

A lattice in V is a free $\mathcal{O}_K$-module $L \subset V$ of rank $d+1$, so it spans V over K. Two lattices L, L' are equivalent ($L \sim L'$) if there exists $\lambda \in K^*$ such that $L' = \lambda L$. We may consider the left action of the group $\mathrm{GL}(V)$ on the set of such lattices in the natural way: if $\gamma \in G$, $\gamma L = \{\gamma x \mid x \in L\}$. Then, $L \sim L'$ if and only if L and L' belong to the same orbit of the center $K^* \subset \mathrm{GL}(V)$, so we get a left action of G on the set of classes of equivalence of lattices.

The Bruhat-Tits building of G is a simplicial complex (not necessarily locally finite) $\mathscr{B}(G)$ whose vertices -which we shall denote by $\mathscr{B}(G)_0$- are the set of lattices in V up to equivalence. We shall denote the equivalence class of L by $[L]$.

There is a metric ρ in $\mathscr{B}(G)_0$. Take $\Lambda = [L], \Lambda' = [L']$ two any vertices in V. By the equivalence, we may assume $L \supset L'$. Therefore we have

$$L/L' \cong \mathcal{O}_K/\mathfrak{m}_K^{m_0} \oplus \cdots \oplus \mathcal{O}_K/\mathfrak{m}_K^{m_d}, \qquad m_i \geq 0$$

and we define $\rho([L],[L']) := \max_i m_i - \min_i m_i$. This is equivalent to take L' in the same class as above, satisfying $L \supset L' \supset \mathfrak{m}_K^r L$, $L' \not\supset \mathfrak{m}_K^{r-1} L$ and defining $\rho([L],[L']) := r$. For any integer $0 \leq q \leq d$, a q-dimensional simplex (or cell) of $\mathscr{B}(G)$ is a subset $\Delta = \{\Lambda_0, \ldots, \Lambda_q\}$ of $\mathscr{B}(G)_0$ such that satisfies any of the next equivalence conditions:

- For any $i \neq j$, $\rho(\Lambda_i, \Lambda_j) = 1$.
- We may choose $\Lambda_i = [L_i]$ such that $L_0 \supsetneq L_1 \supsetneq \cdots \supsetneq L_q \supsetneq \pi_K L_0$ (what we shall call a q-flag in L_0).

(see [Gar97, Ch. 19]). We shall denote the set of q-simplices by $\mathscr{B}(G)_q$, as usual. As already defined, 0-simplices are called vertices, 1-simplices are called edges, and d-simplices are the maximal simplices of $\mathscr{B}(G)$, which will be called chambers. The subsimplices of a simplex are called its faces. The codimension 1 faces ($= d-1$-simpices) will be called panels.

Definition 4.1.1. *Two chambers Δ, Δ' are adjacent if they are distinct and have a common panel $A = \Delta \bigcap \Delta'$.*

A pregallery of length n is a sequence of $n+1$ chambers $S_{\overline{\Delta}} = (\Delta_0, \dots, \Delta_n)$ such that Δ_i and Δ_{i+1} are adjacent or equal for $i = 0, 1, \dots, n-1$. A gallery is a pregallery in which $\Delta_i \neq \Delta_{i+1}$ for all i.

We say that Δ_0 and Δ_n are the ends of the (pre-)gallery, that $S_{\overline{\Delta}}$ is a (pre-)gallery from Δ_0 to Δ_n or that it connects these chambers.

A gallery is minimal if there is no gallery with the same ends and length strictly less than n, and that length is called the gallery distance between Δ_0 and Δ_n.

Notice the cyclic order of the vertices of a q-simplex Δ of $\mathscr{B}(G)$ (and so, its orientation). We shall say that Δ is pointed if it has a distinguished vertex, or equivalently, if we fix an order in it $\Lambda_0 = [L_0] < \cdots < \Lambda_q = [L_q]$ where $L_0 \supsetneq L_1 \supsetneq \cdots \supsetneq L_q \supsetneq \pi_K L_0$. In this case, writing $d_i = \dim_k(L_i/L_{i+1})$ (where $L_{q+1} := \pi_K L_0$), we say that Δ has type $\mathbf{t} = (d_0, \dots, d_q) \in \mathbb{N}^{q+1}_{\leq d+1}$. Note that $\sum_{i=0}^{q} d_i = \dim_k(L_0/\pi_K L_0) = d + 1$ and

$$\sum_{i=j}^{q} d_i = \dim_k(L_j/\pi_K L_0) =: n_j.$$

A pointed q-simplex Δ can be written

$$\Delta = (L_0 \supsetneq L_1 \supsetneq \cdots \supsetneq L_q \supsetneq \pi_K L_0) = (\Lambda_0, \Lambda_1, \dots, \Lambda_q).$$

There are $\binom{d}{q}$ types of q-simplices, of which we shall call the minimal type to $(d+1-q, 1\dots, 1)$. We will denote the set of pointed q-simplices of type $\mathbf{t}$, the minimal q-simplices, and the set of all the pointed q-simplices by

$$\widehat{\mathscr{B}(G)}_q^{\mathbf{t}}, \ \widehat{\mathscr{B}(G)}_q^{min} \text{ and } \widehat{\mathscr{B}(G)}_q$$

respectively, so that

$$\widehat{\mathscr{B}(G)}_q = \bigsqcup_{\mathbf{t}} \mathscr{B}(G)_q^{\mathbf{t}}.$$

Note that $\widehat{\mathscr{B}(G)}_0 = \mathscr{B}(G)_0$.

We will call the distinguished vertex Λ_0 of a pointed simplex Δ its source, and we will denote it as $s(\Delta) = \Lambda_0$, generalizing the edges classical notation. Precisely for an edge e, its another vertex is called its target and denoted by $t(e)$, as usual.

Given two simplices, pointed or not, Δ' and Δ, if Δ is a face of Δ' as q-simplices, we will write $\Delta \leq \Delta'$.

Given a simplex Δ_0, we will denote by $\mathscr{B}(G)_q(\Delta_0)$ the set of q-simplices Δ such that for any vertices $\Lambda_0 \leq \Delta_0$, $\Lambda \leq \Delta$, then $\rho(\Lambda_0, \Lambda) \leq 1$. This is equivalent to say that Δ_0 and Δ are face of a common chamber. We will denote by $\widehat{\mathscr{B}(G)}_q(\Delta_0)$ the set of pointed q-simplices Δ verifying the same property and we will specify the type if we restrict to it accordingly: $\widehat{\mathscr{B}(G)}_q^{\mathbf{t}}(\Delta_0)$. Removing the specifications, the subcomplex with its cells will be denoted by $\mathscr{B}(G)(\Delta_0)$.

For a general $n \in \mathbb{N}$, we will denote the set of q-simplices (resp. pointed, resp. of a given type $\mathbf{t}$) Δ such that $\rho(\Lambda_0, \Lambda) \leq n$ for any vertices $\Lambda_0 \leq \Delta_0$, $\Lambda \leq \Delta$, by

$$\mathscr{B}(G)_q(\Delta_0)^{(n)} \text{ (resp. } \widehat{\mathscr{B}(G)}_q(\Delta_0)^{(n)}, \text{ resp. } \widehat{\mathscr{B}(G)}_q^{\mathbf{t}}(\Delta_0)^{(n)}),$$

and we will denote $\mathscr{B}(G)(\Delta_0)^{(n)}$ the subcomplex generated by them.

We will say that a basis $\{v_0, \ldots, v_d\}$ of V is adapted to a q-simplex $\Delta = \{[L_0], \ldots, [L_q]\}$ with source L_0 if we have

$$L_i = \bigoplus_{j<n_i} \mathcal{O}_K v_j \oplus \bigoplus_{j \geq n_i} \mathcal{O}_K \pi_K v_j \text{ (and then } L_i/\pi_K L_0 = \bigoplus_{j<n_i} k v_j).$$

Such a basis always exists ([Mus78, Lem. 1.1.]).

The apartment associated to a basis $\mathbf{v} = \{v_0, \ldots, v_d\}$ of V is the simplicial subcomplex of $\mathscr{B}(G)$ generated by the vertices of the form $[\bigoplus_i \mathcal{O}_K \pi_K^{m_i} v_i]$, where $m_i \in \mathbb{Z}$. We will denote it by $\mathbb{A}_{\mathbf{v}}$. The same notations for the sets of q-simplices, pointed q-simplices and the ones of a given type introduced for $\mathscr{B}(G)$ apply to the apartments. Clearly, we have a bijection

$$\mathbb{A}_{\mathbf{v}0} \cong \mathbb{Z}^{d+1}/\mathbb{Z} \cdot (1, \ldots, 1)$$

mapping $[\bigoplus_i \mathcal{O}_K \pi_K^{m_i} v_i]$ to $[(m_0, \ldots, m_d)] = (m_0, \ldots, m_d) + \mathbb{Z}(1, \ldots, 1)$, which extends to an isomorphism of cyclic ordered simplicial complexes

$$\mathbb{A}_{\mathbf{v}} \cong \mathbb{Z}^{d+1}/\mathbb{Z} \cdot (1, \ldots, 1)$$

giving to $\mathbb{Z}^{d+1}/\mathbb{Z} \cdot (1, \ldots, 1)$ the suitable structure.

Let us recall some facts from [Mus78, § 1.].

- Fixed a vertex $\Lambda = [L] \in \mathscr{B}(G)_0$, the set of q-simplices Δ with source Λ are in one to one canonical correspondence with the q-flags in L and with the flags of length r in k^{d+1}, preserving the type.

- Given two chambers there exists at least an apartment containing both.

- The action of G on $\mathscr{B}(G)_0$ extends to an action on $\widehat{\mathscr{B}(G)}_q$ on the left, which is simplicial and transitive on simplices with distinguished vertex of the same type.

- $\mathscr{B}(G)$ is a contractible simplicial complex of dimension d. The topological realization of any apartment $\mathbb{A} \subset \mathscr{B}(G)$ is isomorphic to $\mathbb{R}^d$.

- The isomorphisms $|\mathbb{A}_{\mathbf{v}}| \cong \mathbb{R}^d$ induce a G-invariant Euclidean metric on the topological realization of the building $|\mathscr{B}(G)|$.

Given a basis $\mathbf{v} = \{v_0, \ldots, v_d\}$ of V consider the element $v_{d+1} = \sum_{i=0}^{d} v_d$. Let us compute the intersection of $\mathbb{A}_{\mathbf{v}}$ with the apartment associated to the basis $\mathbf{v^0} := \{v_{d+1}, v_1, \ldots, v_d\}$. It is the subcomplex generated by the set of classes of lattices $\bigoplus_i \mathcal{O}_K \pi_K^{m_i} v_i$ which coincide with lattices of the form

$$\mathcal{O}_K \pi_K^{m_0'} v_{d+1} \oplus \bigoplus_{i>1} \mathcal{O}_K \pi_K^{m_i'} v_i.$$

To get the equality we need $m_i = m_i'$ for all i and $m_0 \geq m_i$ for all i, and this condition is enough.

Therefore, if we call $\mathbf{v^i}$ to the basis consequence of replacing v_i by v_{d+1} we get

$$\mathbb{A}_{\mathbf{v}} \cap \bigcap_{i=0}^{d} \mathbb{A}_{\mathbf{v^i}} = \left[\bigoplus_i \mathcal{O}_K v_i \right] =: t([v_0], \ldots, [v_d], [v_{d+1}])$$

where the notation is inspired by the introduced in the previous chapter.

Observe that the apartment $\mathbb{A}_{\mathbf{v}}$ only depends on the classes $[v_i] \in \mathbb{P}(V)$, but the isomorphism with $\mathbb{Z}d+1/\mathbb{Z} \cdot (1, \ldots, 1)$ depends on the given basis ordered. So in this last case we will have to use the introduced notation $\mathbb{A}_{\mathbf{v}}$, but in general, if we do not need so much precision, we can write an apartment as $\mathbb{A}_{\mathcal{P}}$, where $\mathcal{P} \subset \mathbb{P}(V)$ is a set of $d+1$ projective points linearly independent (as $\{[v_0], \ldots, [v_d]\}$).

More generally, for any subset $\mathcal{L} \subset \mathbb{P}(V)$ and any set of $d+1$ points linearly independent $\mathcal{P} \subset \mathcal{L}$ we may consider the apartment $\mathbb{A}_{\mathcal{P}}$. We consider the subcomplex of $\mathscr{B}(G)$ associated to $\mathcal{L}$ defined by

$$\mathscr{B}_{\mathcal{L}} := \bigcup_{\substack{\mathcal{P} \subset \mathcal{L} \\ \mathcal{P} = \{p_0, \ldots, p_d\} \\ \text{l. i.}}} \mathbb{A}_{\mathcal{P}}$$

It is a simplicial complex of dimension d whose maximal cells are chambers of $\mathscr{B}(G)$. In some cases, it is a building in the sense of [AB08, Ch. 4] (it is a convex subcomplex of $\mathscr{B}(G)$), as when $\mathscr{B}_{\mathcal{P}} = \mathbb{A}_{\mathcal{P}}$ and $\mathscr{B}_{\mathbb{P}(V)} = \mathscr{B}(G)$, but generally it is not. For example, assume $d = 2$, let v_0, v_1, v_2 be a basis for V and take $\mathcal{L} = \{[v_0], [v_1], [v_2], [v_0 + v_1 + v_2]\}$. In this case $\mathscr{B}_{\mathcal{L}}$ is not a building, since there is no apartment in $\mathscr{B}_{\mathcal{L}}$ containing the vertices $[\mathcal{O}_K \pi_K v_0 \oplus \mathcal{O}_K v_1 \oplus \mathcal{O}_K v_2]$ and $[\mathcal{O}_K v_0 \oplus \mathcal{O}_K (v_0 + v_1 + v_2) \oplus \mathcal{O}_K \pi_K v_2]$ simultaneously.

If $\mathscr{B}_{\mathcal{L}}$ is a building, we know it is contractible ([AB08, Thm. 4.127]). It seems reasonable to think $\mathscr{B}_{\mathcal{L}}$ is contractible even if it is not a building, but we are not going to use this and then, we will not prove that.

The same notations for the sets of q-simplices, pointed q-simplices and the ones of a given type introduced for $\mathscr{B}(G)$ apply to $\mathscr{B}_{\mathcal{L}}$.

Let us define the covalence of a panel A in $\mathscr{B}_{\mathcal{L}}$ as the number of chambers in $\mathscr{B}_{\mathcal{L}}$ containing that panel. We will denote it by $\mathrm{cov}_{\mathcal{L}}(A)$.

From now on, $\mathcal{L} \subset \mathbb{P}(V)$ is a closed subset not contained in a hyperplane (so $\mathscr{B}_{\mathcal{L}}$ is not empty).

Proposition 4.1.2. *Given a vertex $\Lambda \in \mathscr{B}_{\mathcal{L}0}$ and a point $p \in \mathcal{L}$, there exists a subset $\mathcal{P} \subset \mathcal{L}$ of $d+1$-linearly independent points such that $p \in \mathcal{P}$ and $\Lambda \in \mathbb{A}_{\mathcal{P}0}$.*

Proof. Since $\Lambda = [L] \in \mathscr{B}_{\mathcal{L}0}$, there is a basis $\mathbf{v} = \{v_0, \ldots, v_d\}$ such that $[v_i] \in \mathcal{L}$ for each i and $L = \bigoplus_{i=0}^{d} \mathcal{O}_K v_i \in \mathbb{A}_{\mathbf{v}_0}$. Consider a representant $v = \sum_{i=0}^{d} \lambda^i v_i \in V$ of the point $p = [v]$. Since $v \neq 0$, there exists an i such that $\{v\} \cup \mathbf{v} \setminus \{v_i\}$ is a basis of V. More specifically, we choose this i such that $v_K(\lambda^i) \leq v_K(\lambda^j)$ for all $j \neq i$. Without loss of generality we assume $i = 0$. Further, taking a suitable representant v (multiplying by $\pi_K^{-v_K(\lambda^0)}$), we have that $\mathbf{v}' = \{v, v_1, \ldots, v_d\}$ is a basis for L, and therefore $\Lambda \in \mathbb{A}_{\mathbf{v}'_0}$. $\square$

The minimal subgraph

We have a particular interest in the minimal edges of $\mathscr{B}(G)$, so next we are going to restrict us to an apartment and see how are these when we look at $\mathbb{Z}^{d+1}/\mathbb{Z} \cdot (1, \ldots, 1)$, after the given isomorphism. Then, we will work with edges with a distinguished vertex.

Consider the norm in $\mathbb{Z}^{d+1}$ defined by

$$\|(m_0, \ldots, m_d)\| := \max_{i,j} |m_i - m_j| = \max_{i,j} \{m_i - m_j\} = \max_i m_i - \min_j m_j.$$

It is an easy exercise to check that it is a norm. Further, it factorizes by $\mathbb{Z} \cdot (1, \ldots, 1)$, so we get a norm in $\mathbb{Z}^{d+1}/\mathbb{Z} \cdot (1, \ldots, 1)$. Thus, we shall call it the tropical norm, and denote it by $\| \cdot \|_{trop}$ (as others had already done previously).

Let $\Lambda,\ \Lambda'$ be two vertices, and assume that they correspond to

$$[\underline{m}] = [(m_0, \dots, m_d)] \text{ and } [\underline{m'}] = [(m'_0, \dots, m'_d)]$$

respectively. I claim that $(\Lambda,\ \Lambda')$ is an edge if and only if $\|[\underline{m}]-[\underline{m'}]\|_{trop} = 1$. After subtracting $\underline{m}$, we can assume that Λ corresponds to $[\underline{0}]$. Then, the assertion becomes clear if we remind that $(\Lambda,\ \Lambda')$ is an edge if and only if $\rho(\Lambda,\ \Lambda') = 1$, and we observe that $\rho(\Lambda,\ \Lambda') = \|[\underline{m'}] - [\underline{0}]\|_{trop}$.

Identically, if we have vertices $\Lambda_0, \dots, \Lambda_q$ corresponding to $[\underline{m^{(0)}}], \dots, [\underline{m^{(q)}}]$, they form a simplex if and only if $\|[\underline{m^{(i)}}] - [\underline{m^{(j)}}]\|_{trop} = 1$ for all $i \neq j$.

Now, that an edge $(\Lambda,\ \Lambda')$ corresponding to $([\underline{m}],[\underline{m'}])$ is minimal if and only if $[\underline{m'}] - [\underline{m}] = [(1, \dots, 1, 0, 1, \dots, 1)]$ is just a quick verification.

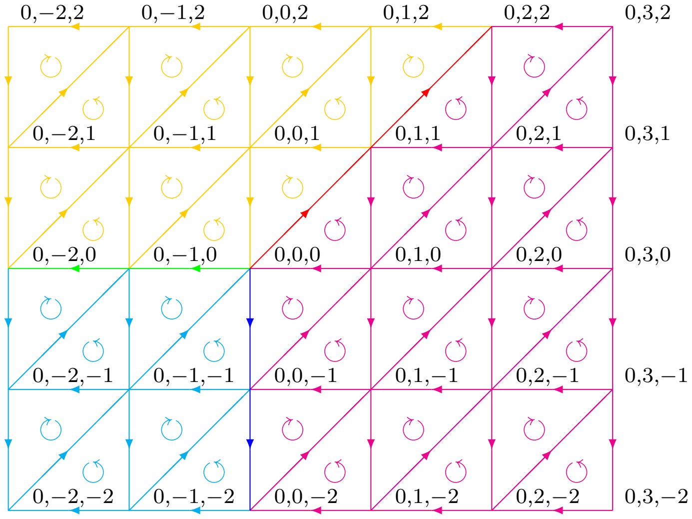


Figure 4.1: A small portion of an apartment $\mathbb{A}$ for $d = 2$ seen inside $\mathbb{Z}^3/\mathbb{Z}(1,1,1)$ with the orientations of the maximal simplices and the minimal edges indicated, and colored following the intersections $\mathbb{A}_{\mathbf{v}} \cap \mathbb{A}_{\mathbf{v}^i}$ studied above.

Definition 4.1.3. *We shall say that two minimal edges $e, e' \in \widehat{\mathscr{B}(G)}_1^{min}$ are straight if either $t(e) = s(e')$ or $t(e') = s(e)$, and there are no chamber containing both edges.*

Proposition 4.1.4. *Let $\mathbb{A}$ be an apartment. For each $e \in \widehat{\mathbb{A}}_1^{min}$ there exists a unique $e' \in \widehat{\mathbb{A}}_1^{min}$ such that they are straight with $t(e) = s(e')$ (and the same applies with $t(e') = s(e)$).*

Proof. We can write $e = ([\underline{m}], [\underline{m'}])$ with $[\underline{m}], [\underline{m'}] \in \mathbb{Z}^{d+1}/\mathbb{Z} \cdot (1, \dots, 1)$. Since we ask for $t(e) = s(e')$, we also can write $e' = ([\underline{m'}], [\underline{m''}])$ with $[\underline{m''}] \in \mathbb{Z}^{d+1}/\mathbb{Z} \cdot (1, \dots, 1)$.

Since e, e' are minimal, we have

$$[\underline{m'}] - [\underline{m}] = [(1, \dots, 1, 0, 1, \dots, 1)]$$

with the 0 in the position i, and

$$[\underline{m''}] - [\underline{m'}] = [(1, \dots, 1, 0, 1, \dots, 1)]$$

with the 0 in the position i'. Recall that $e, [\underline{m}], [\underline{m'}], i$ are given, while $[\underline{m''}], i''$ are the unknowns. If $i' \neq i$ we have

$$\|[\underline{m''}] - [\underline{m}]\|_{trop} = 1$$

so $([\underline{m'}], [\underline{m''}])$ is an edge and $[\underline{m}], [\underline{m'}], [\underline{m''}]$ are contained in a chamber. Otherwise $i' = i$, $[\underline{m''}] = [\underline{m'}] + ([\underline{m'}] - [\underline{m}])$ and

$$\|[\underline{m''}] - [\underline{m}]\|_{trop} = 2,$$

therefore they do not belong to any common chamber. □

Remark 4.1.5. *For some basis $\{v_0, \dots, v_d\}$ we have $s(e) = [\bigoplus_i \mathcal{O}_K v_i]$, and $i_0 \in \{0, 1, \dots, d\}$ such that $t(e) = [\mathcal{O}_K v_{i_0} \oplus \bigoplus_{i \neq i_0} \mathcal{O}_K \pi_K v_i]$. Thus, we obtain*

$$t(e') = \left[\mathcal{O}_K v_{i_0} \oplus \bigoplus_{i \neq i_0} \mathcal{O}_K \pi_K^2 v_i \right].$$

Sum action on an apartment and parallelism

Note that we have a left action

$$\mathbb{Z}^{d+1} \times \mathbb{Z}^{d+1}/\mathbb{Z} \cdot (1, \dots, 1) \longrightarrow \mathbb{Z}^{d+1}/\mathbb{Z} \cdot (1, \dots, 1)$$

given by the sum of vectors

$$(z_0, \dots, z_d) \cdot (m_0, \dots, m_d) := (z_0 + m_0, \dots, z_d + m_d),$$

so we shall denote it $(z_0, \dots, z_d) + (m_0, \dots, m_d)$. It factorizes by $\mathbb{Z} \cdot (1, \dots, 1)$, so we get a left action

$$\mathbb{Z}^{d+1}/\mathbb{Z} \cdot (1, \dots, 1) \times \mathbb{A}_{\mathbf{v}} \longrightarrow \mathbb{A}_{\mathbf{v}}.$$

Definition 4.1.6. *We will say that two minimal edges $e, e' \in \widehat{\mathbb{A}_{\boldsymbol{v}1}}^{min}$ are parallel with respect to $\boldsymbol{v}$ and we shall denote this by $e \parallel_{\boldsymbol{v}} e'$, if they are in the same orbit by the action of $\mathbb{Z}^{d+1}/\mathbb{Z} \cdot (1, \dots, 1)$, that is, if there exists $[\underline{n}] \in \mathbb{Z}^{d+1}/\mathbb{Z} \cdot (1, \dots, 1)$ such that $e' = [\underline{n}] + e$.*

Note that, at least initially, this definition depends on the basis. In fact we see that it only depends on the points $[v_i] \in \mathbb{P}(V)$, where v_i are the vectors of the basis, but neither on the representants v_i (whose change is a translation of the vectors), nor on their order (whose change gives a reorder of the coordinates), that is, it only depends on the apartment.

Note also that this notion of parallelism restricted to an apartment is an equivalence relation, since it is defined by the orbit of an action.

Another consequence of the proof of the proposition 4.1.4, is that two straight edges are parallel with respect to any basis.

Proposition 4.1.7. *The notion of parallelism between minimal edges does not depend on the apartment. In particular, two minimal edges are parallel in an apartment containing both if and only if they are parallel in any apartment containing them.*

Proof. Let e, e' edges both contained in two apartments $\mathbb{A}, \mathbb{A}'$ of $\mathscr{B}(G)$. Since $\mathscr{B}(G)$ is a building, it verifies the well known property for such objects that given two simplices in two apartments, there is a simplicial isomorphism of the apartments fixing both simplices (cf. [AB08, Def. 4.1]). Now consider a simplicial isomorphism $\mathbb{Z}^{d+1}/\mathbb{Z} \cdot (1, \dots, 1) \cong \mathbb{A}$. Next, let us compose the simplicial isomorphisms $\mathbb{A}' \cong \mathbb{A} \cong \mathbb{Z}^{d+1}/\mathbb{Z} \cdot (1, \dots, 1)$, so we get another simplicial isomorphism between $\mathbb{A}'$ and $\mathbb{Z}^{d+1}/\mathbb{Z} \cdot (1, \dots, 1)$. Since e, e' are in the intersection of both apartments, they are mapped to the same image, and since parallelism does not depend on these isomorphisms, they are parallel with respect to $\mathbb{A}$ if and only if they are parallel with respect to $\mathbb{A}'$. □

4.2 The open sets associated to the minimal edges of $\mathscr{B}_{\mathcal{L}}$

We are going to study open sets associated to the minimal edges of $\mathscr{B}(G)$, which, when $\mathcal{L} \subset \mathbb{P}(V)$ is compact, induce a basis for the topology of $\mathcal{L}$ by open compacts. We also will introduce some "simplicial" maps and obtain some important properties related to the structure of partitions by these open sets, which we will use following repeatedly and that will become key for later proofs.

Lemma 4.2.1. *Let V be a K-vector space and let L be a lattice in V, so it is a free $\mathcal{O}_K$-module with a natural isomorphism $L \otimes_{\mathcal{O}_K} K \cong V$. Let $Z \subset V$ be a K-vector subspace and $L' := \pi_K L + (Z \cap L)$. Then*

$$\dim_K(Z) = \dim_k(L'/\pi_K L).$$

Proof. First, recall the well known facts that $Z \cap L$ is a free $\mathcal{O}_K$-module of rank less than $\mathrm{rank}_{\mathcal{O}_K}(L)$ and in fact,

$$\mathrm{rank}_{\mathcal{O}_K}(Z \cap L) = \dim_k \left(\frac{Z \cap L}{\pi_K (Z \cap L)} \right) = \dim_k \left(\frac{L'}{\pi_K L} \right).$$

Next, we are going to prove $\mathrm{rank}_{\mathcal{O}_K}(Z \cap L) = \dim_K(Z)$. Let us write $s := \mathrm{rank}_{\mathcal{O}_K}(Z \cap L)$, so $Z \cap L = \langle w_1, \dots, w_s \rangle_{\mathcal{O}_K}$. Then, clearly $s \leq \dim_K(Z)$. If $s < \dim_K(Z)$, there would be a $v \in Z \setminus \langle w_1, \dots, w_s \rangle_K$ and an $r \in \mathbb{Z}$ such that $\pi_K^r v \in L \cap Z$, so we would get a contradiction. Therefore, $s = \dim_K(Z)$. □

Following, let $Z \subset V$ be a 1-dimensional K-vector subspace. As we have shown in the previous proof, there exists $w \in L$ such that $Z \cap L = \mathcal{O}_K w$. Further, we see that $w \in L \setminus \pi_K L$ and it is unique up to $\mathcal{O}_K^*$. Therefore we have $\pi_K L + (Z \cap L) = \pi_K L + \mathcal{O}_K w$.

Observe that for any lattice L we have

$$L/\pi_K L \cong k^{d+1}.$$

Then, for each vertex $\Lambda = [L] \in \mathscr{B}(G)_0$ there is a reduction map

$$\mathrm{r}_\Lambda : \mathbb{P}(V) \longrightarrow \mathbb{P}(L \otimes_{\mathcal{O}_K} k) \cong \mathbb{P}^d(k)$$

defined as follows: for any $Z \in \mathbb{P}(V)$ we have just seen that there is $\omega \in Z \cap L$ such that $Z \cap (L \setminus \pi_K L) = \mathcal{O}_K^* \omega$, and therefore we can define $\mathrm{r}_\Lambda(Z)$ as the class of this element in

$$((L \setminus \pi_K L)/\pi_K L) / \mathcal{O}_K^* \cong (k^{d+1} \setminus \{0\})/k^* = \mathbb{P}^d(k).$$

Recall that a 1-flag in V is a linear subspace of dimension 1, that is a point of $\mathbb{P}(V)$. Let $e = (\Lambda, \Lambda')$ be a minimal edge and take representants $\Lambda = [L]$, $\Lambda' = [L']$ verifying $L \supsetneq L' \supsetneq \pi_K L$. Since e is minimal, $\dim_k(L'/\pi_K L) = 1$. We define the open set associated to e by

$$\mathcal{B}(e) := \{ Z \subset V \mid \dim_K(Z) = 1, \ L' = \pi_K L + (Z \cap L) \} \subset \mathbb{P}(V),$$

and more generally, given $\mathcal{L} \subset \mathbb{P}(V)$ closed, one defines

$$\mathcal{B}_{\mathcal{L}}(e) := \mathcal{B}(e) \cap \mathcal{L}.$$

Note that if $\mathcal{L}' \subset \mathcal{L}$, then $\mathcal{B}_{\mathcal{L}'}(e) \subset \mathcal{B}_{\mathcal{L}}(e)$.

Lemma 4.2.2. *We have $Z \in \mathcal{B}(e)$ if and only if $Z \cap L \subset L'$. In particular $[u] \in \mathcal{B}(e)$ if and only if the representant of $[u]$ in $L \setminus \pi_K L$ belongs to $L' \setminus \pi_K L$.*

Proof. One implication is obvious and the opposite follows from previous considerations. We have seen that $Z \cap L = \mathcal{O}_K w$ for a $w \in L \setminus \pi_K L$. If $L' \supsetneq \pi_K L + (Z \cap L)$, any w' in the difference would be K-linearly independent with w, what would imply $\dim_k(L'/\pi_K L) \geq 2$, therefore it cannot exist, and $L' = \pi_K L + (Z \cap L)$. $\square$

Thus, the open set can also be defined as

$$\mathcal{B}(e) = \{Z \in \mathbb{P}(V) \mid \mathrm{r}_\Lambda(Z) = L'/\pi_K L\} = \mathrm{r}_\Lambda^{-1}(L'/\pi_K L).$$

The minimal star maps

Observe that given $\Lambda = [L]$, the point $Z =: z \in \mathbb{P}(V)$ determines $\Lambda' = [L']$ and (Λ, Λ') is a minimal edge, and so, given $\Lambda, \Lambda'_1, \Lambda'_2$ such that $e_1 = (\Lambda, \Lambda'_1)$ and $e_2 = (\Lambda, \Lambda'_2)$ are minimal edges, $\mathcal{B}(e_1) \cap \mathcal{B}(e_2) = \emptyset$. Therefore, for each $\Lambda \in \mathscr{B}(G)_0$ we get

$$\mathbb{P}(V) = \bigsqcup_{\substack{e \in \widehat{\mathscr{B}(G)}_1^{min} \\ s(e) = \Lambda}} \mathcal{B}(e)$$

Given a closed set $\mathcal{L} \subset \mathbb{P}(V)$, we shall call the minimal star of Λ in $\mathscr{B}_\mathcal{L}$ the set of edges

$$\mathrm{St}_\mathcal{L}^{min}(\Lambda) = \{e \in \widehat{\mathscr{B}_{\mathcal{L}1}}^{min} \mid s(e) = \Lambda\}$$

and when $\mathcal{L} = \mathbb{P}(V)$ we will write $\mathrm{St}^{min} := \mathrm{St}^{min}_{\mathbb{P}(V)}$.

Then, another way to tell the previous discussion is that the map r_Λ induces an injective map

$$\mathrm{r}_\Lambda^{min} : \mathrm{St}^{min}(\Lambda) \longrightarrow \mathbb{P}^d(k),$$

which maps $e = (\Lambda, \Lambda')$ with $\Lambda = [L], \Lambda = [L']$ and $L \supsetneq L' \supsetneq \pi_K L$, to $L'/\pi_K L \in \mathbb{P}(L \otimes_{\mathcal{O}_K} k)$. Then, for $e = (\Lambda, \Lambda')$ we have

$$\mathcal{B}(e) = \{Z \in \mathbb{P}(V) \mid \ \mathrm{r}_\Lambda(Z) = \mathrm{r}_\Lambda^{min}(e)\} = \mathrm{r}_\Lambda^{-1}(\mathrm{r}_\Lambda^{min}(e))$$

and

$$\mathbb{P}(V) = \bigsqcup_{e \in \mathrm{St}^{min}(\Lambda)} \mathrm{r}_\Lambda^{-1}(\mathrm{r}_\Lambda^{min}(e)) = \bigsqcup_{e \in \mathrm{St}^{min}(\Lambda)} \mathcal{B}(e).$$

It is not immediate that we have the similar equality

$$\mathcal{L} = \bigsqcup_{\mathrm{St}_\mathcal{L}^{min}(\Lambda)} \mathcal{B}_\mathcal{L}(e),$$

since it means that for any point $p \in \mathcal{L}$ and any $\Lambda \in \mathscr{B}_{\mathcal{L}0}$ there is $e \in \mathrm{St}_\mathcal{L}^{min}(\Lambda)$ such that $p \in \mathcal{B}(e)$. We know that there is a unique $e \in \mathrm{St}(\Lambda)$ such that $p \in \mathcal{B}(e)$, so we have to see that actually $e \in \widehat{\mathscr{B}_{\mathcal{L}1}}^{min}$.

Proposition 4.2.3. *Let $\mathcal{L} = \mathcal{P}$ be a set of $d+1$ points linearly independents, so $\mathscr{B}_{\mathcal{L}} = \mathbb{A}_{\mathcal{P}}$. Then, for any minimal edge of $\mathbb{A}_{\mathcal{P}}$, $\mathcal{B}_{\mathcal{P}}(e)$ consists of exactly one point of $\mathcal{P}$ and for each point $p \in \mathcal{P}$ and each vertex $\Lambda \in \mathbb{A}_{\mathcal{P}0}$ there is an edge $e \in \mathrm{St}_{\mathcal{P}}^{min}(\Lambda)$ such that $\{p\} = \mathcal{B}_{\mathcal{P}}(e)$.*

Proof. Without loss of generality we assume $\mathcal{P} = \{p_0, \dots, p_d\}$, with $p_i = [v_i]$, where $\mathbf{v} = \{v_0, \dots, v_d\}$ is a basis of V, such that

$$\mathcal{P} = \{(1:0:\dots:0), (0:1:0:\dots:0), \dots, (0:\dots:0:1)\}$$

and $s(e) = [\underline{0}]$. Then this minimal edge has the form

$$e = ([\underline{0}], [(1, \dots, 1, 0, 1, \dots, 1)])$$

after going through the isomorphism $\mathbb{A}_{\mathbf{v}} \cong \mathbb{Z}^{d+1}/\mathbb{Z} \cdot (1, \dots, 1)$. Now, writing

$$e = \left(\bigoplus_{i=0}^{d} \mathcal{O}_K v_i \supsetneq \bigoplus_{j \neq i} \mathcal{O}_K \pi_K v_j \oplus \mathcal{O}_K v_i \supsetneq \bigoplus_{i=0}^{d} \mathcal{O}_K \pi_K v_i \right),$$

we claim that $\mathcal{B}_{\mathcal{P}}(e) = \{p_i\}$. Indeed, it is clear, since

$$\bigoplus_{j \neq i} \mathcal{O}_K \pi_K v_j \oplus \mathcal{O}_K v_i = \bigoplus_{i=0}^{d} \mathcal{O}_K \pi_K v_i + \left(\bigoplus_{i=0}^{d} \mathcal{O}_K v_i \cap \langle v_h \rangle \right) \Longleftrightarrow i = h$$

Reciprocally, we can assume $\Lambda = [\underline{0}]$ and we have seen that for each p_i there is an edge $e \in \mathrm{St}_{\mathcal{P}}^{min}(\Lambda)$ of the chosen form such that $\{p_i\} = \mathcal{B}_{\mathcal{P}}(e)$. □

Remark 4.2.4. *Note that if we have $[\underline{n}] \in \mathbb{Z}^{d+1}/\mathbb{Z} \cdot (1, \dots, 1)$ and*

$$e' := [\underline{n}] + e = ([\underline{n}], [(n_0 + 1, \dots, n_{i-1} + 1, n_i, n_{i+1} + 1, \dots, n_d + 1)]),$$

after taking the basis $u_j := \pi_K^{n_j} v_j$ for the same apartment we can apply the same proof to e' and we get $\mathcal{B}_{\mathcal{P}}(e') = \mathcal{B}_{\mathcal{P}}(e)$.

Corollary 4.2.5. *Given $\mathcal{L} \subset \mathbb{P}(V)$ closed and e minimal edge in $\mathscr{B}_{\mathcal{L}}$, the open set $\mathcal{B}_{\mathcal{L}}(e)$ is not empty. Further, if $\mathcal{P}_0 \subset \mathcal{L}$ is a set of $d+1$ points linearly independents such that $e \in \widehat{\mathbb{A}_{\mathcal{P}_0 1}}^{min}$, then $\mathcal{P}_0 \cap \mathcal{B}_{\mathcal{L}}(e)$ is exactly one point.*

Proof. The minimal edge e is contained in an apartment $\mathbb{A}_{\mathcal{P}}$ with $\mathcal{P} \subset \mathcal{L}$, therefore $\emptyset \neq \mathcal{B}_{\mathcal{P}}(e) \subset \mathcal{B}_{\mathcal{L}}(e)$. The second claim follows from the fact that, since $e \in \widehat{\mathbb{A}_{\mathcal{P}_0 1}}^{min}$ we get $\mathcal{P}_0 \cap \mathcal{B}_{\mathcal{L}}(e) = \mathcal{B}_{\mathcal{P}_0}(e)$. □

Corollary 4.2.6. *Let e be a minimal edge in $\mathscr{B}(G)$ and $p \in \mathcal{B}(e)$. Then there exists a set $\mathcal{P}$ of $d+1$ points linearly independents containing p, such that e belongs to the apartment $\mathbb{A}_{\mathcal{P}}$. If $s(e) \in \mathscr{B}_{\mathcal{L}0}$ and $p \in \mathcal{L}$ we can choose $\mathcal{P} \subset \mathcal{L}$.*

Proof. Take $\Lambda = s(e)$ and apply the proposition 4.1.2 to get a set $\mathcal{P}$ and the corresponding apartment $\mathbb{A}_{\mathcal{P}} \subset \mathscr{B}(G)$. The same lemma gives us the last claim under the added hypotheses. Since there is a unique edge $e \in \mathrm{St}^{min}(\Lambda)$ such that $p \in \mathcal{B}(e)$ and there is an edge $e' \in \mathrm{St}^{min}_{\mathcal{P}}(\Lambda) \subset \mathrm{St}^{min}(\Lambda)$ such that $\{p\} = \mathcal{B}_{\mathcal{P}}(e')$, we conclude $e = e'$. $\square$

Corollary 4.2.7. *Given an edge $e \in \widehat{\mathscr{B}(G)}_1^{min}$ and a closed subset $\mathcal{L} \subset \mathbb{P}(V)$ such that $s(e), t(e) \in \mathscr{B}_{\mathcal{L}}$, e is in $\mathscr{B}_{\mathcal{L}}$ if and only if $\mathcal{B}_{\mathcal{L}}(e) = \mathcal{B}(e) \bigcap \mathcal{L} \neq \emptyset$.*

Proof. It is a consequence of the two last corollaries put together. $\square$

Corollary 4.2.8. *Given $\mathcal{L} \subset \mathbb{P}(V)$ closed, a vertex $\Lambda \in \mathscr{B}_{\mathcal{L}0}$ and a point $p \in \mathcal{L}$, there exists $e \in \mathrm{St}^{min}_{\mathcal{L}}(\Lambda)$ such that $p \in \mathcal{B}_{\mathcal{L}}(e)$. Therefore*

$$\mathcal{L} = \bigsqcup_{\mathrm{St}^{min}_{\mathcal{L}}(\Lambda)} \mathcal{B}_{\mathcal{L}}(e).$$

Corollary 4.2.9. *Given $\mathcal{L} \subset \mathbb{P}(V)$ closed, the sets $\mathcal{B}_{\mathcal{L}}(e)$ are open and closed.*

Proof. Consider the minimal star of $s(e)$. The union of all the open sets associated to the edges in $\mathrm{St}^{min}_{\mathcal{L}}(s(e)) \setminus \{e\}$ is open, therefore, its complementary $\mathcal{B}_{\mathcal{L}}(e)$ is closed. $\square$

Corollary 4.2.10. *If $\mathcal{L} \subset \mathbb{P}(V)$ is compact, for all vertex Λ_0 in $\mathscr{B}_{\mathcal{L}}$, $\mathrm{St}^{min}_{\mathcal{L}}(\Lambda_0)$ is finite.*

Proof. The edges in $\mathrm{St}^{min}_{\mathcal{L}}(\Lambda_0)$ provide a disjoint union by open sets of $\mathcal{L}$, which has to be finite when this set is compact. $\square$

Then, we can think the minimal star as a map

$$\mathrm{St}^{min}_{\mathcal{L}} : \mathbb{Z}[\mathscr{B}_{\mathcal{L}0}] \longrightarrow \mathbb{Z}[\widehat{\mathscr{B}}_{\mathcal{L}1}^{min}]$$

defined by

$$\mathrm{St}^{min}_{\mathcal{L}}(\Lambda) = \sum_{\substack{e \in \widehat{\mathscr{B}}_{\mathcal{L}1}^{min} \\ s(e)=\Lambda}} e.$$

Corollary 4.2.11. *If $\mathcal{L} \subset \mathbb{P}(V)$ is compact, the sets $\mathcal{B}_{\mathcal{L}}(e)$ are compact.*

Proof. Since they are closed sets in a compact set, they are compact. □

We may restrict the maps r_Λ and r_Λ^{min} to

$$\mathrm{r}_\Lambda^{\mathcal{L}} : \mathcal{L} \longrightarrow \mathbb{P}^d(k)$$

and

$$\mathrm{r}_\Lambda^{min} : \mathrm{St}_{\mathcal{L}}^{min} \longrightarrow \mathbb{P}^d(k)$$

respectively. Then, for $e = (\Lambda, \Lambda') \in \widehat{\mathscr{B}_{\mathcal{L}1}}^{min}$ we have

$$\mathcal{B}_{\mathcal{L}}(e) = \mathcal{B}(e) \cap \mathcal{L} = \mathrm{r}_\Lambda^{-1}(\mathrm{r}_\Lambda^{min}(e)) \cap \mathcal{L} = \mathrm{r}_\Lambda^{\mathcal{L}^{-1}}(\mathrm{r}_\Lambda^{min}(e)).$$

Corollary 4.2.12. *Given $\mathcal{L} \subset \mathbb{P}(V)$ closed, for any $\Lambda \in \mathscr{B}_{\mathcal{L}0}$,*

$$\mathrm{r}_\Lambda^{min}(\mathrm{St}_{\mathcal{L}}^{min}(\Lambda)) = \mathrm{Im}(\mathrm{r}_\Lambda^{\mathcal{L}}) = \mathrm{r}_\Lambda(\mathcal{L}).$$

Corollary 4.2.13. *For any vertex $\Lambda \in \mathscr{B}(G)_0$, if $\mathcal{L} \subset \mathbb{P}(V)$ is compact, its reduction $\mathrm{r}_\Lambda(\mathcal{L}) = \mathrm{r}_\Lambda^{\mathcal{L}}(\Lambda)$ is finite.*

Proof. Indeed the map r_Λ^{min} is injective and $\mathrm{St}_{\mathcal{L}}^{min}(\Lambda)$ is finite, therefore, $\mathrm{r}_\Lambda(\mathcal{L}) = \mathrm{r}_\Lambda^{min}(\mathrm{St}_{\mathcal{L}}^{min}(\Lambda))$ is finite. □

Proposition 4.2.14. *If $\mathcal{L} \subset \mathbb{P}(V)$ is compact, the complex $\mathscr{B}_{\mathcal{L}}$ is locally finite.*

Proof. All we have to show is that every vertex Λ is contained in a finite number of cells. Since each cell is contained in a chamber, and the number of faces of chambers is finite, it is enough to show that every vertex is contained in a finite number of chambers.

Note that a cell of $\mathscr{B}_{\mathcal{L}}$ is contained in some apartment of the complex, therefore in some chamber of $\mathscr{B}(G)$ in $\mathscr{B}_{\mathcal{L}}$, which, in turn, has dimension d.

Next, on one hand, we have just remarked that the number of minimal edges with source a given vertex is finite. On the other hand, every minimal edge is contained in a chamber of $\mathscr{B}_{\mathcal{L}}$. Therefore, it is equivalent to prove that the number of chambers containing a minimal edge is finite.

Fix a vertex Λ and a minimal edge $e \in \mathrm{St}_{\mathcal{L}}^{min}(\Lambda)$. Assume that there are an infinite number of chambers which contain e, and note that each of them also includes a minimal edge with source $t(e)$. Since $\mathrm{St}_{\mathcal{L}}^{min}(t(e))$ is finite, there is a minimal edge $e_1 \in \mathrm{St}_{\mathcal{L}}^{min}(t(e))$ such that the number of chambers containing e and e_1 is infinite, and all of them comprise $t(e_1)$. After applying the same reasoning we get $e_2 \in \mathrm{St}_{\mathcal{L}}^{min}(t(e_2))$ such that there are infinite chambers including e, e_1 and e_2. Recursively we get $d+1$ minimal edges contained in infinite chambers, but they determine a unique chamber, so we have arrived to a contradiction.

Thus, the number of chambers to which Λ belongs is finite. □

Proposition 4.2.15. *Let $\mathcal{L} = \mathcal{P}$ be a set of $d+1$ points linearly independents $p_i = [v_i]$, so $\mathscr{B}_{\mathcal{L}} = \mathbb{A}_{\mathcal{P}}$, and write $\boldsymbol{v} = \{v_0, \dots, v_d\}$. Then, given minimal edges e, e' of $\mathbb{A}_{\boldsymbol{v}}$, we have $\mathcal{B}_{\mathcal{P}}(e) = \mathcal{B}_{\mathcal{P}}(e')$ if and only if $e \parallel_{\boldsymbol{v}} e'$.*

Proof. $\Longleftarrow|$

This is what we have shown in the remark 4.2.4.

$\Longrightarrow|$

We can write $e = ([\underline{m^{(0)}}], [\underline{m^{(1)}}]), e' = ([\underline{m'^{(0)}}], [\underline{m'^{(1)}}])$ with

$$[\underline{m^{(0)}}], [\underline{m^{(1)}}], [\underline{m'^{(0)}}], [\underline{m'^{(1)}}] \in \mathbb{Z}^{d+1}/\mathbb{Z} \cdot (1, \dots, 1).$$

Since e, e' are minimal, we have

$$[\underline{m^{(1)}}] - [\underline{m^{(0)}}] = [(1, \dots, 1, 0, 1, \dots, 1)]$$

with the 0 in the position i_e, and

$$[\underline{m'^{(1)}}] - [\underline{m'^{(0)}}] = [(1, \dots, 1, 0, 1, \dots, 1)]$$

with the 0 in the position $i_{e'}$. Again, by the remark 4.2.4 and by the proposition 4.2.3, $\mathcal{B}_{\mathcal{P}}(e) = \mathcal{B}_{\mathcal{P}}(e')$ implies $i_e = i_{e'}$, that is,

$$[\underline{m^{(1)}}] - [\underline{m^{(0)}}] = [\underline{m'^{(1)}}] - [\underline{m'^{(0)}}].$$

Now, define $[\underline{n}] := [\underline{m'^{(0)}}] - [\underline{m^{(0)}}]$. Then we get

$$[\underline{m'^{(1)}}] = [\underline{m'^{(0)}}] + ([\underline{m'^{(1)}}] - [\underline{m'^{(0)}}]) = [\underline{m'^{(1)}}] + ([\underline{m^{(1)}}] - [\underline{m^{(0)}}]) = [\underline{m^{(1)}}] + [\underline{n}]$$

So we have $[\underline{m'^{(0)}}] = [\underline{n}] + [\underline{m^{(0)}}]$ and $[\underline{m'^{(1)}}] = [\underline{n}] + [\underline{m^{(1)}}]$, therefore $e' = [\underline{n}] + e$, as we claimed. □

Corollary 4.2.16. *Let $\mathcal{L} \subset \mathbb{P}(V)$ be a closed set. For any two edges e, e' in $\mathscr{B}_{\mathcal{L}}$ such that they are parallel in any apartment containing both, and these coincide, $\mathcal{B}_{\mathcal{L}}(e) = \mathcal{B}_{\mathcal{L}}(e')$.*

Proof. A point $p \in \mathcal{B}_{\mathcal{L}}(e)$ gives an apartment $\mathbb{A} = \mathbb{A}_{\mathcal{P}}$ such that $p \in \mathcal{P}$ and $e \in \widehat{\mathbb{A}}_1^{min}$, as we have shown in the corollary 4.2.6. By hypothesis, this apartment contains e', and since they are parallel, again by the proposition 4.2.15, $\{p\} = \mathcal{B}_{\mathcal{P}}(e) = \mathcal{B}_{\mathcal{P}}(e') \subset \mathcal{B}_{\mathcal{L}}(e')$, as we wanted to show. □

Let us do a small excursus based on the proposition 4.2.3. Let $\mathcal{L} \subset \mathbb{P}(V)$ be any closed subset, let Λ be any vertex in $\mathscr{B}_{\mathcal{L}}$ and let $\mathbb{A} = \mathbb{A}_{\mathbf{v}}$ be an apartment in $\mathscr{B}_{\mathcal{L}}$ containing Λ, where $\mathbf{v} = \{v_0, \dots, v_d\}$ is a basis of V such that

$$\Lambda = \left[\bigoplus_{i=0}^{d} \mathcal{O}_K v_i \right].$$

In fact, we can work with $\mathcal{L} = \mathbb{P}(V)$ and $\mathscr{B}_{\mathcal{L}} = \mathscr{B}(G)$ and later restrict the discussion to the corresponding subcomplex. We consider the edges in $\mathrm{St}^{min}(\Lambda) \cap \mathbb{A}$ (actually, since $\mathbb{A}$ is any apartment containing Λ, we are taking into account all the edges in $\mathrm{St}^{min}(\Lambda)$). Let

$$e_i = \left(\bigoplus_{i=0}^{d} \mathcal{O}_K v_i \supsetneq \bigoplus_{j \neq i} \mathcal{O}_K \pi_K v_j \oplus \mathcal{O}_K v_i \supsetneq \bigoplus_{i=0}^{d} \mathcal{O}_K \pi_K v_i \right),$$

so, if $p_i := [v_i]$ and $\mathcal{P} = \{p_0, \dots, p_d\}$, we proved through the proposition 4.2.3 $\mathcal{B}_{\mathcal{P}}(e_i) = \{p_i\}$ Let us compute $\mathcal{B}(e_i)$. It is the set of points $[v]$ such that

$$\bigoplus_{j \neq i} \mathcal{O}_K \pi_K v_j \oplus \mathcal{O}_K v_i = \bigoplus_{i=0}^{d} \mathcal{O}_K \pi_K v_i + \left(\bigoplus_{i=0}^{d} \mathcal{O}_K v_i \cap \langle v \rangle \right).$$

Then, writing $v = \sum_{i=0}^{d} \lambda_i v_i$, one gets

$$\mathcal{B}(e_i) = \{[v] \mid v_K(\lambda_i) < v_K(\lambda_j) \ \forall\ j \neq i\}.$$

Finally, observe that if $[v] \in \mathcal{B}(e_i)$, the set of vectors $(\mathbf{v} \setminus \{v_i\}) \cup \{v\}$ is a basis for V, and if v is a representant of V in $\bigoplus_{j=0}^{d} \mathcal{O}_K v_j \setminus \bigoplus_{j=0}^{d} \mathcal{O}_K \pi_K v_j$, then $\bigoplus_{j=0}^{d} \mathcal{O}_K v_j = \bigoplus_{j \neq i} \mathcal{O}_K v_j \oplus \mathcal{O}_K v$. As a consequence, Λ is in the apartment obtained from changing $p_i \in \mathcal{B}(e_i)$ by any other $p \in \mathcal{B}(e_i)$.

Proposition 4.2.17. *Given $\mathcal{L} \subset \mathbb{P}(V)$ closed, for all vertex Λ in $\mathscr{B}_{\mathcal{L}}$ and all apartment $\mathbb{A} = \mathbb{A}_{\mathcal{P}} \leq \mathscr{B}_{\mathcal{L}}$ containing Λ, given any points $\widetilde{p}_0 \in \mathcal{B}_{\mathcal{L}}(e_i)$ where e_i are the edges in $\mathrm{St}^{min}_{\mathcal{P}}(\Lambda)$, the vertex Λ belongs to the apartment $\mathbb{A}_{\{\widetilde{p}_0, \dots, \widetilde{p}_d\}}$.*

Proof. The previous discussion applied reiteratively to all the edges in $\mathrm{St}^{min}_{\mathcal{P}}(\Lambda)$. □

The minimal differential map

Proposition 4.2.18. *Let $\mathcal{L} \subset \mathbb{P}(V)$ be a closed set. Consider a chamber*

$$\Delta = (L_0 \supsetneq L_1 \supsetneq \cdots \supsetneq L_d \supsetneq \pi_K L_0) \in \widehat{\mathscr{B}_{\mathcal{L}d}}$$

and a collection of vectors verifying $v_i \in L_i \setminus L_{i+1}$, where $L_{d+1} := \pi_K L_0$. Then, this set of vectors is a basis adapted to the chamber after reversing the order of the vectors. Thus, if $[v_i] \in \mathcal{L}$ for all i, we got $\Delta \leq \mathbb{A}_{\mathcal{P}} \leq \mathscr{B}_{\mathcal{L}}$ where $\mathcal{P} = \{[v_0], \dots, [v_d]\}$.

Let us denote $e_0 := (L_0 \supsetneq L_d \supsetneq \pi_K L_0)$ and $e_i := (L_i \supsetneq \pi_K L_{i-1} \supsetneq \pi_K L_I)$ for $i \geq 1$. Then

$$\mathcal{B}_{\mathcal{L}}(e_i) = \{[u] \in \mathcal{L} | \ u \in L_i \setminus L_{i+1}\}.$$

so we get

$$\mathcal{L} = \bigsqcup_{i=0}^{d} \mathcal{B}_{\mathcal{L}}(e_i).$$

Proof. Consider the minimal edge $L_0 \supsetneq L_d \supsetneq \pi_K L_0$. We have $v_d \in L_d \setminus \pi_K L_0$ and such as we have seen in the lemma 4.2.2 we get

$$L_d = \pi_K L_0 + \mathcal{O}_K v_d.$$

For $i < d$, consider the minimal edge $\pi_K^{-1} L_{i+1} \supsetneq L_i \supsetneq L_{i+1}$. Identically as above we get $L_i = L_{i+1} + \mathcal{O}_K v_i$, so inductively we obtain

$$L_i = \sum_{j=i}^{d} \mathcal{O}_K v_j + \pi_K L_0$$

and

$$L_0 = \sum_{i=0}^{d} \mathcal{O}_K v_i + \pi_K L_0 = \sum_{i=0}^{d} \mathcal{O}_K v_i + \pi_K^m L_0 \ \forall m \in \mathbb{Z}_{\geq 1}$$

Next we have to see $\pi_K^m L_0 \subset \sum_{i=0}^{d} \mathcal{O}_K v_i$ for some $m \in \mathbb{Z}_{\geq 1}$. Take an $\mathcal{O}_K$-basis $\{u_0, \ldots, u_d\}$ for L_0 and let us write $u_i = \sum_j \lambda_i^j v_j$ with $\lambda_i^j \in K$, and $m = \max\{-\min_{i,j}\{v_K(\lambda_i^j)\}, 1\}$. Then we get $\pi_K^m L_0 \subset \sum_{i=0}^{d} \mathcal{O}_K v_i$, therefore $L_0 = \sum_{i=0}^{d} \mathcal{O}_K v_i$ and so $L_0 = \bigoplus_{i=0}^{d} \mathcal{O}_K v_i$. As a consequence, we obtain

$$L_i = \bigoplus_{j=0}^{i-1} \mathcal{O}_K \pi_K v_j \oplus \bigoplus_{j=i}^{d} \mathcal{O}_K v_j$$

The second assert follows again from the lemma 4.2.2 combined with the definition of $\mathcal{B}_{\mathcal{L}}(e)$ from $\mathcal{B}(e)$ for any edge e. $\square$

Remark 4.2.19. *In the previous proposition we have used all the minimal edges contained in a chamber. Further, we see that for each vertex there is one minimal edge in the chamber having that vertex as source, another having it as target and there are no more minimal edges in that cell passing through that vertex.*

With the notation of the previous proposition, we define the map

$$\partial^{min} : \mathbb{Z}[\mathscr{B}_{\mathcal{L}d}] \longrightarrow \mathbb{Z}[\widehat{\mathscr{B}_{\mathcal{L}1}}^{min}]$$

(and also

$$\widehat{\partial}^{min} : \mathbb{Z}[\widehat{\mathscr{B}_{\mathcal{L}d}}] \longrightarrow \mathbb{Z}[\widehat{\mathscr{B}_{\mathcal{L}1}}^{min}])$$

by $\partial^{min}(\Delta) = \sum_{i=0}^{d} e_i$. By abuse of notation, in some ocasion we also shall write $\partial^{min}(\Delta) = \{e_0, \ldots, e_d\}$.

Then, we rewrite the last result as

$$\mathcal{L} = \bigsqcup_{\partial^{min}(\Delta)} \mathcal{B}_{\mathcal{L}}(e).$$

Corollary 4.2.20. *Two minimal edges e, e' are straight with $t(e) = s(e')$ if and only if $t(e) = s(e')$ and $\mathcal{B}(e') \subset \mathcal{B}(e)$.*

Proof. Since $t(e) = s(e')$ we can denote

$$e = (L_0 \supsetneq L_1 \supsetneq \pi_K L_0) \qquad e' = (L_1 \supsetneq L_2 \supsetneq \pi_K L_1).$$

If there exists a chamber containing both, then $\mathcal{B}(e) \cap \mathcal{B}(e') = \emptyset$, so we conclude the "if". For the opposite implication, take a point $[u] \in \mathcal{B}(e')$ with $u \in L_2 \setminus \pi_K L_1 \subset L_1$. We want to see $u \notin \pi_K L_0$. Assume this is not the case. Then we have

$$\pi_K L_1 \subsetneq L_2 = \pi_K L_1 + \mathcal{O}_K u \subset \pi_K L_0 \subsetneq L_1$$

and we get a contradiction with the fact that there is no chamber containing both edges. $\square$

Proposition 4.2.21. *Let $e_0, e_1, \ldots, e_d$ be minimal edges such that*

$$s(e_i) = t(e_{i+1}) \text{ for all } i = 0, \ldots, d-1 \text{ and } s(e_d) = t(e_0).$$

Then, there exists a chamber Δ such that $\partial^{min}(\Delta) = \sum_{i=0}^{d} e_i$.

Proof. Let us denote the vertices of the edges verifying the following rule: $\Lambda_i = s(e_i)$. Then $e_0 = (\Lambda_0, \Lambda_d)$ and for all $i \geq 1$, $e_i = (\Lambda_i, \Lambda_{i-1})$.

Let us write $\Lambda_i = [L_i]$. We can take the lattices satifying

$$\begin{gathered} L_0 \supsetneq L_d \supsetneq \pi_K L_0, \\ L_1 \supsetneq \pi_K L_0 \supsetneq \pi_K L_1, \\ \vdots \\ L_d \supsetneq \pi_K L_{d-1} \supsetneq \pi_K L_d. \end{gathered}$$

Then we have

$$L_0 \supsetneq L_1 \supsetneq L_2 \supsetneq \cdots \supsetneq L_{d-1} \supsetneq L_d \supsetneq \pi_K L_0,$$

which gives a chamber Δ as we claimed. $\square$

The behaviour of the open sets by galleries of chambers

Lemma 4.2.22. *Let Δ and Δ' be two chambers intersecting in a panel or maximal face of both $A = \Delta \cap \Delta'$. Let e^{Δ}_{A} be the minimal edge of Δ whose target vertex is the opposite to A, that is the vertex of Δ not contained in A. Let $e^{A}_{\Delta'}$ be the minimal edge of Δ' whose source vertex is the opposite to A in Δ'. Then $\mathcal{B}(e^{\Delta}_{A}) \subset \mathcal{B}(e^{A}_{\Delta'})$.*

Proof. We may denote

$$\Delta = (L_0 \supsetneq L_1 \supsetneq \cdots \supsetneq L_d \supsetneq \pi_K L_0)$$

and

$$\Delta' = (L'_0 \supsetneq L'_1 \supsetneq \cdots \supsetneq L'_d \supsetneq \pi_K L'_0)$$

with $L_i = L'_i$ for each $i \neq d$ and $L_d \neq L'_d$, so

$$A = (L_0 \supsetneq L_1 \supsetneq \cdots \supsetneq L_{d-1} \supsetneq \pi_K L_0)$$

and

$$e^{\Delta}_{A} = (L_0 \supsetneq L_d \supsetneq \pi_K L_0) \text{ and } e^{A}_{\Delta'} = (L'_d \supsetneq \pi_K L_{d-1} \supsetneq \pi_K L'_d)$$

Let $[u] \in \mathcal{B}(e^{\Delta}_{A})$ with $u \in L_d \setminus \pi_K L_0$, as we know we can assume. Then $\pi_K u \in \pi_K L_d \subset \pi_K L_{d-1}$. Suppose $\pi_K u \in \pi_K L'_d$, so that $u \in L'_d \setminus \pi_K L_0$ and $[u] \in \mathcal{B}((L_0 \supsetneq L_d \supsetneq \pi_K L_0)) \cap \mathcal{B}((L_0 \supsetneq L'_d \supsetneq \pi_K L_0)) = \emptyset$ as we have shown above, so that we would get a contradiction. Therefore, $\pi_K u \in \pi_K L_{d-1} \setminus \pi_K L'_d$ and thus, $[u] \in \mathcal{B}(e^{A}_{\Delta'})$. $\square$

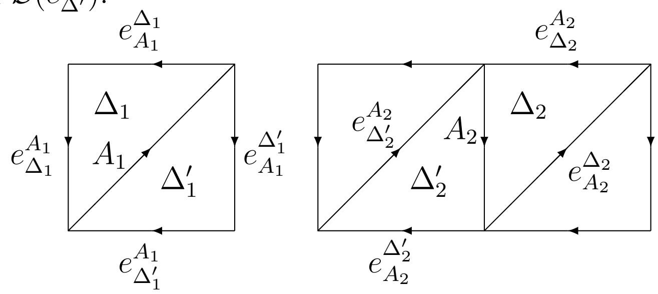


Figure 4.2: Distinct configurations examples of the lemma.

Remark 4.2.23. *Note that* $\mathbb{P}(V) = \bigsqcup_{\partial^{min}(\Delta)} \mathcal{B}(e) =$

$$= \mathcal{B}(e^{\Delta}_{A}) \sqcup \mathcal{B}(e^{A}_{\Delta}) \sqcup \left(\bigsqcup_{\substack{e \in \widehat{\mathscr{B}(G)}^{min}_1 \\ e \leq A}} \mathcal{B}(e) \right) = \mathcal{B}(e^{\Delta'}_{A}) \sqcup \mathcal{B}(e^{A}_{\Delta'}) \sqcup \left(\bigsqcup_{\substack{e \in \widehat{\mathscr{B}(G)}^{min}_1 \\ e \leq A}} \mathcal{B}(e) \right)$$

and so $\mathcal{B}(e_A^\Delta) \sqcup \mathcal{B}(e_\Delta^A) = \mathcal{B}(e_A^{\Delta'}) \sqcup \mathcal{B}(e_{\Delta'}^A)$.

Remark 4.2.24. *The edges* e_A^Δ, $e_{\Delta'}^A$ *are parallel in any apartment containing* Δ *and* Δ'. *Indeed, let us take any apartment* $\mathbb{A}$ *containing both chambers, and consider the unique isomorphism* $\mathbb{A} \cong \mathbb{Z}^{d+1}/\mathbb{Z} \cdot (1, \dots, 1)$ *such that* Δ *goes to*

$$([(0, \dots, 0)], [(0, \dots, 0, 1)], \dots, [(0, 1, \dots, 1)])$$

with e_A^Δ *mapping to* $([(0, \dots, 0)], [(0, 1, \dots, 1)])$. *Then* A *applies to*

$$([(0, \dots, 0)], [(0, \dots, 0, 1)], \dots, [(0, 0, 1, \dots, 1)])\,,$$

Δ' *to*

$$([(0, \dots, 0)], [(0, \dots, 0, 1)], \dots, [(0, 0, 1, \dots, 1)], [(1, 0, 1, \dots, 1)])$$

and $e_{\Delta'}^A$ *to* $([(1, 0, 1, \dots, 1)], [(0, 0, 1, \dots, 1)]) = [(1, 0, 1 \dots, 1)] + e$.

The reciprocal is also true.

Proposition 4.2.25. *Two minimal edges* e, e' *are parallel in a common apartment* $\mathbb{A}$ *if and only if there exist minimal edges* $e_0 := e, e, \dots, e_r := e'$ *and chambers* Δ_i *for* $i = 0, \dots, r$ *in* $\mathbb{A}$ *such that* $e_i \le \Delta_i$, $A_{i+1} := \Delta_i \bigcap \Delta_{i+1}$ *are panels, and* $e_i = e_{A_{i+1}}^{\Delta_i}$ *and* $e_{i+1} = e_{\Delta_{i+1}}^{A_{i+1}}$, *or* $e_i = e_{\Delta_i}^{A_{i+1}}$ *and* $e_{i+1} = e_{A_{i+1}}^{\Delta_{i+1}}$.

Proof. $\Longleftarrow|$

This is the previous remark together with the fact that the parallelism relation restricted to an apartment is an equivalent relation, and, in particular, transitive.

$\Longrightarrow|$

Let $e \parallel_{\mathbb{A}} e'$ and fix an isomorphism $\mathbb{A} \cong \mathbb{Z}^{d+1}/\mathbb{Z} \cdot (1, \dots, 1)$. We divide the proof in different steps.

We suppose first $e' = [(0, \dots, 0, 1, 0, \dots, 0)] + e$ with the 1 in the coordinate i_0. We may assume

$$e = ([(0, \dots, 0)], [(1, \dots, 1, 0, 1, \dots, 1)]) =$$

$$= \left(\bigoplus_{j=0}^{d} \mathcal{O}_K v_j \supsetneq \bigoplus_{j \ne i} \mathcal{O}_K \pi_K v_j \oplus \mathcal{O}_K v_i \supsetneq \bigoplus_{j=0}^{d} \mathcal{O}_K \pi_K v_j \right)$$

with the 0 in the position i, without loss of generality. If $i \ne i_0$, let Δ be any chamber as follows

$$\bigoplus_{j=0}^{d} \mathcal{O}_K v_j \supsetneq \bigoplus_{j \ne i_0} \mathcal{O}_K v_j \oplus \mathcal{O}_K \pi_K v_{i_0} \supsetneq \cdots \supsetneq \bigoplus_{j \ne i} \mathcal{O}_K \pi_K v_j \oplus \mathcal{O}_K v_i \supsetneq \bigoplus_{j=0}^{d} \mathcal{O}_K \pi_K v_j.$$

Let A be the panel opposite to the vertex $\bigoplus_{j=0}^{d}\mathcal{O}_K v_j$ in Δ:

$$\bigoplus_{j\neq i_0}\mathcal{O}_K v_j \oplus \mathcal{O}_K\pi_K v_{i_0} \supsetneq \cdots \supsetneq \bigoplus_{j\neq i}\mathcal{O}_K\pi_K v_j \oplus \mathcal{O}_K v_i \supsetneq \bigoplus_{j\neq i_0}\mathcal{O}_K\pi_K v_j \oplus \mathcal{O}_K\pi_K^2 v_{i_0}.$$

and let Δ' be the chamber containing A given by

$$\bigoplus_{j\neq i_0}\mathcal{O}_K v_j \oplus \mathcal{O}_K\pi_K v_{i_0} \supsetneq \cdots \supsetneq \bigoplus_{j\neq i}\mathcal{O}_K\pi_K v_j \oplus \mathcal{O}_K v_i \supsetneq$$

$$\supsetneq \bigoplus_{j\neq i,i_0}\mathcal{O}_K\pi_K v_j \oplus \mathcal{O}_K v_i \oplus \mathcal{O}_K\pi_K^2 v_{i_0}.$$

Then, we have $e = e_{\Delta}^{A}$ and $e' = e_{A}^{\Delta'}$. Next, if $e' = [(n_0,\ldots,n_d)] + e$ with $n_i = 0$ and $n_j \geq 0$ for all $j \neq i$, we get this case by induction. But we also want to allow $n_j < 0$. Again by induction, we reduce to the case $e' = [(0,\ldots,0,-1,0,\ldots,0)] + e$ with -1 in the position $i_0 \neq i$. Observe that this is the same that $e' = [(1,\ldots,1,0,1,\ldots,1)] + e$ with the 0 in the same marked position. We have e' given by

$$\bigoplus_{j\neq i_0}\mathcal{O}_K\pi_K v_j \oplus \mathcal{O}_K v_{i_0} \supsetneq \bigoplus_{j\neq i_0,i}\mathcal{O}_K\pi_K^2 v_j \oplus \mathcal{O}_K\pi_K v_i \oplus \mathcal{O}_K\pi_K v_{i_0} \supsetneq$$

$$\supsetneq \bigoplus_{j\neq i_0}\mathcal{O}_K\pi_K^2 v_j \oplus \mathcal{O}_K\pi_K v_{i_0}.$$

Now let Δ be any chamber given by

$$\bigoplus_{j}\mathcal{O}_K v_j \supsetneq \cdots \supsetneq \bigoplus_{j\neq i,i_0}\mathcal{O}_K\pi_K v_j \oplus \mathcal{O}_K v_{i_0} \oplus \mathcal{O}_K v_i \supsetneq \bigoplus_{j\neq i}\mathcal{O}_K\pi_K v_j \oplus \mathcal{O}_K v_i,$$

let A be the face obtained removing $\bigoplus_{j\neq i}\mathcal{O}_K\pi_K v_j \oplus \mathcal{O}_K v_i$, and let Δ' be the chamber which we get joining the vertex $\bigoplus_{j\neq i_0}\mathcal{O}_K\pi_K v_j \oplus \mathcal{O}_K v_{i_0}$ to A. Thus, we have $e = e_{A}^{\Delta}$ and $e' = e_{\Delta'}^{A}$.

□

Proposition 4.2.26. *Let $\mathcal{L} \subset \mathbb{P}(V)$ be a closed set. We recall the situation of lemma 4.2.22. Let Δ and Δ' be two chambers in $\mathscr{B}_{\mathcal{L}}$ intersecting in a panel or maximal face of both $A = \Delta \cap \Delta'$. Let e_{A}^{Δ} be the minimal edge of Δ whose target vertex is the opposite to A, that is the vertex of Δ not contained in A. Let $e_{\Delta'}^{A}$ be the minimal edge of Δ' whose source vertex is the opposite to A in Δ'. In this situation, $\mathrm{cov}_{\mathcal{L}}(A) = 2$ if and only if $\mathcal{B}_{\mathcal{L}}(e_{A}^{\Delta}) = \mathcal{B}_{\mathcal{L}}(e_{\Delta'}^{A})$ and more generally,*

$$\mathcal{B}_{\mathcal{L}}(e_{\Delta'}^{A}) = \bigsqcup_{A\leq\Delta}\mathcal{B}_{\mathcal{L}}(e_{A}^{\Delta}).$$

Proof. One inclusion is given by lemma 4.2.22. For the other, recall the same notation as there:

$$\Delta = (L_0 \supsetneq L_1 \supsetneq \cdots \supsetneq L_d \supsetneq \pi_K L_0),$$
$$\Delta' = (L'_0 \supsetneq L'_1 \supsetneq \cdots \supsetneq L'_d \supsetneq \pi_K L'_0),$$
$$A = (L_0 \supsetneq L_1 \supsetneq \cdots \supsetneq L_{d-1} \supsetneq \pi_K L_0),$$
$$e^{\Delta}_A = (L_0 \supsetneq L_d \supsetneq \pi_K L_0) \text{ and } e^{A}_{\Delta'} = (L'_d \supsetneq \pi_K L_{d-1} \supsetneq \pi_K L'_d)$$

with $L_i = L'_i$ for each $i \neq d$ and $L_d \neq L'_d$.

Recall also the notation of the proposition 4.2.18:

$$e_0 = (L_0 \supsetneq L_d \supsetneq \pi_K L_0), \qquad e_i = (L_i \supsetneq \pi_K L_{i-1} \supsetneq \pi_K L_i),$$

and

$$e'_0 = (L'_0 \supsetneq L'_d \supsetneq \pi_K L'_0), \qquad e'_i = (L'_i \supsetneq \pi_K L'_{i-1} \supsetneq \pi_K L'_i),$$

and observe that $e_i = e'_i$ for all $i \neq 0, d$.

Assume $\mathrm{cov}_{\mathcal{L}}(A) = 2$.

Take now $[u] \in \mathcal{B}_{\mathcal{L}}(e^{A}_{\Delta'})$, so we may assume $u \in L_{d-1} \backslash L'_d$. Then $u \notin \pi_K L_0$. Suppose $u \notin L_d$ and consider the lattice

$$L_u := \pi_K L_0 + \mathcal{O}_K u \subsetneq L_{d-1}.$$

Then we have a chamber

$$\Delta'' = (L_0 \supsetneq L_1 \supsetneq \cdots \supsetneq L_{d-1} \supsetneq L_u \supsetneq \pi_K L_0)\,.$$

Denote $e''_0 = (L_0 \supsetneq L_u \supsetneq \pi_K L_0)$ and $e''_d = (L_u \supsetneq \pi_K L_{d-1} \supsetneq \pi_K L_u)$. By the proposition 4.2.18, we have

$$\mathcal{L} = \mathcal{B}_{\mathcal{L}}(e'_0) \sqcup \mathcal{B}_{\mathcal{L}}(e'_d) \sqcup \bigsqcup_{i \neq 0,d} \mathcal{B}_{\mathcal{L}}(e'_i) = \mathcal{B}_{\mathcal{L}}(e''_0) \sqcup \mathcal{B}_{\mathcal{L}}(e''_d) \sqcup \bigsqcup_{i \neq 0,d} \mathcal{B}_{\mathcal{L}}(e'_i),$$

so that

$$\mathcal{B}_{\mathcal{L}}(e'_0) \sqcup \mathcal{B}_{\mathcal{L}}(e'_d) = \mathcal{B}_{\mathcal{L}}(e''_0) \sqcup \mathcal{B}_{\mathcal{L}}(e''_d).$$

Since $\mathcal{B}_{\mathcal{L}}(e''_0) \subset \mathcal{B}_{\mathcal{L}}(e^{A}_{\Delta'}) = \mathcal{B}_{\mathcal{L}}(e'_d)$, by the lemma 4.2.22, the intersection $\mathcal{B}_{\mathcal{L}}(e'_d) \cap \mathcal{B}_{\mathcal{L}}(e''_d) \subset \mathcal{B}_{\mathcal{L}}(e''_d)$ is non empty, therefore, applying again the proposition 4.2.18, $\Delta'' \leq \mathscr{B}_{\mathcal{L}}$.

Moreover, the assumption, $L_u \neq L_d$ implies $\Delta'' \neq \Delta$. Since $u \notin L'_d$, $L_u \neq L'_d$, so $\Delta'' \neq \Delta'$. But A is contained in Δ, Δ' and Δ'', so $\mathrm{cov}_{\mathcal{L}}(A) \geq 3$, which contradicts the hypothesis. Therefore, $u \in L_d \setminus \pi_K L_0$, and thus $[u] \in \mathcal{B}_{\mathcal{L}}(e^{\Delta}_A)$.

If $\mathrm{cov}_{\mathcal{L}}(A) \geq 3$, there exists such a chamber

$$\Delta'' = (L_0 \supsetneq L_1 \supsetneq \cdots \supsetneq L_{d-1} \supsetneq L''_d \supsetneq \pi_K L_0)$$

as above in $\mathscr{B}_{\mathcal{L}}$. The minimal edge $e'' = e^{\Delta''}_A = (L_0 \supsetneq L''_d \supsetneq \pi_K L_0)$ verifies $\emptyset \neq \mathcal{B}_{\mathcal{L}}(e'') \subset \mathcal{B}_{\mathcal{L}}(e^{A}_{\Delta'})$ and $\mathcal{B}_{\mathcal{L}}(e'') \bigcap \mathcal{B}_{\mathcal{L}}(e^{\Delta}_A) = \emptyset$, and thus, we conclude. $\square$

Ends on $\mathscr{B}(G)$

Definition 4.2.27. *We shall call a ray in $\mathscr{B}(G)$ an oriented infinite sequence of straight minimal edges contained in some apartment. That is*

$$r = (e_0, e_1, e_2, \dots)_\infty \text{ such that } t(e_i) = s(e_i + 1) \text{ and } \mathcal{B}(e_{i+1}) \subset \mathcal{B}(e_i).$$

For each $n \in \mathbb{N}$, the n-truncation of r is the ray $\tau_n(r) := (e_n, e_{n+1}, e_{n+2}, \dots)_\infty$.

Remark 4.2.28. *For any n consider an apartment $\mathbb{A}_\mathcal{P}$ containing e_0 and e_n. Then, e_i is in $\mathbb{A}_\mathcal{P}$ for all $i = 0, \dots, n$. It is enough to see this for $i = 1$. As we have shown in the proposition 4.2.3, the open sets $\mathcal{B}(e_n) \subset \mathcal{B}(e_1)$ contain exactly one point p of $\mathcal{P}$. The same result tells us that the unique edge in $\mathrm{St}^{min}(t(e_0))$ containing p is in $\mathbb{A}_\mathcal{P}$, since $t(e_0)$ is in the apartment. But that edge is e_1.*

Take a ray r in an apartment $\mathbb{A}_\mathbf{v}$. We observe that $e_i \parallel_\mathbf{v} e_j$ for any i, j, since parallelism with respect to $\mathbf{v}$ is an equivalence relation, e_i, e_{i+1} are straight, and then they are parallel.

Definition 4.2.29. *We shall say that two rays r, r' in an apartment $\mathbb{A}_\mathbf{v}$ are parallel with respect to $\mathbf{v}$ and we shall denote this by $r \parallel_\mathbf{v} r'$, if $e_i \parallel_\mathbf{v} e'_j$ for any i, j. This has sense by the previous consideration.*

We shall say that two rays are equivalent if they have truncations which are contained in a common apartment and parallel with respect to it.

Remark 4.2.30. *Let $r = (e_0, e_1, e_2, \dots)_\infty$ be a ray. Taking $e = e_0$ in the remark 4.1.5, and applying it to all the edges e_i in a ray simultaneously (since the ray is contained in an apartment by hypothesis), we get*

$$s(e_i) = \left[\mathcal{O}_K v_{i_0} \oplus \bigoplus_{j \neq i_0} \mathcal{O}_K \pi_K^i v_j \right].$$

and

$$\bigcap_{i \in \mathbb{N}} \left(\mathcal{O}_K v_{i_0} \oplus \bigoplus_{j \neq i_0} \mathcal{O}_K \pi_K^i v_j \right) = \mathcal{O}_K v_{i_0}.$$

Let us project this point to $\mathbb{P}(V)$, so we deal with $[v_{i_0}]$. We formalize this with the next proposition. Note that given a parallel ray in the same apartment, we obtain the same projective point.

Proposition 4.2.31. *Given a ray $r = (e_0, e_1, e_2, \dots)_\infty$, we can assume that*

$$s(e_i) = \left[\mathcal{O}_K v_{i_0} \oplus \bigoplus_{j \neq i_0} \mathcal{O}_K \pi_K^i v_j \right].$$

and therefore

$$\bigcap_{i\in\mathbb{N}}\left(\mathcal{O}_K v_{i_0}\oplus\bigoplus_{j\neq i_0}\mathcal{O}_K\pi_K^i v_j\right)=\mathcal{O}_K v_{i_0}.$$

Then

$$\bigcap_{i\in\mathbb{N}}\mathcal{B}(e_i)=[v_{i_0}].$$

Proof. That the point is included in the intersection is clear. Recall that each e_i is given by

$$\mathcal{O}_K v_{i_0}\oplus\bigoplus_{j\neq i_0}\mathcal{O}_K\pi_K^i v_j\supsetneq\mathcal{O}_K v_{i_0}\oplus\bigoplus_{j\neq i_0}\mathcal{O}_K\pi_K^{i+1} v_j\supsetneq\mathcal{O}_K\pi_K v_{i_0}\oplus\bigoplus_{j\neq i_0}\mathcal{O}_K\pi_K^{i+1} v_j.$$

For any point $z\in\bigcap_{i\in\mathbb{N}}\mathcal{B}(e_i)$ and for each $i\in\mathbb{N}$ it has to have a representant

$$v\in\left(\mathcal{O}_K v_{i_0}\oplus\bigoplus_{j\neq i_0}\mathcal{O}_K\pi_K^{i+1} v_j\right)\setminus\left(\mathcal{O}_K\pi_K v_{i_0}\oplus\bigoplus_{j\neq i_0}\mathcal{O}_K\pi_K^{i+1} v_j\right).$$

Note that this implies that the coefficient of v_{i_0} in v will have valuation 0.

Since $z\in\mathcal{B}(e_0)$, it has a representant with the form $v=v_{i_0}+\sum_{j\neq i_0}\lambda_j v_j$, with $v_K(\lambda_j)\geq 1$ for all $j\neq i_0$. Because of the previous comment, this is the representant satisfying the previous inclusion for each $i\in\mathbb{N}$. Therefore, $z\in\mathcal{B}(e_i)$ implies $v_K(\lambda_j)\geq i+1$ for all $j\neq i_0$ and for all $i\in\mathbb{N}$, but then, the unique possibility is $\lambda_j=0$ for all $j\neq i_0$, so $z=[v_{i_0}]$. □

Definition 4.2.32. *We shall call an end on $\mathscr{B}(G)$ a ray up to equivalence. We shall denote the set of ends by $\mathcal{E}=\mathcal{E}(\mathscr{B}(G))$. We take into consideration the set of ends classes of rays starting from a minimal edge e*

$$\mathcal{E}(e)=\{\eta|\ \exists\ r=(e,\dots)_\infty,[r]=\eta\}=\{[(e,\dots)_\infty]\}\subset\mathcal{E}.$$

By the remark 4.2.30, we have an exhaustive map

$$\widetilde{\varepsilon}:\mathcal{E}(\mathscr{B}(G))\longrightarrow\mathbb{P}(V).$$

defined by $\widetilde{\varepsilon}([(e_0,e_1,\dots)_\infty])=\bigcap_{i\in\mathbb{N}}\mathcal{B}(e_i)$.

For any point $p\in\mathbb{P}(V)$ and any vertex $\Lambda\in\mathscr{B}(G)_0$, by the proposition 4.1.2, there is $\Lambda\in\mathbb{A}_{\mathcal{P}}$ with $p\in\mathcal{P}$. The remarks 4.2.30 together with 4.1.5 give us a unique ray r in $\mathbb{A}_{\mathcal{P}}$ with $e_0\in\mathrm{St}^{min}(\Lambda)$ such that $\widetilde{\varepsilon}([r])=p$.

Proposition 4.2.33. *For each minimal edge e $\widetilde{\varepsilon}(\mathcal{E}(e))=\mathcal{B}(e)$.*

Proof. The proposition 4.2.31 implies $\widetilde{\varepsilon}(\mathcal{E}(e)) \subset \mathcal{B}(e)$. On the other hand, by the corollary 4.2.6, given $p \in \mathcal{B}(e)$ there exists an apartment $\mathbb{A}_{\mathcal{P}}$ such that $e \leq \mathbb{A}_{\mathcal{P}}$ and $p \in \mathcal{P}$. Then the unique sequence of straight edges in $\mathbb{A}_{\mathcal{P}}$ starting by e is a ray whose associated end is mapped to p by $\widetilde{\varepsilon}$. □

Remark 4.2.34. *Actually we expect the map $\widetilde{\varepsilon}$ is a bijection, which seems to be provable using the methods introduced in [AB08, Ch. 11]. Indeed, the idea is that if two ends go to the same projective point, two rays representing them should have truncations in a common apartment. (cf. [AB08, Lem. 11.77]).*

4.3 Properties for dimension $d = 2$

Even what we are going to do seems adaptable to any dimension, we need to restrict us to consider the 2-dimensional case as an assumption to avoid difficulties provided by the fact that, generally, our objects $\mathscr{B}_{\mathcal{L}}$ are not buildings. Thus, henceforth $V \cong K^3$ and $\mathscr{B}(G)$ is the 2-dimensional Bruhat-Tits building, unless otherwise stated. Beyond that, we go on with the study of the structure of $\mathscr{B}(G)$ and the subcomplexes $\mathscr{B}_{\mathcal{L}}$ that we started in the previous sections. Later on, we will introduce a notion of convexity with a suitable behaviour of galleries in "convex complexes" with respect to the associated open sets, and finally we will give a smaller basis for the topology of a closed set $\mathcal{L} \subset \mathbb{P}(V)$ in terms of the rays from a given vertex.

Through this section $\mathcal{L} \subset \mathbb{P}(V)$ will be a closed set, unless otherwise specified, even if we are thinking mainly in the cases $\mathcal{L} = \mathbb{P}(V)$ or $\mathcal{L}$ compact.

4.3.1 On the rays in $\mathscr{B}_{\mathcal{L}}$ and a number of consequences

Proposition 4.3.1. *Given two minimal edges $e, e' \in \widehat{\mathscr{B}_{\mathcal{L}1}}^{min}$ such that their associated open sets verify $\mathcal{B}_{\mathcal{L}}(e) = \mathcal{B}_{\mathcal{L}}(e')$, there exists an apartment $\mathbb{A} \leq \mathscr{B}_{\mathcal{L}}$ containing both and they are parallel in it.*

Proof. Consider chambers Δ, Δ' in $\mathscr{B}_{\mathcal{L}}$ containing e and e' respectively and write $\partial^{min}(\Delta) = e + e_1 + e_2$, $\partial^{min}(\Delta') = e' + e'_1 + e'_2$. We have $\mathcal{B}_{\mathcal{L}}(e_i), \mathcal{B}_{\mathcal{L}}(e'_i)$ are non empty for any i and

$$\mathcal{L} = \mathcal{B}_{\mathcal{L}}(e) \sqcup \mathcal{B}_{\mathcal{L}}(e_1) \sqcup \mathcal{B}_{\mathcal{L}}(e_2) = \mathcal{B}_{\mathcal{L}}(e') \sqcup \mathcal{B}_{\mathcal{L}}(e'_1) \sqcup \mathcal{B}_{\mathcal{L}}(e'_2)$$

Therefore, $\mathcal{B}_{\mathcal{L}}(e_1) \sqcup \mathcal{B}_{\mathcal{L}}(e_2) = \mathcal{B}_{\mathcal{L}}(e'_1) \sqcup \mathcal{B}_{\mathcal{L}}(e'_2)$.

Let p_0 be a point in $\mathcal{B}_{\mathcal{L}}(e) = \mathcal{B}_{\mathcal{L}}(e')$.

If $\mathcal{B}_{\mathcal{L}}(e_1) \subset \mathcal{B}_{\mathcal{L}}(e'_1)$, then $\mathcal{B}_{\mathcal{L}}(e'_2) \subset \mathcal{B}_{\mathcal{L}}(e_2) \neq \emptyset$. Then, choose $p_1 \in \mathcal{B}_{\mathcal{L}}(e_1)$ and $p_2 \in \mathcal{B}_{\mathcal{L}}(e'_2)$ and let $\mathcal{P}$ be $\{p_0, p_1, p_2\}$. If $\mathcal{B}_{\mathcal{L}}(e_i) \subset \mathcal{B}_{\mathcal{L}}(e'_j)$ or $\mathcal{B}_{\mathcal{L}}(e'_i) \subset \mathcal{B}_{\mathcal{L}}(e_j)$

for some i, j, after a change of notation we are in the same case. Otherwise, $\mathcal{B}_{\mathcal{L}}(e_i) \cap \mathcal{B}_{\mathcal{L}}(e'_j) \neq \emptyset$ for any i, j, then choose $p_i \in \mathcal{B}_{\mathcal{L}}(e_i) \cap \mathcal{B}_{\mathcal{L}}(e'_i)$ for each $i = 1, 2$ and let $\mathcal{P}$ be $\{p_0, p_1, p_2\}$.

In both cases, by the proposition 4.2.18, both chambers Δ, Δ', and so the edges e, e', are in the apartment $\mathbb{A}_{\mathcal{P}} \leq \mathscr{B}_{\mathcal{L}}$, as we claimed.

The parallelism has been already proved, after we know $\mathcal{B}_{\mathcal{P}}(e) = \mathcal{B}_{\mathcal{P}}(e')$. □

Let $\mathcal{L} \subset \mathbb{P}(V)$ be a closed set and let $\mathcal{P}, \mathcal{P}' \subset \mathcal{L}$ be sets of $d+1$ linearly independent points such that $\mathcal{P} \cap \mathcal{P}' \neq \emptyset$.

Let us write now $\mathcal{L}' := \mathcal{P} \cup \mathcal{P}'$. Take $p \in \mathcal{P} \cap \mathcal{P}'$ and consider rays $r = (e_0, e_1, \dots)_\infty$ and $r' = (e'_0, e'_1, \dots)_\infty$ in $\mathbb{A}_{\mathcal{P}}$ and $\mathbb{A}_{\mathcal{P}'}$ respectively, such that $\widetilde{\varepsilon}([r]) = \widetilde{\varepsilon}([r']) = p$, that is, $\cap_{e_i} \mathcal{B}_{\mathcal{L}'}(e_i) = \cap_{e'_i} \mathcal{B}_{\mathcal{L}'}(e'_i) = \{p\}$. Now, as $\mathcal{L}'$ is finite, there are i_0 and i'_0 such that $\mathcal{B}_{\mathcal{L}'}(e_i) = \mathcal{B}_{\mathcal{L}'}(e'_{i'}) = \{p\}$ for all $i \geq i_0$, $i' \geq i'_0$.

Next, we have seen in the proposition 4.3.1 that for such i, i', $e_i, e'_{i'}$ are contained in a common apartment $\mathbb{A}_{\mathcal{P}^o}$ of $\mathscr{B}_{\mathcal{L}'}$.

Proposition 4.3.2. *The complex of chambers $\mathscr{B}_{\mathcal{L}}$ is path-connected.*

Proof. Let $\mathbb{A}_{\mathcal{P}}, \mathbb{A}_{\mathcal{P}'}$ be two apartments in $\mathscr{B}_{\mathcal{L}}$ and let $\mathcal{L}' := \mathcal{P} \cup \mathcal{P}' \subset \mathcal{L}$. Consider a sequence of apartments $\mathbb{A}_{\mathcal{P}_0}, \dots, \mathbb{A}_{\mathcal{P}_r}$ where $\mathcal{P}_0 = \mathcal{P}$, $\mathcal{P}_r = \mathcal{P}'$, $\mathcal{P}_i \subset \mathcal{L}'$ for all i and $\mathcal{P}_i \cap \mathcal{P}_{i+1} \neq \emptyset$.

Then, our claim reduces to the fact that $\mathbb{A}_{\mathcal{P}}$ can be connected with $\mathbb{A}_{\mathcal{P}'}$ when $\mathcal{P} \cap \mathcal{P}' \neq \emptyset$. Indeed, in the previous discussion we have seen that both apartments have non empty intersection with a third apartment $\mathbb{A}_{\mathcal{P}^o}$, through which we can connect them. □

Corollary 4.3.3. *The complex $\mathscr{B}_{\mathcal{L}}$ is a chamber complex in the sense of [AB08], that is, the maximal cells are chambers of $\mathscr{B}(G)$ and any two of them are gallery connected.*

Proof. As in the previous proof we can consider two apartments $\mathbb{A}_{\mathcal{P}}, \mathbb{A}_{\mathcal{P}'}$ such that $\mathcal{P} \cap \mathcal{P}' \neq \emptyset$ and write $\mathcal{L}' := \mathcal{P} \cup \mathcal{P}' \subset \mathcal{L}$. In the discussion above the precedent proposition we have seen there are minimal edges $e \leq \mathbb{A}_{\mathcal{P}}$, $e' \leq \mathbb{A}_{\mathcal{P}'}$ such that $\mathcal{B}_{\mathcal{L}'}(e) = \mathcal{B}_{\mathcal{L}'}(e')$, therefore an apartment $\mathbb{A}_{\mathcal{P}^o}$ containing both. But, in the course of the proof of the proposition 4.3.1 we have seen that this intersection in $\mathbb{A}_{\mathcal{P}^o}$ contains chambers containing both edges. Thus we can connect build a gallery of chambers from $\mathbb{A}_{\mathcal{P}}$ to $\mathbb{A}_{\mathcal{P}'}$ through $\mathbb{A}_{\mathcal{P}^o}$. □

Proposition 4.3.4. *The map $\widetilde{\varepsilon} : \mathcal{E}(\mathscr{B}(G)) \longrightarrow \mathbb{P}(V)$ is a bijection.*

Proof. Indeed, take two ends $[r], [r']$ such that $\widetilde{\varepsilon}([r]) = \widetilde{\varepsilon}([r']) = p$. They are represented by rays r, r' in apartments $\mathbb{A}_{\mathcal{P}}, \mathbb{A}_{\mathcal{P}'}$ respectively, with $p \in \mathcal{P} \cap \mathcal{P}'$. In addition, by the previous discussion, we can assume (after taking suitable truncations) r and r' are in some common apartment, in which they are parallel, therefore they are equivalent and $[r] = [r']$. □

Proposition 4.3.5. *Given different minimal edges $e, e' \in \widehat{\mathscr{B}_{\mathcal{L}}}_1^{min}$, there exists a chamber containing both in $\mathscr{B}_{\mathcal{L}}$ if and only if it exists in $\mathscr{B}(G)$.*

Proof. We just have to prove that if there exists a chamber containing both in $\mathscr{B}(G)$, there exists a chamber containing both in $\mathscr{B}_{\mathcal{L}}$, since the opposite is trivial. Thus, we assume that e, e' are contained in a common chamber.

Recall that the 2-dimensional chambers have three vertices and three edges. Therefore, e and e' have a common vertex, which we assume without loss of generality to be $s(e') = t(e)$. Further, since there are three vertices in e together with e', there is a unique chamber in $\mathscr{B}(G)$ containing them. Let us denote it by $\hat{\Delta}$ and its other edge by $\hat{e}$. We want to see that this chamber is in $\mathscr{B}_{\mathcal{L}}$. Since e and e' are in this complex, $\mathcal{B}_{\mathcal{L}}(e)$ and $\mathcal{B}_{\mathcal{L}}(e')$ are non empty, therefore, by the proposition 4.2.18 (applied on $\mathscr{B}(G)$) it is enough to see that $\mathcal{B}_{\mathcal{L}}(\hat{e}) \neq \emptyset$. Indeed, we will be able to choose points $p \in \mathcal{B}_{\mathcal{L}}(e)$, $p' \in \mathcal{B}_{\mathcal{L}}(e')$ and $\hat{p} \in \mathcal{B}_{\mathcal{L}}(\hat{e})$, obtaining by the lemma $\hat{\Delta} \leq \mathbb{A}_{\{p,p',\hat{p}\}} \leq \mathscr{B}_{\mathcal{L}}$ as we claimed.

Let Δ be a chamber in $\mathscr{B}_{\mathcal{L}}$ containing e and assume it is not $\hat{\Delta}$. Let us denote the edge of Δ with source $t(e)$ by e_0. Since it is in $\mathscr{B}_{\mathcal{L}}$, $\mathcal{B}_{\mathcal{L}}(e_0)$ is non empty, and by the lemma 4.2.22 applied to the chambers $\Delta, \hat{\Delta}$ with common panel e we have $\emptyset \neq \mathcal{B}_{\mathcal{L}}(e_0) \subset \mathcal{B}_{\mathcal{L}}(\hat{e})$, obtaining a point in $\mathcal{L} \cap \mathcal{B}(\hat{e})$, as we expected.

□

Remark 4.3.6. *The previous result is valid for any dimension when $\mathscr{B}_{\mathcal{L}}$ is a building, since then, given two edges there exists an apartment containing both, and then, the existence of a chamber in it is characterized, as in $\mathscr{B}(G)$, by the tropical distances among the vertices.*

Corollary 4.3.7. *Two minimal edges $e, e' \in \widehat{\mathscr{B}_{\mathcal{L}}}_1^{min}$ are straight if and only if either $t(e) = s(e')$ or $t(e') = s(e)$, and there are no chamber in $\mathscr{B}_{\mathcal{L}}$ containing them.*

Proof. This is just the definition of straight edges restricted to $\mathscr{B}_{\mathcal{L}}$ after taking into consideration the previous proposition. □

Proposition 4.3.8. *Two minimal edges $e, e' \in \widehat{\mathscr{B}_{\mathcal{L}}}_1^{min}$ are straight with $t(e) = s(e')$ if and only if $t(e) = s(e')$ and $\mathcal{B}_{\mathcal{L}}(e') \subset \mathcal{B}_{\mathcal{L}}(e)$.*

Proof. The "if" is clear as in corollary 4.2.20. For the other implication we use the same result and the fact that there is no chamber containing them in $\mathscr{B}(G)$, therefore

$$\mathcal{B}_{\mathcal{L}}(e') = \mathcal{B}(e') \cap \mathcal{L} \subset \mathcal{B}(e) \cap \mathcal{L} = \mathcal{B}_{\mathcal{L}}(e)$$

□

Definition 4.3.9. *We shall call a ray in $\mathscr{B}_{\mathcal{L}}$ an oriented infinite sequence of straight minimal edges contained in some apartment of $\mathscr{B}_{\mathcal{L}}$. That is*

$$r = (e_0, e_1, e_2, \dots)_\infty \text{ such that } t(e_i) = s(e_i + 1) \text{ and } \mathcal{B}_{\mathcal{L}}(e_{i+1}) \subset \mathcal{B}_{\mathcal{L}}(e_i).$$

Proposition 4.3.10. *A ray in $\mathscr{B}_{\mathcal{L}}$ is the same that a ray in $\mathscr{B}(G)$ contained in $\mathscr{B}_{\mathcal{L}}$.*

Proof. Clearly, a ray in $\mathscr{B}_{\mathcal{L}}$ is nothing other than a ray in $\mathscr{B}(G)$ contained in some apartment of $\mathscr{B}_{\mathcal{L}}$. Thus, it is enough to prove that a ray in $\mathscr{B}(G)$ contained in $\mathscr{B}_{\mathcal{L}}$ is contained in some apartment of $\mathscr{B}_{\mathcal{L}}$.

Indeed, let $r = (e_0, e_1, \dots)_\infty$ be such a ray. Since r is contained in $\mathscr{B}_{\mathcal{L}}$ its edges verify $\mathcal{B}_{\mathcal{L}}(e_i) \neq \emptyset$. On the other hand we now that $\bigcap_i \mathcal{B}(e_i) = \{p_r\}$. Then, as $\mathcal{L}$ is closed, $p_r \in \mathcal{L}$ and there is an apartment $\mathbb{A}_{\mathcal{P}} \leq \mathscr{B}_{\mathcal{L}}$ with $p_r \in \mathcal{P}$ and $e_0 \leq \mathbb{A}_{\mathcal{P}}$. Therefore r is contained in that apartment. □

Recall the notions of truncation of a ray and of parallelism of rays in an apartment.

Definition 4.3.11. *We will say that two rays in $\mathscr{B}_{\mathcal{L}}$ are $\mathcal{L}$-equivalent if they have truncations which are contained in a common apartment of $\mathscr{B}_{\mathcal{L}}$ and they are parallel with respect to it. We shall call an end on $\mathscr{B}_{\mathcal{L}}$ a ray in $\mathscr{B}_{\mathcal{L}}$ up to $\mathcal{L}$-equivalence. We shall denote the set of ends on $\mathscr{B}_{\mathcal{L}}$ by $\mathcal{E}_{\mathcal{L}} = \mathcal{E}(\mathscr{B}_{\mathcal{L}})$. We take into consideration the set of ends classes of rays starting from a minimal edge e*

$$\mathcal{E}_{\mathcal{L}}(e) = \{\eta \mid \exists\ r = (e, \dots)_\infty \subset \mathscr{B}_{\mathcal{L}}, [r] = \eta\} = \{[(e, \dots)_\infty]\}.$$

Proposition 4.3.12. *Two rays in $\mathscr{B}_{\mathcal{L}}$ are $\mathcal{L}$-equivalent if and only if they are equivalent in $\mathscr{B}(G)$.*

Proof. That $\mathcal{L}$-equivalence implies equivalence in $\mathscr{B}(G)$ is clear. In the other direction, take two rays r, r' in $\mathscr{B}_{\mathcal{L}}$ which are equivalent in $\mathscr{B}(G)$. By the discussion shortly after the beginning of this section, there is an apartment $\mathbb{A}$ in $\mathscr{B}_{\mathcal{L}}$ containing truncations of both rays. Since truncations of them are parallel in some apartment of $\mathscr{B}(G)$ by hypothesis, they are parallel in $\mathbb{A}$, so that they are $\mathcal{L}$-equivalent. □

Thus, we get that the equivalence relation in $\mathscr{B}_{\mathcal{L}}$ is compatible with that in $\mathscr{B}(G)$, and then we get $\mathcal{E}_{\mathcal{L}} = \mathcal{E} \bigcap \mathcal{L}$ and $\mathcal{E}_{\mathcal{L}}(e) = \mathcal{E}(e) \bigcap \mathcal{L}$ for all e.

Corollary 4.3.13. *There is a bijection* $\widetilde{\varepsilon} : \mathcal{E}_{\mathcal{L}} \longrightarrow \mathcal{L}$.

Proof. It is a consequence of the same result for $\mathcal{L} = \mathbb{P}(V)$ and of the compatibility previously commented. □

Definition 4.3.14. *Let us call the set of minimal edges in* $\mathscr{B}_{\mathcal{L}}$ *straight with an edge* e *at* $t(e)$ *the flow of* e *and denote it by* $\mathrm{Flow}_{\mathcal{L}}(e)$ *(or also* $\mathrm{Flow}(e)$ *when* $\mathcal{L} = \mathbb{P}(V)$*). Reciprocally, the edges straight at* $s(e)$ *will be called the preflow of* e *and denoted* $\mathrm{PFlow}_{\mathcal{L}}(e)$ *(resp.* $\mathrm{PFlow}(e)$*).*

We also will call the cling of e *at* $t(e)$ *the set of minimal edges with source at* $t(e)$ *and contained in a common chamber with* e*, that is*

$$\mathrm{Cling}_{\mathcal{L}}^{min}(e) := \mathrm{St}_{\mathcal{L}}^{min}(t(e)) \setminus \mathrm{Flow}_{\mathcal{L}}(e)$$

and the cling of e *at* $s(e)$ *wil be the set of minimal edges with target at* $s(e)$ *and contained in a common chamber with* e*, that is*

$$\mathrm{Cling}_{\mathcal{L}}^{-min}(e) := \mathrm{St}_{\mathcal{L}}^{-min}(s(e)) \setminus \mathrm{PFlow}_{\mathcal{L}}(e)$$

where $\mathrm{St}_{\mathcal{L}}^{-min}(\Lambda)$ *is the set of minimal edges with target* Λ.

As above for the $\mathrm{St}_{\mathcal{L}}^{min}$, when these sets are finite we may identify them with the sum of their edges by a small abuse of notation.

Remark 4.3.15. *Note that an edge* e' *belongs to the preflow of* e *if and only if* $t(e') = s(e)$ *and they are not contained in a common chamber, if and only if* e *is in the flow of* e'*. Thus, in some way they are inverse concepts.*

Corollary 4.3.16. *For any* e *in* $\mathscr{B}_{\mathcal{L}}$ *(resp. in* $\mathscr{B}(G)$*) we have*

$$\mathcal{B}_{\mathcal{L}}(e) = \bigsqcup_{e' \in \mathrm{Flow}_{\mathcal{L}}(e)} \mathcal{B}_{\mathcal{L}}(e').$$

Proof. Since $\mathrm{Flow}(e) \subset \mathrm{St}^{min}(t(e))$, we get the disjointness of the union. Now, consider a point $p \in \mathcal{B}_{\mathcal{L}}(e)$. Then, we have an apartment $\mathbb{A}_{\mathcal{P}} \leq \mathscr{B}_{\mathcal{L}}$ containing e with $p \in \mathcal{P}$, and we know that there is a unique straight edge with e at $t(e)$ in $\mathbb{A}_{\mathcal{P}}$, that is a unique edge e' in $\widehat{\mathbb{A}_{\mathcal{P}1}} \bigcap \mathrm{Flow}_{\mathcal{L}}(e)$. But then, $\{p\} = \mathcal{B}_{\mathcal{P}}(e') \subset \mathcal{B}_{\mathcal{L}}(e')$. □

4.3.2 Open sets relations on an apartment and chamber-convexity

Let us consider for any apartment $\mathbb{A}$ the isomorphism with $\mathbb{Z}^3/\mathbb{Z}\cdot(1,1,1)$. Let $([(m_0,m_1,m_2)],[(m'_0,m'_1,m'_2)])$ be an edge, so

$$\max\{(m'_i - m_i) - (m'_j - m_j\} = 1.$$

Then, we can express $[(m'_0,m'_1,m'_2)] - [(m_0,m_1,m_2)]$ as the class of a vector with 0's and $1's$. If it has exactly one zero, then it is minimal; if it has two zeros, the opposite edge has exactly one zero; therefore, each edge of $\mathbb{A}$ is either minimal or the opposite of a minimal edge.

We simplify even more, considering the isomorphism

$$\begin{array}{ccc} \mathbb{Z}^3/\mathbb{Z}\cdot(1,1,1) & \xrightarrow{\cong} & \mathbb{Z}^2 \\ (m_0,m_1,m_1) & \longmapsto & (m_1-m_0, m_2-m_0). \end{array}$$

Definition 4.3.17. *Let us call a minimal edge* $([\underline{m}],[\underline{m}'])$ *horizontal if*

$$[\underline{m}'] - [\underline{m}] = [(1,0,1)] = [(0,-1,0)],$$

vertical if

$$[\underline{m}'] - [\underline{m}] = [(1,1,0)] = [(0,0,-1)]$$

and diagonal if

$$[\underline{m}'] - [\underline{m}] = [(0,1,1)].$$

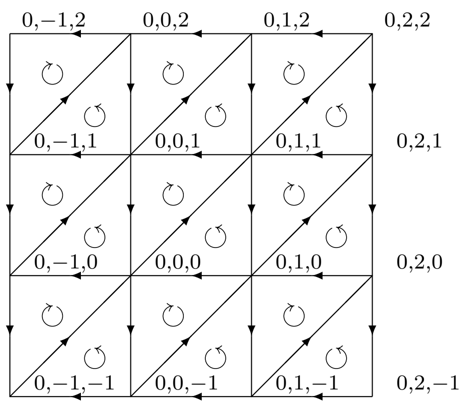


Figure 4.3: Horizontal, vertical and diagonal minimal edges.

Now we will use two coordinates. The $\mathbb{Z}^2\,(\cong \mathbb{Z}^3/\mathbb{Z}\cdot(1,1,1))$-action on a vertex $\Lambda=(m_1,m_2)$ is given by $(n_1,n_2)+\Lambda=(0,n_1+m_1,n_2+m_2)$, while the action on an horizontal edge $e=((m_1+1,m_2),(m_1,m_2))$ is given by

$$(n_1,n_2)+e=((0,n_1+m_1+1,n_2+m_2),(0,n_1+m_1,n_2+m_2)).$$

Observe that applying the lemma 4.2.22, given e horizontal edge, we have

$$\begin{array}{ccccc} & & & & \mathcal{B}((1,1)+e) \\ & & & \rotatebox{45}{$\subset$} & \cap \\ & & \mathcal{B}(e) & & \mathcal{B}((1,0)+e), \\ & & \cap & \rotatebox{45}{$\subset$} & \\ & \mathcal{B}((0,-1)+e) & & & \end{array}$$

and so $\mathcal{B}(e)\subset\mathcal{B}((l+n_1,l-n_2)+e)$ for any $l,n_1,n_2\in\mathbb{N}$. If e is vertical we have

$$\begin{array}{ccccc} & & \mathcal{B}((0,1)+e) & \supset & \mathcal{B}((1,1)+e) \\ & \rotatebox{45}{$\subset$} & & \rotatebox{45}{$\subset$} & \\ \mathcal{B}((-1,0)+e) & \supset & \mathcal{B}(e), & & \end{array}$$

and so $\mathcal{B}(e)\subset\mathcal{B}((l-n_1,l+n_2)+e)$ for any $l,n_1,n_2\in\mathbb{N}$. And finally, if e is diagonal we get

$$\begin{array}{ccc} \mathcal{B}((-1,0)+e) & \supset & \mathcal{B}(e) \\ \cap & & \cap \\ \mathcal{B}((-1,-1)+e) & \supset & \mathcal{B}((0,-1)+e), \end{array}$$

and so $\mathcal{B}(e)\subset\mathcal{B}((-n_1,-n_2)+e)$ for any $n_1,n_2\in\mathbb{N}$.

Proposition 4.3.18. *Let e be a horizontal edge. Then*

$$\mathcal{B}(e)=\mathcal{B}((0,-1)+e)\bigcap\mathcal{B}((1,1)+e)$$

and

$$\mathcal{B}((1,0)+e)=\mathcal{B}((0,-1)+e)\bigcup\mathcal{B}((1,1)+e).$$

Proof. After a change of basis (let us denote it by v_0,v_1,v_2), we may assume $e=((0,0),(-1,0))=([(0,0,0)],[(1,0,1)])$, and so it is given by

$$e=(\mathcal{O}_Kv_0\oplus\mathcal{O}_Kv_1\oplus\mathcal{O}_Kv_2\supsetneq\mathcal{O}_K\pi_Kv_0\oplus\mathcal{O}_Kv_1\oplus\mathcal{O}_K\pi_Kv_2)$$

and

$$\mathcal{B}(e)=\{z=[z_0:z_1:z_2]\in\mathbb{P}(V)|\ v_K(z_1)<v_K(z_0),v_K(z_2)\}\,.$$

In the same way,

$$(0,-1)+e = \left(\mathcal{O}_K v_0 \oplus \mathcal{O}_K v_1 \oplus \mathcal{O}_K \pi_K^{-1} v_2 \supsetneq \mathcal{O}_K \pi_K v_0 \oplus \mathcal{O}_K v_1 \oplus \mathcal{O}_K v_2\right),$$

$$\mathcal{B}((0,-1)+e) = \{z=[z_0:z_1:z_2] \in \mathbb{P}(V) |\ v_K(z_1) \le v_K(z_2), v_K(z_1) < v_K(z_0)\}.$$

$$(1,1)+e = \left(\mathcal{O}_K \pi_K^{-1} v_0 \oplus \mathcal{O}_K v_1 \oplus \mathcal{O}_K v_2 \supsetneq \mathcal{O}_K v_0 \oplus \mathcal{O}_K v_1 \oplus \mathcal{O}_K \pi_K v_2\right),$$

$$\mathcal{B}((1,1)+e) = \{z=[z_0:z_1:z_2] \in \mathbb{P}(V) |\ v_K(z_1) \le v_K(z_0), v_K(z_1) < v_K(z_2)\},$$

$$(1,0)+e = \left(\mathcal{O}_K \pi_K^{-1} v_0 \oplus \mathcal{O}_K v_1 \oplus \mathcal{O}_K \pi_K^{-1} v_2 \supsetneq \mathcal{O}_K v_0 \oplus \mathcal{O}_K v_1 \oplus \mathcal{O}_K v_2\right),$$

and

$$\mathcal{B}((1,0)+e) = \{z=[z_0:z_1:z_2] \in \mathbb{P}(V) |\ v_K(z_1) \le v_K(z_0), v_K(z_1) \le v_K(z_2)\}.$$

Thus, we get the claim. □

Let us introduce some notation. We will write

$$e = ((0,-1)+e) \wedge ((1,1)+e) \text{ and } ((1,0)+e) = ((0,-1)+e) \vee ((1,1)+e)$$

for the previous configuration, and with the given names of the corresponding edges. Generally, when $\mathcal{B}(e) = \mathcal{B}(e') \bigcap \mathcal{B}(e'')$ we will write $e = e' \wedge e''$ (or $\vee$ when we have union, respectively). We also will write

$$e \prec (1,1)+e \text{ and } e \prec (0,-1)+e$$

and in general, $e \prec e'$ if $\mathcal{B}(e) \subset \mathcal{B}(e')$. For instance, $(1,1)+e \prec (1,0)+e$.

We state similar results when e is not horizontal, even when they reduce to that one.

Proposition 4.3.19. *Let e be a vertical edge. Then*

$$\mathcal{B}(e) = \mathcal{B}((-1,0)+e) \bigcap \mathcal{B}((1,1)+e)$$

and

$$\mathcal{B}((0,1)+e) = \mathcal{B}((-1,0)+e) \bigcup \mathcal{B}((1,1)+e).$$

Proof. If we consider the basis and we reorder the two last vectors v_1 and v_2, the edges become horizontal and we apply the previous proposition. □

Proposition 4.3.20. *Let e be a diagonal edge. Then*

$$\mathcal{B}(e) = \mathcal{B}((-1,0)+e) \bigcap \mathcal{B}((0,-1)+e)$$

and

$$\mathcal{B}((-1,-1)+e) = \mathcal{B}((-1,0)+e) \bigcup \mathcal{B}((0,-1)+e).$$

Proof. Now reorder v_0 and v_1. Then, the edges become horizontal again and we apply the proposition. □

Next, we will make an ad-hoc definition of the convex hull of two horizontal edges. From now on we restrict us to these edges since, as we have seen just above, the other cases are isomorphic after a reordering of the basis.

Definition 4.3.21. *Consider two horizontal edges*

$$e = ((m_1 + 1, m_2), (m_1, m_2)) \quad \text{and} \quad e' = ((m'_1 + 1, m'_2), (m'_1, m'_2)) .$$

We define its chamber-convex hull in $\mathbb{A}$ as follows:

- *if $m_2 \neq m'_2$, the chamber-convex hull of e and e' is the full simplicial complex generated by the set of edges $e'' = ((m''_1 + 1, m''_2), (m''_1, m''_2))$ verifying:*

$$\min\{m_1, m'_1\} \le m''_1 \le \max\{m_1, m'_1\}$$
$$\min\{m_2, m'_2\} \le m''_2 \le \max\{m_2, m'_2\}$$

  and

$$\min\{m_2 - m_1, m'_2 - m'_1\} \le m''_2 - m''_1 \le \max\{m_2 - m_1, m'_2 - m'_1\}.$$

- *if $m_2 = m'_2$, the first and the last sequence of inequaities as before, but the next one instead of the middle above:*

$$m_2 - 1 \le m'' \le m_2 + 1.$$

  If we assume $m_1 < m'_1$, this is the same that considering the chamber-convex hull of e and $(0, 1) + e'$ together with the chamber-convex hull of e and $(-1, -1) + e'$ together with the chamber-convex hull of $(0, 1) + e'$ and $(-1, -1) + e'$.

Remark 4.3.22. *We expect this is the convex hull in the sense of [AB08, 3.139 (c)], in the first case, but we will not make use of this fact, so we will not prove that here. In the other case, we take a different definition since we need the convex hull including chambers. This is the reason why we call it the chamber-convex hull.*

Definition 4.3.23. *The chamber-convex hull of two edges in $\mathscr{B}(G)$ is the union of all the chamber-convex hulls of e, e' in $\mathbb{A}$ for all the apartments $\mathbb{A}$ containing both edges. The chamber-convex hull of two edges in $\mathscr{B}_{\mathfrak{L}}$ is the intersection of $\mathscr{B}_{\mathfrak{L}}$ with the chamber-convex hull in $\mathscr{B}(G)$.*

Proposition 4.3.24. *If e, e', e'' are parallel edges in $\mathbb{A}$, and e'' is in the chamber-convex-hull of e and e' then*

$$\mathcal{B}(e)\bigcap\mathcal{B}(e') \subset \mathcal{B}(e'') \subset \mathcal{B}(e)\bigcup\mathcal{B}(e'),$$

that is, $e \wedge e' \prec e'' \prec e \vee e'$. Moreover, there are edges $e'' = e \vee e'$, $e''' = e \wedge e'$ inside of it.

Proof. First of all, we will assume $e = ((1,0),(0,0))$ without loss of generality through all the proof.

Recall next that if $e' = (l+n_1, l-n_2) + e$ with $l, n_1, n_2 \in \mathbb{N}$, then $e \prec e'$, and we claim that any edge e'' in the convex hull verifies $e \prec e'' \prec e'$ since it verifies a similar expression. Indeed, in this case e' can be expressed as $(l+n, l) + e$ or as $(n_1, -n_2) + e$.

If $e' = (l+n, l)+e$ and $e'' = (n_1'', n_2'')+e$, then $0 \le n_1'' \le l+n$, $0 \le n_2'' \le l$ and $-n \le n_2'' - n_1'' \le 0$. Therefore, $n_2'' \le n_1''$, and so $e'' = (n_2'' + (n_1'' - n_2''), n_2'') + e$ with $n_2'', n_1'' - n_2'' \in \mathbb{N}$, that is, $e \prec e''$. Further

$$e' = (l+n, l)+e = (l+n-n_1'', l-n_2'')+e'' = ((l-n_2'')+n+n_2''-n_1'', l-n_2'')+e''$$

with $-n_2'', n + n_2'' - n_1'' \in \mathbb{N}$, so $e'' \prec e'$.

If $e' = (n_1, -n_2)+e$ and $e'' = (n_1'', n_2'')+e$, then $0 \le n_1'' \le n_1$, $-n_2 \le n_2'' \le 0$ and $-n_2 - n_1 \le n_2'' - n_1'' \le 0$. Thus we have $e'' = (n_1'', -(-n_2'')) + e$ with $n_1'', -n_2'' \in \mathbb{N}$, and so $e \prec e''$. In addition,

$$e' = (n_1, -n_2) + e = (n_1 - n_1'', -n_2 - n_2'') + e''$$

with $n_1 - n_1'', n_2 + n_2'' \in \mathbb{N}$, so $e'' \prec e'$ concluding the proof of our assertion at the beginning. It can be observed that we have not taken into consideration the case in which l or n_2 are 0, for which there is a different definition of the chamber-convex-hull, but this reduces to the other, since it is a union of chamber-convex-hulls whose definition edges verify $(0,1) + e' \prec e'$ and $(-1,-1) + e' \prec e'$.

Next, let $e' = (n_1, n_2) + e$ be such that $n_1 \ge 0$ and $n_2 > n_1$ (if $n_2 \le n_1$, we are in the previous case). Let us write $e^i_{-j} = (i, i+j) + e$ for $i, j \in \mathbb{N}$. Observe that $e'' = (n_1'', n_2'') + e$ belongs to the chamber-convex-hull of e and e' if and only if $0 \le n_1'' \le n_1$, $0 \le n_2'' \le n_2$ and $0 \le n_2'' - n_1'' \le n_2 - n_1$, if and only if $e'' = e^i_{-j}$ with $0 \le i \le n_1$ and $0 \le j \le n_2 - n_1$. Note that we have $e = e^0_{-0}$, $e' = e^{n_1}_{-(n_2-n_1)}$,

$$e^i_{-j} \prec e^{i+1}_{-j} \quad \text{and} \quad e^i_{-(j+1)} \prec e^i_{-j}$$

We will prove $e \wedge e' \prec e^i_{-j} \prec e \vee e'$ by double induction on i and j. For $i = 0$ and $j = 0$ it is trivial ($e = e^0_{-0}$). Assume that it is satisfied for j, and $j + 1 \leq n_2 - n_1$. We have

$$e \vee e' \succ e \succ e^0_{-j} \succ e^0_{-(j+1)} = e^0_{-j} \wedge e^1_{-(j+1)} =$$

$$= e^0_{-j} \wedge (e^1_{-j} \wedge e^2_{-(j+1)}) = e^0_{-j} \wedge e^2_{-(j+1)} = \cdots = e^0_{-j} \wedge e^{n_1}_{-(j+1)} \succ e \wedge e'$$

by the previous inequalities. In particular,

$$e \wedge e' \prec e^0_{-(n_2-n_1)} = e^0_{-(n_2-n_1-1)} \wedge e^{n_1}_{-(n_2-n_1)} = e^0_{-(n_2-n_1-1)} \wedge e' \prec e \wedge e'$$

and so $e^0_{-(n_2-n_1)} = e \wedge e'$.

Next, assume we have the result for i and suppose again $j = 0$:

$$e \wedge e' \prec e^i_{-0} \prec e^{i+1}_{-0} = e^i_0 \vee e^{i+1}_{-1} =$$

$$= e^i_0 \vee (e^i_{-1} \vee e^{i+1}_{-2}) = e^i_0 \vee e^{i+1}_{-2} = \cdots = e^i_0 \vee e^{i+1}_{-(n_2-n_1)} \prec e \vee e'.$$

In particular,

$$e \vee e' = e^0_{-0} \vee e^{n_1}_{-(n_2-n_1)} \prec e^{n_1}_0 \prec e \vee e',$$

therefore, $e^{n_1}_0 = e \vee e'$.

Finally, if this is verified by j, and $j + 1 \leq n_2 - n_1$:

$$e \wedge e' \prec e^i_{-j} \wedge e^{n_1}_{-(j+1)} = \cdots = e^i_{-j} \wedge e^{i+2}_{-(j+1)} = e^i_{-j} \wedge (e^{i+1}_{-j} \wedge e^{i+2}_{-(j+1)}) =$$

$$= e^i_{-j} \wedge e^{i+1}_{-(j+1)} = e^i_{-(j+1)} \prec e^i_0 \prec e \vee e',$$

as we wanted to prove.

Finally, if $n_1 < 0$, we change e with e' to get the result. □

Remark 4.3.25. *It is easy to check that the reciprocal is not true, since the set of edges whose associated open sets verify those inclusions is generally quite bigger. This would lead to another kind of "expanded" convex-hull.*

Corollary 4.3.26. *Given two parallel edges e, e' in $\mathscr{B}_{\mathcal{L}}$ such that $\mathcal{B}_{\mathcal{L}}(e) = \mathcal{B}_{\mathcal{L}}(e')$, any e'' in the chamber-convex-hull of e and e' verifies $\mathcal{B}_{\mathcal{L}}(e'') = \mathcal{B}(e)$.*

Proof. This is a corollary of the previous proposition. □

4.3.3 A basis from the edges on the rays

Let us consider a vertex $\Lambda_0 \in \mathscr{B}(G)_0$ and an apartment $\mathbb{A}$ in $\mathscr{B}(G)$ containing it, and assume it has the coordinates $(0,0)$ by the isomorphism $\mathbb{A} \cong \mathbb{Z}^2$. Consider also the full subcomplex $\mathbb{A}(\Lambda_0)^{(n)}$ generated by the vertices at distance less or equal than n for a given $n \in \mathbb{N}_{\geq 1}$. Observe that the edge $((-(n-1),0),(-n,0))$ belongs to it.

Lemma 4.3.27. *For any horizontal edge e in $\mathbb{A}(\Lambda_0)^{(n)}$,*

$$\mathcal{B}(((-(n-1),0),(-n,0))) \subset \mathcal{B}(e).$$

Proof. Recall that the

$$\rho((0,0),(m_1,m_2)) = \|(0,m_1,m_2)\|_{trop} = \max\{|m_1|,|m_2|,|m_1-m_2|\}.$$

Let us write $e_n := ((-(n-1),0),(-n,0))$ and $e = ((m_1+1,m_2),(m_1,m_2))$. We know $\mathcal{B}(e_n) \subset \mathcal{B}((l+n_1, l-n_2)+e_n)$ for $l, n_1, n_2 \in \mathbb{N}$ and also

$$\max\{|m_1+1|,|m_1|,|m_2|,|m_1+1-m_2|,|m_1-m_2|\} \leq n.$$

Now, if $0 \leq m_2 (\leq n)$, take $l := m_2$, $n_1 := m_1 - m_2 + n$ and $n_2 := 0$, and note that $n_1 \geq -n+n = 0$. If $m_2 < 0$, take $n_2 := -m_2$, $n_1 := m_1 + n$ and $l := 0$ and note that $n_1 \geq 0$. In any case we have $e = (l+n_1, l-n_2) + e_n$, so thus, we finish the proof. □

Remark 4.3.28. *With the notation of the previous proof, e_n is the n-th edge in the ray $r = (e_1, e_2, e_3, \dots)_\infty$ in $\mathbb{A}$ where $e_1 := ((0,0),(-1,0))$. Reciprocally, the n-th edge of any ray from Λ_0 can be represented as e_n choosing an apartment containing the ray and a suitable isomorphism $\mathbb{A} \cong \mathbb{Z}^2$.*

In addition, consider the edges in $\mathrm{St}_{\mathcal{L}}^{min}(\Lambda_0)$, and given a set of minimal edges $E \subset \widehat{\mathscr{B}_{\mathcal{L}1}}^{min}$ let $\mathrm{Flow}_{\mathcal{L}}(E)$ denote the set of edges in $\mathrm{Flow}_{\mathcal{L}}(e)$ where $e \in E$. Then, for any $n \in \mathbb{N}_{\geq 1}$ the set of edges e_n in $\mathscr{B}_{\mathcal{L}}$ coincides with

$$\mathrm{Flow}_{\mathcal{L}}^{n-1}\left(\mathrm{St}_{\mathcal{L}}^{min}(\Lambda_0)\right).$$

Let us denote this set by $\mathscr{R}_{\mathcal{L}}^{(n)}(\Lambda_0)$. The corollary 4.3.16 together with the minimal star decomposition imply

$$\mathcal{L} = \bigsqcup_{e \in \mathscr{R}_{\mathcal{L}}^{(n)}(\Lambda_0)} \mathcal{B}_{\mathcal{L}}(e).$$

Lemma 4.3.29. *For each $\widetilde{e} \in \mathscr{R}_{\mathcal{L}}^{(n)}(\Lambda_0)$ there are lattices $L_0, \widetilde{L_0}, \widetilde{L_1}$ such that $\Lambda_0 = [L_0]$, $\widetilde{e}$ is given by $\widetilde{L_0} \supsetneq \widetilde{L_1} \supsetneq \pi_K \widetilde{L_0}$, and for any $p \in \mathcal{B}(\widetilde{e})$ there is a representant $v \in V$ veryfying $v \in L_0 \setminus \pi_K L_0$, $\mathcal{O}_K v + \pi_K^{n-1} L_0 = \widetilde{L_0}$, $\mathcal{O}_K v + \pi_K^n L_0 = \widetilde{L_1}$ and*

$$\mathcal{B}(\widetilde{e}) = \{[\mu v + \pi_K^n v'] \in \mathbb{P}(V) \mid \mu \in \mathcal{O}_K^*, v' \in L_0\}.$$

Proof. Let us write $\Lambda_0 = [L_0]$ and $p = [v]$ and assume that $\widetilde{e}$ is given by

$$\widetilde{L_0} \supsetneq \widetilde{L_1} \supsetneq \pi_K \widetilde{L_0}.$$

Consider for a moment an apartment $\mathbb{A}$ containing $\widetilde{e}$ and Λ_0 in the way explained in the previous remark, so Λ_0 corresponds to $[(0,0,0)]$ and $\widetilde{e}$ corresponds to $e_n = ([(0, -(n-1), 0)], [(0, -n, 0)]) = ([(n-1, 0, n-1)], [(n, 0, n)])$. That, is, we are taking into account a basis v_0, v_1, v_2 of $V \cong K^3$ such that we can define the lattices as

$$L_0 = \mathcal{O}_K v_0 \oplus \mathcal{O}_K v_1 \oplus \mathcal{O}_K v_2,$$

$$\widetilde{L_0} = \mathcal{O}_K \pi_K^{n-1} v_0 \oplus \mathcal{O}_K v_1 \oplus \mathcal{O}_K \pi_K^{n-1} v_2$$

and

$$\widetilde{L_1} = \mathcal{O}_K \pi_K^{n} v_0 \oplus \mathcal{O}_K v_1 \oplus \mathcal{O}_K \pi_K^{n} v_2$$

Further $p \in \mathcal{B}(\widetilde{e})$ means that

$$\widetilde{L_1} = \pi_K \widetilde{L_0} + \left(\widetilde{L_0} \cap \langle v \rangle\right),$$

what, writing $v = \sum_{i=0}^{2} \lambda_i v_i$, is equivalent to $v_K(\lambda_1) + n \leq v_K(\lambda_0), v_K(\lambda_2)$. Since $v \neq 0$ we assume without loss of generality $v_K(\lambda_0), v_K(\lambda_2) \geq n$ and $v_K(\lambda_1) = 0$, in particular $v \in L_0 \setminus \pi_K L_0$. Therefore, $\mathcal{O}_K v + \pi_K^{n-1} L_0 = \widetilde{L_0}$, $\mathcal{O}_K v + \pi_K^n L_0 = \widetilde{L_1}$ and

$$\mathcal{B}(\widetilde{e}) = \{[\mu v + \pi_K^n v'] \in \mathbb{P}(V) \mid \mu \in \mathcal{O}_K^*, v' \in L_0\}.$$

□

Lemma 4.3.30. *Let $e = (\Lambda, \Lambda')$ be a minimal edge in $\widehat{\mathscr{B}_{\mathcal{L}_1}}^{min}(\Lambda_0)^{(n)}$. Then we can choose representing lattices such that $[L_0] = \Lambda_0$, $[L] = \Lambda$, $[L'] = \Lambda'$ and*

$$L_0 \supset L' \supset \pi_K L \supset \pi_K^n L_0.$$

Proof. Take an apartment $\mathbb{A}$ in $\mathscr{B}(G)$ containing Λ_0 and e. There are two possibilities: either the two vertices of e are at distance n of Λ_0 or one is at distance $n-1$.

In the first case, by the well known structure of the apartment, there is a vertex $\widetilde{\Lambda} = [\widetilde{L}]$ at distance $n-1$ from Λ_0, which together with e form a chamber. Indeed, take a basis for $\mathbb{A}$ for which e is horizontal, so it is represented by $((m_1+1, m_2), (m_1, m_2))$, and Λ_0 is represented by $(0,0)$. It is a quick check that if the two vertices of e are at distance n of Λ_0, then m_2 has to be $\pm n$. Assume without loss of generality $m_2 = n$. Now the vertex we claim its existence is $(m_1, n-1)$.

Next, we can take $\widetilde{L}$ such that $L_0 \supsetneq \widetilde{L} \supsetneq \pi_K^{n-1} L_0$ and L, L' such that $L \supsetneq \widetilde{L} \supsetneq L' \supsetneq \pi_K L$. Then

$$L_0 \supsetneq \widetilde{L} \supsetneq L' \supsetneq \pi_K L \supsetneq \pi_K \widetilde{L} \supsetneq \pi_K^n L_0.$$

In the second case, if the vertex at distance $n-1$ is Λ, we can take representants L, L' such that $L_0 \supsetneq L \supsetneq \pi_K^{n-1} L_0$, and also

$$L_0 \supsetneq L \supsetneq L' \supsetneq \pi_K L \supsetneq \pi_K^n L_0,$$

while, if the vertex at distance $n-1$ is Λ', we can take representants L, L' such that $L_0 \supsetneq L' \supsetneq \pi_K^{n-1} L_0$, and also

$$L_0 \supsetneq L' \supsetneq \pi_K L \supsetneq \pi_K L' \supsetneq \pi_K^n L_0.$$

$\square$

Proposition 4.3.31. *Let e be a minimal edge in $\widehat{\mathscr{B}_{\mathcal{L}1}}^{min}(\Lambda_0)^{(n)}$. Then, there is a subset $\mathscr{R}_e$ of $\mathscr{R}_{\mathcal{L}}^{(n)}(\Lambda_0)$ such that*

$$\mathcal{B}_{\mathcal{L}}(e) = \bigsqcup_{\widetilde{e} \in \mathscr{R}_e} \mathcal{B}_{\mathcal{L}}(\widetilde{e}).$$

Proof. We already know that the sets in the right-hand side of the equality are disjoint. Then, it is enough to prove that for each $p \in \mathcal{B}_{\mathcal{L}}(e)$ there is $\widetilde{e} \in \mathscr{R}_{\mathcal{L}}^{(n)}(\Lambda_0)$ such that

$$p \in \mathcal{B}_{\mathcal{L}}(\widetilde{e}) \subset \mathcal{B}_{\mathcal{L}}(e).$$

In fact, by the remark 4.3.28, there is a unique $\widetilde{e} \in \mathscr{R}_{\mathcal{L}}^{(n)}(\Lambda_0)$ such that $p \in \mathcal{B}_{\mathcal{L}}(\widetilde{e})$. Now, under these conditions, one only need to show $\mathcal{B}(\widetilde{e}) \subset \mathcal{B}(e)$.

Let us write $\Lambda_0 = [L_0]$ and $p = [v]$. By the lemma 4.3.29 we assume $\widetilde{e}$ is given by

$$\widetilde{L_0} \supsetneq \widetilde{L_1} \supsetneq \pi_K \widetilde{L_0},$$

this sequence satisfies

$$L_0 \supsetneq \widetilde{L_0} \supsetneq \widetilde{L_1} \supsetneq \pi_K \widetilde{L_0} \supsetneq \pi_K^n L_0,$$

and v is chosen veryfying $v \in L_0 \setminus \pi_K L_0$ and

$$\mathcal{B}(\widetilde{e}) = \{[\mu v + \pi_K^n v'] \in \mathbb{P}(V) \mid \mu \in \mathcal{O}_K^*, v' \in L_0\}.$$

Next, let e be given by

$$L \supsetneq L' \supsetneq \pi_K L.$$

By the previous lemma we can take these lattices such that

$$L_0 \supsetneq L' \supsetneq \pi_K L \supsetneq \pi_K^n L_0$$

so we do it. Recall that $p \in \mathcal{B}(e)$, so that there exists $\lambda_e \in K$ such that $\lambda_e v \in L' \setminus \pi_K L$, and similarly, $\mu\lambda_e v \in L' \setminus \pi_K L$ for any $\mu \in \mathcal{O}_K^*$.

Thus, one gets $v \in L_0 \setminus \pi_K L_0$ and $\lambda_e v \in L_0 \setminus \pi_K^n L_0$, and so

$$-1 < v_K(\lambda_e) < n.$$

Therefore $v_K(\lambda_e \pi_K^n) \geq n$, so for any $v' \in L_0$ we get, $\lambda_e \pi_K^n v' \in \pi_K^n L_0 \subset \pi_K L$ and as a consequence

$$\lambda_e(\mu v + \pi_K^n v') = \mu\lambda_e v + \lambda_e \pi_K^n v' \in L' \setminus \pi_K L.$$

for any $v' \in L_0$. But these vectors represent all the points in $\mathcal{B}(\widetilde{e})$, so that implies $\mathcal{B}(\widetilde{e}) \subset \mathcal{B}(e)$.

□

Remark 4.3.32. *Note that we are not saying that we could obtain all these edges $\widetilde{e}$ in $\mathscr{R}_e$ as the edges e_n in the apartments containing e and Λ_0. That is, given an apartment $\mathbb{A}$ containing e (horizontally) and Λ_0, the corresponding e_n as in the lemma 4.3.27 is in $\mathscr{R}_e$. But, even it seems natural, we are not able to ensure the reciprocal.*

Corollary 4.3.33. *The sets $\mathcal{B}_{\mathcal{L}}(e)$ with e in $\bigcup_{n\in\mathbb{N}_{\geq 1}} \mathscr{R}_{\mathcal{L}}^{(n)}(\Lambda_0)$ form a basis for the topology of $\mathcal{L}$.*

Further, each open compact $\mathcal{U} \subset \mathcal{L}$ has associated a unique minimum covering by sets $\mathcal{B}_{\mathcal{L}}(e)$ with $e \in \mathscr{R}_{\mathcal{U}} \subset \bigcup_{n\in\mathbb{N}_{\geq 1}} \mathscr{R}_{\mathcal{L}}^{(n)}(\Lambda_0)$, that is

$$\mathcal{U} = \bigsqcup_{e\in\mathscr{R}_{\mathcal{U}}} \mathcal{B}_{\mathcal{L}}(e).$$

In particular, any covering of $\mathcal{U}$ by sets $\mathcal{B}_{\mathcal{L}}(e)$ with $e \in \bigcup_{n\in\mathbb{N}_{\geq 1}} \mathscr{R}^{(n)}_{\mathcal{L}}(\Lambda_0)$ is a refinement of the covering given by $\mathscr{R}_{\mathcal{U}}$ and, as a consequence, the former induces a covering of $\mathcal{B}_{\mathcal{L}}(e)$ for each $e \in \mathscr{R}_{\mathcal{U}}$. Moreover, given two coverings of $\mathcal{U}$ "by edges"

$$\{e\}_{e\in I}, \{e'\}_{e'\in I'} \subset \bigcup_{n\in\mathbb{N}_{\geq 1}} \mathscr{R}^{(n)}_{\mathcal{L}}(\Lambda_0),$$

there is another covering "by edges" $\{e''\}_{e''\in I''}$ finer than both.

Proof. Since the open sets $\mathcal{B}_{\mathcal{L}}(e)$ with e minimal edge in $\mathscr{B}_{\mathcal{L}}$ form a basis for the topology of $\mathcal{L}$, the previous result allows us to take just the edges in the rays from Λ_0, so they form the claimed basis.

For the second claim, observe that given $e, e' \in \bigcup_{n\in\mathbb{N}_{\geq 1}} \mathscr{R}^{(n)}_{\mathcal{L}}(\Lambda_0)$, the associated open sets verify either $\mathcal{B}_{\mathcal{L}}(e) \cap \mathcal{B}_{\mathcal{L}}(e') = \emptyset$ or some inclussion, maybe $\mathcal{B}_{\mathcal{L}}(e) \subset \mathcal{B}_{\mathcal{L}}(e')$. In fact, the inclusion is satisfied if there is a common ray $r = (e_1, e_2, \dots)_\infty$ from Λ_0, and $e = e_n, e' = e_m$ with $m \leq n$, while the intersection is empty if and only if there is no common ray from Λ_0 containing both edges.

Take $p \in \mathcal{U}$. Then, by the first part of the corollary, there exists a ray r from Λ_0 (for example, the ray such that $\widetilde{\varepsilon}(r) = p$) and an edge e_n in r such that $p \in \mathcal{B}_{\mathcal{L}}(e) \subset \mathcal{U}$. Now we have $\mathcal{B}_{\mathcal{L}}(e_n) \subset \mathcal{B}_{\mathcal{L}}(e_{n-1})$. If $\mathcal{B}_{\mathcal{L}}(e_{n-1}) \not\subset \mathcal{U}$, let e_n be in $\mathscr{R}_{\mathcal{U}}$, otherwise apply this rule to e_{n-1}. This proceeding is finite, since, either there is an n such that $\mathcal{B}_{\mathcal{L}}(e_n) \subset \mathcal{U}$ and $\mathcal{B}_{\mathcal{L}}(e_{n-1}) \not\subset \mathcal{U}$, or $\mathcal{B}_{\mathcal{L}}(e_1) \subset \mathcal{U}$; in this situation let $e_1 \in \mathscr{R}_{\mathcal{U}}$.

Thus, for each $e \in \mathscr{R}_{\mathcal{U}}$ there is no $e' \in \bigcup_{n\in\mathbb{N}_{\geq 1}} \mathscr{R}^{(n)}_{\mathcal{L}}(\Lambda_0)$ with bigger associated open contained in $\mathcal{U}$, what shows the minimality of the covering, and so its uniqueness. Thus concludes the proof of the first claim and its immediate consequence.

Now, given two coverings by edges $\{e\}_{e\in I}, \{e'\}_{e'\in I'}$, for each $e \in I$, either there is $e' \in I'$ such that $\mathcal{B}_{\mathcal{L}}(e) \subset \mathcal{B}_{\mathcal{L}}(e')$ or there are $e' \in I'$ such that the opposite inclusion verifies. In any case, we take the edges $e'' \in I''$ being the ones among the edges of $I \cup I'$ inducing the smaller open sets, so that they are inside all the open sets of the given coverings, and so they generate a finer covering. □

4.4 Harmonic cochains on $\mathscr{B}_{\mathcal{L}}$ and its isomorphism with harmonic measures on $\mathcal{L}$ when $d = 2$ and $\mathcal{L} \subset \mathbb{P}(V)$ is compact

In the first paragraph of this section we introduce global and local harmonic cochains on the minimal 1-skeletons of the subcomplexes $\mathscr{B}_{\mathcal{L}}$ of the Bruhat-Tits building of dimension 2, some related morphisms, and we study some properties. The next two paragraphs are devoted to the proof of the isomorphism between the harmonic cochains and the harmonic measures on $\mathcal{L}$, which is the main theorem of this chapter. Finally we will see that the harmonic cochains are homotopically invariant.

Through this section, we keep the hypothesis of dimension $d = 2$ for the Bruhat-Tits building.

4.4.1 Harmonic cochains on $\mathscr{B}_{\mathcal{L}}$

Let $\mathcal{L} \subset \mathbb{P}(V)$ be a compact set.

We introduce a definition of harmonic cochains inspired directly by the space of cochains obtained by Schneider and Stuhler in [SS91, §4 Cor. 17]. We restrict to the cochains on edges, since they are the ones we need, but the definition generalizes to any dimension.

Definition 4.4.1. *A map $c : \mathbb{Z}[\widehat{\mathscr{B}_{\mathcal{L}1}}^{min}] \longrightarrow \mathbb{Z}$ is called a harmonic cochain if it satisfies the following properties:*

- $c \circ \mathrm{St}_{\mathcal{L}}^{min} = 0$.
- *For any minimal edge $e \in \widehat{\mathscr{B}_{\mathcal{L}1}}^{min}$, $c(e) = c(\mathrm{Flow}_{\mathcal{L}}(e))$, that is*

$$c \circ \left(\mathbb{1}_{\widehat{\mathscr{B}_{\mathcal{L}1}}^{min}} - \mathrm{Flow}_{\mathcal{L}}\right) = 0.$$

- $c \circ \partial^{min} = 0$.

The set of harmonic cochains is denoted by $\mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}}, \mathbb{Z})$.

Remark 4.4.2. *Recall that*

$$\mathrm{St}_{\mathcal{L}}^{min}(t(e)) = \mathrm{Flow}_{\mathcal{L}}(e) \bigsqcup \mathrm{Cling}_{\mathcal{L}}^{min}(e)$$

Then, the two first conditions imply

$$c(e) = -c(\mathrm{Cling}_{\mathcal{L}}^{min}(e)).$$

Actually, under the assumption of the star condition, the flow property is equivalent to this one.

Lemma 4.4.3. *We recall the situation of the lemma 4.2.22 and the proposition 4.2.26. Let Δ and Δ' be two chambers in $\mathscr{B}_{\mathcal{L}}$ intersecting in a panel or maximal face of both $A = \Delta \cap \Delta'$. Let e_A^{Δ} be the minimal edge of Δ whose target vertex is the opposite to A, that is the vertex of Δ not contained in A. Let $e_{\Delta'}^{A}$ be the minimal edge of Δ' whose source vertex is the opposite to A in Δ'. In this situation, $\mathrm{cov}_{\mathcal{L}}(A) = 2$ implies $c(e_{\Delta'}^{A}) = c(e_A^{\Delta})$ and in general*

$$c(e_{\Delta'}^{A}) = \sum_{\substack{\Delta \in \mathscr{B}_{\mathcal{L}d} \\ \Delta \cap \Delta' = A}} c(e_A^{\Delta}).$$

Proof. Observe that

$$\partial^{min}(\Delta) = e_A^{\Delta} + e_{\Delta}^{A} + A \quad \text{ and } \quad \partial^{min}(\Delta') = e_A^{\Delta'} + e_{\Delta'}^{A} + A$$

with A properly oriented, and so $c(e_A^{\Delta'}) + c(e_{\Delta'}^{A}) + c(A) = 0$. Since $\mathrm{cov}_{\mathcal{L}}(A) = 2$ we have $\mathrm{Cling}_{\mathcal{L}}^{min}(A) = \{e_A^{\Delta}, e_A^{\Delta'}\}$, and so

$$0 = \sum_{e \in \mathrm{St}_{\mathcal{L}}^{min}(t(A))} c(e) = \sum_{e \in \mathrm{Flow}_{\mathcal{L}}(A)} c(e) + \sum_{\mathrm{Cling}_{\mathcal{L}}^{min}(A)} c(e) = c(A) + c(e_A^{\Delta}) + c(e_A^{\Delta'}).$$

We put all together:

$$0 = c(e_A^{\Delta'}) + c(e_{\Delta'}^{A}) + c(A) = c(e_A^{\Delta'}) + c(e_{\Delta'}^{A}) - c(e_A^{\Delta}) - c(e_A^{\Delta'}) = c(e_{\Delta'}^{A}) - c(e_A^{\Delta})$$

as we wanted to proof. In general,

$$0 = \sum_{e \in \mathrm{Flow}_{\mathcal{L}}(A)} c(e) + \sum_{\mathrm{Cling}_{\mathcal{L}}^{min}(A)} c(e) = c(A) + c(e_A^{\Delta'}) + \sum_{\substack{\Delta \in \mathscr{B}_{\mathcal{L}d} \\ \Delta \cap \Delta' = A}} c(e_A^{\Delta}).$$

concluding as above. □

Definition 4.4.4. *Given a simplex Δ_0, a local harmonic cochain on Δ_0 is a map $c : \mathbb{Z}[\widehat{\mathscr{B}_{\mathcal{L}1}}^{min}(\Delta_0)] \longrightarrow \mathbb{Z}$ verifying:*

- $c(\mathrm{St}_{\mathcal{L}}^{min}(\Lambda)) = 0$ *when* $\mathrm{St}_{\mathcal{L}}^{min}(\Lambda) \leq \widehat{\mathscr{B}_{\mathcal{L}1}}^{min}(\Delta_0)$ *respectively,*
- $c(e) = -c(\mathrm{Cling}_{\mathcal{L}}^{min}(e))$ *when* $e, \mathrm{Cling}_{\mathcal{L}}^{min}(e) \leq \widehat{\mathscr{B}_{\mathcal{L}1}}^{min}(\Delta_0)$, *and*
- $c(\partial^{min}(\Delta)) = 0$ *when* $\partial^{min}(\Delta) \leq \widehat{\mathscr{B}_{\mathcal{L}1}}^{min}(\Delta_0)$ *(if and only if* $\Delta \leq \widehat{\mathscr{B}_{\mathcal{L}d}}^{min}(\Delta_0)$*).*

The set of local harmonic cochains on Δ_0 in $\mathscr{B}_{\mathcal{L}}$ is denoted by $\mathrm{C}^1_{\mathrm{har}}(\Delta_0, \mathbb{Z})_{\mathcal{L}}$.

A face relation of simplices $\Delta_1 \leq \Delta_0$ induces an inclusion

$$\widehat{\mathscr{B}_{\mathcal{L}1}}^{min}(\Delta_0) \subset \widehat{\mathscr{B}_{\mathcal{L}1}}^{min}(\Delta_1),$$

and so, a restriction map $\mathrm{C}^1_{\mathrm{har}}(\Delta_1, \mathbb{Z})_{\mathcal{L}} \longrightarrow \mathrm{C}^1_{\mathrm{har}}(\Delta_0, \mathbb{Z})_{\mathcal{L}}$ which is a kind of coface map.

Remark 4.4.5. *Given a vertex $\Lambda \in \mathscr{B}_{\mathcal{L}0}$, we define its (minimal) 1-link as*

$$\mathrm{Lk}_1^{min}(\Lambda) := \widehat{\mathscr{B}_{\mathcal{L}1}}^{min}(\Lambda) \setminus \left(\mathrm{St}_{\mathcal{L}}^{min}(\Lambda) \bigsqcup \mathrm{St}_{\mathcal{L}}^{-min}(\Lambda)\right)$$

Then, the next equalities are straightforward:

$$\mathrm{Lk}_1^{min}(\Lambda) = \bigsqcup_{e \in \mathrm{St}_{\mathcal{L}}^{min}(\Lambda)} \mathrm{Cling}_{\mathcal{L}}^{min}(e)$$

$$\widehat{\mathscr{B}_{\mathcal{L}1}}^{min}(e) = \widehat{\mathscr{B}_{\mathcal{L}1}}^{min}(s(e)) \bigcap \widehat{\mathscr{B}_{\mathcal{L}1}}^{min}(t(e)) = \{e\} \bigsqcup \mathrm{Cling}_{\mathcal{L}}^{min}(e) \bigsqcup \mathrm{Cling}_{\mathcal{L}}^{-min}(e).$$

Take a vertex $\Lambda \in \mathscr{B}_{\mathcal{L}0}$. Then $\mathrm{C}^1_{\mathrm{har}}(\Lambda, \mathbb{Z})_{\mathcal{L}}$ is the set of maps

$$c : \mathbb{Z}[\widehat{\mathscr{B}_{\mathcal{L}1}}^{min}(\Lambda)] \longrightarrow \mathbb{Z}$$

such that $c(\mathrm{St}_{\mathcal{L}}^{\pm min}(\Lambda)) = 0$, $c(\partial^{min}(\Delta)) = 0$ for all $\Delta \in \widehat{\mathscr{B}_{\mathcal{L}d}}^{min}(\Lambda)$ and $c(e) = -c(\mathrm{Cling}_{\mathcal{L}}^{min}(e))$ for all $e \in \mathrm{St}_{\mathcal{L}}^{\pm min}(\Lambda)$. In addition, we have a restriction map

$$\mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}}, \mathbb{Z}) \longrightarrow \mathrm{C}^1_{\mathrm{har}}(\Lambda, \mathbb{Z})_{\mathcal{L}}.$$

Given an edge $e \in \mathscr{B}_{\mathcal{L}1}$, $\mathrm{C}^1_{\mathrm{har}}(e, \mathbb{Z})_{\mathcal{L}}$ is the set of maps

$$c : \mathbb{Z}[\widehat{\mathscr{B}_{\mathcal{L}1}}^{min}(e)] \longrightarrow \mathbb{Z}$$

such that $c(e) = -c(\mathrm{Cling}_{\mathcal{L}}^{min}(e))$ (with e oriented as a minimal edge), and $c(\partial^{min}(\Delta)) = 0$ for all $\Delta \in \widehat{\mathscr{B}_{\mathcal{L}d}}^{min}(e)$, that is all chambers $\Delta \geq e$. From this description it is easy to obtain $\mathrm{C}^1_{\mathrm{har}}(e, \mathbb{Z})_{\mathcal{L}} \cong \mathbb{Z}[\mathrm{Cling}_{\mathcal{L}}^{min}(e)] \cong \mathbb{Z}^{\mathrm{cov}_{\mathcal{L}}(e)}$.

And a chamber $\Delta \in \mathscr{B}_{\mathcal{L}2}$ verifies $\widehat{\mathscr{B}_{\mathcal{L}1}}^{min}(\Delta) = \partial^{min}(\Delta) = \{e_0, e_1, e_2\}$. Therefore $\mathrm{C}^1_{\mathrm{har}}(\Delta, \mathbb{Z})_{\mathcal{L}}$ is the set of maps

$$c : \mathbb{Z}[\widehat{\mathscr{B}_{\mathcal{L}1}}^{min}(\Delta)] \longrightarrow \mathbb{Z}$$

such that $c(\partial^{min}(\Delta)) = 0$. But then we get $\mathrm{C}^1_{\mathrm{har}}(\Delta, \mathbb{Z})_{\mathcal{L}} \cong \mathbb{Z}^2$.

Thus, the coface maps previously defined as restrictions rise to a map

$$\partial^1 : \prod_{\Lambda \in \mathscr{B}_{\mathcal{L}0}} \mathrm{C}^1_{\mathrm{har}}(\Lambda, \mathbb{Z})_{\mathcal{L}} \longrightarrow \prod_{e \in \mathscr{B}_{\mathcal{L}1}} \mathrm{C}^1_{\mathrm{har}}(e, \mathbb{Z})_{\mathcal{L}}$$

given by $\partial^1((c_\Lambda)_\Lambda)_e = (c_{t(e)} - c_{s(e)})_e$, resulting in a complex

$$0 \longrightarrow \mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}}, \mathbb{Z}) \longrightarrow \prod_{\Lambda \in \mathscr{B}_{\mathcal{L}0}} \mathrm{C}^1_{\mathrm{har}}(\Lambda, \mathbb{Z})_{\mathcal{L}} \xrightarrow{\partial^1} \prod_{e \in \mathscr{B}_{\mathcal{L}1}} \mathrm{C}^1_{\mathrm{har}}(e, \mathbb{Z})_{\mathcal{L}}$$

since each composition vanishes. Moreover, the restriction map is clearly injective.

Proposition 4.4.6. *The complex*

$$0 \longrightarrow \mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}}, \mathbb{Z}) \longrightarrow \prod_{\Lambda \in \mathscr{B}_{\mathcal{L}0}} \mathrm{C}^1_{\mathrm{har}}(\Lambda, \mathbb{Z})_{\mathcal{L}} \xrightarrow{\partial^1} \prod_{e \in \mathscr{B}_{\mathcal{L}1}} \mathrm{C}^1_{\mathrm{har}}(e, \mathbb{Z})_{\mathcal{L}}$$

is exact.

Proof. Take $(c_\Lambda)_\Lambda \in \mathrm{Ker}(\partial^1)$. Then, each c_Λ agrees with the other maps on nonempty intersections on $\mathscr{B}_{\mathcal{L}}$, which allows us to define a map c on all $\widehat{\mathscr{B}_{\mathcal{L}1}}^{min}$. It is harmonic, since each of the conditions of harmonicity is inside some $\widehat{\mathscr{B}_{\mathcal{L}1}}^{min}(\Lambda)$ for $\Lambda \in \mathscr{B}_{\mathcal{L}0}$. □

4.4.2 Relating harmonic cochains on $\mathscr{B}_{\mathcal{L}}$ and harmonic measures on $\mathcal{L}$

Let $\mathcal{L} \subset \mathbb{P}(V)$ be a compact set. Our goal is to get an isomorphism of abelian groups between the harmonic cochains on $\mathscr{B}_{\mathcal{L}}$ and the harmonic measures on $\mathcal{L}$. Through this paragraph we will relate them and we will introduce the tools to get the isomorphism in the next one.

Proposition 4.4.7. *There is an injective map*

$$\kappa : \mathscr{M}(\mathcal{L}, \mathbb{Z})_0 \longrightarrow \mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}}, \mathbb{Z}).$$

defined by $\kappa(\mu)(e) := \mu(\mathcal{B}(e))$.

Proof. The only thing we have to proof is that it is well defined, that is to see that $\kappa(\mu)$ is a harmonic cochain. But we know that

$$\mathcal{L} = \bigsqcup_{\mathrm{St}^{min}_{\mathcal{L}}(\Lambda)} \mathcal{B}_{\mathcal{L}}(e) = \bigsqcup_{\partial^{min}(\Delta)} \mathcal{B}_{\mathcal{L}}(e),$$

so $\kappa(\mu)\left(\mathrm{St}^{min}_{\mathcal{L}}(\Lambda)\right) = \mu(\mathcal{L}) = 0 = \kappa(\mu)\left(\partial^{min}(\Delta)\right)$. Moreover, we also have

$$\mathcal{B}_{\mathcal{L}}(e) = \bigsqcup_{e' \in \mathrm{Flow}_{\mathcal{L}}(e)} \mathcal{B}_{\mathcal{L}}(e'),$$

therefore $\kappa(\mu)(e) = \sum_{e' \, \mathrm{Flow}_{\mathcal{L}}(e)} \kappa(\mu)(e')$.

The injectivity reduces to the fact that the sets $\mathcal{B}_{\mathcal{L}}(e)$ are a basis for the open compact subsets of $\mathcal{L}$. □

Given a simplex Δ in $\mathscr{B}_{\mathcal{L}}$, consider the kernel of the composition

$$\mathscr{M}(\mathcal{L}, \mathbb{Z})_0 \xrightarrow{\kappa} \mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}}, \mathbb{Z}) \longrightarrow \mathrm{C}^1_{\mathrm{har}}(\Delta, \mathbb{Z})_{\mathcal{L}}.$$

which is $I_\Delta := \left\{\mu \in \mathscr{M}(\mathcal{L}, \mathbb{Z})_0 \mid \mu(\mathcal{B}_{\mathcal{L}}(e)) = 0 \ \forall \ e \in \widehat{\mathscr{B}_{\mathcal{L}1}}^{min}(\Delta)\right\}$. Then, let us consider the quotients

$$\mathscr{M}_\Delta(\mathcal{L}, \mathbb{Z})_0 := \mathscr{M}(\mathcal{L}, \mathbb{Z})_0 / \operatorname{Ker}\left(\mathscr{M}(\mathcal{L}, \mathbb{Z})_0 \longrightarrow \mathrm{C}^1_{\mathrm{har}}(\Delta, \mathbb{Z})_{\mathcal{L}}\right).$$

so we get injective maps

$$\kappa_\Lambda : \mathscr{M}_\Delta(\mathcal{L}, \mathbb{Z})_0 \longrightarrow \mathrm{C}^1_{\mathrm{har}}(\Delta, \mathbb{Z})_{\mathcal{L}}.$$

Thus, given a face relation $\Delta_1 \leq \Delta_0$ we have coface maps

$$\mathscr{M}_{\Delta_1}(\mathcal{L}, \mathbb{Z})_0 \longrightarrow \mathscr{M}_{\Delta_0}(\mathcal{L}, \mathbb{Z})_0,$$

which give a complex

$$0 \longrightarrow \mathscr{M}(\mathcal{L}, \mathbb{Z})_0 \longrightarrow \prod_{\Lambda \in \mathscr{B}_{\mathcal{L}0}} \mathscr{M}_\Lambda(\mathcal{L}, \mathbb{Z})_0 \xrightarrow{\partial^1} \prod_{e \in \mathscr{B}_{\mathcal{L}1}} \mathscr{M}_e(\mathcal{L}, \mathbb{Z})_0$$

where $\partial^1((\mu_\Lambda)_\Lambda)_e = (\mu_{t(e)} - \mu_{s(e)})_e$.

Therefore, we got an injective map of complexes

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathscr{M}(\mathcal{L}, \mathbb{Z})_0 & \longrightarrow & \prod_{\Lambda \in \mathscr{B}_{\mathcal{L}0}} \mathscr{M}_\Lambda(\mathcal{L}, \mathbb{Z})_0 & \xrightarrow{\partial^1} & \prod_{e \in \mathscr{B}_{\mathcal{L}1}} \mathscr{M}_e(\mathcal{L}, \mathbb{Z})_0 \\
& \downarrow \kappa & & \downarrow \kappa_\Lambda & & \downarrow \kappa_e \\
0 \longrightarrow & \mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}}, \mathbb{Z}) & \longrightarrow & \prod_{\Lambda \in \mathscr{B}_{\mathcal{L}0}} \mathrm{C}^1_{\mathrm{har}}(\Lambda, \mathbb{Z})_{\mathcal{L}} & \xrightarrow{\partial^1} & \prod_{e \in \mathscr{B}_{\mathcal{L}1}} \mathrm{C}^1_{\mathrm{har}}(e, \mathbb{Z})_{\mathcal{L}}
\end{array}$$

of which we already know the below sequence is exact. If we see that the local maps κ_Λ are isomorphisms and the above sequence is exact, then we will conclude that κ is an isomorphism.

From a finite set $\mathcal{L}_f$ to a compact set $\mathcal{L}$

Proposition 4.4.8. *If $\mathcal{L}_f \subset \mathbb{P}(V)$ is finite, the map*

$$\kappa : \mathscr{M}(\mathcal{L}_f, \mathbb{Z})_0 \longrightarrow \mathrm{C}^1_{\text{har}}(\mathscr{B}_{\mathcal{L}_f}, \mathbb{Z}).$$

is an isomorphism.

Proof. Since $\mathcal{L}_f$ is finite, it is discrete, and so each point $p \in \mathcal{L}_f$ is open and compact. Therefore we have an isomorphism

$$\begin{aligned} \mathscr{M}(\mathcal{L}_f, \mathbb{Z})_0 &\xrightarrow{\cong} \mathbb{Z}[\mathcal{L}_f]_0 \\ \mu &\longmapsto \sum_{p \in \mathcal{L}_f} \mu(\{p\})p \end{aligned}$$

We already know that κ is injective, therefore, we only have to see that it is surjective.

Take $c \in \mathrm{C}^1_{\text{har}}(\mathscr{B}_{\mathcal{L}_f}, \mathbb{Z})$. We want to construct a harmonic measure $\mu_c \in \mathscr{M}(\mathcal{L}_f, \mathbb{Z})_0$ such that $\kappa(\mu_c) = c$. Next, we define $\mu_c(p)$ for $p \in \mathcal{L}_f$. There is an apartment $\mathbb{A} = \mathbb{A}_{\mathcal{P}}$ with $p \in \mathcal{P} \subset \mathcal{L}_f$ and thus, there are rays $r = (e_0, e_1, e_2, \dots)_\infty$ in $\mathbb{A}$ such that $\bigcap_{e_i} \mathcal{B}_{\mathcal{L}_f}(e_i) = \{p\}$ by the proposition 4.2.31 and the proposition 4.2.3. Since $\mathcal{L}_f$ is finite, $\mathcal{B}_{\mathcal{L}_f}(e_i)$ is finite for all i, therefore, there is an i_0 such that for all i greater than i_0, $\mathcal{B}_{\mathcal{L}_f}(e_i) = \{p\}$. We define $\mu_c(p) := c(e)$ for any minimal edge e such that $\mathcal{B}_{\mathcal{L}_f}(e) = \{p\}$.

Now, given e, e' verifying that condition, we only have to prove that $c(e) = c(e')$. Since $\mathcal{B}_{\mathcal{L}_f}(e) = \mathcal{B}_{\mathcal{L}_f}(e')$, by the proposition 4.3.1 there is an apartment $\mathbb{A} = \mathbb{A}_{\mathcal{P}}$ containing both edges and they are parallel. Further, the condition implies that $p \in \mathcal{P}$. Next, any edge in the chamber-convex-hull has the same associated open set by the proposition 4.3.24 and the chamber-convex-hull is gallery connected, that is, there is a sequence of chambers $(\Delta_0, \dots, \Delta_r)$ inside it such that $A_{i+1} := \Delta_i \bigcap \Delta_{i+1}$ is a panel, $e \leq \Delta_0,\ e' \leq \Delta_r$. This tell us that the edges $e_0 = e, e_1, \dots, e_r = e'$ obtained in the proposition 4.2.25 are in the convex hull. Since the $\mathcal{B}_{\mathcal{L}_f}(e_i)$ are all equal, by the proposition 4.2.26, $\mathrm{cov}_{\mathcal{L}_f}(A_i) = 2$ for all i, and by the lemma 4.4.3, the value of c is constant on those edges.

Finally, we have to show that $\kappa(\mu_c) = c$. Note that by the flow condition, the values of c and $\kappa(\mu_c)$ on an edge e are determined by their values on edges e_i on rays from e. As above, due to the finiteness of $\mathcal{L}_f$, we can take these edges e_i such that $\mathcal{B}_{\mathcal{L}_f}(e_i)$ consist of exactly a point and for these edges both harmonic cochains coincide by construction. □

Corollary 4.4.9. *If* $\mathcal{L}_f \subset \mathbb{P}(V)$ *is finite and* $\Lambda \in \mathscr{B}_{\mathcal{L}_f 0}$, *the restriction map*

$$\mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}_f}, \mathbb{Z}) \longrightarrow \mathrm{C}^1_{\mathrm{har}}(\Lambda, \mathbb{Z})_{\mathcal{L}_f}.$$

is an epimorphism.

Proof. We start describing the isomorphism

$$\mathbb{Z}[\mathcal{L}_f]_0 \cong \mathscr{M}(\mathcal{L}_f, \mathbb{Z})_0 \cong \mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}_f}, \mathbb{Z}).$$

The corresponding divisor to a harmonic cochain c is $\sum_{p\in\mathcal{L}_f} c_p p$, where if e is a minimal edge such that $\mathcal{B}_{\mathcal{L}_f}(e) = \{p\}$, then $c(e) = c_p$. Further, the isomorphism tells us that

$$c(e) = \sum_{p\in\mathcal{B}_{\mathcal{L}_f}(e)} c_p.$$

for any minimal edge $e \in \widehat{\mathscr{B}_{\mathcal{L}_f 1}}^{min}$.

We are going to see that the composition of those isomorphisms with the restriction map

$$\mathbb{Z}[\mathcal{L}_f]_0 \longrightarrow \mathrm{C}^1_{\mathrm{har}}(\Lambda, \mathbb{Z})_{\mathcal{L}_f}$$

is surjective.

Consider a local harmonic cochain $c \in \mathrm{C}^1_{\mathrm{har}}(\Lambda, \mathbb{Z})_{\mathcal{L}_f}$. Note that its values on the edges of $\mathrm{St}^{min}_{\mathcal{L}_f}(\Lambda)$ determine its values on the edges of $\mathrm{St}^{-min}_{\mathcal{L}_f}(\Lambda)$ by the "flow" condition (which here is equivalent to the "cling" condition). After this, since chambers have three minimal edges, by the vanishing condition on their closed minimal paths, the values previously fixed also determine the values of c on the other edges of $\widehat{\mathscr{B}_{\mathcal{L}_f 1}}^{min}(\Lambda)$. Therefore, if we have another local harmonic cochain which coincides with c on $\mathrm{St}^{min}_{\mathcal{L}_f}(\Lambda)$, they are equal.

For each $e \in \mathrm{St}^{min}_{\mathcal{L}_f}(\Lambda)$, and for each $p \in \mathcal{B}_{\mathcal{L}_f}(e)$ choose a value $c_p \in \mathbb{Z}$ in such a way that

$$c(e) = \sum_{p\in\mathcal{B}_{\mathcal{L}_f}(e)} c_p.$$

Since $\sum_{e\in\mathrm{St}^{min}_{\mathcal{L}_f}(\Lambda)} c(e) = 0$, the divisor $\sum_{p\in\mathcal{L}_f} c_p$ has degree 0 and the local harmonic cochain associated by restriction coincides with c on $\mathrm{St}^{min}_{\mathcal{L}_f}(\Lambda)$, thus, they are equal, as we wanted to proof. $\square$

Corollary 4.4.10. *For any compact subset* $\mathcal{L} \subset \mathbb{P}(V)$ *and any vertex* $\Lambda \in \mathscr{B}_{\mathcal{L}}$, *the group of local harmonic cochains on* Λ *verifies*

$$\mathrm{C}^1_{\mathrm{har}}(\Lambda, \mathbb{Z})_{\mathcal{L}} \cong \mathbb{Z}[\mathrm{St}^{min}_{\mathcal{L}}(\Lambda)]_0.$$

Proof. We already knew that $\mathrm{C}^1_{\mathrm{har}}(\Lambda,\mathbb{Z})_{\mathcal{L}}$ is isomorphic to a subgroup of $\mathbb{Z}[\mathrm{St}^{min}_{\mathcal{L}}(\Lambda)]_0$, and by the previous proof we can assign arbitrary values to the different edges while their sum be zero, therefore we get the claim. $\square$

Remark 4.4.11. *Let us describe now the map*

$$\partial^1 : \prod_{\Lambda\in\mathscr{B}_{\mathcal{L}0}} \mathrm{C}^1_{\mathrm{har}}(\Lambda,\mathbb{Z})_{\mathcal{L}} \longrightarrow \prod_{e\in\mathscr{B}_{\mathcal{L}1}} \mathrm{C}^1_{\mathrm{har}}(e,\mathbb{Z})_{\mathcal{L}}$$

given by $\partial^1((c_\Lambda)_\Lambda)_e = (c_{t(e)} - c_{s(e)})_e$ *as*

$$\partial^1 : \prod_{\Lambda\in\mathscr{B}_{\mathcal{L}0}} \mathbb{Z}[\mathrm{St}^{min}_{\mathcal{L}}(\Lambda)]_0 \longrightarrow \prod_{e\in\mathscr{B}_{\mathcal{L}1}} \mathbb{Z}[\mathrm{Cling}^{min}_{\mathcal{L}}(e)].$$

Consider an element $(c_\Lambda)_\Lambda$*, which is a degree zero divisor* $c_\Lambda = \sum_{e\in\mathrm{St}^{min}_{\mathcal{L}}(\Lambda)} m_e e$ *for each* Λ *and fix an edge* $e \in \mathscr{B}_{\mathcal{L}1}$*. We want to compute* $\partial^1((c_\Lambda)_\Lambda)_e$ *with our present description. There are only two vertices which contribute to this element,* $t(e)$ *and* $s(e)$*. Then, the below arrow in the square*

$$\begin{array}{ccc} \mathrm{C}^1_{\mathrm{har}}(t(e),\mathbb{Z})_{\mathcal{L}} & \twoheadrightarrow & \mathrm{C}^1_{\mathrm{har}}(e,\mathbb{Z})_{\mathcal{L}} \\ \cong\downarrow & & \cong\downarrow \\ \mathbb{Z}[\mathrm{St}^{min}_{\mathcal{L}}(t(e))]_0 & \twoheadrightarrow & \mathbb{Z}[\mathrm{Cling}^{min}_{\mathcal{L}}(e)] \end{array}$$

projects $\sum_{e'\in\mathrm{St}^{min}_{\mathcal{L}}(t(e))} m_{e'}e'$ *to* $\sum_{e'\in\mathrm{Cling}^{min}_{\mathcal{L}}(e)} m_{e'}e'$*, while the below arrow in the square*

$$\begin{array}{ccc} \mathrm{C}^1_{\mathrm{har}}(s(e),\mathbb{Z})_{\mathcal{L}} & \twoheadrightarrow & \mathrm{C}^1_{\mathrm{har}}(e,\mathbb{Z})_{\mathcal{L}} \\ \cong\downarrow & & \cong\downarrow \\ \mathbb{Z}[\mathrm{St}^{min}_{\mathcal{L}}(s(e))]_0 & \twoheadrightarrow & \mathbb{Z}[\mathrm{Cling}^{min}_{\mathcal{L}}(e)] \end{array}$$

projects $\sum_{e'\in\mathrm{St}^{min}_{\mathcal{L}}(s(e))} m_{e'}e'$ *to*

$$\sum_{e'\in\mathrm{Cling}^{min}_{\mathcal{L}}(e)} \left(\sum_{e''\in\mathrm{St}^{min}_{\mathcal{L}}(s(e))\cap\widehat{\mathscr{B}_{\mathcal{L}1}}^{min}(t(e'))\setminus\{e\}} m_{e''} \right) e'.$$

This can be seen applying the lemma 4.4.3.

Finally we get

$$\partial^1((c_\Lambda)_\Lambda)_e = \sum_{e'\in\mathrm{Cling}^{min}_{\mathcal{L}}(e)} \left(m_{e'} - \sum_{e''\in\mathrm{St}^{min}_{\mathcal{L}}(s(e))\cap\widehat{\mathscr{B}_{\mathcal{L}1}}^{min}(t(e'))\setminus\{e\}} m_{e''} \right) e'.$$

Corollary 4.4.12. *If $\mathcal{L}_f \subset \mathbb{P}(V)$ is finite and $e \in \mathscr{B}_{\mathcal{L}_f 1}$, the restriction map*

$$\mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}_f}, \mathbb{Z}) \longrightarrow \mathrm{C}^1_{\mathrm{har}}(e, \mathbb{Z})_{\mathcal{L}_f}.$$

is an epimorphism.

Proof. As in the proof for the local cochains on a vertex, we are going to see that the map

$$\mathbb{Z}[\mathcal{L}_f]_0 \longrightarrow \mathrm{C}^1_{\mathrm{har}}(e, \mathbb{Z})_{\mathcal{L}_f}$$

is surjective.

Consider a local harmonic cochain $c \in \mathrm{C}^1_{\mathrm{har}}(e, \mathbb{Z})_{\mathcal{L}_f}$. As we have observed in the discussion after the definition of the local harmonic cochains, to give c is the same that assign freely values on the edges of $\mathrm{Cling}^{min}_{\mathcal{L}}(e)$ and defining

$$c(e) = - \sum_{e' \in \mathrm{Cling}^{min}_{\mathcal{L}}(e)} c(e').$$

Now, for each $e' \in \mathrm{Cling}^{min}_{\mathcal{L}_f}(e) \cup \{e\}$, and for each $p \in \mathcal{B}_{\mathcal{L}_f}(e')$ choose a value $c_p \in \mathbb{Z}$ in such a way that

$$c(e') = \sum_{p \in \mathcal{B}_{\mathcal{L}_f}(e')} c_p.$$

Since $c(e) + \sum_{e' \in \mathrm{Cling}^{min}_{\mathcal{L}_f}(e)} c(e') = 0$, the divisor $\sum_{p \in \mathcal{L}_f} c_p$ has degree 0 and the local harmonic cochain associated by restriction is c, as we wanted to proof. □

Corollary 4.4.13. *If $\mathcal{L}_f \subset \mathbb{P}(V)$ is finite and $\Delta \in \mathscr{B}_{\mathcal{L}_f 2}$, the restriction map*

$$\mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}_f}, \mathbb{Z}) \longrightarrow \mathrm{C}^1_{\mathrm{har}}(\Delta, \mathbb{Z})_{\mathcal{L}_f}.$$

is an epimorphism.

Proof. The proof follows the same proceeding as for vertices and edges, taking into account the three minimal edges of Δ. □

Corollary 4.4.14. *If $\mathcal{L}_f \subset \mathbb{P}(V)$ is finite and $\Delta \leq \mathscr{B}_{\mathcal{L}_f}$ is a simplex, the map*

$$\kappa_\Delta : \mathscr{M}_\Delta(\mathcal{L}_f, \mathbb{Z})_0 \longrightarrow \mathrm{C}^1_{\mathrm{har}}(\Delta, \mathbb{Z})_{\mathcal{L}_f}.$$

is an isomorphism.

Proof. By some of the previous corollaries, we have a commutative square

$$\begin{array}{ccc} \mathscr{M}(\mathcal{L}_f,\mathbb{Z})_0 & \xrightarrow[\cong]{\kappa} & \mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}_f},\mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathscr{M}_\Delta(\mathcal{L}_f,\mathbb{Z})_0 & \xhookrightarrow{\kappa_\Delta} & \mathrm{C}^1_{\mathrm{har}}(\Delta,\mathbb{Z})_{\mathcal{L}_f}. \end{array}$$

which let us to deduce the surjectivity of κ_Δ. □

Corollary 4.4.15. *If $\mathcal{L}_f \subset \mathbb{P}(V)$ is finite, there is an isomorphism of exact sequences*

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathscr{M}(\mathcal{L}_f,\mathbb{Z})_0 & \longrightarrow & \prod_{\Lambda\in\mathscr{B}_{\mathcal{L}_f 0}} \mathscr{M}_\Lambda(\mathcal{L}_f,\mathbb{Z})_0 & \xrightarrow{\partial^1} & \prod_{e\in\mathscr{B}_{\mathcal{L}_f 1}} \mathscr{M}_e(\mathcal{L}_f,\mathbb{Z})_0 \\ & \cong\downarrow\kappa & & \cong\downarrow\kappa_\Lambda & & \cong\downarrow\kappa_e \\ 0 \longrightarrow & \mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}_f},\mathbb{Z}) & \longrightarrow & \prod_{\Lambda\in\mathscr{B}_{\mathcal{L}_f 0}} \mathrm{C}^1_{\mathrm{har}}(\Lambda,\mathbb{Z})_{\mathcal{L}_f} & \xrightarrow{\partial^1} & \prod_{e\in\mathscr{B}_{\mathcal{L}_f 1}} \mathrm{C}^1_{\mathrm{har}}(e,\mathbb{Z})_{\mathcal{L}_f} \end{array}$$

Proof. It consists of recalling that the below complex is exact and the previous results give the isomorphisms in each column. □

Now, recall that given $\mathcal{L}' \subset \mathcal{L}$ compacts, we have an injection

$$\begin{array}{rcl} \mathrm{ext}^{\mathcal{L}',\mathcal{L}} : \mathscr{M}(\mathcal{L}',\mathbb{Z})_0 & \hookrightarrow & \mathscr{M}(\mathcal{L},\mathbb{Z})_0 \\ \mu & \longmapsto & \mu^e : \mathcal{U} \mapsto \mu^e(\mathcal{U}) := \mu(\mathcal{U}\bigcap\mathcal{L}'), \end{array}$$

and, if $\widehat{\mathscr{B}_{\mathcal{L}_1}}^{min}(\Delta) = \widehat{\mathscr{B}_{\mathcal{L}'1}}^{min}(\Delta)$ it descends to another injection

$$\begin{array}{rcl} \mathrm{ext}_\Delta^{\mathcal{L}',\mathcal{L}} : \mathscr{M}_\Delta(\mathcal{L}',\mathbb{Z})_0 & \hookrightarrow & \mathscr{M}_\Delta(\mathcal{L},\mathbb{Z})_0 \\ [\mu] & \longmapsto & [\mu^e]. \end{array}$$

Corollary 4.4.16. *Let $\Delta \leq \mathscr{B}_\mathcal{L}$ be a simplex and let $\mathcal{L}_f \subset \mathcal{L}$ be a finite set such that $\Delta \leq \mathscr{B}_{\mathcal{L}_f}$ and $\widehat{\mathscr{B}_{\mathcal{L}_1}}^{min}(\Delta) = \widehat{\mathscr{B}_{\mathcal{L}_f 1}}^{min}(\Delta)$. Then,*

$$\mathrm{ext}_\Delta^{\mathcal{L}_f,\mathcal{L}} : \mathscr{M}_\Delta(\mathcal{L}_f,\mathbb{Z})_0 \longrightarrow \mathscr{M}_\Delta(\mathcal{L},\mathbb{Z})_0$$

and

$$\kappa_\Delta : \mathscr{M}_\Delta(\mathcal{L},\mathbb{Z})_0 \longrightarrow \mathrm{C}^1_{\mathrm{har}}(\Delta,\mathbb{Z})_\mathcal{L}.$$

are isomorphisms.

Proof. We just have to note that $\mathrm{C}^1_{\mathrm{har}}(\Delta,\mathbb{Z})_{\mathcal{L}_f} \cong \mathrm{C}^1_{\mathrm{har}}(\Delta,\mathbb{Z})_{\mathcal{L}}$ by the hypothesis $\widehat{\mathscr{B}_{\mathcal{L}1}}^{min}(\Delta) = \widehat{\mathscr{B}_{\mathcal{L}_f 1}}^{min}(\Delta)$, and consider the commutative square

$$\begin{array}{ccc}
\mathscr{M}_\Delta(\mathcal{L}_f,\mathbb{Z})_0 & \xhookrightarrow{\mathrm{ext}_\Delta^{\mathcal{L}_f,\mathcal{L}}} & \mathscr{M}_\Delta(\mathcal{L},\mathbb{Z})_0 \\
\cong \downarrow \kappa_\Delta & & \downarrow \kappa_\Delta \\
\mathrm{C}^1_{\mathrm{har}}(\Delta,\mathbb{Z})_{\mathcal{L}_f} & \xrightarrow[\cong]{} & \mathrm{C}^1_{\mathrm{har}}(\Delta,\mathbb{Z})_{\mathcal{L}}.
\end{array}$$

□

Corollary 4.4.17. *For $\mathcal{L} \subset \mathbb{P}(V)$ compact and for all $\Delta \leq \mathscr{B}_{\mathcal{L}}$, the restriction map*

$$\kappa_\Delta : \mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}},\mathbb{Z}) \longrightarrow \mathrm{C}^1_{\mathrm{har}}(\Delta,\mathbb{Z})_{\mathcal{L}}$$

is surjective.

Proof. Consider the commutative square

$$\begin{array}{ccc}
\mathscr{M}(\mathcal{L},\mathbb{Z})_0 & \xhookrightarrow{\kappa} & \mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}},\mathbb{Z}) \\
\downarrow & & \downarrow \\
\mathscr{M}_\Delta(\mathcal{L},\mathbb{Z})_0 & \xhookrightarrow[\cong]{\kappa_\Delta} & \mathrm{C}^1_{\mathrm{har}}(\Delta,\mathbb{Z})_{\mathcal{L}}.
\end{array}$$

□

4.4.3 The isomorphism $\mathscr{M}(\mathcal{L},\mathbb{Z})_0 \cong \mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}},\mathbb{Z})$

Let $\mathcal{L} \subset \mathbb{P}(V)$ be a compact set. Recall the diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathscr{M}(\mathcal{L},\mathbb{Z})_0 & \longrightarrow & \prod_{\Lambda\in\mathscr{B}_{\mathcal{L}0}} \mathscr{M}_\Lambda(\mathcal{L},\mathbb{Z})_0 & \xrightarrow{\partial^1} & \prod_{e\in\mathscr{B}_{\mathcal{L}1}} \mathscr{M}_e(\mathcal{L},\mathbb{Z})_0 \\
& \downarrow \kappa & & \cong \downarrow \kappa_\Lambda & & \cong \downarrow \kappa_e \\
0 \longrightarrow & \mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}},\mathbb{Z}) & \longrightarrow & \prod_{\Lambda\in\mathscr{B}_{\mathcal{L}0}} \mathrm{C}^1_{\mathrm{har}}(\Lambda,\mathbb{Z})_{\mathcal{L}} & \xrightarrow{\partial^1} & \prod_{e\in\mathscr{B}_{\mathcal{L}1}} \mathrm{C}^1_{\mathrm{har}}(e,\mathbb{Z})_{\mathcal{L}}
\end{array}$$

We know all in it is exact except in the middle of the above complex. Let us take

$$(\mu_\Lambda)_\Lambda \in \mathrm{Ker}\left(\prod_{\Lambda\in\mathscr{B}_{\mathcal{L}0}} \mathscr{M}_\Lambda(\mathcal{L},\mathbb{Z})_0 \longrightarrow \prod_{e\in\mathscr{B}_{\mathcal{L}1}} \mathscr{M}_e(\mathcal{L},\mathbb{Z})_0\right).$$

Fix a vertex $\Lambda_0 \in \mathscr{B}_{\mathcal{L}0}$, a number $n \in \mathbb{N}_{\geq 1}$ and let $\mathcal{L}_n \subset \mathcal{L}$ a finite set such that $\mathscr{B}_{\mathcal{L}}(\Lambda_0)^{(n)} = \mathscr{B}_{\mathcal{L}_n}(\Lambda_0)^{(n)}$. By the proposition 4.2.17 it is enough to form $\mathcal{L}_n$ with a point of $\mathcal{B}_{\mathcal{L}}(e)$ for each e in

$$\bigcup_{\Lambda \in \mathscr{B}_{\mathcal{L}0}(\Lambda_0)^{(n)}} \mathrm{St}_{\mathcal{L}}^{min}(\Lambda).$$

Now, let us consider a (connected) chamber-complex $\mathcal{C}(n)$ in $\mathscr{B}_{\mathcal{L}}(\Lambda_0)^{(n)}$ for each n in such a way that $\bigcup_{n\in\mathbb{N}_{\geq 1}} \mathcal{C}(n) = \mathscr{B}_{\mathcal{L}}$,

$$\mathscr{B}_{\mathcal{L}}(\Lambda_0) = \mathscr{B}_{\mathcal{L}}(\Lambda_0)^{(1)} = \mathcal{C}(1) \subset \mathcal{C}(2) \subset \cdots \subset \mathcal{C}(n) \subset \mathcal{C}(n+1) \subset \cdots .$$

and $\mathscr{R}_{\mathcal{L}}^{(n)}(\Lambda_0) \subset \mathcal{C}(n)_1^{min}$. Indeed, let $\mathcal{C}(n)$ be the chamber complex generated by the vertices which are separated from Λ_0 by a path with n edges (without orientation) in $\mathscr{B}_{\mathcal{L}}$.

Observe that in this construction there is implicit a local distance $\rho_{\mathcal{L}}$ in $\mathscr{B}_{\mathcal{L}}$ which we call $\mathcal{L}$-distance and that the global (tropical) distance on the building $\mathscr{B}(G)$ verifies $\rho \leq \rho_{\mathcal{L}}$. Then, if $\Lambda \in \mathcal{C}(n)_0$ is at $\mathcal{L}$-distance n from Λ_0, by definition it is at $\mathcal{L}$-distance 1 of another vertex Λ' which is at $\mathcal{L}$-distance $n-1$ of Λ_0, therefore it is contained in a chamber inside $\mathcal{C}(n)$. Observe another way of describing $\mathcal{C}(n)$ inductively:

$$\mathcal{C}(1) := \mathscr{B}_{\mathcal{L}}(\Lambda_0), \qquad \mathcal{C}(n) := \bigcup_{\Lambda \in \mathcal{C}(n-1)} \mathscr{B}_{\mathcal{L}}(\Lambda)$$

Consider the commutative squares

$$\begin{array}{ccc}
\prod_{\Lambda \in \mathscr{B}_{\mathcal{L}0}} \mathscr{M}_{\Lambda}(\mathcal{L},\mathbb{Z})_0 & \xrightarrow{\partial^1} & \prod_{e \in \mathscr{B}_{\mathcal{L}1}} \mathscr{M}_{e}(\mathcal{L},\mathbb{Z})_0 \\
\downarrow & & \downarrow \\
\prod_{\Lambda \in \mathcal{C}(n-1)_0} \mathscr{M}_{\Lambda}(\mathcal{L},\mathbb{Z})_0 & \xrightarrow{\partial^1} & \prod_{e \in \mathcal{C}(n-1)_1} \mathscr{M}_{e}(\mathcal{L},\mathbb{Z})_0 \\
\cong \downarrow \kappa_{\Lambda} & & \cong \downarrow \kappa_{e} \\
\prod_{\Lambda \in \mathcal{C}(n-1)_0} \mathrm{C}^1_{\mathrm{har}}(\Lambda,\mathbb{Z})_{\mathcal{L}} & \xrightarrow{\partial^1} & \prod_{e \in \mathcal{C}(n-1)_1} \mathrm{C}^1_{\mathrm{har}}(e,\mathbb{Z})_{\mathcal{L}} \\
\kappa_{\Lambda} \uparrow \cong & & \kappa_{e} \uparrow \cong \\
\prod_{\Lambda \in \mathcal{C}(n-1)_0} \mathscr{M}_{\Lambda}(\mathcal{L}_n,\mathbb{Z})_0 & \xrightarrow{\partial^1} & \prod_{e \in \mathcal{C}(n-1)_1} \mathscr{M}_{e}(\mathcal{L}_n,\mathbb{Z})_0
\end{array}$$

and the projection of $(\mu_\Lambda)_\Lambda = (\mu_\Lambda)_{\Lambda\in\mathscr{B}_{\mathcal{L}0}}$ via

$$\begin{array}{ccc} \prod_{\Lambda\in\mathscr{B}_{\mathcal{L}0}} \mathscr{M}_\Lambda(\mathcal{L},\mathbb{Z})_0 & \longrightarrow & \prod_{\Lambda\in\mathcal{C}(n-1)_0} \mathscr{M}_\Lambda(\mathcal{L}_n,\mathbb{Z})_0 \\ (\mu_\Lambda)_{\Lambda\in\mathscr{B}_{\mathcal{L}0}} & \longmapsto & (\mu_\Lambda)_{\Lambda\in\mathcal{C}(n-1)_0}. \end{array}$$

Since the diagram is commutative we have $\partial^1\left((\mu_\Lambda)_{\Lambda\in\mathcal{C}(n-1)_0}\right) = 0$, which implies that for each $e'\in\mathcal{C}(n-1)_1$,

$$\mu_{s(e')}(\mathcal{B}_{\mathcal{L}_n}(e')) = \mu_{t(e')}(\mathcal{B}_{\mathcal{L}_n}(e')) = \sum_{e\in\mathrm{Flow}_{\mathcal{L}_n}(e')} \mu_{t(e')}(\mathcal{B}_{\mathcal{L}_n}(e)).$$

The key result is to get a global harmonic measure μ_n on $\mathcal{L}_n$ coinciding with $(\mu_\Lambda)_\Lambda$ with Λ in $\mathcal{C}(n-1)$ along the rays from Λ_0. We divide it in smaller steps.

Proposition 4.4.18. *There exists a harmonic measure $\mu\in\mathscr{M}(\mathcal{L}_n,\mathbb{Z})_0$ such that*

$$[\mu]_{\Lambda_0} = \mu_{\Lambda_0} \in \mathscr{M}_{\Lambda_0}(\mathcal{L}_n,\mathbb{Z})_0.$$

Proof. First, recall the isomorphism $\mathscr{M}(\mathcal{L}_n,\mathbb{Z})_0 \cong \mathbb{Z}[\mathcal{L}_n]_0$. Since

$$\mathcal{L}_n = \bigsqcup_{e\in\mathscr{R}^{(n)}_{\mathcal{L}_n}(\Lambda_0)} \mathcal{B}_{\mathcal{L}_n}(e),$$

we define μ as a divisor $\sum_{p\in\mathcal{L}_n} c_p p$ such that

$$\mu(e) := \sum_{p\in\mathcal{B}_{\mathcal{L}_n}(e)} c_p = \mu_{s(e)}(\mathcal{B}_{\mathcal{L}_n}(e))$$

for each $e\in\mathscr{R}^{(n)}_{\mathcal{L}}(\Lambda_0) = \mathscr{R}^{(n)}_{\mathcal{L}_n}(\Lambda_0)$. Now we will see that it has degree zero to be certain that it corresponds to a harmonic measure. Recall that

$$\mathscr{R}^{(n)}_{\mathcal{L}_n}(\Lambda_0) = \mathrm{Flow}^{n-1}_{\mathcal{L}_n}\left(\mathrm{St}^{min}_{\mathcal{L}_n}(\Lambda_0)\right)$$

are the edges e_n in the rays $(e_1,e_2,e_3,\dots)_\infty$ in $\mathscr{B}_{\mathcal{L}_n}$ from Λ_0. Then we have

$$\deg(\mu) = \sum_{e\in\mathscr{R}^{(n)}_{\mathcal{L}_n}(\Lambda_0)} \mu_{s(e)}(\mathcal{B}_{\mathcal{L}_n}(e)) = \sum_{e'\in\mathscr{R}^{(n-1)}_{\mathcal{L}_n}(\Lambda_0)} \sum_{e\in\mathrm{Flow}_{\mathcal{L}_n}(e')} \mu_{s(e)}(\mathcal{B}_{\mathcal{L}_n}(e)) =$$

$$= \sum_{e'\in\mathscr{R}^{(n-1)}_{\mathcal{L}_n}(\Lambda_0)} \sum_{e\in\mathrm{Flow}_{\mathcal{L}_n}(e')} \mu_{s(e)}(\mathcal{B}_{\mathcal{L}_n}(e)) = \sum_{e'\in\mathscr{R}^{(n-1)}_{\mathcal{L}_n}(\Lambda_0)} \mu_{s(e')}(\mathcal{B}_{\mathcal{L}_n}(e')).$$

Iterating this process we get

$$\deg(\mu) = \sum_{e \in \mathscr{R}^{(1)}_{\mathcal{L}_n}(\Lambda_0)} \mu_{s(e)}(\mathcal{B}_{\mathcal{L}_n}(e)) = \sum_{e \in \mathrm{St}^{min}_{\mathcal{L}_n}(\Lambda_0)} \mu_{\Lambda_0}(\mathcal{B}_{\mathcal{L}_n}(e)) = 0,$$

as we desired to prove to begin. Thus, μ is a global harmonic measure on $\mathcal{L}_n$ such that, since it verifies the flow condition, the previous proceeding shows that

$$\mu(\mathcal{B}_{\mathcal{L}_n}(e)) = \mu_{s(e)}(\mathcal{B}_{\mathcal{L}_n}(e))$$

for all $e \in \mathscr{R}^{(m)}_{\mathcal{L}_n}(\Lambda_0)$ for each $1 \leq m \leq n$. In particular μ coincides with μ_{Λ_0} in $\mathrm{St}^{min}_{\mathcal{L}_n}(\Lambda_0)$ and, therefore, $[\mu]_{\Lambda_0} = \mu_{\Lambda_0} \in \mathscr{M}_{\Lambda_0}(\mathcal{L}_n, \mathbb{Z})_0$. □

Proposition 4.4.19. *Let $n > 1$. The measure μ obtained in the previous proposition verifies $[\mu]_\Lambda = \mu_\Lambda$ for all $\Lambda \in \mathcal{C}(n-1)_0$ in the rays from Λ_0.*

Proof. Let $\Lambda \neq \Lambda_0$ be a vertex in $\mathcal{C}(n-1)$ inside a ray from Λ_0. Then, there are edges e', e in that ray such that $e \in \mathrm{Flow}_{\mathcal{L}_n}(e')$, $t(e') = s(e) = \Lambda$ and $e \in \mathscr{R}^{(m)}_{\mathcal{L}_n}(\Lambda_0)$ for some $2 \leq m \leq n$. We have already shown in the proof of the previous proposition that $\mu(\mathcal{B}_{\mathcal{L}_n}(e)) = \mu_\Lambda(\mathcal{B}_{\mathcal{L}_n}(e))$ for each $e \in \mathrm{Flow}_{\mathcal{L}_n}(e')$.

For the other edges we will proceed by induction on $m - 1 = \rho_{\mathcal{L}_n}(\Lambda_0, \Lambda)$. Recall that

$$\mathrm{St}^{min}_{\mathcal{L}_n}(t(e')) \setminus \mathrm{Flow}_{\mathcal{L}_n}(e') = \mathrm{Cling}^{min}_{\mathcal{L}_n}(e') \subset \mathscr{B}_{\mathcal{L}_n}(s(e'))^{min}_1.$$

Let $e'' \in \mathrm{Cling}^{min}_{\mathcal{L}_n}(e')$. If $m = 2$, $s(e') = \Lambda_0$, and so

$$\mu(\mathcal{B}_{\mathcal{L}_n}(e'')) = \mu_{\Lambda_0}(\mathcal{B}_{\mathcal{L}_n}(e'')) = \mu_\Lambda(\mathcal{B}_{\mathcal{L}_n}(e'')).$$

Therefore μ coincides with μ_Λ in $\mathrm{St}^{min}_{\mathcal{L}_n}(\Lambda)$, what implies $[\mu]_\Lambda = \mu_\Lambda$. Next, assume we know this for $m-1$ and note that we have $\rho_{\mathcal{L}_n}(\Lambda_0, s(e')) = m-2$. Then, by induction hypothesis

$$\mu(\mathcal{B}_{\mathcal{L}_n}(e'')) = \mu_{s(e')}(\mathcal{B}_{\mathcal{L}_n}(e'')) = \mu_\Lambda(\mathcal{B}_{\mathcal{L}_n}(e'')),$$

concluding as above $[\mu]_\Lambda = \mu_\Lambda$. □

Keep $n > 1$. Consider an edge $e_m \in \mathscr{R}^{(m)}_{\mathcal{L}_n}(\Lambda_0)$ for some $1 \leq m < n-1$ and write $\Lambda_m := t(e_m)$. By the two preceding propositions, we have $[\mu]_{\Lambda_0} = \mu_{\Lambda_0}$, $[\mu]_{\Lambda_m} = \mu_{\Lambda_m}$ and $[\mu]_{t(e_{m+1})} = \mu_{t(e_{m+1})}$ for each $e_{m+1} \in \mathrm{Flow}_{\mathcal{L}_n}(e_m)$. Now, if

$$e \in \mathrm{St}^{min}_{\mathcal{L}_n}(e_m) \setminus \mathrm{Flow}_{\mathcal{L}_n}(e_m) = \mathrm{Cling}^{min}_{\mathcal{L}_n}(e_m).$$

we are going to see that $[\mu]_{t(e)} = \mu_{t(e)}$.

First we will do it when $m = 1$, so we have $e_1 \in \mathscr{R}^{(1)}_{\mathcal{L}_n}(\Lambda_0) = \mathrm{St}^{min}_{\mathcal{L}_n}(\Lambda_0)$ and then, $e_{-1} := (t(e), \Lambda_0) \in \mathrm{St}^{-min}_{\mathcal{L}_n}(\Lambda_0)$ is the other edge in the chamber which contains e_1 and e, and reciprocally, given any edge with target vertex Λ_0, it can be obtained in this way from some e_1 as above.

In order to get this, first we prove a fundamental lemma. Let us assume $n > 2$, or we also can take on the convention $\mathcal{C}(0) := \{\Lambda_0\}$ to apply the next lemma when $n = 1$.

Lemma 4.4.20. *For any vertex $\Lambda \in \mathcal{C}(n-2)_0$, if $[\mu]_{\Lambda_1} = \mu_{\Lambda_1}$ for all $\Lambda_1 = t(e_1)$ with $e_1 \in \mathrm{St}^{min}_{\mathcal{L}_n}(\Lambda)$, then $[\mu]_{\Lambda_{-1}} = \mu_{\Lambda_{-1}}$ for all $\Lambda_{-1} = s(e_{-1})$ with $e_{-1} \in \mathrm{St}^{-min}_{\mathcal{L}_n}(\Lambda)$.*

Proof. Fix a vertex $\Lambda_{-1} = s(e_{-1})$ for some $e_{-1} \in \mathrm{St}^{-min}_{\mathcal{L}_n}(\Lambda)$.

First, observe that Λ_{-1} is contained in a chamber formed by edges e_{-1}, $e_1^0 \in \mathrm{St}^{min}_{\mathcal{L}_n}(\Lambda)$ and $e = (\Lambda_1^0, \Lambda_{-1})$ where $\Lambda_1^0 = t(e_1^0)$.

Since $[\mu]_{\Lambda_1^0} = \mu_{\Lambda_1^0}$, for each $e' \in \mathrm{Cling}^{min}_{\mathcal{L}_n}(e) \subset \mathrm{St}^{min}_{\mathcal{L}_n}(\Lambda_{-1})$

$$\mu(\mathcal{B}_{\mathcal{L}_n}(e')) = \mu_{\Lambda_1^0}(\mathcal{B}_{\mathcal{L}_n}(e')) = \mu_{\Lambda_{-1}}(\mathcal{B}_{\mathcal{L}_n}(e')),$$

in particular for $e' = e_{-1}$. Therefore, our claim reduces to show the same equality for each

$$e' \in \mathrm{St}^{min}_{\mathcal{L}_n}(\Lambda_{-1}) \setminus \mathrm{Cling}^{min}_{\mathcal{L}_n}(e) = \mathrm{Flow}_{\mathcal{L}_n}(e).$$

Fix one of those edges $e' \in \mathrm{Flow}_{\mathcal{L}_n}(e)$ and consider points $p_{-1} \in \mathcal{B}_{\mathcal{L}_n}(e_{-1})$, $p_1^0 \in \mathcal{B}_{\mathcal{L}_n}(e_1^0)$ and $p \in \mathcal{B}_{\mathcal{L}_n}(e') \subset \mathcal{B}_{\mathcal{L}_n}(e)$. Thus, the chamber formed by e_{-1}, e_1^0 and e is inside of the apartment $\mathbb{A} := \mathbb{A}_{\{p_{-1}, p_1^0, p\}} \leq \mathscr{B}_{\mathcal{L}_n}$, which also contains e' since it is the edge in $\mathrm{St}^{min}_{\mathcal{L}_n}(\Lambda_{-1})$ whose open set contains p.

Let us consider the isomorphism $\mathbb{A} \cong \mathbb{Z}^2$ making Λ and e correspond to $(0,0)$ and $((1,1),(1,0))$ respectively. Then, e' corresponds to $((1,0),(1,-1))$. Let us denote by $\hat{e}, \widetilde{e}$ and e_1^1 the minimal edges corresponding to $((1,-1),(0,-1))$, $((0,-1),(1,0))$ and $((0,0),(0,-1))$ respectively, and $\Lambda_1^1 := t(e_1^1)$. Observe that $e', \widetilde{e}$ and $\hat{e}$ form a chamber, $e_1^1 \leq \mathscr{B}_{\mathcal{L}_n}(\Lambda_1^1) \cap \mathscr{B}_{\mathcal{L}_n}(\Lambda_{-1}) = \mathscr{B}_{\mathcal{L}_n}(\widetilde{e})$, $t(\hat{e}) = \Lambda_1^1$ and $e_1^1 \in \mathrm{St}^{min}_{\mathcal{L}_n}(\Lambda)$. In particular, we know $[\mu]_{\Lambda_1^1} = \mu_{\Lambda_1^1}$, and

$$\mu_{\Lambda_{-1}}(\mathcal{B}_{\mathcal{L}_n}(e')) = \mu_{\Lambda_1^1}(\mathcal{B}_{\mathcal{L}_n}(e')) = \mu(\mathcal{B}_{\mathcal{L}_n}(e')),$$

as we wanted to prove. □

Corollary 4.4.21. *For any $\Lambda \in \mathscr{B}_{\mathcal{L}_n}(\Lambda_0)_0$, $[\mu]_\Lambda = \mu_\Lambda$.*

Proof. By the proposition 4.4.19, we know $[\mu]_{t(e_1)} = \mu_{t(e_1)}$ for all $e_1 \in \mathrm{St}^{min}_{\mathcal{L}_n}(\Lambda_0)$. Then, the last lemma implies the claim. □

Corollary 4.4.22. *Let* $\Lambda_m := t(e_m) \in \mathcal{C}(n-1)_0$ *be a vertex in a ray* $(e_1, e_2, \ldots, e_m, \ldots)_\infty$ *from* Λ_0. *Then,* $[\mu]_\Lambda = \mu_\Lambda$ *for all* $\Lambda \in \mathscr{B}_{\mathcal{L}_n}(\Lambda_m)_0$.

Proof. If $\Lambda = t(e_{m+1})$ with $e_{m+1} \in \mathrm{Flow}_{\mathcal{L}_n}(e_m)$, we have already pointed out that $[\mu]_\Lambda = \mu_\Lambda$ since it belongs to a ray from Λ_0. For the other possibilites we proceed by induction.

If $m = 1$ and $\Lambda = t(e)$ for some $e \in \mathrm{Cling}^{min}_{\mathcal{L}_n}(e_1)$, then $[\mu]_\Lambda = \mu_\Lambda$ is what was proven in the previous corollary. Thus, we proved the result for all $\Lambda = t(e)$ with $e \in \mathrm{St}^{min}_{\mathcal{L}_n}(\Lambda_1)$, and we finish by the lemma.

Assume we know the claim for $m-1$. Now, let $\Lambda = t(e)$ with

$$e \in \mathrm{St}^{min}_{\mathcal{L}_n}(\Lambda_m) \setminus \mathrm{Flow}_{\mathcal{L}_n}(e_m) = \mathrm{Cling}^{min}_{\mathcal{L}_n}(e_m).$$

Then, since $e_m = (\Lambda_{m-1}, \Lambda_m)$, the minimal edge $e' = (\Lambda, \Lambda_{m-1})$ belongs to $\mathrm{St}^{-min}_{\mathcal{L}_n}(\Lambda_{m-1})$ and, as a consequence, $[\mu]_\Lambda = \mu_\Lambda$ by induction hypothesis. Again by the lemma, as we proved the result for all $\Lambda = t(e)$ with $e \in \mathrm{St}^{min}_{\mathcal{L}_n}(\Lambda_m)$, we conclude. □

Let us denote the measure $\mu \in \mathscr{M}(\mathcal{L}_n, \mathbb{Z})_0$ obtained in the last results by μ_n.

Consider the extension morphism

$$\mathrm{ext}^{\mathcal{L}_n,\mathcal{L}} : \mathscr{M}(\mathcal{L}_n, \mathbb{Z})_0 \hookrightarrow \mathscr{M}(\mathcal{L}, \mathbb{Z})_0$$

and apply it to μ_n. Thus, one obtains a sequence of global harmonic measures $\mathrm{ext}^{\mathcal{L}_n,\mathcal{L}}(\mu_n)$ such that for all $n \geq n_0$,

$$\mathrm{ext}^{\mathcal{L}_n,\mathcal{L}}(\mu_n)(\mathcal{B}_{\mathcal{L}}(e)) = \mathrm{ext}^{\mathcal{L}_{n_0},\mathcal{L}}(\mu_{n_0})(\mathcal{B}_{\mathcal{L}}(e))$$

for all $e \in \mathscr{R}^m_{\mathcal{L}}(\Lambda_0)$ for all $m \leq n_0$, since they restrict to μ_Λ for all Λ in the rays from Λ_0 by the previous corollary.

Next, we define a harmonic measure μ giving it values on the basis by open compact sets $\mathcal{B}_{\mathcal{L}}(e)$ with e in the rays from Λ_0 (cf. corollary 4.3.33). In fact, given such an e, it is in $\mathscr{B}_{\mathcal{L}}(\Lambda_0)^{(n)}$ for some n. Then, we define

$$\mu(\mathcal{B}_{\mathcal{L}}(e)) = \mathrm{ext}^{\mathcal{L}_n,\mathcal{L}}(\mu_n)(\mathcal{B}_{\mathcal{L}}(e)).$$

By the preceding considerations, this value is independent on n, therefore μ is a well defined map and only rests to proof that it is a harmonic measure. Indeed, given two disjoint open compact sets, they can be covered by a finite set of open compact sets of the form $\mathcal{B}_{\mathcal{L}}(e)$ whose definition edges are in rays from Λ_0 and belong to a common $\mathscr{B}_{\mathcal{L}}(\Lambda_0)^{(n)}$ for some n big enough, and there, μ behaves like the measure $\mathrm{ext}^{\mathcal{L}_n,\mathcal{L}}(\mu_n)$. More specifically, given two coverings of a compact open set by these edges, by the corollary 4.3.33 there is a common finer covering which is also inside $\mathscr{B}_{\mathcal{L}}(\Lambda_0)^{(n)}$ for some n big enough, so that μ is well defined, since its value on the first two coverings coincides with its value on the third. As a consequence, it is additive with respect to disjoint open compact sets and $\mu(\mathcal{L}) = 0$, then it is a harmonic measure.

In addition, if Λ is in a ray from Λ_0, μ restricts to $\mu_{\Lambda'}$ in $\mathscr{M}_{\Lambda'}(\mathcal{L}, \mathbb{Z})_0$ for all $\Lambda' \in \mathscr{B}_{\mathcal{L}}(\Lambda)_0$.

Lemma 4.4.23. *Let $\mathbb{A} = \mathbb{A}_{\mathcal{P}} \leq \mathscr{B}_{\mathcal{L}}$ be an apartment containing Λ_0. Consider an isomorphism $\mathbb{A} \cong \mathbb{Z}^2$ by which Λ_0 corresponds to the vertex $(0,0)$ and let e_1 be the edge corresponding to $((0,0),(-1,0))$. Then, each horizontal edge $e = ((m_1,m_2),(m_1-1,m_2))$ such that $\mathcal{B}_{\mathcal{L}}(e) \subset \mathcal{B}_{\mathcal{L}}(e_1)$ verifies the equality*

$$\mu_{s(e)}(\mathcal{B}_{\mathcal{L}}(e)) = \mu(\mathcal{B}_{\mathcal{L}}(e)).$$

Proof. We divide the proof in different cases according to the coordinates of e.

First note that if $m_2 = 0 \geq m_1$, e is in the ray of $\mathbb{A}$ starting by e_1, so that in this case we know the result. If $|m_2| = 1 > m_1$ we also know the result, since in this case $s(e)$ is at distance 1 of a vertex in the ray.

Second, assume $m_1 \geq \max\{0, m_2\}$. Then we have

$$\mathcal{B}_{\mathcal{L}}(e_1) \subset \mathcal{B}_{\mathcal{L}}(e) \subset \mathcal{B}_{\mathcal{L}}(e_1),$$

that is, its equality. Consider the chamber-convex hull between them (recall definition 4.3.21). By the corollary 4.3.26, all the horizontal edges in it have associated the same open set $\mathcal{B}_{\mathcal{L}}(e)$. Consider a gallery from e_1 to e in its chamber-convex hull. By hypothesis on the coordinates of e, it can be given by a sequence of horizontal edges $e_1, e_2, \ldots, e_r = e$ such that each e_i is the sum of e_{i-1} plus $(1,1)$ or $(0,-1)$, and chambers Δ_i for $i = 1, \ldots, r$ such that $e_i \leq \Delta_i$ and

$$\Delta_{i-1} \cap \Delta_i = e^1_{i-1} := (t(e_i), s(e_{i-1})).$$

The equality $\mathcal{B}_{\mathcal{L}}(e_{i-1}) = \mathcal{B}_{\mathcal{L}}(e_i)$, implies that $\operatorname{cov}_{\mathcal{L}}(e^1_{i-1}) = 2$ meaning that $\operatorname{Cling}^{min}_{\mathcal{L}}(e^1_{i-1})$ has only two elements, in Δ_{i-1} and Δ_i respectively. Now we have

$$\mu_{s(e_i)}(\mathcal{B}_{\mathcal{L}}(e_i)) = \mu_{t(e_i)}(\mathcal{B}_{\mathcal{L}}(e_i)) = \mu_{s(e_{i-1})}(\mathcal{B}_{\mathcal{L}}(e_i)) = \mu_{s(e_{i-1})}(\mathcal{B}_{\mathcal{L}}(e_{i-1}))$$

so that $\mu_{s(e)}(\mathcal{B}_{\mathcal{L}}(e)) = \mu_{\Lambda_0}(\mathcal{B}_{\mathcal{L}}(e_1)) = \mu(\mathcal{B}_{\mathcal{L}}(e_1)) = \mu(\mathcal{B}_{\mathcal{L}}(e))$.

With the two first cases, we have proved the result always that $|m_2| \leq 1$ (under the hypothesis $\mathcal{B}_{\mathcal{L}}(e) \subset \mathcal{B}_{\mathcal{L}}(e_1)$).

In the third place assume $m_2 > \max\{0, m_1\}$. Let $p_0 \in \mathcal{P}$ be the point such that

$$\mathcal{B}_{\mathcal{P}}(e) = \mathcal{B}_{\mathcal{P}}(e_1) = \{p_0\},$$

let $\Delta \leq \mathbb{A}$ be the chamber generated by e and the vertex $(m_1 - 1, m_2 - 1)$, which we will denote by $\Lambda_{\Delta,e}$ and write $e^1 = (t(e), \Lambda_{\Delta,e}), e^2 = (\Lambda_{\Delta,e}, s(e))$. for the other two edges of the chamber. Next recall that by the proposition 4.2.26,

$$\mathcal{B}_{\mathcal{L}}(e) = \bigsqcup_{\Delta' \cap \Delta = e^1} \mathcal{B}_{\mathcal{L}}(e^{\Delta'}_{e^1}).$$

In particular, in the apartment $\mathbb{A}$, the corresponding edge $e^{\Delta'}_{e^1}$ is

$$((m_1 - 1, m_2 - 1), (m_1 - 2, m_2 - 1)).$$

Observe also that Δ' is determined by $e^{\Delta'}_{e^1}$ and e^1. Next, we will see that all of these edges and so, all the chambers Δ' intersecting with Δ through e^1 are contained in apartments containing Λ_0 and Δ. By the previous comment, we only have to see that the corresponding edge $e^{\Delta'}_{e^1}$ is in such apartment. Indeed, fix such a Δ', consider

$$p \in \mathcal{B}_{\mathcal{L}}(e^{\Delta'}_{e^1}) \subset \mathcal{B}_{\mathcal{L}}(e) \subset \mathcal{B}_{\mathcal{L}}(e_1).$$

and define $\mathcal{P}' := \mathcal{P} \setminus \{p_0\} \cup \{p\}$. By the proposition 4.2.18, it defines an apartment $\mathbb{A}_{\mathcal{P}'}$ containing Δ, and by the proposition 4.2.17 the vertex Λ_0 belongs to $\mathbb{A}_{\mathcal{P}'}$. Further, there is an edge in $\mathrm{St}^{min}_{\mathcal{P}'}(\Lambda_{\Delta,e})$ whose associated open set is p, therefore, this edge is $e^{\Delta'}_{e^1}$ and it and Δ' are contained in $\mathbb{A}_{\mathcal{P}'}$. Consider the isomorphism of $\mathbb{A}'_{\mathcal{P}}$ with $\mathbb{A}_{\mathcal{P}}$ which leaves fixed the intersection. Then, by the composition $\mathbb{A}_{\mathcal{P}'} \cong \mathbb{A}_{\mathcal{P}} \cong \mathbb{Z}^2$, the edge $e^{\Delta'}_{e^1}$ corresponds again to $((m_1 - 1, m_2 - 1), (m_1 - 2, m_2 - 1))$. Let us denote $E(e; m_1 - 1, m_2 - 1)$ the set of edges $e^{\Delta'}_{e^1}$, which coincides with $\mathrm{Cling}^{min}_{\mathcal{L}}(e^1) \setminus \{e^2\}$, and let us write with this notation

$$\mathcal{B}_{\mathcal{L}}(e) = \bigsqcup_{e' \in E(e; m_1 - 1, m_2 - 1)} \mathcal{B}_{\mathcal{L}}(e').$$

Now we have

$$\mu_{s(e)}(\mathcal{B}_{\mathcal{L}}(e)) = \mu_{\Lambda_{\Delta,e}}(\mathcal{B}_{\mathcal{L}}(e)) = -\mu_{\Lambda_{\Delta,e}}(\mathcal{B}_{\mathcal{L}}(e^1)) - \mu_{\Lambda_{\Delta,e}}(\mathcal{B}_{\mathcal{L}}(e^2)) =$$

$$= \sum_{e' \in \mathrm{Cling}^{min}_{\mathcal{L}}(e^1) \setminus \{e^2\}} \mu_{\Lambda_{\Delta,e}}(\mathcal{B}_{\mathcal{L}}(e')) = \sum_{e' \in E(e; m_1 - 1, m_2 - 1)} \mu_{\Lambda_{\Delta,e}}(\mathcal{B}_{\mathcal{L}}(e'))$$

To finish the proof, we proceed by induction on m_2. Since if $m_2 = 1$ we know the result, we can make the induction hypothesis

$$\mu(\mathcal{B}_{\mathcal{L}}(e')) = \mu_{s(e)}(\mathcal{B}_{\mathcal{L}}(e'))$$

for the edges e' with second coordinate $m_2 - 1$ (and verifying the assumption $\mathcal{B}_{\mathcal{L}}(e') \subset \mathcal{B}_{\mathcal{L}}(e_1)$). Then, note that each $e' \in E(e; m_1 - 1, m_2 - 1)$ verifies both conditions, since

$$\mathcal{B}_{\mathcal{L}}(e') \subset \mathcal{B}_{\mathcal{L}}(e) \subset \mathcal{B}_{\mathcal{L}}(e_1).$$

Therefore, we conclude

$$\mu_{s(e)}(\mathcal{B}_{\mathcal{L}}(e)) = \sum_{e' \in E(e; m_1-1, m_2-1)} \mu_{\Lambda_{\Delta,e}}(\mathcal{B}_{\mathcal{L}}(e')) =$$

$$= \sum_{e' \in E(e; m_1-1, m_2-1)} \mu(\mathcal{B}_{\mathcal{L}}(e')) = \mu(\mathcal{B}_{\mathcal{L}}(e)).$$

Finally assume $0 > \max\{m_1, m_2\}$. The process is identical to the previous one up to rewrite some coordinates.Now, let $\Delta \leq \mathbb{A}$ be the chamber generated by e and the vertex $\Lambda_{\Delta,e} := (m_1, m_2 + 1)$. We define e^1, e^2 exactly as above. We have again

$$\mathcal{B}_{\mathcal{L}}(e) = \bigsqcup_{\Delta' \cap \Delta = e^1} \mathcal{B}_{\mathcal{L}}(e_{e^1}^{\Delta'}),$$

but, now the corresponding edge $e_{e^1}^{\Delta'}$ in $\mathbb{A}$ is

$$((m_1, m_2 + 1), (m_1 - 1, m_2 + 1)).$$

The construction of $\mathbb{A}_{\mathcal{P}'}$ above applies again, so all these edges, which are those of the set $\mathrm{Cling}_{\mathcal{L}}^{min}(e^1) \setminus \{e^2\}$, can be represented with the given coordinates, so we denote it by $E(e; m_1, m_2 + 1)$. As above

$$\mu_{s(e)}(\mathcal{B}_{\mathcal{L}}(e)) = \sum_{e' \in E(e; m_1, m_2+1)} \mu_{\Lambda_{\Delta,e}}(\mathcal{B}_{\mathcal{L}}(e'))$$

and we proceed by induction on $-m_2$. The case $-m_2 = 1$ is known. The hypothesis induction is

$$\mu(\mathcal{B}_{\mathcal{L}}(e')) = \mu_{s(e)}(\mathcal{B}_{\mathcal{L}}(e'))$$

for all $e' = ((m_1', m_2 + 1), (m_1' - 1, m_2 + 1))$ with $-m_1' \in \mathbb{N}$ and for a fixed $-m_2 \in \mathbb{N}_{\geq 2}$, which is the case for all the edges e' in $E(e; m_1, m_2 + 1)$, which, moreover, verify, as above, the condition $\mathcal{B}_{\mathcal{L}}(e') \subset \mathcal{B}_{\mathcal{L}}(e) \subset \mathcal{B}_{\mathcal{L}}(e_1)$ and we conclude in the same way. □

Proposition 4.4.24. *Let $\mathcal{L} \subset \mathbb{P}(V)$ be a compact set such that $\mathscr{B}_{\mathcal{L}}$ is a building. Let e be a minimal edge in $\mathscr{B}_{\mathcal{L}}$ such that there exists $e_1 \in \mathrm{St}_{\mathcal{L}}^{min}(\Lambda_0)$ satisfying $\mathcal{B}_{\mathcal{L}}(e) \subset \mathcal{B}_{\mathcal{L}}(e_1)$. Then*

$$\mu_{s(e)}(\mathcal{B}_{\mathcal{L}}(e)) = \mu(\mathcal{B}_{\mathcal{L}}(e))$$

Proof. Since $\mathscr{B}_{\mathcal{L}}$ is a building, there exists an apartment $\mathbb{A}_{\mathcal{P}} \leq \mathscr{B}_{\mathcal{L}}$ containing e and e_1. Then we have $\emptyset \neq \mathcal{B}_{\mathcal{P}}(e) \subset \mathcal{B}_{\mathcal{P}}(e_1)$ which consists of a unique point, therefore, this inclusion is an equality and the edges are parallel. Then, we consider an isomorphism $\mathbb{A} \cong \mathbb{Z}^2$ such that e_1 corresponds to $((0,0),(-1,0))$ and we apply the previous lemma. □

Remark 4.4.25. *Observe that we only use that $\mathscr{B}_{\mathcal{L}}$ is a building to ensure that for any edge e such that that there exists $e_1 \in \mathrm{St}_{\mathcal{L}}^{min}(\Lambda_0)$ satisfying $\mathcal{B}_{\mathcal{L}}(e) \subset \mathcal{B}_{\mathcal{L}}(e_1)$, there is an apartment in $\mathscr{B}_{\mathcal{L}}$ containing e and Λ_0. Therefore, if we have this for some vertex of $\mathscr{B}_{\mathcal{L}}$, even if it is not a building, we can choose this vertex as Λ_0 and we get the same result.*

Theorem 4.4.26. *Let $\mathcal{L} \subset \mathbb{P}(V)$ be a compact set such that $\mathscr{B}_{\mathcal{L}}$ is a building. Then*

$$\begin{array}{ccccccc} 0 \longrightarrow \mathscr{M}(\mathcal{L},\mathbb{Z})_0 & \longrightarrow & \prod_{\Lambda\in\mathscr{B}_{\mathcal{L}0}} \mathscr{M}_\Lambda(\mathcal{L},\mathbb{Z})_0 & \xrightarrow{\partial^1} & \prod_{e\in\mathscr{B}_{\mathcal{L}1}} \mathscr{M}_e(\mathcal{L},\mathbb{Z})_0 \\ \mu & \longmapsto & ([\mu]_\Lambda)_\Lambda & & \end{array}$$

is exact and, as a consequence $\mathscr{M}(\mathcal{L},\mathbb{Z})_0 \cong \mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}},\mathbb{Z})$.

Proof. To get the exactness of the sequence, the only step rests to do is to get a global harmonic measure which projects to

$$(\mu_\Lambda)_\Lambda \in \mathrm{Ker}\left(\prod_{\Lambda\in\mathscr{B}_{\mathcal{L}0}} \mathscr{M}_\Lambda(\mathcal{L},\mathbb{Z})_0 \longrightarrow \prod_{e\in\mathscr{B}_{\mathcal{L}1}} \mathscr{M}_e(\mathcal{L},\mathbb{Z})_0\right).$$

We have already built a harmonic measure $\mu \in \mathscr{M}(\mathcal{L},\mathbb{Z})_0$ such that $[\mu]_\Lambda = \mu_\Lambda$ through the rays from Λ_0 and we are going to check that it verifies the expected property.

Let Λ be any vertex in $\mathscr{B}_{\mathcal{L}}$, for which we want to prove the same equality. It is sufficient that μ and any representant of μ_Λ coincide on the edges of $\mathrm{St}_{\mathcal{L}}^{min}(\Lambda)$, that is

$$\mu_\Lambda(\mathcal{B}_{\mathcal{L}}(e)) = \mu(\mathcal{B}_{\mathcal{L}}(e))$$

for each $e \in \mathrm{St}_{\mathcal{L}}^{min}(\Lambda)$, so let us fix any of these edges. Recall the corollary 4.3.33, by which there is a minimum finite (since the corresponding open sets are compact) set

$$\mathscr{R}_e \subset \bigcup_{n\in\mathbb{N}_{\geq 1}} \mathscr{R}_{\mathcal{L}}^{(n)}(\Lambda_0)$$

such that

$$\mathcal{B}_{\mathcal{L}}(e) = \bigsqcup_{e'\in\mathscr{R}_e} \mathcal{B}_{\mathcal{L}}(e').$$

Observe that $\mathscr{R}_e$ depends on Λ_0, even we did not specify this until now since it is a given fixed vertex from the begining. Now, let us write $\mathscr{R}_{\Lambda_0}(e) := \mathscr{R}_e$, so we can work with different of these sets, while we change the "base" vertex.

Let us enumerate the edges in $\mathscr{R}_{\Lambda_0}(e)$:

$$\mathscr{R}_{\Lambda_0}(e) = \{e^1, \ldots, e^{m_e}\}.$$

Each one of them is $e^i = \mathrm{Flow}_{\mathcal{L}}^{j_i}(e_1^i)$ for some $e_1^i \in \mathrm{St}_{\mathcal{L}}^{min}(\Lambda_0)$ and $j_i \in \mathbb{N}$. Observe that the set formed by the edges e_1^i coincides with

$$S_{\Lambda_0}(e) := \{e' \in \mathrm{St}_{\mathcal{L}}^{min}(\Lambda_0) \mid \mathcal{B}_{\mathcal{L}}(e) \cap \mathcal{B}_{\mathcal{L}}(e') \neq \emptyset\}.$$

If $|S_{\Lambda_0}(e)| = 1$ we are under the hypotheses of the previous proposition, and so we have apply it. Fix any edge of this set, for example e_1^1 and consider the minimal decomposition of the associated open set on the rays from $\Lambda = s(e)$, that is

$$\mathcal{B}_{\mathcal{L}}(e_1^1) = \bigsqcup_{e'\in\mathscr{R}_{\Lambda}(e_1^1)} \mathcal{B}_{\mathcal{L}}(e').$$

Now, since if $e' \in \mathscr{R}_{\Lambda}(e_1^1)$, $\mathcal{B}_{\mathcal{L}}(e') \subset \mathcal{B}_{\mathcal{L}}(e_1^1)$, we can apply the previous proposition again, so that

$$\mu_{s(e')}(\mathcal{B}_{\mathcal{L}}(e')) = \mu(\mathcal{B}_{\mathcal{L}}(e')).$$

Observe that the set of edges in the decomposition of e_1^1 which are in rays starting by e is

$$\mathscr{R}_e(e_1^1) := \left\{e' \in \mathscr{R}_{\Lambda}(e_1^1) : \ \mathcal{B}_{\mathcal{L}}(e') \subset \mathcal{B}_{\mathcal{L}}(e)\right\} \subset \mathscr{R}_{\Lambda}(e_1^1)$$

and gives the equality

$$\mathcal{B}_{\mathcal{L}}(e) \bigcap \mathcal{B}_{\mathcal{L}}(e_1^1) = \bigsqcup_{e'\in\mathscr{R}_e(e_1^1)} \mathcal{B}_{\mathcal{L}}(e').$$

Let us denote $i^1 := \max\left\{i \in \mathbb{N} \mid \mathscr{R}_e(e_1^1) \cap \mathrm{Flow}_{\mathcal{L}}^i(e) \neq \emptyset\right\} \in \mathbb{N}$, so that we have $\mathscr{R}_e(e_1^1) \subset \bigcup_{j \leq i^1} \mathrm{Flow}_{\mathcal{L}}^j(e)$.

Next, we do this process for the rest of edges $e_1^2, \ldots, e_1^{m_e}$ in $S_{\Lambda_0}(e)$ and we also consider the rest of indices $i^2, \ldots, i^{m_1}$. We define

$$i_e := \max\{i^1, i^2, \ldots, i^{m_e}\}.$$

so that $\mathscr{R}_e(e_1^i) \subset \bigcup_{j \leq i_e} \mathrm{Flow}_{\mathcal{L}}^j(e)$ for each $i = 1, \ldots, m_e$. Note also that we have

$$\mathcal{B}_{\mathcal{L}}(e) = \bigsqcup_{e_1 \in S_{\Lambda_0}(e)} (\mathcal{B}_{\mathcal{L}}(e_1) \cap \mathcal{B}_{\mathcal{L}}(e)) =$$

$$= \bigsqcup_{e_1 \in S_{\Lambda_0}(e)} \left(\bigsqcup_{e' \in \mathscr{R}_e(e_1)} \mathcal{B}_{\mathcal{L}}(e') \right) = \bigsqcup_{e' \in \mathrm{Flow}_{\mathcal{L}}^{i_e}(e)} \mathcal{B}_{\mathcal{L}}(e')$$

and the edges appearing in the last two expressions are in rays strating by e, therefore its open sets are disjoint or verify some inclusion. In addition, since they are equal, given any edge of an expression, there is another edge of the other expression with nonempty intersection of the associated open sets. Then, since $\mathscr{R}_e(e_1) \subset \bigcup_{j \leq i_e} \mathrm{Flow}_{\mathcal{L}}^j(e)$ for each $e_1 \in S_{\Lambda_0}(e)$, given any $e' \in \mathrm{Flow}_{\mathcal{L}}^{i_e}(e)$ there is an edge $e_1 \in S_{\Lambda_0}(e)$, a number $j \leq i_e$ and another edge $\widetilde{e} \in \mathscr{R}_e(e_1) \cap \mathrm{Flow}_{\mathcal{L}}^j(e)$ such that $e' \in \mathrm{Flow}_{\mathcal{L}}^{i_e - j}(\widetilde{e})$ and so,

$$\mathcal{B}_{\mathcal{L}}(e') \subset \mathcal{B}_{\mathcal{L}}(\widetilde{e}) \subset \mathcal{B}_{\mathcal{L}}(e_1).$$

Therefore we can apply the previous proposition to each $e' \in \mathrm{Flow}_{\mathcal{L}}^{i_e}(e)$, obtaining

$$\mu_{s(e')}(\mathcal{B}_{\mathcal{L}}(e')) = \mu(\mathcal{B}_{\mathcal{L}}(e')).$$

Next, recall the formula

$$\mu_{s(e_0)}(\mathcal{B}_{\mathcal{L}}(e_0)) = \mu_{t(e_0)}(\mathcal{B}_{\mathcal{L}}(e_0)) =$$

$$= \sum_{e_1 \in \mathrm{Flow}_{\mathcal{L}}(e_0)} \mu_{t(e_0)}(\mathcal{B}_{\mathcal{L}}(e_1)) = \sum_{e_1 \in \mathrm{Flow}_{\mathcal{L}}(e_0)} \mu_{s(e_1)}(\mathcal{B}_{\mathcal{L}}(e_1)),$$

apply it to e and iterate it to the edges appearing on the successive expressions:

$$\mu_{\Lambda}(\mathcal{B}_{\mathcal{L}}(e)) = \mu_{s(e)}(\mathcal{B}_{\mathcal{L}}(e)) = \sum_{e_1 \in \mathrm{Flow}_{\mathcal{L}}(e)} \mu_{s(e_1)}(\mathcal{B}_{\mathcal{L}}(e_1)) =$$

$$= \sum_{e_1 \in \mathrm{Flow}_{\mathcal{L}}(e)} \sum_{e_2 \in \mathrm{Flow}_{\mathcal{L}}(e_1)} \mu_{s(e_2)}(\mathcal{B}_{\mathcal{L}}(e_2)) = \sum_{e_2 \in \mathrm{Flow}_{\mathcal{L}}^2(e)} \mu_{s(e_2)}(\mathcal{B}_{\mathcal{L}}(e_2))$$

what, inductively, gives

$$\mu_\Lambda(\mathcal{B}_{\mathcal{L}}(e)) = \sum_{e' \in \mathrm{Flow}^{i}_{\mathcal{L}}(e)} \mu_{s(e')}(\mathcal{B}_{\mathcal{L}}(e'))$$

for all i, and applied to $i = i_e$:

$$\mu_\Lambda(\mathcal{B}_{\mathcal{L}}(e)) = \sum_{e' \in \mathrm{Flow}^{i_e}_{\mathcal{L}}(e)} \mu_{s(e')}(\mathcal{B}_{\mathcal{L}}(e')) = \sum_{e' \in \mathrm{Flow}^{i_e}_{\mathcal{L}}(e)} \mu(\mathcal{B}_{\mathcal{L}}(e')) = \mu(\mathcal{B}_{\mathcal{L}}(e)).$$

Since we have obtained this for all $e \in \widehat{\mathscr{B}_{\mathcal{L}1}}^{min}$, we have finished the proof of the exactness of the sequence.

The isomorphism $\mathscr{M}(\mathcal{L},\mathbb{Z})_0 \cong \mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}},\mathbb{Z})$ follows from an easy hunt of elements in the diagram of exact sequences locally isomorphic

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathscr{M}(\mathcal{L},\mathbb{Z})_0 & \longrightarrow & \prod_{\Lambda \in \mathscr{B}_{\mathcal{L}0}} \mathscr{M}_\Lambda(\mathcal{L},\mathbb{Z})_0 & \xrightarrow{\partial^1} & \prod_{e \in \mathscr{B}_{\mathcal{L}1}} \mathscr{M}_e(\mathcal{L},\mathbb{Z})_0 \\
& \Big\downarrow{\scriptstyle \kappa} & & \cong \Big\downarrow{\scriptstyle \kappa_\Lambda} & & \cong \Big\downarrow{\scriptstyle \kappa_e} \\
0 \longrightarrow & \mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}},\mathbb{Z}) & \longrightarrow & \prod_{\Lambda \in \mathscr{B}_{\mathcal{L}0}} \mathrm{C}^1_{\mathrm{har}}(\Lambda,\mathbb{Z})_{\mathcal{L}} & \xrightarrow{\partial^1} & \prod_{e \in \mathscr{B}_{\mathcal{L}1}} \mathrm{C}^1_{\mathrm{har}}(e,\mathbb{Z})_{\mathcal{L}}.
\end{array}$$

□

4.4.4 Invariance of the harmonic cochains with respect to homotopy

Through this subsection $\mathscr{B}_{\mathcal{L}}$ is a building of dimension $d = 2$ with $\mathcal{L} \subset \mathbb{P}(V)$ compact.

Simplicial homology on $\mathscr{B}_{\mathcal{L}}$

Let us recall first how we compute the homology of a locally finite simplicial complex.

Let $\mathcal{K}$ be a finite simplicial complex of dimension d and let $\Delta \in \mathcal{K}$ be a simplex. Two (total) orderings of the vertex set of Δ are said to be equivalent if they differ from one another by an even permutation. An orientation of Δ is an equivalence class of the orderings of the vertex set of Δ. Note that if Δ consists of a vertex, there is a unique orientation, and otherwise there are two different orientatons. An oriented simplex is a simplex Δ together with an orientation of Δ. An orientation of $\mathcal{K}$ is a choice of an orientation of each of its simplices. Then, $\mathcal{K}$ is said to be oriented if an orientation of $\mathcal{K}$ is fixed.

Then, for each $q \in \mathbb{N}_{\leq d}$ the abelian group of q-dimensional chains is the free abelian group generated by the oriented simplices of $\mathcal{K}$ of dimension q

$$C_q(\mathcal{K}) := \bigoplus_{\substack{\Delta \in \mathcal{K} \\ \dim(\Delta)=q}} \mathbb{Z}[\Delta].$$

We introduce the q-simplex Δ with the opposite orientation to the one chosen as $-\Delta$. Thus, we define a border operator $\partial_q : C_q(\mathcal{K}) \longrightarrow C_{q-1}(\mathcal{K})$ in the usual way: $\partial_q(\Delta) = \sum_{i=0}^{q} (-1)^i \Delta_i$ (where Δ_i has an orientation induced by the one of Δ) for $q \geq 1$ and $\partial_0 \equiv 0$. They form a complex of abelian groups and the simplicial homology is defined as

$$H_q(\mathcal{K}, \mathbb{Z}) := \operatorname{Ker}(\partial_q) / \operatorname{Im}(\partial_{q+1}).$$

If we take the singular homology of its topological realization, $H_q(|\mathcal{K}|, \mathbb{Z})$, it is well known that there is a natural isomorphism

$$H_q(\mathcal{K}, \mathbb{Z}) \cong H_q(|\mathcal{K}|, \mathbb{Z}).$$

Next, let us consider the simplicial complex $\mathscr{B}_{\mathcal{L}}$. Recall that each edge in $\mathscr{B}_{\mathcal{L}}$ has exactly one minimal orientation (as we have commented at the begining of subsection 4.3.2), and choose it to define the group of 1-chains of $\mathscr{B}_{\mathcal{L}}$, so that

$$C_1(\mathscr{B}_{\mathcal{L}}) \cong \mathbb{Z}[\widehat{\mathscr{B}_{\mathcal{L}1}}^{min}].$$

Then, for any oriented chamber Δ, the second differential verifies either $\partial_2(\Delta) = \partial^{min}(\Delta)$ or $\partial_2(\Delta) = -\partial^{min}(\Delta)$. Indeed, if Δ is defined by vertices $\Lambda_i = [L_i]$ where

$$L_0 \supsetneq L_1 \supsetneq L_2 \supsetneq \pi_K L_0$$

and if we choose the orientation given by $\Lambda_0 < \Lambda_1 < \Lambda_2$, then

$$\partial_2(\Delta) = (\Lambda_1, \Lambda_2) - (\Lambda_0, \Lambda_2) + (\Lambda_0, \Lambda_1) = -\partial^{min}(\Delta),$$

while if we choose the other orientation the two differential maps coincide. We choose for each chamber the orientation which makes these two maps coincide.

In addition, since $\mathscr{B}_{\mathcal{L}}$ is contractible the 1-homology group vanishes, therefore $\operatorname{Ker}(\partial_1) = \operatorname{Im}(\partial_2) = \operatorname{Im}(\partial^{min})$.

Homotopy invariance of the harmonic cochains

Given a harmonic cochain $c \in \mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}}, \mathbb{Z})$, it vanishes on ∂^{min} by definition, therefore the previous discussion implies that it vanishes on the closed paths on $\mathscr{B}_{\mathcal{L}}$.

Lemma 4.4.27. *Given two vertices $\Lambda, \Lambda' \in \mathscr{B}_{\mathcal{L}0}$, there exists an oriented path from Λ to Λ' through minimal edges.*

Proof. If the two vertices are in a chamber, since there exists a closed oriented path formed by minimal edges through all its vertices, then there exists an oriented path through these minimal edges from one vertex to the other.

Otherwise, since $\mathscr{B}_{\mathcal{L}}$ is connected, there is a sequence $\Delta_0, \dots, \Delta_r$ of chambers such that $\Lambda \in \Delta_0$, $\Lambda' \in \Delta_r$ and for each $i = 0, \dots, r-1$, there is a vertex $\Lambda_i \in \Delta_i \cap \Delta_{i+1}$. Since all these vertices are connected as claimed by the first part of the proof, the assertion follows. □

Lemma 4.4.28. *Given a harmonic cochain $c \in \mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}}, \mathbb{Z})$, two vertices $\Lambda, \Lambda' \in \mathscr{B}_{\mathcal{L}0}$ and two oriented paths between them through minimal edges, determined by the sums $P_1(\Lambda, \Lambda') \in \mathbb{Z}[\widehat{\mathscr{B}_{\mathcal{L}1}}^{min}]$ and $P_2(\Lambda, \Lambda') \in \mathbb{Z}[\widehat{\mathscr{B}_{\mathcal{L}1}}^{min}]$ respectively (as their support), we have*

$$c\left(P_1(\Lambda, \Lambda')\right) = c\left(P_2(\Lambda, \Lambda')\right)$$

Proof. Choose an oriented path P_3 from Λ' to Λ through minimal edges. The union (sum in $\mathbb{Z}[\widehat{\mathscr{B}_{\mathcal{L}1}}^{min}]$) of the path $P_i(\Lambda, \Lambda')$ with the path P_3 is a closed oriented path for each $i = 1, 2$, therefore it is in the kernel of ∂_1 and c vanishes on it. Then,

$$c\left(P_1(\Lambda, \Lambda')\right) = -c(P_3) = c\left(P_2(\Lambda, \Lambda')\right).$$

□

Corollary 4.4.29. *There is a map*

$$\mathbb{Z}[\mathscr{B}_{\mathcal{L}0}]_0 \longrightarrow \mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}}, \mathbb{Z})^\vee$$

given by mapping $\Lambda' - \Lambda$ to the evaluation of a harmonic cochain on any oriented path from Λ to Λ' on $\mathbb{Z}[\widehat{\mathscr{B}_{\mathcal{L}1}}^{min}]$.

Remark 4.4.30. *This is just to say that any harmonic cochain factorizes as follows:*

$$\begin{array}{ccc} \mathbb{Z}[\widehat{\mathscr{B}_{\mathcal{L}1}}^{min}] & \xrightarrow{\quad c \quad} & \mathbb{Z} \\ \downarrow & \nearrow & \\ \mathbb{Z}[\widehat{\mathscr{B}_{\mathcal{L}1}}^{min}]/\operatorname{Ker}(\partial_1) \cong \mathbb{Z}[\mathscr{B}_{\mathcal{L}0}]_0 & & \end{array}$$

4.5 The construction of the expected Albanese variety of a non-Archimedean uniformized variety by a “generalized Schottky group” and some steps to prove that it is a torus when $d = 2$

We introduce the elements which allow to define the non-Archimedean uniformized varieties introduced by Mustafin. They are a group being a generalization to any dimension of Schottky groups, an associated compact set and the related rigid analytic space whose quotient by the given group is the uniformized variety. We recall the Drinfeld reduction map and some definitions given in the section 3.4 of this thesis which can be done without any change in any dimension, and we prove some properties about them. With these elements, we show the construction of an object which we conjecture that it has the universal property of the Albanese variety in the category of abeloid varieties. Then, using the isomorphism between harmonic measures and harmonic cochains proved in the previous section, we reduce the proof of that such object is a torus to that a certain map involving the reduction of the uniformized variety is injective.

Through this section it is not necessary to assume that the dimension is $d = 2$ unless the parts involving harmonic cochains.

4.5.1 Integration on a compact set $\mathcal{L} \subset \mathbb{P}(V)$ and the analytic reduction

Let $\mathcal{L} \subset \mathbb{P}(V)$ be a closed subset. The analytic space associated to $\mathcal{L}$ is

$$\Omega_{\mathcal{L}} = \mathbb{P}_{V^*} \setminus \bigcup_{z \in \mathcal{L}} H_z.$$

The retraction map

As observed by Bruhat and Tits in [BT72, Note ajoutée sur épreuves, pp. 238-239], we may identify $|\mathscr{B}(G)|$ with the set of homothety classes of diagonalizable real norms on V, where a norm

$$\alpha : V \longrightarrow \mathbb{R}_{\geq 0}$$

is diagonalizable if there exists a basis $v_0, \dots, v_d$ of V such that

$$\alpha\left(\sum_{i=0}^{d} \lambda_i v_i\right) = \max_{i=0,\dots,d}\{\alpha(v_i)|\lambda_i|\}.$$

Remark 4.5.1. *We use the term diagonalizable instead of decomposable by coherence with the section 3.1 and the references cited there, mainly [RTW15, §1.2].*

Further, given a vertex $[L] \in \mathscr{B}(G)_0$, the corresponding norm (up to homothety depending on the chosen representant of the lattice) is defined by

$$\alpha_L(v) := \min\{|\lambda| : \ \lambda^{-1}v \in L\}$$

and reciprocally, the lattice associated to a norm α is

$$L_\alpha := \{v \in V| \ \alpha(v) \leq 1\}.$$

Then, for any complete extension $L|K$ we have a retraction map

$$\mathrm{r}_V(L) : \Omega_{\mathbb{P}(V)}(L) \longrightarrow |\mathscr{B}(G)|$$

given by $\mathrm{r}_V(L)(\omega)(v) = |\omega(v)|$ for $\omega \in \Omega_{\mathbb{P}(V)}$ and $v \in V$. Note that these maps are the evaluation on L of a map

$$\mathrm{r}_V : \Omega_{\mathbb{P}(V)} \longrightarrow |\mathscr{B}(G)|$$

Proposition 4.5.2. *The retraction map* $\mathrm{r}_V : \Omega_{\mathbb{P}(V)} \longrightarrow |\mathscr{B}(G)|$ *and its restriction* $\mathrm{r}_V : \Omega^{nr}_{\mathbb{P}(V)} \longrightarrow \mathscr{B}(G)_0$ *are surjective and* $GL(V)$*-equivariant.*

The integration map

Let $\mathcal{L} \subset \mathbb{P}(V)$ be a compact set, and let $\Omega_{\mathcal{L}}$ be its associated analytic space and $\mathscr{B}_{\mathcal{L}}$ the corresponding chamber complex.

We want to build an integration map

$$\times\!\!\!\!\!\!\int_{\bullet} d : \mathbb{Z}[\Omega_{\mathcal{L}}]_0 \longrightarrow \mathrm{Hom}(\mathscr{M}(\mathcal{L}, \mathbb{Z})_0, \mathbb{G}_{m,K}).$$

To do that we may copy verbatim adapted to higher dimension (and we do) the construction made in the section 3.4 of this work.

Let $L|K$ be an arbitrary compete extension of fields.

Definition 4.5.3. *Let $\mathcal{P}$ be a finite set of points in $\Omega_{\mathcal{L}}(L)$, and consider $D := \sum_{p\in\mathcal{P}} m_p p$ a divisor of degree zero. We denote by f_D the element of* $\mathrm{Maps}(\mathcal{L}, L^*)/L^*$ *which is defined up to scalars as follows: if we choose representatives $w_p \in V^*$ for any $p \in \mathcal{P}$ and $v_q \in V$ for q, then*

$$f_D(q) := \prod_{p\in\mathcal{P}} w_p(v_q)^{m_p}$$

does not depend on v_q. Any other election of the vectors v_p change f_D to λf_D for some $\lambda \in L^*$.

Remark 4.5.4. *Given divisors $D, D' \in \mathbb{Z}[\Omega_{\mathcal{L}}(L)]_0$ we have $f_{D+D'} = f_D f_{D'}$ and $f_{-D} = f_D^{-1}$.*

In particular, for any points $p, p', p'' \in \Omega_{\mathcal{L}}$ we have

$$f_{p'-p} = f_{p'-p''} f_{p''-p}.$$

Remark 4.5.5. *We can see the degree zero divisor 0 as the divisor $0p$ for any $p \in \Omega_{\mathcal{L}}(L)$. Therefore, as $m_p = 0$, we get $f_0 \equiv 1$.*

Let us apply the valuation map to these functions:

$$\mathcal{L} \xrightarrow{f_D} L^* \xrightarrow{v_K} \mathbb{Q}.$$

Lemma 4.5.6. *Let $p, q \in \Omega^{nr}_{\mathbb{P}(V)}$ be points such that $(\mathrm{r}_V(p), \mathrm{r}_V(q)) = e \in \widehat{\mathscr{B}_{\mathcal{L}}}^{min}_{1}$. Then*

$$v_K(f_{q-p}) = \chi_{\mathcal{B}_{\mathcal{L}}(e)} - 1.$$

Proof. Let us write $e = (L_0 \supsetneq L_1 \supsetneq \pi_K L_0)$ and recall that $z \in \mathcal{B}(e)$ if and only if $L_1 = \pi_K L_0 + (z \bigcap L_0)$, that is, for any representant $v_z \in L_0 \setminus \pi_K L_0$ of z it is in $L_1 \setminus \pi_K L_0$. We assume that there is an unramified complete extension $L|K$ such that $p, q \in \Omega_{\mathbb{P}(V)}(L)$. Let us take $z \in \mathcal{L}$, consider any representants $\omega_p, \omega_q \in \mathbb{P}(V_L^*)$ of p and q respectively, and observe that

$$L_0 = \{v \in V : \ |\omega_p(v)| \le 1\} = \{v \in V : \ v_K(\omega_p(v)) \ge 0\},$$
$$L_1 = \{v \in V : \ |\omega_q(v)| \le 1\} = \{v \in V : \ v_K(\omega_q(v)) \ge 0\}$$

and

$$\pi_K L_0 = \{v \in V : \ v_K(\omega_p(v)) \ge 1\}.$$

Then we have $0 \le v_K(\omega_p(v_z)) < 1$ and so $v_K(\omega_p(v_z)) = 0$, and, if $z \in \mathcal{B}(e)$, then $v_K(\omega_q(v_z)) = 0$, while if $z \notin \mathcal{B}(e)$, then $v_K(\omega_q(v_z)) = -1$.

Now we compute

$$v_K(f_{q-p}(z)) = v_K\left(\frac{\omega_q}{\omega_p}(v_z)\right) = \begin{cases} 0 & \text{if } z \in \mathcal{B}_{\mathcal{L}}(e) \\ -1 & \text{if } z \notin \mathcal{B}_{\mathcal{L}}(e). \end{cases}$$

and we get that we claimed. □

Definition 4.5.7. *Given any degree 0 divisor $D = \sum_{i\in I} m_i p_i$ with support in $\Omega_{\mathcal{L}}(L)$ (i.e. $m_i \in \mathbb{Z}$, $p_i \in \Omega_{\mathcal{L}}(L)$, being I a finite set and with $\sum_{i\in I} m_i = 0$) we choose v_i in V^* representatives of the $p_i \in \mathbb{P}_{V^*}(L)$ and consider the map up to scalars $f_D \in \mathrm{Maps}(\mathcal{L}, L^*)/K^*$ given by a representant $\prod_{i\in I} w_i(x)^{m_i}$ (which depends on the w_i's). Let $\mu \in \mathscr{M}(\mathcal{L},\mathbb{Z})_0$ be a $\mathbb{Z}$-valued harmonic measure on $\mathcal{L}$.*

We define

$$\times\!\!\!\!\!\!\int_{\mathcal{L},D} d\mu := \times\!\!\!\!\!\!\int_{\mathcal{L}} f_D d\mu \in L^*,$$

which is well defined since the integral does not depend on f_D but only on D. Indeed, although the representant of f_D depends on the elections of the representatives in V^* of the points in $\mathbb{P}_{V^*}(L)$, the multiplicative integral does not, since the measure is harmonic.

In general, when some $\mathcal{L}$ was fixed previously, we will omit its corresponding set, writing

$$\times\!\!\!\!\!\!\int_{D} d\mu := \times\!\!\!\!\!\!\int_{\mathcal{L},D} d\mu,$$

meanwhile we will specify the other sets over which we will integrate.

Note also that when $D = 0$, we have $\times\!\!\!\!\!\!\int_{0} d\mu = 1$, since $f_0 \equiv 1$.

Therefore, this definition gives us a morphism of groups

$$\begin{array}{ccc}
\mathbb{Z}[\Omega_{\mathcal{L}}(L)]_0 & \xrightarrow{\times\!\!\!\!\!\!\int_{\bullet} d} & \mathrm{Hom}(\mathscr{M}(\mathcal{L},\mathbb{Z})_0, L^*) \\
D & \longmapsto & \times\!\!\!\!\!\!\int_{D} d : \mu \longmapsto \times\!\!\!\!\!\!\int_{D} d\mu
\end{array}$$

The lemma 3.4.10 generalizes to any group $\Gamma \subset \mathrm{PGL}(V)$ acting on $\mathcal{L}$, so this map is Γ-equivariant.

Lemma 4.5.8. *Let $p, q \in \Omega^{nr}_{\mathbb{P}(V)}$ be points such that $(\mathrm{r}_V(p), \mathrm{r}_V(q)) = e \in \widehat{\mathscr{B}}_{\mathcal{L}1}^{min}$ and let $\mu \in \mathscr{M}(\mathcal{L},\mathbb{Z})_0$. Then*

$$v_K\left(\times\!\!\!\!\!\!\int_{q-p} d\mu\right) = \mu(\mathcal{B}_{\mathcal{L}}(e)).$$

Proof. This is a corollary of lemma 4.5.6. Indeed, we have

$$v_K\left(\times\!\!\!\!\!\!\int_{q-p} d\mu\right) = \int_{\mathcal{L}} v_K\left(f_{q-p}\right) d\mu = \int_{\mathcal{L}} (\chi_{\mathcal{B}_{\mathcal{L}}(e)} - 1) d\mu = \mu(\mathcal{B}_{\mathcal{L}}(e)).$$

where we are also using the additive integrals theory introduced in section 2.3.

□

Remark 4.5.9. *Note that given $p, p' \in \Omega^{nr}_{\mathbb{P}(V)}$ such that $\mathrm{r}_V(p), \mathrm{r}_V(p') \in \mathscr{B}_{\mathcal{L}0}$, we have an oriented path $\sum_{i=1}^r e_i \in \mathbb{Z}[\widehat{\mathscr{B}_{\mathcal{L}1}}^{min}]$ from $\mathrm{r}_V(p)$ to $\mathrm{r}_V(p')$ and points $p_i \in \Omega^{nr}_{\mathbb{P}(V)}$, for $i = 0, \ldots, r$, such that $p_0 = p$, $p_r = p'$, $\mathrm{r}_V(p_0) = s(e_1)$, $\mathrm{r}_V(p_r) = t(e_r)$ and $\mathrm{r}_V(p_i) = t(e_i) = s(e_{i+1})$ for $i = 1, \ldots, r-1$, and then*

$$v_K\left(\times\!\!\!\!\!\!\int_{p'-p} d\mu\right) = v_K\left(\prod_{i=1}^r \times\!\!\!\!\!\!\int_{p_i - p_{i-1}} d\mu\right) =$$

$$= \sum_{i=1}^r v_K\left(\times\!\!\!\!\!\!\int_{p_i - p_{i-1}} d\mu\right) = \sum_{i=1}^r \mu(\mathcal{B}_{\mathcal{L}}(e_i))$$

If we restrict to dimension $d = 2$, we have defined harmonic cochains and the map $\kappa : \mathscr{M}(\mathcal{L}, \mathbb{Z})_0 \longrightarrow \mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}}, \mathbb{Z})$, so that applying the remark 4.4.30, we get

$$v_K\left(\times\!\!\!\!\!\!\int_{p'-p} d\mu\right) = \sum_{i=1}^r \mu(\mathcal{B}_{\mathcal{L}}(e_i)) = \kappa(\mu)\left(\sum_{i=1}^r e_i\right) = \kappa(\mu)(\mathrm{r}_V(p') - \mathrm{r}_V(p)).$$

4.5.2 Generalized Schottky groups in $\mathrm{PGL}(V)$

Hyperbolic subgroups of $\mathrm{PGL}(V)$

Definition 4.5.10. *An element $\gamma \in \mathrm{PGL}(V)$ is said to be hyperbolic if for any representant $\tilde{\gamma} \in \mathrm{GL}(V)$ all its eigenvalues are in K and two of them have different valuation. It is said strictly hyperbolic if, moreover, $\tilde{\gamma}$ diagonalizes.*

Lemma 4.5.11. *If $\gamma \in \mathrm{PGL}(V)$ is hyperbolic, it has no torsion.*

Proof. Since a matrix representing γ can be taken triangular superior, and we only need its diagonal, we can take its diagonal $(\lambda^0_\gamma, \ldots, \lambda^d_\gamma)$ (as if γ would be strictly diagonal). We also can assume that the two eigenvalues with different valuation are $\lambda^0_\gamma, \lambda^1_\gamma$. The eigenvalues of the n-th power of γ are $(\lambda^i_\gamma)^n$, and the difference between the valuations of $(\lambda^0_\gamma)^n$ and $(\lambda^1_\gamma)^n$ is n times the initial difference, therefore, each time higher. Thus, none power of γ can be the identity. □

Definition 4.5.12. *A subgroup $H \subset \mathrm{PGL}(V)$ is said to be hyperbolic (resp. strictly hyperbolic) if all its elements distinct of the identity are hyperbolic (resp. strictly hyperbolic).*

Recall the next well known result.

Theorem 4.5.13. *If $A_1, \dots, A_r$ are linear operators on V, each of them diagonalizable, they are simultaneously diagonalizabe (there exists a basis for which all of them diagonalize) if and only if they commute.*

These property characterizes elements being in the same maximal torus of $GL(V)$. The conjugation matrix which relates it to the "canonical torus" $\mathbb{G}_{m,K}^{d+1}$ gives the matrix of change of basis diagonalizing the given matrices.

Let us denote the set of fixed points in $\mathbb{P}(V)$ of a subgroup $\Gamma \subset \mathrm{PGL}(V)$ by

$$\mathcal{L}_\Gamma := \mathbb{P}(V)^\Gamma.$$

Remark 4.5.14. *The torus $T = \mathbb{G}_{m,K}^{d+1}/\mathbb{G}_{m,K} \subset \mathrm{PGL}(V)$ is the centralizer of the "reference points" $p_0, \dots, p_d$, and reciprocally, they form the set of fixed points of the torus: $\mathcal{L}_T = \{p_0, \dots, p_d\}$. Since each other maximal K-split torus is conjugate to it, it is the centralizer in $\mathrm{PGL}(V)$ of a set of $d+1$ linearly independent points which is its set of fixed points, and the sets of maximal K-split tori and of these centralizers coincide. Further, we have*

$$\gamma \mathcal{L}_T = \mathcal{L}_{\gamma T \gamma^{-1}}$$

for any $\gamma \in \mathrm{PGL}(V)$.

MAH subgroups

Lemma 4.5.15. *Let $T \subset \mathrm{PGL}(V)$ be a maximal K-split torus and let $H \subset T$ be a strictly hyperbolic subgroup. Then H is free as abelian group and $\mathrm{rank}_{\mathbb{Z}}(H) \leq d$.*

Proof. By conjugation, we assume that $H \subset \mathbb{G}_{m,K}^{d+1}/\mathbb{G}_{m,K}$, so the elements of H are diagonal matrices up to scalars with all the diagonal elements distinct of zero. Therefore, we can lift H to a subgroup $\tilde{H} \subset \mathbb{G}_{m,K}^{d+1} \subset \mathrm{GL}(V)$ as follows: if the diagonal of $\gamma \in H$ is $(\lambda_\gamma^0, \dots, \lambda_\gamma^d)$, multiply it by $(\lambda_\gamma^0)^{-1}$ and take that representant which we denote $\tilde{\gamma}$, that is the one having $\lambda_{\tilde{\gamma}}^0 = 1$. Thus we have a morphism of abelian groups

$$\begin{array}{ccc} H & \xrightarrow{\quad v_K \quad} & \mathbb{Z}^d \\ \gamma = \left[\begin{pmatrix} 1 & & & \\ & \lambda_\gamma^1 & & 0 \\ & & \ddots & \\ & 0 & & \lambda_\gamma^d \end{pmatrix}\right] & \longmapsto & (v_K(\lambda_\gamma^1), \dots, v_K(\lambda_\gamma^d)) \end{array}$$

Since any $\gamma \neq \mathbb{1}_H$ is strictly hyperbolic, there is an index $i \geq 1$ such that $v_K(\lambda^i_\gamma) \neq v_K(1) = 0$, so this map is injective. Thus, H is isomorphic to its image, a subgroup of a finite generated free abelian group of rank d, therefore H is a finite generated free abelian group with $\mathrm{rank}(H) \leq d$. □

Proposition 4.5.16. *Let $T \subset \mathrm{PGL}(V)$ be a maximal K-split torus, and let $H \subset T$ (and so $\mathcal{L}_T \subset \mathcal{L}_H$) be another subgroup. Let $v_0, \dots, v_d \in V$ the basis given by choosing representants of the points of $\mathcal{L}_T = \{[v_0], \dots, [v_d]\}$. Then, the following are equivalent:*

1. *H is a strictly hyperbolic subgroup and $H \cong \mathbb{Z}^d$.*
2. *There exist generators $\gamma_1, \dots, \gamma_d \in H$ such that representants $\tilde{\gamma}_i \in \mathrm{GL}(V)$ verify $\tilde{\gamma}_i v_j = \lambda^j_i v_j$ for $\lambda^j_i \in K^*$ and, the matrix*

$$M = \left(v_K(\lambda^j_i) - v_K(\lambda^0_i)\right)_{i,j\geq 1} =$$

$$= \begin{pmatrix} v_K(\lambda^1_1) - v_K(\lambda^0_1) & \dots & v_K(\lambda^1_d) - v_K(\lambda^0_d) \\ \vdots & & \vdots \\ v_K(\lambda^d_1) - v_K(\lambda^0_1) & \dots & v_K(\lambda^d_d) - v_K(\lambda^0_d) \end{pmatrix}$$

has rank d.

Proof. 1. $\Longrightarrow$ 2.| Since $H \cong \mathbb{Z}^d$, it is generated by $\gamma_1, \dots, \gamma_d \in H$, and any representants $\tilde{\gamma}_i \in \mathrm{GL}(V)$ satisfy equalities of the form $\tilde{\gamma}_i v_j = \lambda^j_i v_j$ with $\lambda^j_i \in K^*$, due to the points $[v_j]$ are fixed by the γ_i. Consider the matrix

$$M = \left(v_K(\lambda^j_i) - v_K(\lambda^0_i)\right)_{i,j\geq 1} = \left(v_K\left(\frac{\lambda^j_i}{\lambda^0_i}\right)\right)_{i,j\geq 1}.$$

Note that this is the matrix of the linear map $v_{K|H} : H \longrightarrow \mathbb{Z}^d$ that we introduced during the proof of the previous lemma, in the basis $\gamma_1, \dots, \gamma_d$. Since we have seen that it is injective, then $\mathrm{rank}(M) = \mathrm{rank}_{\mathbb{Z}}(H) = d$.

2. $\Longrightarrow$ 1.|

The hypothesis tells us that the elements of H diagonalize in the basis given by $v_0, \dots, v_d$. Then, as we have just noted, M is the matrix of the images of the elements γ_i by the map $v_{K|H}$ of the previous lemma. Since $\mathrm{rank}(M) = d$, these elements are linearly independent in H and there are no torsion elements in H, therefore $H \cong \mathbb{Z}^d$ and it is strictly hyperbolic.

□

Definition 4.5.17. *We will say that a subgroup $H \subset \mathrm{PGL}(V)$ verifying any of the equivalent conditions of the previous proposition is maximal abelian hyperbolic (MAH for short), or also maximal toric hyperbolic.*

Corollary 4.5.18. *If $H \subset \mathrm{PGL}(V)$ is MAH, its set of fixed points $\mathcal{L}_H$ consists of $d+1$ linearly independent points.*

Proof. With the notation of the previous proposition, we already have a basis $v_0, \dots, v_d$ such that

$$\{[v_0], \dots, [v_d]\} \subset \mathcal{L}_H$$

so, the only thing we have to prove is that there are no more points. Let $v = \sum_{i=0}^{d} \alpha_i v_i$ with $\alpha_i \in K$ such that for each $i = 1, \dots, d$, $\tilde{\gamma}_i v = \beta_i v$ (with the same representants of the previous proof) for some $\beta_i \in K^*$. Then, for each i we have

$$\sum_{j=0}^{d} \lambda_i^j \alpha_j v_j = \sum_{j=0}^{d} \alpha_j \tilde{\gamma}_i v_j = \tilde{\gamma}_i v = \beta_i v = \sum_{j=0}^{d} \beta_i \alpha_j v_j.$$

Thus $\alpha_j \neq 0$ implies that $\lambda_i^j = \beta_i$. If $v \neq v_i$ for all i, there are $j_1 \neq j_2$ such that $\alpha_{j_1}, \alpha_{j_2} \neq 0$, so $\lambda_i^{j_1} = \lambda_i^{j_2}$ for each i, giving place to two equal columns in the matrix M of the valuations, which we have seen that has rank d. Therefore, $[v]$ has to be one of the points $[v_i]$, as we wanted to show. $\square$

Remark 4.5.19. *Since, if H is MAH, $\mathcal{L}_H$ is finite, we have*

$$\overline{\Gamma \mathcal{L}_H} = \bigcup_{p \in \mathcal{L}_H} \overline{\Gamma p}.$$

Let $H \subset \mathrm{PGL}(V)$ be MAH and fix a basis $\mathbf{v} = \{v_0, \dots, v_d\}$ of V for which H consists of diagonal matrices up to K^*. Consider also the standard symmetric bilinear form in V with respect to the coordinates in this basis

$$\Phi : V \times V \longrightarrow K$$

$$\left(\sum_{i=0}^{d} \alpha_i v_i, \sum_{i=0}^{d} \beta_i v_i \right) \longmapsto \Phi \left(\sum_{i=0}^{d} \alpha_i v_i, \sum_{i=0}^{d} \beta_i v_i \right) = \sum_{i=0}^{d} \alpha_i \beta_i.$$

Note that given $\lambda \in K^*$ we have $\Phi(v, v') = 0$ if and only if $\Phi(v, \lambda v') = 0$. We can define for any subspace $W \subset V$, its orthogonal with respect to $\mathbf{v}$ as

$$W^{\perp_{\mathbf{v}}} := \{u \in V \mid \Phi(w, u) = 0 \ \forall w \in W\}$$

Since this is another linear subspace, we can projectivize this definition.

Remark 4.5.20. *Consider the dual basis of* $\boldsymbol{v}$ *in* V^* *which is* $\boldsymbol{w} = \{\omega_0, \ldots, \omega_d\}$ *determined by the relations* $\omega_i(v_j) = \delta_{ij}$. *Then we define naturally the orthogonal of* $W \subset V$ *in* V^* *as*

$$W^\perp := \{\omega \in V^* \mid \omega(w) = 0 \ \forall w \in W\}$$

If we apply the isomorphism $\varphi_{\boldsymbol{v}} : V^* \longrightarrow V$ *defined by* $\varphi_{\boldsymbol{v}}(\omega_i) = v_i$ *we have*

$$W^{\perp_{\boldsymbol{v}}} = \varphi_{\boldsymbol{v}}(W^\perp).$$

Given a point $p \in \mathbb{P}(V)$, we denote its orthogonal hyperplane with respect to the basis $\mathbf{v}$ as $\mathbb{H}_p^{\mathbf{v}} \subset \mathbb{P}(V)$.

Remark 4.5.21. *For a point* $p \in \{[v_0], \ldots, [v_d]\} =: \mathcal{P}$, *the hyperplane* $\mathbb{H}_p^{\boldsymbol{v}}$ *is independent of the representant* v_i *chosen for* $[v_i]$ *so we can write* $\mathbb{H}_p^{\mathcal{P}} := \mathbb{H}_p^{\boldsymbol{v}}$, *and, concretely*

$$\mathbb{H}_{p_i}^{\mathcal{P}} = \left\{ \left[\sum_{j \neq i} \alpha_j v_j \right] \in \mathbb{P}(V) \right\}.$$

Note also that

$$\bigcap_{i=0}^{d} \mathbb{H}_{p_i}^{\mathcal{P}} = \emptyset.$$

Lemma 4.5.22. *Let* $H \subset \mathrm{PGL}(V)$ *be MAH and* $p \in \mathcal{L}_H$. *Then, there exists* $\gamma \in H$ *such that for all* $p' \notin \mathbb{H}_p^{\mathcal{L}_H}$

$$\lim_{n \to \infty} \gamma^n p' = p.$$

Proof. After conjugation, we can suppose that H are classes of diagonalizable matrices and $\mathcal{L}_H = \{p_0, \ldots, p_d\}$ are the "reference" points for the canonical basis, and we take without loss of generality $p = p_0 = (1 : 0 \cdots : 0)$. Since the map $v_{K|H} : H \longrightarrow \mathbb{Z}^d$ has rank d, there is an element

$$(v_1, \ldots, v_d) \in \mathrm{Im}(v_{K|H}) \cap \mathbb{Z}_{>0}.$$

Its preimage is the set of elements $\gamma \in H$ such that $v_K(\lambda_\gamma^0) < v_K(\lambda_\gamma^i)$ for all $i \neq 0$. Take any of them and the representant $\tilde{\gamma}$ with $\lambda_{\tilde{\gamma}}^0 = 1$. Since $p' \notin \mathbb{H}_{p_0}^{\mathcal{L}_H}$, p' can be represented by a vector with the form $v' = v_0 + \sum_{i=1}^d \alpha_i v_i$ and we have

$$\tilde{\gamma}^n v' = v_0 + \sum_{i=1}^{d} (\lambda_{\tilde{\gamma}}^i)^n \alpha_i v_i$$

Since the valuations $v_K(\lambda_{\tilde{\gamma}}^i)$ are strictly positive, their powers tend to infinity, and so,

$$\lim_{n \to \infty} \gamma^n p' = p_0.$$

□

Generalized Schottky groups

Let $\Gamma \subset \mathrm{PGL}(V)$ be any subgroup. Define its set of limit points as

$$\mathcal{L}_\Gamma := \overline{\bigcup_{\substack{H \subset \Gamma \\ H \text{MAH}}} \mathcal{L}_H}.$$

Definition 4.5.23. *A subgroup $\Gamma \subset \mathrm{PGL}(V)$ is a generalized Schottky group if:*

1. *It is hyperbolic.*
2. *It is finitely generated.*
3. *There exists a MAH subgroup $H \subset \Gamma$.*
4. *$\overline{\Gamma \cdot p}$ is compact for all $p \in \mathbb{P}(V)$.*
5. *$\mathscr{B}_{\mathcal{L}_\Gamma}$ is a building.*
6. *$\Gamma \backslash \mathscr{B}_{\mathcal{L}_\Gamma}$ is a finite simplicial complex and*

   $$\pi_\Gamma : \mathscr{B}_{\mathcal{L}_\Gamma} \longrightarrow \Gamma \backslash \mathscr{B}_{\mathcal{L}_\Gamma}$$

   is a universal covering.

Proposition 4.5.24. *The action of Γ on $\mathscr{B}_{\mathcal{L}_\Gamma}$ is free.*

Proof. This is due to the fact that all the elements of Γ are hyperbolic, and so torsion-free (cf. [Gar73, Lem. 2.6] and [Mus78, Prop. 1.4]). □

Remark 4.5.25. *Let us enumerate a number of immediate facts from the definition:*

1. *After a finite complete extension $L|K$, $\Gamma \backslash \mathscr{B}_{\mathcal{L}_\Gamma}$ becomes always a simplicial complex.*
2. *Since $\mathscr{B}_{\mathcal{L}_\Gamma}$ is a universal cover, then*

   $$\Gamma^{ab} \cong H_1(\Gamma \backslash \mathscr{B}_{\mathcal{L}_\Gamma}, \mathbb{Z}).$$

3. *The quotient being a simplicial complex means, in particular, that it has no loops, what implies that for each vertex $\Lambda \leq \mathscr{B}_{\mathcal{L}_\Gamma}$ and each $\gamma \in \Gamma \setminus \{\mathbb{1}_\Gamma\}$: $\rho(\Lambda, \gamma\Lambda) \neq 1$. Further, since Γ acts freely on $\mathscr{B}_{\mathcal{L}_\Gamma}$, we get $\rho(\Lambda, \gamma\Lambda) \geq 2$, which is the assumption made in [dS01, § 9.1]. A consequence of this is that given a minimal edge $e \in \widehat{\mathscr{B}_{\mathcal{L}_\Gamma 1}}^{min}$, it is not identified with any edge in $\mathrm{Flow}_{\mathcal{L}_\Gamma}(e)$.*

4. *Since the action of* $\mathrm{PGL}(V)$ *on* $\mathscr{B}_{\mathcal{L}_\Gamma}$ *preserves the cyclic order of its cells and the types of the pointed simplices, these notions go down to the quotient. Thus,*

$$(\Gamma\backslash\mathscr{B}_{\mathcal{L}_\Gamma})_q := \Gamma\backslash\mathscr{B}_{\mathcal{L}_\Gamma q} \quad \textit{and} \quad \widehat{\Gamma\backslash\mathscr{B}_{\mathcal{L}_\Gamma q}}^{\,t} := \Gamma\backslash\widehat{\mathscr{B}_{\mathcal{L}_\Gamma q}}^{\,t}.$$

In particular, the maps $\mathrm{St}^{min}_{\mathcal{L}_\Gamma}$ *and* ∂^{min} *also go down to the quotient. By the last comment of the previous item, the map* $\mathrm{Flow}_{\mathcal{L}_\Gamma}$ *also can be well defined on the quotient, over an edge* $\overline{e}$ *as the edges in* $\mathrm{St}^{min}_{\mathcal{L}_\Gamma}(t(\overline{e}))$ *which do not share any chamber with* $\overline{e}$*, it verifies*

$$\mathrm{Flow}_{\mathcal{L}_\Gamma}(\overline{e}) = \left\{\overline{e'} \mid e' \in \mathrm{Flow}_{\mathcal{L}_\Gamma}(e)\right\}$$

and $\overline{e} \notin \mathrm{Flow}_{\mathcal{L}_\Gamma}(\overline{e}) \subset \mathrm{St}^{min}_{\mathcal{L}_\Gamma}(t(\overline{e}))$*.*

Conjecture 1. *The first four items imply the condition 5) and, after a finite complete extension* $L|K$*, they also imply 6), in the definition of generalized Schottky group.*

Lemma 4.5.26. *If* $\Gamma \subset \mathrm{PGL}(V)$ *is a generalized Schottky group and* $\mathcal{P} \subset \mathcal{L}_\Gamma$ *is a subset of* $d+1$ *points linearly independent, then*

$$\mathcal{L}_\Gamma = \overline{\Gamma\mathcal{P}}.$$

In particular, this is the case for $\mathcal{P} = \mathcal{L}_H$ *where* $H \subset \Gamma$ *is MAH.*

Proof. We only have to show that any point $p \in \mathcal{L}_{H'}$ for $H' \subset \Gamma$ MAH is in the closure of $\Gamma\mathcal{P}$.

By hypothesis, there is no hyperplane $\mathbb{H}$ such that $\mathcal{P} \subset \mathbb{H}$. Then, there is at least a point $q \in \mathcal{P}$ such that $q \notin \mathbb{H}_p^{\mathcal{L}_{H'}}$, and we know by the lemma 4.5.22 that there is $\gamma \in H' \subset \Gamma$ such that

$$p = \lim_{n\to\infty} \gamma^n q \in \overline{\Gamma\mathcal{P}}.$$

□

Corollary 4.5.27. *If* Γ *is a generalized Schottky group, the set* $\mathcal{L}_\Gamma$ *is compact, so that* $\mathscr{B}_{\mathcal{L}_\Gamma}$ *is locally finite.*

Remark 4.5.28. *We got this without using the condition 6) in the definition of generalized Schotty group.*

Proposition 4.5.29. *The quotient* $X_\Gamma := \Gamma\backslash\Omega_{\mathcal{L}_\Gamma}$ *is a rigid analytic variety, and its reduction has dual complex* $\Gamma\backslash\mathscr{B}_{\mathcal{L}_\Gamma}$*.*

Proof. This is the result given in [Mus78, Thm. 3.1]. □

Note that the examples A) and B) in [Mus78, § 4] also apply here.

4.5.3 The conjectural Albanese variety and Abel-Jacobi map

Let $\Gamma \subset \mathrm{PGL}(V)$ be a generalized Schottky group, and let $\mathcal{L} := \mathcal{L}_\Gamma \subset \mathbb{P}(V)$ be its associated compact set.

As in the section 3.9 of the chapter 3 for dimension 1, we have a map

$$\Gamma^{ab} \cong H_1(\Gamma, \mathbb{Z}) \longrightarrow H_0(\Gamma, \mathbb{Z}[\Omega_{\mathcal{L}}]_0) = \mathbb{Z}[\Omega_{\mathcal{L}}]_{0\Gamma}$$

obtained from applying Γ-coinvariants to the short exact sequence

$$0 \longrightarrow \mathbb{Z}[\Omega_{\mathcal{L}}]_0 \longrightarrow \mathbb{Z}[\Omega_{\mathcal{L}}] \longrightarrow \mathbb{Z} \longrightarrow 0.$$

After composing that connecting morphism with the integration map built before, we get the morphism

$$\mathscr{I} : \Gamma^{ab} \longrightarrow \mathrm{Hom}(\mathscr{M}(\mathcal{L}, \mathbb{Z})_0^\Gamma, \mathbb{G}_{m,K}) =: T$$

given by

$$\mathscr{I}(\gamma) = \times\!\!\!\!\!\!\int_{\gamma p - p} d : \mu \mapsto \times\!\!\!\!\!\!\int_{\gamma p - p} d\mu \text{ for any } p \in \Omega_{\mathcal{L}}.$$

Note that $\mathrm{Im}(\mathscr{I}) \subset T(K) = \mathrm{Hom}(\mathscr{M}(\mathcal{L}, \mathbb{Z})_0^\Gamma, K^*)$ as in dimension 1.

Let $L|K$ be a complete extension such that the analytic variety $X_\Gamma := \Gamma \backslash \Omega_{\mathcal{L}}$ has some L-point $p \in X_\Gamma(L)$. Then, there is a map

$$\begin{array}{ccc} X_\Gamma & \xrightarrow{\iota_p} & T/\mathscr{I}(\Gamma^{ab}) \\ q & \longmapsto & \iota_p(q) : \mu \longmapsto \times\!\!\!\!\!\!\int_{q-p} d\mu \end{array}$$

We recall next the definition of abeloid variety from [Lüt09]:

Definition 4.5.30. *An abeloid variety is a group object in the category of rigid analytic spaces whose underlying variety is smooth, proper and connected.*

Remark 4.5.31. *The main result of [Lüt95] tells that all the abeloid varieties are, after a suitably extension of the base field, an analytic quotient by a lattice of an abeloid group with good reduction (whatever it be) by an affine torus. In particular, rigid analytic torus are abeloid varieties.*

Conjecture 2. *The object* $A(X_\Gamma) := T^{an}/\mathscr{I}(\Gamma)$ *is a rigid analytic torus and for any complete extension* $L|K$ *such that* $X_\Gamma(L) \neq \emptyset$ *and for any* $p \in X_\Gamma(L)$*, the map*

$$\begin{array}{ccc} X_{\Gamma,L} & \xrightarrow{\iota_p} & A(X_\Gamma)_L \\ q & \longmapsto & \iota_p(q) : \mu \longmapsto \displaystyle\times\!\!\!\!\!\!\int_{q-p} d\mu \end{array}$$

is a morphism of analytic varieties such that $i_p(p) = 0$*, and for any such other map* $\varphi : X_{\Gamma,L} \longrightarrow A_L$*, where* A_L *is an abeloid variety and* $\varphi(p) = 0$*, there exists a unique morphism of abeloid varieties*

$$\phi : A(X_\Gamma)_L \longrightarrow A_L$$

such that $\varphi = \phi \circ \iota_p$*.*

Remark 4.5.32. *It seems reasonable to expect that the universal property in the conjecture can be reduced to the cases in which* A_L *is a rigid analytic torus, by means of topological arguments.*

Remark 4.5.33. *We just are stating a generalization of the universal property of the Albanese torus in the category of abeloid varieties.*

In the previous chapter we proved it when the varieties are curves, in which case they are always algebraic, so that $A(X_\Gamma)$ *is the Albanese torus for* $d = 1$*.*

4.5.4 Towards a proof of the conjecture 2 when $d = 2$, I: harmonic cochains on the simplicial complex quotient of $\mathscr{B}_{\mathcal{L}_\Gamma}$ and an equivalent formulation of being an analytic torus for $A(X_\Gamma)$

Let $\Gamma \subset \mathrm{PGL}(V)$ be a generalized Schottky group, and let $\mathcal{L} := \mathcal{L}_\Gamma \subset \mathbb{P}(V)$ be its associated compact set and let $\mathcal{K}_\Gamma := \Gamma \backslash \mathscr{B}_{\mathcal{L}_\Gamma}$. Let the dimension be $d = 2$.

To finish, we show that under these assumptions, proving that $T/\mathscr{I}(\Gamma^{ab})$ is an analytic torus reduces to see that a certain map involving harmonic cochains on $\mathcal{K}_\Gamma$ instead of harmonic measures is an isogeny.

Definition 4.5.34. *A map* $c : \mathbb{Z}[\widehat{\mathcal{K}_{\Gamma 1}}^{min}] \longrightarrow \mathbb{Z}$ *is called a harmonic cochain if it satisfies the following properties:*

- $c \circ \mathrm{St}_{\mathcal{L}}^{min} = 0$.

- *For any minimal edge* $e \in \widehat{\mathcal{K}_{\Gamma 1}}^{min}$, $c(e) = c(\mathrm{Flow}_{\mathcal{L}}(e))$, *that is*

$$c \circ \left(\mathbb{1}_{\widehat{\mathcal{K}_{\Gamma 1}}^{min}} - \mathrm{Flow}_{\mathcal{L}} \right) = 0.$$

- $c \circ \partial^{min} = 0$.

The set of harmonic cochains is denoted by $\mathrm{C}^1_{\mathrm{har}}(\mathcal{K}_\Gamma, \mathbb{Z})$.

Remark 4.5.35. *The facts that all these conditions are local, and that for each vertex* $\Lambda \leq \mathscr{B}_{\mathcal{L}_\Gamma}$ *and for each* $\gamma \in \Gamma \setminus \{\mathbb{1}_\Gamma\}$, $\rho(\Lambda, \gamma\Lambda) \geq 2$, *imply*

$$\mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}_\Gamma}, \mathbb{Z})^\Gamma \cong \mathrm{C}^1_{\mathrm{har}}(\mathcal{K}_\Gamma, \mathbb{Z}).$$

Lemma 4.5.36. *The map*

$$\begin{array}{ccc} \Gamma^{ab} & \xrightarrow{\quad\psi\quad} & \mathrm{Hom}(\mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}}, \mathbb{Z})^\Gamma, \mathbb{Z}) \\ \gamma & \longmapsto & \psi_\gamma : c \longmapsto \psi_\gamma(c) = c\left(P(\Lambda, \gamma\Lambda)\right) \end{array}$$

is well defined, independent of $\Lambda \in \mathscr{B}_{\mathcal{L}0}$ *and a morphism of abelian groups.*

Proof. The map is well defined if it is independent of Λ and of the path $P(\Lambda, \gamma\Lambda)$. The independence of the path was proved in lemma 4.4.28. To get the independence of Λ, for any other Λ' consider the oriented closed path union (sum in $\mathbb{Z}[\widehat{\mathscr{B}_{\mathcal{L}1}}^{min}]$) of oriented paths $P(\Lambda, \gamma\Lambda)$, $P(\gamma\Lambda, \gamma\Lambda')$, $P(\gamma\Lambda', \Lambda')$ and $P(\Lambda', \Lambda)$. Since it is in $\mathrm{Ker}(\partial_1)$, c vanishes on it. Moreover, since c is Γ-invariant,

$$c(P(\gamma\Lambda, \gamma\Lambda')) = c(\gamma P(\Lambda, \Lambda')) = c(P(\Lambda, \Lambda')) = -c(P(\Lambda', \Lambda)),$$

and therefore

$$c(P(\Lambda, \gamma\Lambda)) = -c(P(\gamma\Lambda', \Lambda')) = c(P(\Lambda', \gamma\Lambda')).$$

To see that the map is a morphism, note that given $\gamma, \gamma' \in \Gamma$, we get an oriented path from Λ to $\gamma\gamma'\Lambda$ by adjoining a path from Λ to $\gamma'\Lambda$ with a path from $\gamma'\Lambda$ to $\gamma\gamma'\Lambda$. □

Remark 4.5.37. *We can reinterpret this lemma as follows:* Γ *acts on* $\mathscr{B}_{\mathcal{L}}$ *simplicially, and so, acts in the short exact sequence*

$$0 \longrightarrow \mathbb{Z}[\mathscr{B}_{\mathcal{L}0}]_0 \longrightarrow \mathbb{Z}[\mathscr{B}_{\mathcal{L}0}] \longrightarrow \mathbb{Z} \longrightarrow 0$$

what gives to us a connecting morphism

$$\Gamma^{ab} \cong H_1(\Gamma, \mathbb{Z}) \longrightarrow H_0(\Gamma, \mathbb{Z}[\mathscr{B}_{\mathcal{L}0}]_0) = \mathbb{Z}[\mathscr{B}_{\mathcal{L}0}]_{0\Gamma}.$$

The corresponding lemma would say that the map

$$\mathbb{Z}[\mathscr{B}_{\mathcal{L}0}]_0 \longrightarrow \mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}}, \mathbb{Z})^\vee$$

of corollary 4.4.29 is Γ*-equivariant, so we could compose*

$$\Gamma^{ab} \longrightarrow \mathbb{Z}[\mathscr{B}_{\mathcal{L}0}]_{0\Gamma} \longrightarrow \mathrm{Hom}(\mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}}, \mathbb{Z}), \mathbb{Z})_\Gamma = \mathrm{Hom}(\mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}}, \mathbb{Z})^\Gamma, \mathbb{Z})$$

to get ψ*.*

Let us consider the isomorphism

$$\kappa : \mathscr{M}(\mathcal{L}, \mathbb{Z})_0 \longrightarrow \mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}}, \mathbb{Z}),$$

restricted to Γ-invariant harmonic measures and cochains, and the projection

$$\pi_\Gamma : \mathscr{B}_{\mathcal{L}} \longrightarrow \Gamma\backslash\mathscr{B}_{\mathcal{L}}.$$

Proposition 4.5.38. *Given* $\gamma \in \Gamma$*,* $\mu \in \mathscr{M}(\mathcal{L}, \mathbb{Z})^\Gamma_0$*,* $p \in \Omega^{nr}_{\mathbb{P}(V)}$*,* $\Lambda \in \mathscr{B}_{\mathcal{L}0}$*, we have*

$$v_K\left(\times\!\!\!\!\!\!\int_{\gamma p - p} d\mu\right) = \kappa(\mu)(P(\Lambda, \gamma\Lambda)) = \psi_\gamma(\kappa(\mu)).$$

where $P(\Lambda, \gamma\Lambda) \in \mathbb{Z}[\widehat{\mathscr{B}_{\mathcal{L}1}}^{min}]$ *is any path from* Λ *to* $\gamma\Lambda$*.*

Proof. Apply the remark 4.5.9 and recall that r_V is $GL(V)$-equivariant and the expression in the middle is independent of Λ. □

Theorem 4.5.39. *If*

$$\psi : \Gamma^{ab} \longrightarrow \mathrm{Hom}(\mathrm{C}^1_{\mathrm{har}}(\mathcal{K}_\Gamma, \mathbb{Z}), \mathbb{Z})$$

is an isogeny, $A(X_\Gamma) = T/\mathscr{I}(\Gamma^{ab})$ *is an analytic torus.*

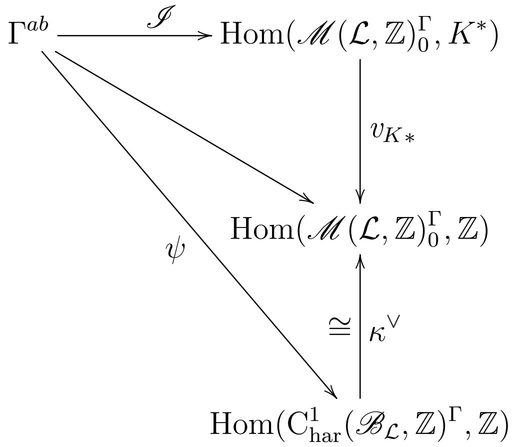
Proof. Compose the integration map

$$\mathscr{I} : \Gamma^{ab} \longrightarrow \mathrm{Hom}(\mathscr{M}(\mathcal{L}, \mathbb{Z})^\Gamma_0, \mathbb{G}_{m,K}) = T$$

with the induced by the valuation,

$$v_{K*} : \mathrm{Hom}(\mathscr{M}(\mathcal{L}, \mathbb{Z})^\Gamma_0, K^*) \longrightarrow \mathrm{Hom}(\mathscr{M}(\mathcal{L}, \mathbb{Z})^\Gamma_0, \mathbb{Z}) \cong \mathbb{Z}^g$$

(where g is the rank of Γ), so by the last proposition we get a commutative diagram



By [FvdP04, §6.4], $T/\mathscr{I}(\Gamma^{ab})$ is an analytic torus if and only if

$$v_{K*} : \mathscr{I}(\Gamma^{ab}) \longrightarrow \mathrm{Hom}(\mathscr{M}(\mathcal{L},\mathbb{Z})_0^\Gamma, \mathbb{Z}), \mathbb{Z})$$

is an isogeny, what, since κ is an isomorphism, is equivalent to that

$$v_{K*} : \mathscr{I}(\Gamma^{ab}) \longrightarrow \mathrm{Hom}(\mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}},\mathbb{Z})^\Gamma, \mathbb{Z}) \cong \mathrm{Hom}(\mathrm{C}^1_{\mathrm{har}}(\mathcal{K}_\Gamma,\mathbb{Z}),\mathbb{Z})$$

is an isogeny. But $\psi = \kappa^{-1} \circ v_{K*} \circ \mathscr{I}$ and $\mathscr{I} : \Gamma^{ab} \longrightarrow \mathscr{I}(\Gamma^{ab})$ is surjective, therefore $\mathrm{Coker}(\psi) \cong \mathrm{Coker}(v_{K*})$ and if ψ is injective, then $v_{K*|\mathscr{I}(\Gamma^{ab})}$ is also injective. $\square$

Then, to get that $A(X_\Gamma)$ is an analytic torus when $d = 2$ we just have to see that ψ is an isogeny. We will prove that ψ has finite cokernel, so that we can formulate the belief that $A(X_\Gamma)$ is an analytic torus as:

Conjecture 3. *The map ψ is injective.*

4.5.5 Towards a proof of the conjecture 2 when $d = 2$, II: the map ψ has finite cokernel

We mantain the same hypotheses and notation of the previous paragraph. First, recall that $\Gamma^{ab} \cong H_1(\mathcal{K}_\Gamma, \mathbb{Z})$ and ψ is the natural map

$$\psi : H_1(\mathcal{K}_\Gamma,\mathbb{Z}) \longrightarrow \mathrm{Hom}(\mathrm{C}^1_{\mathrm{har}}(\mathcal{K}_\Gamma,\mathbb{Z}),\mathbb{Z}) = \mathrm{C}^1_{\mathrm{har}}(\mathcal{K}_\Gamma,\mathbb{Z})^\vee$$

which maps a 1-chain to the evaluation of a harmonic cochain on it.

Second, let us rewrite the definition of the harmonic cochains on $\mathcal{K}_\Gamma$. Observe that $\mathrm{C}^1_{\mathrm{har}}(\mathcal{K}_\Gamma, \mathbb{Z}) \subset \mathbb{Z}[\widehat{\mathcal{K}_{\Gamma 1}}^{min}]^\vee$, and

$$\mathrm{C}^1_{\mathrm{har}}(\mathcal{K}_\Gamma, \mathbb{Z}) := \mathrm{Ker}(\mathrm{St}^{min}_{\mathcal{L}}) \bigcap \mathrm{Ker}(\partial^{min}) \bigcap \mathrm{Ker}(\mathbb{1}_{\widehat{\mathcal{K}_{\Gamma 1}}^{min}} - \mathrm{Flow}_{\mathcal{L}}) =$$

$$= \mathrm{Ker}\left(\mathrm{St}^{min}_{\mathcal{L}} \oplus \partial^{min} \oplus \left(\mathbb{1}_{\widehat{\mathcal{K}_{\Gamma 1}}^{min}} - \mathrm{Flow}_{\mathcal{L}}\right)\right)$$

Let us denote by $\mathscr{H} := \mathrm{St}^{min}_{\mathcal{L}} \oplus \partial^{min} \oplus \left(\mathbb{1}_{\widehat{\mathcal{K}_{\Gamma 1}}^{min}} - \mathrm{Flow}_{\mathcal{L}}\right)$ the map

$$\mathbb{Z}[\widehat{\mathcal{K}_{\Gamma 1}}^{min}]^\vee \longrightarrow \mathbb{Z}[\widehat{\mathcal{K}_{\Gamma 0}}]^\vee \oplus \mathbb{Z}[\widehat{\mathcal{K}_{\Gamma 2}}]^\vee \oplus \mathbb{Z}[\widehat{\mathcal{K}_{\Gamma 1}}^{min}]^\vee$$

defining the harmonic cochains, and observe that these modules are free and finite generated, therefore $\mathrm{C}^1_{\mathrm{har}}(\mathcal{K}_\Gamma, \mathbb{Z})$ and $\mathrm{Im}(\mathscr{H})$ are also finite generated free abelian groups. As a consequence, dualizing we get a short exact sequence

$$0 \longrightarrow \mathrm{Im}(\mathscr{H})^\vee \longrightarrow \mathbb{Z}[\widehat{\mathcal{K}_{\Gamma 1}}^{min}] \longrightarrow \mathrm{C}^1_{\mathrm{har}}(\mathcal{K}_\Gamma, \mathbb{Z})^\vee \longrightarrow 0.$$

Let us denote by η the map $\mathbb{Z}[\widehat{\mathcal{K}_{\Gamma 1}}^{min}] \longrightarrow \mathrm{C}^1_{\mathrm{har}}(\mathcal{K}_\Gamma, \mathbb{Z})^\vee$, which maps z to the linear map that evaluated on a harmonic cochain is $\eta(z) := z^*(c) = c(z)$, and note that it factorizes as the composition of two surjective maps:

$$\mathbb{Z}[\widehat{\mathcal{K}_{\Gamma 1}}^{min}] \longrightarrow \mathrm{Ker}(\mathrm{St}^{min}_{\mathcal{L}})^\vee \longrightarrow \mathrm{C}^1_{\mathrm{har}}(\mathcal{K}_\Gamma, \mathbb{Z})^\vee.$$

Observe that by means of restriction and factorization we recover ψ from η as we show in the next commutative diagram.

$$\begin{array}{ccc}
\mathbb{Z}[\widehat{\mathcal{K}_{\Gamma 1}}^{min}] & \xrightarrow{\quad \eta \quad} & \mathrm{C}^1_{\mathrm{har}}(\mathcal{K}_\Gamma, \mathbb{Z})^\vee \\
\uparrow & \nearrow^{\varphi} & \uparrow{\scriptstyle \psi} \\
\mathrm{Ker}(\partial_1) & \longrightarrow & \dfrac{\mathrm{Ker}(\partial_1)}{\mathrm{Im}(\partial^{min})} =: H_1(\mathcal{K}_\Gamma, \mathbb{Z})
\end{array}$$

In addition, because of the various surjective maps we get

$$\mathrm{Coker}(\psi) = \mathrm{Coker}(\varphi) \cong \frac{\mathbb{Z}[\widehat{\mathcal{K}_{\Gamma 1}}^{min}]}{\mathrm{Ker}(\eta) + \mathrm{Ker}(\partial_1)}.$$

Consider now the composition

$$\mathbb{Z}[\mathcal{K}_{\Gamma 0}] \xrightarrow{\mathrm{St}^{min}_{\mathcal{L}}} \mathbb{Z}[\widehat{\mathcal{K}_{\Gamma 1}}^{min}] \xrightarrow{\partial_1} \mathbb{Z}[\mathcal{K}_{\Gamma 0}]$$

Remark 4.5.40. *Recall that in the lemma 4.4.27 we have proved that the map* $\partial_1 : \mathbb{Z}[\widehat{\mathcal{K}_{\Gamma 1}}^{min}] \longrightarrow \mathbb{Z}[\mathcal{K}_{\Gamma 0}]$ *applies onto* $\mathbb{Z}[\mathbb{G}_0]_0$. *We will say that the* 1*-skeleton of* $\mathcal{K}_\Gamma$ *is a strongly connected graph.*

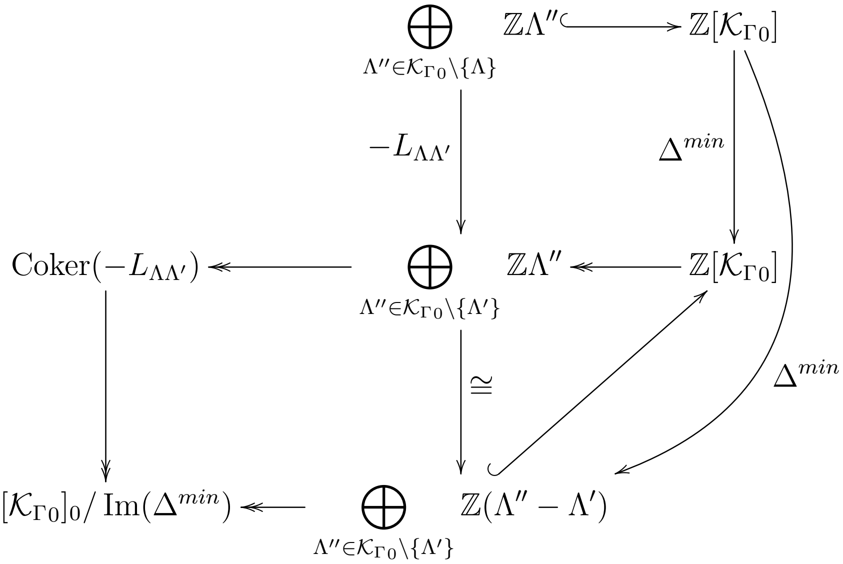
Let us compute:

$$(\partial_1 \circ \mathrm{St}^{min}_{\mathcal{L}})(\Lambda) = \partial_1 \left(\sum_{s(e)=\Lambda} e \right) = \sum_{s(e)=\Lambda} t(e) - d^+_\Lambda \Lambda$$

where d^+_Λ is the out-degree of Λ, equal to the number of edges having it as source. Let us denote this composition by Δ^{min} and call it the oriented Laplacian of $\mathcal{K}_\Gamma$. Note that $\mathrm{Im}(\Delta^{min}) \subset \mathbb{Z}[\mathcal{K}_{\Gamma 0}]_0$. Observe that its matrix in the basis of vertices is given as follows: if D^+ is the diagonal matrix having the element d^+_Λ in the row corresponding to the vertex Λ and $A^{\mathcal{K}_\Gamma} = (a^{\mathcal{K}_\Gamma})_{\Lambda\Lambda'}$ is the adjacency matrix,

$$a^{\mathcal{K}_\Gamma}_{\Lambda\Lambda'} = \begin{cases} 1 & \text{if there is an edge } e \text{ verifying } s(e) = \Lambda,\ t(e) = \Lambda', \\ 0 & \text{else,} \end{cases}$$

then, Δ^{min} is represented by $A^{\mathcal{K}_\Gamma} - D^+$. Let $L = D^+ - A^{\mathcal{K}_\Gamma}$, and let $L_{\Lambda\Lambda'}$ be the matrix result of removing the Λ-column and the Λ'-row, whose additive opposite fits in the next diagram as follows:



in the specified basis, where the isomorphism is given by the identity matrix, we are taking into account that

$$\mathbb{Z}[\mathcal{K}_{\Gamma 0}]_0 = \bigoplus_{\Lambda'' \in \mathcal{K}_{\Gamma 0} \setminus \{\Lambda'\}} \mathbb{Z}(\Lambda'' - \Lambda'),$$

and the projection from $\mathbb{Z}[\mathcal{K}_{\Gamma 0}]$ onto the submodule generated by all the vertices except Λ' is the identity matrix on $\mathbb{Z}[\mathcal{K}_{\Gamma 0} \setminus \{\Lambda'\}]$ together with the column zero for Λ'. Observe also that if $\det(L_{\Lambda\Lambda'}) \neq 0$, then

$$|\operatorname{Coker}(-L_{\Lambda\Lambda'})| = |\operatorname{Coker}(L_{\Lambda\Lambda'})| = |\det(L_{\Lambda\Lambda'})|.$$

Definition 4.5.41. *Let $G = (V, E)$ be a directed graph. A vertex $\Lambda \in V$ is called a root if for every vertex $\Lambda' \neq \Lambda$ there is an oriented path from Λ' to Λ. A graph G is called an in-tree if it is a tree and it contains a root. It is said also to be an arborescence to the root.*

The out-degree of a vertex is the number of edges whose source is that vertex. We denote it by d^+.

Remark 4.5.42. *If G is an in-tree with root Λ ("rooted at Λ"), then $d^+(\Lambda) = 0$ and $d^+(\Lambda') = 1$ for all $\Lambda' \neq \Lambda$. In fact, this is an "if and only if".*

Definition 4.5.43. *Let $G = (V, E)$ be a directed graph. An spanning in-tree or in-branching is a subgraph $T \subset G$ being an in-tree and such that $V(T) = V(G)$.*

Theorem 4.5.44 (Kirchhoff-Tutte Matrix-Tree Theorem)**.** *The number of spanning in-trees rooted at Λ is* $\det(L_{\Lambda\Lambda})$.

Corollary 4.5.45. *The abelian group $\mathbb{Z}[\mathcal{K}_{\Gamma 0}]_0/\operatorname{Im}(\Delta^{min})$ is finite.*

Proof. It is enough to show that $\operatorname{Coker}(L_{\Lambda\Lambda})$ is finite for some $\Lambda \in \mathcal{K}_{\Gamma 0}$, and so, it is also enough to see that $\det(L_{\Lambda\Lambda}) \neq 0$. Now, by the Kirchhoff-Tutte theorem this is equivalent to the existence of some spanning in-tree. Finally, since the 1-skeleton of $\mathcal{K}_\Gamma$ is a strongly connected graph, as we noted in the remark 4.5.40, we conclude the existence of an in-branching rooted at Λ for every Λ by the Edmonds branching theorem ([Edm73]). □

Next note that

$$\operatorname{Ker}(\partial_1) \bigcap \operatorname{Im}(\operatorname{St}_{\mathcal{L}}^{min}) = \operatorname{St}(\operatorname{Ker}(\Delta^{min})).$$

Theorem 4.5.46. *The map*

$$\psi : H_1(\mathcal{K}_\Gamma, \mathbb{Z}) \longrightarrow \operatorname{Hom}(\mathrm{C}^1_{\mathrm{har}}(\mathcal{K}_\Gamma, \mathbb{Z}), \mathbb{Z}) = \mathrm{C}^1_{\mathrm{har}}(\mathcal{K}_\Gamma, \mathbb{Z})^\vee$$

has finite cokernel.

Proof. Recall that we have

$$\operatorname{Coker}(\psi) = \operatorname{Coker}(\varphi) \cong \frac{\mathbb{Z}[\widehat{\mathcal{K}_{\Gamma 1}}^{min}]}{\operatorname{Ker}(\eta) + \operatorname{Ker}(\partial_1)}.$$

Moreover, the inclusion $\operatorname{Ker}(\eta) + \operatorname{Ker}(\partial_1) \supset \operatorname{Im}(\operatorname{St}_{\mathcal{L}}^{min}) + \operatorname{Ker}(\partial_1)$ induces a surjective map

$$\mathbb{Z}[\widehat{\mathcal{K}_{\Gamma 1}}^{min}]/\left(\operatorname{Im}(\operatorname{St}_{\mathcal{L}}^{min}) + \operatorname{Ker}(\partial_1)\right) \longrightarrow \mathbb{Z}[\widehat{\mathcal{K}_{\Gamma 1}}^{min}]/\left(\operatorname{Ker}(\eta) + \operatorname{Ker}(\partial_1)\right).$$

Therefore, it is enough to see that $\mathbb{Z}[\widehat{\mathcal{K}_{\Gamma 1}}^{min}]/\left(\operatorname{Im}(\operatorname{St}_{\mathcal{L}}^{min}) + \operatorname{Ker}(\partial_1)\right)$ is finite. Recall the remark 4.5.40, from which we deduce the next diagram

$$\begin{array}{ccc}
\mathbb{Z}[\widehat{\mathcal{K}_{\Gamma 1}}^{min}] & \xrightarrow{\quad\partial_1\quad} & \mathbb{Z}[\mathcal{K}_{\Gamma 0}]_0 \\
\downarrow & \nearrow_{\partial_1,\ \cong} & \downarrow \\
\mathbb{Z}[\widehat{\mathcal{K}_{\Gamma 1}}^{min}]/\operatorname{Ker}(\partial_1) & & \\
\downarrow & & \\
\mathbb{Z}[\widehat{\mathcal{K}_{\Gamma 1}}^{min}]/\left(\operatorname{Im}(\operatorname{St}_{\mathcal{L}}^{min}) + \operatorname{Ker}(\partial_1)\right) & \xrightarrow[\cong]{\partial_1} & \mathbb{Z}[\mathcal{K}_{\Gamma 0}]_0/\operatorname{Im}(\Delta^{min})
\end{array}$$

Thus, we are reduced to see that $\mathbb{Z}[\mathcal{K}_{\Gamma 0}]_0/\operatorname{Im}(\Delta^{min})$ is finite, but this is the corollary 4.5.45 $\square$

Chapter 5

Conclusions and open questions

We have developed in the chapter 3 a construction of the Jacobian of Mumford curves with all generality in a very natural way, obtaining on the way a variety of results related to them. Further, a version of that construction gives us the Jacobian of a finite metric graph, as we have shown in the chapter 2, and the same process of multiplicative integration with harmonic measures on a compact set in the ends of a building can be done in higher dimension, once we have the uniformized varieties built by Mustafin, as we did in the chapter 4.

We also know the definition of harmonic cochains and the isomorphism with harmonic measures in several cases. When $\mathcal{L}$ is a set of $d+1$ points in general position, $\mathscr{B}_{\mathcal{L}}$ is an apartment and the uniformized variety is an abelian variety; when K is local, Γ is a torsion-free, discrete, cocompact subgroup of $\mathrm{PGL}_(K)$ an $\mathcal{L} = \mathbb{P}^d(K)$; when $\mathcal{L} \subset \mathbb{P}^(K)$ is a compact set and the dimension is $d = 1, 2$.

As we have already told, when the base field is p-adic and the given uniformized variety is algebraic, Raskind and Xarles have related to the uniformized varieties what they called their p-adic intermediate Jacobians by means of their cohomology groups, therefore, by the works of Schneider, Stuhler, de Shalit and Alon, in the cocompact case they can be computed through groups of harmonic cochains or harmonic measures (actually, for general intermediate Jacobians, we should say harmonic distributions, following the terminology of de Shalit and Alon in [AdS02]).

All these developments together lead us to believe that the construction of the Jacobian of Mumford curves by using harmonic measures and multiplicative integrals, and using the isomorphism with harmonic cochains to make the proofs is generalizable to any dimension of the uniformized variety, to build any intermediate Jacobian, and over more general complete non-Archimedean fields.

In the chapter 4 we have given a conjectural construction of the Albanese variety of the Mustafin uniformized varieties as we introduced them in the proposition 4.5.29, with the little licence that we refer to the universal property of the Albanese variety in the category of abeloid varieties, and none of them have to be algebraic. Further, in dimension 2 we have reduced the fact that our construction gives an analytic torus to the injectivity of a map related to a finite simplicial complex.

Thus, several questions remain open when one wants to extend the construction of the Jacobians of Mumford curves to higher dimension.

The Albanese variety of a Mustafin uniformized variety

The first open questions which stand out are the conjectures we proposed in the final part of this thesis.

We have given a construction in any dimension of an object that we expect it is an analytic torus and it verifies the universal property of the Albanese variety in the category of abeloid varieties. In dimension 2, we have reduced the fact that it is an analytic torus to:

Conjecture 3. *The map*

$$\psi : H_1(\mathcal{K}_\Gamma, \mathbb{Z}) \longrightarrow \mathrm{Hom}(\mathrm{C}^1_{\mathrm{har}}(\mathcal{K}_\Gamma, \mathbb{Z}), \mathbb{Z})$$

is injective, where $\mathcal{K}_\Gamma = \Gamma \backslash \mathscr{B}_{\mathcal{L}_\Gamma}$.

This would have as a consequence:

Corollary 5.0.1. *Given a generalized Schottky group* $\Gamma \subset \mathrm{PGL}(V)$*, if the associated non-Archimedean uniformized variety* $X_\Gamma = \Gamma \backslash \Omega_{\mathcal{L}_\Gamma}$ *has dimension* 2*,*

$$A(X_\Gamma) := \frac{\mathrm{Hom}(\mathscr{M}(\mathcal{L}_\Gamma, \mathbb{Z})_0^\Gamma, \mathbb{G}_{m,K})^{an}}{\mathscr{I}(\Gamma^{ab})}$$

is an analytic torus.

Then we could abbreviate our second conjecture in the 2-dimensional case or, more generally, if we know that $A(X_\Gamma)$ is an analytic torus:

Conjecture 2'. *For any complete extension* $L|K$ *such that* $X_\Gamma(L) \neq \emptyset$ *and for any* $p \in X_\Gamma(L)$*, the map*

$$\begin{array}{ccc} X_{\Gamma,L} & \xrightarrow{\iota_p} & A(X_\Gamma)_L \\ q & \longmapsto & \iota_p(q) : \mu \longmapsto \displaystyle\oint_{q-p} d\mu \end{array}$$

is a morphism of analytic varieties such that $i_p(p) = 0$, and for any such other map $\varphi : X_{\Gamma,L} \longrightarrow A_L$, where A_L is an abeloid variety and $\varphi(p) = 0$, there exists a unique morphism of abeloid varieties

$$\phi : A(X_\Gamma)_L \longrightarrow A_L$$

such that $\varphi = \phi \circ \iota_p$.

If, as we told in the remark 4.5.32, it is enough to deal with analytic torus, it can be stated as follows:

Conjecture 2". *For any complete extension $L|K$ such that $X_\Gamma(L) \neq \emptyset$ and for any $p \in X_\Gamma(L)$, the map*

$$\begin{array}{rcl} X_{\Gamma,L} & \xrightarrow{\iota_p} & A(X_\Gamma)_L \\ q & \longmapsto & \iota_p(q) : \mu \longmapsto \displaystyle\times\!\!\!\!\!\!\int_{q-p} d\mu \end{array}$$

is a morphism of analytic varieties such that $i_p(p) = 0$, and for any such other map $\varphi : X_{\Gamma,L} \longrightarrow A_L$, where A_L is an analytic torus and $\varphi(p) = 0$, there exists a unique morphism of analytic torus

$$\phi : A(X_\Gamma)_L \longrightarrow A_L$$

such that $\varphi = \phi \circ \iota_p$.

Next, we reproduce this conjecture as in the previous chapter, without any assumption:

Conjecture 2. *The object $A(X_\Gamma) := T^{an}/\mathscr{I}(\Gamma)$ is a rigid analytic torus and for any complete extension $L|K$ such that $X_\Gamma(L) \neq \emptyset$ and for any $p \in X_\Gamma(L)$, the map*

$$\begin{array}{rcl} X_{\Gamma,L} & \xrightarrow{\iota_p} & A(X_\Gamma)_L \\ q & \longmapsto & \iota_p(q) : \mu \longmapsto \displaystyle\times\!\!\!\!\!\!\int_{q-p} d\mu \end{array}$$

is a morphism of analytic varieties such that $i_p(p) = 0$, and for any such other map $\varphi : X_{\Gamma,L} \longrightarrow A_L$, where A_L is an abeloid variety and $\varphi(p) = 0$, there exists a unique morphism of abeloid varieties

$$\phi : A(X_\Gamma)_L \longrightarrow A_L$$

such that $\varphi = \phi \circ \iota_p$.

Generalizing Schottky groups to any dimension and the chamber subcomplexes $\mathscr{B}_{\mathcal{L}}$ of the Bruhat-Tits building

Another conjecture we have stated refers to the definition of generalized Schottky group, to which we impose several conditions:

Conjecture 1. *Let K a complete, discrete, valued field and V a K-vector space. Let $\Gamma \subset \mathrm{PGL}(V)$ be a finitely generated, hyperbolic subgroup containing a MAH subgroup H and such that $\overline{\Gamma \cdot p}$ is compact for all $p \in \mathbb{P}(V)$. Then*

- *$\mathscr{B}_{\mathcal{L}_\Gamma}$ is a building.*
- *After a finite complete extension $L|K$, $\Gamma \backslash \mathscr{B}_{\mathcal{L}_\Gamma}$ is a finite simplicial complex and*

$$\pi_\Gamma : \mathscr{B}_{\mathcal{L}_\Gamma} \longrightarrow \Gamma \backslash \mathscr{B}_{\mathcal{L}_\Gamma}$$

  is a universal covering.

More specifically, the question should be on the appropiate conditions to generalize Schottky groups and obtaining Mustafin uniformized varieties. This author also imposes as a condition that the quotient is finite, but he takes a building by definition as a convex envelope in the Bruhat-Tits building, while we prefer to associate first a compact set $\mathcal{L}_\Gamma \subset \mathbb{P}(V)$ to the given group and then to take the associated chamber subcomplex $\mathscr{B}_{\mathcal{L}_\Gamma} \leq \mathscr{B}(G)$. From this the next question arises:

Question 1. *For which compact (or more generally, closed) sets $\mathcal{L} \subset \mathbb{P}(V)$, the chamber subcomplex $\mathscr{B}_{\mathcal{L}} \leq \mathscr{B}(G)$ is a building? Equivalently, we are asking for, that given two chambers in $\mathscr{B}_{\mathcal{L}}$ there is an apartment in $\mathscr{B}_{\mathcal{L}}$ containing both.*

It is important, among other matters, because in the 2-dimensional case we have proved the isomorphism $\mathscr{M}(\mathcal{L}, \mathbb{Z})_0 \cong \mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}}, \mathbb{Z})$ only when $\mathscr{B}_{\mathcal{L}}$ is a building or $\mathcal{L}$ is finite. Nevertheless, we only have used that $\mathscr{B}_{\mathcal{L}}$ is a building in one step in the proof of the propsition 4.4.24. For dimension 2, where we have defined harmonic cochains, we can formulate the next conjecture:

Conjecture 4. *For any compact set $\mathcal{L} \subset \mathbb{P}(V)$, the map*

$$\kappa : \mathscr{M}(\mathcal{L}, \mathbb{Z})_0 \longrightarrow \mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}}, \mathbb{Z})$$

is an isomorphism.

Another conjecture on the generalization to any dimension of Schottky groups relates to MAH subgroups as an extension to any dimension of cyclic subgroups in dimension 1.

Conjecture 5. *A MAH subgroup is a generalized Schottky group, or equivalently,* $\overline{H \cdot p}$ *is compact for all* $p \in \mathbb{P}(V)$.

Remark 5.0.2. *If* Γ *is a generalized Schottky group and* $H \subset \Gamma$ *is a MAH group, H also is a generalized Schottky group, since* $\overline{\Gamma \cdot p}$ *is compact for any* $p \in \mathbb{P}(V)$ *and* $\overline{H \cdot p} \subset \overline{\Gamma \cdot p}$ *is also compact as a closed subset of a compact set.*

Question 2. *The previous conjecture is related to imposing the condition that the closure of the orbit of any point is compact. We use it mainly to prove that* $\mathcal{L}_\Gamma$ *is compact, and therefore, that* $\mathscr{B}_{\mathcal{L}}$ *is locally finite; further, Mustafin does not impose any other similar condition but the finiteness of the quotient. Nevertheless, in dimension* 1 *we have used it to extend the set of limit points to other compact sets in order to make some proofs.*

Is this condicion necessary? Is it enough to impose a weaker condition (at least, $\mathscr{B}_{\mathcal{L}}$ *locally finite) to develope a suitable theory?*

Another question on these groups uniformizing Mustafin varieties is on the quotient finite simplicial complex?

Question 3. *Which conditions characterize the finite simplicial complexes* $\mathcal{K}$ *realizing as the degeneration complexes of Mustafin uniformized varieties, so that they can be expressed as* $\mathcal{K} = \Gamma \backslash \mathscr{B}_{\mathcal{L}_\Gamma}$ *over a suitable complete extension of* K*?*

In dimension 1 we know there is no condition, but it does not seem to be the general case. Moreover, in dimension 1 we have shown the construction of the Jacobian of a graph. Then a related question appears:

Question 4. *How can be defined the Albanese variety of a finite simplicial complex, maybe under some assumptions as those which answer the previous question? In which categories should we work to get these definitions?*

Constructions in the Berkovich setting and tropical geometry in any dimension

As in dimension 1 the base fields are not necessarily discrete, the quotients $\Gamma \backslash \mathscr{B}_{\mathcal{L}_\Gamma}$ are finite metric graphs, and thus, tropical curves. Then we can ask:

Question 5. *Is there in higher dimension an equivalence between certain metric objects including the finite simplicial complexes of the form* $\Gamma\backslash\mathscr{B}_{\mathcal{L}_\Gamma}$ *and some kind of tropical varieties as in dimension* 1*?*

This would be more useful if the next question has an affirmative answer:

Question 6. *Can Mustafin uniformization be extended to varieties over any complete non-Archimedean field (without the discreteness condition), maybe by means of the Berkovich analytic geometry?*

On the side of Bruhat-Tits buildings in this setting, which have been recently studied in [RTW10], [RTW12] and [RTW15], the next question arises:

Question 7. *How to give them a suitable combinatorial structure? How to describe the types of cells and, in particular, the minimal cells?*

On the extension to any dimension of the properties for dimension 2

The isomorphism that we have proved in dimension 2 between harmonic cochains and harmonic measures rests in properties and constructions that we could make in dimension 2, but not in general. Further, through the section 4.3 we could prove other results. Are these generalizable to any dimension?

For example, we could prove the bijection between the ends of $\mathscr{B}_{\mathcal{L}}$ as defined by us and the points of $\mathcal{L}$, a result that we told to expect for $\mathcal{L} = \mathbb{P}(V)$ in the remark 4.2.34:

Conjecture 6. *The map* $\widetilde{\varepsilon} : \mathcal{E}_{\mathcal{L}} \longrightarrow \mathcal{L}$ *is a bijection.*

Other results generalize to any dimension under the assumption that $\mathscr{B}_{\mathcal{L}}$ is a building because of the convexity condition, as explained for a particular proposition in the remark 4.3.6.

Later, we define a certain chamber-convex hull and we told in the remark 4.3.22 to expect that it is equivalent to the given in [AB08]. Further, our interest is in a result of the kind of the proposition 4.3.24. so that by the commented in the remark 4.3.25 different definitions could be given. One can ask which is better to generalize to higher dimension.

On more general harmonic cochains, harmonic distributions and intermediate Jacobians

Next, we ask again on the harmonic cochains:

Question 8. *Given a compact set $\mathcal{L} \subset \mathbb{P}(V)$ on any dimension, how harmonic cochains on the minimal edges of $\mathscr{B}_{\mathcal{L}}$ should be defined?*

Once we have this definition, the known isomorphisms in dimensions 1 and 2, and in the abelian and local cocompact cases lead us to the conjecture:

Conjecture 7. *For any compact set $\mathcal{L} \subset \mathbb{P}(V)$, the map*

$$\kappa : \mathscr{M}(\mathcal{L}, \mathbb{Z})_0 \longrightarrow \mathrm{C}^1_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}}, \mathbb{Z})$$

is an isomorphism (maybe, under the assumption that $\mathscr{B}_{\mathcal{L}}$ is a building).

Actually, one can expect that a definition of harmonic cochains such that κ is an isomorphism for finite sets $\mathcal{L} \subset \mathbb{P}(V)$ would be good in the sense that the previous conjecture would be satisfied (since, as we did in dimension 2, the proof could be done by means of a restriction to local isomorphisms which could be seen for finite sets $\mathcal{L}$).

A problem beyond that we do not know how to define harmonic cochains, is that our given proofs depend strongly on the dimension, so that they are not generalizable.

More widely, we can ask:

Question 9. *Given a compact set $\mathcal{L} \subset \mathbb{P}(V)$ on any dimension d and given $1 \leq q \leq d$, how harmonic cochains on the minimal q-cells of $\mathscr{B}_{\mathcal{L}}$ should be defined?*

We think the answer to these questions can be approached in the same way that Schneider and Stuhler do in [SS91, §4], which is also described in [dS01, §8.3], after comparing the combinatorial and the group theoretical presentations of the buildings.

Let us denote the conjectural group of A-valued harmonic cochains on the minimal q-cells of $\mathscr{B}_{\mathcal{L}}$ by $\mathrm{C}^{\mathrm{q}}_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}}, A)$. In [AdS02], given a certain space of flags $\mathcal{B}^{min}_{q-1}$ which corresponds to the total Bruhat-Tits building in the local, cocompact case, the authors define a certain space of harmonic distributions $\mathcal{D}(\mathcal{B}^{min}_{q-1}, K)_{har}$ whose harmonicity condition only coincides with that we defined in chapter 2 for $q = 1$. In this case they show $\mathcal{D}(\mathcal{B}^{min}_{q-1}, K)_{har} \cong \mathrm{C}^{\mathrm{q}}_{\mathrm{har}}(\mathscr{B}(G), K)$. Now, given a compact set $\mathcal{L} \subset \mathbb{P}(V)$ such that $\mathscr{B}_{\mathcal{L}}$ is a building over a complete field with a discrete valuation, assume we can associate to it a certain subset $\mathcal{L}_q \subset \mathcal{B}^{min}_{q-1}$, also in a certain set of ends of the building (bigger than as we have defined it). Then we could extend the conjecture on harmonic cochains on the minimal edges to:

Conjecture 8. *Given a compact set $\mathcal{L} \subset \mathbb{P}(V)$ such that $\mathscr{B}_{\mathcal{L}}$ is a building, there is an isomorphism $\mathcal{D}(\mathcal{L}_q, \mathbb{Z})_{har} \cong \mathrm{C}^{\mathrm{q}}_{\mathrm{har}}(\mathscr{B}_{\mathcal{L}}, \mathbb{Z})$.*

These objects describe certain cohomology groups which give other intermediate Jacobians beyond the Albanese varieties. Under the assumption that given $\mathcal{L}$ we know what is $\mathcal{L}_q$, we could try to make a similar construction to that we do for the Albanese variety. But a first question appears:

Question 10. *How to build functions on $\mathcal{L}_q$ on which apply multiplicative integrals?*

Analytic constructions for other varieties with totally degenerate reduction

Given a product of Mumford curves, it is also a variety with totally degenerate reduction. The final and more general question which stands out is how to build analytically the intermediate Jacobians defined by Raskind and Xarles for any variety with totally degenerate reduction or for Berkovich analytic generalizations.

Bibliography

[AB08] Peter Abramenko and Kenneth S. Brown, *Buildings. Theory and applications.*, Graduate Texts in Mathematics, vol. 248, Springer, Berlin-Heideber-New York, 2008 (English). MR 2439729

[AdS02] Gil Alon and Ehud de Shalit, *On the cohomology of Drinfel′d's p-adic symmetric domain*, Israel J. Math. **129** (2002), 1–20 (English). MR 2003i:14019

[AdS03] ———, *Cohomology of discrete groups in harmonic cochains on buildings*, Israel J. Math. **135** (2003), 355–380 (English). MR 2005b:14037

[AVM10] Peter Abramenko and Hendrik Van Maldeghem, *Intersections of apartments*, J. Combin. Theory Ser. A **117** (2010), no. 4, 440–453 (English). MR 2592893

[BdlHN97] Roland Bacher, Pierre de la Harpe, and Tatiana Nagnibeda, *The lattice of integral flows and the lattice of integral cuts on a finite graph*, Bull. Soc. Math. France **125** (1997), no. 2, 167–198 (English). MR 99c:05111

[Bak08] Matthew Baker, *An introduction to Berkovich analytic spaces and non-Archimedean potential theory on curves*, p-adic geometry. Lectures from the 2007 10th Arizona winter school, Tucson, AZ, USA, March 10–14, 2007, Univ. Lecture Ser., vol. 45, Amer. Math. Soc., Providence, RI, 2008, pp. 123–174 (English). MR 2010g:14029

[BF11] Matthew Baker and Xander Faber, *Metric properties of the tropical Abel-Jacobi map*, J. Algebraic Combin. **33** (2011), no. 3, 349–381 (English). MR 2012c:14124

[BN07] Matthew Baker and Serguei Norine, *Riemann-Roch and Abel-Jacobi theory on a finite graph*, Adv. Math. **215** (2007), no. 2, 766–788 (English). MR 2008m:05167

[BN09] ______, *Harmonic morphisms and hyperelliptic graphs*, Int. Math. Res. Not. IMRN (2009), no. 15, 2914–2955 (English). MR 2010e:14031

[BPR13] Matthew Baker, Sam Payne, and Joseph Rabinoff, *On the structure of non-Archimedean analytic curves*, Tropical and non-Archimedean geometry. Bellairs workshop in number theory, tropical and non-Archimedean geometry, Bellairs Research Institute, Holetown, Barbados, USA, May 6–13, 2011, Contemp. Math., vol. 605, Amer. Math. Soc., Providence, RI, 2013, pp. 93–121 (English). MR 3204269

[BPR16] ______, *Nonarchimedean geometry, tropicalization, and metrics on curves*, Algebr. Geom. **3** (2016), no. 1, 63–105 (English). MR 3455421

[BR15] Matthew Baker and Joseph Rabinoff, *The skeleton of the Jacobian, the Jacobian of the skeleton, and lifting meromorphic functions from tropical to algebraic curves*, Int. Math. Res. Not. IMRN (2015), no. 16, 7436–7472 (English). MR 3428970

[Ber90] Vladimir G. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, Mathematical Surveys and Monographs, vol. 33, American Mathematical Society, Providence, RI, 1990 (English). MR 91k:32038

[BDG04] Massimo Bertolini, Henri Darmon, and Peter Green, *Periods and points attached to quadratic algebras*, Heegner points and Rankin L-series. Papers from the workshop on special values of Rankin L-series, Berkeley, CA, USA, December 2001, Math. Sci. Res. Inst. Publ., vol. 49, Cambridge Univ. Press, Cambridge, 2004, pp. 323–367 (English). MR 2005e:11062

[BGS97] S. Bloch, H. Gillet, and C. Soulé, *Algebraic cycles on degenerate fibers*, Arithmetic geometry (Cortona, 1994) (Cambridge), Sympos. Math., XXXVII, Cambridge Univ. Press, 1997, pp. 45–69 (English). MR 98i:14011

[Bou02] Nicolas Bourbaki, *Lie groups and Lie algebras. Chapters 4–6*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2002

(English), Translated from the 1968 French original by Andrew Pressley. MR 1890629

[BMV11] Silvia Brannetti, Margarida Melo, and Filippo Viviani, *On the tropical Torelli map*, Adv. Math. **226** (2011), no. 3, 2546–2586 (English). MR 2012e:14121

[BT72] François Bruhat and Jacques Tits, *Groupes réductifs sur un corps local: I. Données radicielles valuées*, Publ. Math. IHÉS **41** (1972), no. 1, 5–251 (French). MR 0327923

[CV10] Lucia Caporaso and Filippo Viviani, *Torelli theorem for graphs and tropical curves*, Duke Math. J. **153** (2010), no. 1, 129–171 (English). MR 2011j:14013

[CA14] Eduardo Casas-Alvero, *Analytic projective geometry*, EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich, 2014 (English). MR 3236321

[Cha12] Melody Chan, *Combinatorics of the tropical Torelli map*, Algebra Number Theory **6** (2012), no. 6, 1133–1169 (English). MR 2968636

[CF67] George E. Cooke and Ross L. Finney, *Homology of cell complexes (Based on lectures by Norman E. Steenrod)*, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1967 (English). MR 0219059

[Dar01] Henri Darmon, *Integration on $\mathcal{H}_p \times \mathcal{H}$ and arithmetic applications*, Ann. of Math. (2) **154** (2001), no. 3, 589–639 (English). MR 2003j:11067

[Dar06] ———, *Heegner points, Stark-Heegner points, and values of L-series*, Proceedings of the International Congress of Mathematicians (ICM), Madrid, Spain, August 22–30, 2006. Volume II: Invited lectures, European Mathematical Society (EMS), Zürich, 2006, pp. 313–345 (English). MR 2275600

[Das04] Samit Dasgupta, *Gross-Stark units, Stark-Heegner points, and class fields of real quadratic fields*, Ph.D. thesis, University of California, Berkeley, 2004.

[Das05] Samit Dasgupta, *Stark-Heegner points on modular Jacobians*, Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 3, 427–469 (English). MR 2006e:11080

[dS01] Ehud de Shalit, *Residues on buildings and de Rham cohomology of p-adic symmetric domains*, Duke Math. J. **106** (2001), no. 1, 123–191 (English). MR 2002a:14019

[dS05] ______, *The p-adic monodromy-weight conjecture for p-adically uniformized varieties*, Compositio Math. **141** (2005), no. 1, 101–120 (English). MR 2005h:14049

[Dri74] V.G. Drinfel'd, *Elliptic modules.*, Math. USSR, Sb. **23** (1974), no. 4, 561–592 (English. Russian original).

[Edm73] Jack Edmonds, *Edge-disjoint branchings*, Combinatorial algorithms (Courant Comput. Sci. Sympos. 9, New York Univ., New York, 1972), Algorithmics Press, New York, 1973, pp. 91–96. MR 0351889

[FvdP81] Jean Fresnel and Marius van der Put, *Géométrie analytique rigide et applications*, Progress in Mathematics, vol. 18, Birkhäuser Boston Inc., Boston, Mass., 1981 (French). MR 644799 (83g:32001)

[FvdP04] ______, *Rigid analytic geometry and its applications*, Progress in Mathematics, vol. 218, Birkhäuser Boston Inc., Boston, MA, 2004 (English). MR 2004i:14023

[Gar73] Howard Garland, *p-adic curvature and the cohomology of discrete subgroups of p-adic groups*, Ann. of Math. (2) **97** (1973), 375–423 (English). MR 47:8719

[Gar97] Paul Garrett, *Buildings and classical groups*, Chapman & Hall, London, 1997 (English). MR 1449872

[GR96] E.-U. Gekeler and M. Reversat, *Jacobians of Drinfeld modular curves*, J. Reine Angew. Math. **476** (1996), 27–93 (English). MR 97f:11043

[Ger72] L. Gerritzen, *On non-Archimedean representations of abelian varieties*, Math. Ann. **196** (1972), 323–346 (English). MR 0308132

[GvdP80] Lothar Gerritzen and Marius van der Put, *Schottky groups and Mumford curves*, Lecture Notes in Mathematics, vol. 817, Springer, Berlin, 1980 (English). MR 82j:10053

[GI63] O. Goldman and N. Iwahori, *The space of p-adic norms*, Acta Math. **109** (1963), 137–177 (English). MR 0144889

[GK05] Elmar Grosse-Klönne, *Acyclic coefficient systems on buildings*, Compositio Math. **141** (2005), no. 3, 769–786 (English). MR 2005m:20071

[GK11] ______, *On the p-adic cohomology of some p-adically uniformized varieties*, J. Algebraic Geom. **20** (2011), no. 1, 151–198 (English). MR 2012b:14037

[Gub07] Walter Gubler, *Tropical varieties for non-Archimedean analytic spaces*, Invent. Math. **169** (2007), no. 2, 321–376 (English). MR 2318559

[Gub10] ______, *Non-Archimedean canonical measures on abelian varieties*, Compos. Math. **146** (2010), no. 3, 683–730 (English). MR 2644932

[Har01] Urs T. Hartl, *Semi-stability and base change*, Arch. Math. (Basel) **77** (2001), no. 3, 215–221 (English). MR 2002h:14021

[Inf06] Carlos A. Infante, *Ciclos algebraicos y reducción semiestable*, Ph.D. thesis, Universitat Autònoma de Barcelona, 2006.

[Ito05] Tetsushi Ito, *Weight-monodromy conjecture for p-adically uniformized varieties*, Invent. Math. **159** (2005), no. 3, 607–656 (English). MR 2005m:14033

[Kap42] Irving Kaplansky, *Maximal fields with valuations*, Duke Math. J. **9** (1942), 303–321 (English). MR 0006161

[KS00] Motoko Kotani and Toshikazu Sunada, *Jacobian tori associated with a finite graph and its abelian covering graphs*, Adv. in Appl. Math. **24** (2000), no. 2, 89 – 110 (English). MR 2002d:14068

[KS08] ______, *Graph-theoretic albanese maps revisited*, Tech. report, Meiji Institute for Advanced Study of Mathematical Sciences, 2008.

[Kru32] Wolfgang Krull, *Allgemeine Bewertungstheorie*, J. Reine Angew. Math. **167** (1932), 160–196 (German). MR 1581334

[Kün98] Klaus Künnemann, *Projective regular models for abelian varieties, semistable reduction, and the height pairing*, Duke Math. J. **95** (1998), no. 1, 161–212 (English). MR 99m:14043

[Lon02] Ignazio Longhi, *Non-Archimedean integration and elliptic curves over function fields*, J. Number Theory **94** (2002), no. 2, 375–404 (English). MR 2003h:11051

[Lor89] Dino J. Lorenzini, *Arithmetical graphs*, Math. Ann. **285** (1989), no. 3, 481–501 (English). MR 91b:14026

[Lor91] ______, *A finite group attached to the Laplacian of a graph*, Discrete Math. **91** (1991), no. 3, 277–282 (English). MR 93a:05091

[Lüt95] Werner Lütkebohmert, *The structure of proper rigid groups*, J. Reine Angew. Math. **468** (1995), 167–219 (English). MR 1361790

[Lüt09] W. Lütkebohmert, *From Tate's elliptic curve to abeloid varieties*, Pure Appl. Math. Q. **5** (2009), no. 4, 1385–1427 (English), Special Issue: In honor of John Tate. Part 1. MR 2560320

[MD73] Yu. Manin and V. Drinfeld, *Periods of p-adic Schottky groups*, J. Reine Angew. Math. **262/263** (1973), 239–247 (English), Collection of articles dedicated to Helmut Hasse on his seventy-fifth birthday. MR 53:444

[MZ08] Grigory Mikhalkin and Ilia Zharkov, *Tropical curves, their Jacobians and theta functions*, Curves and abelian varieties. Proceedings of the international conference, Athens, GA, USA, March 30–April 2, 2007, Contemp. Math., vol. 465, Amer. Math. Soc., Providence, RI, 2008, pp. 203–230 (English). MR 2011c:14163

[Mum72a] ______, *An analytic construction of degenerating curves over complete local rings*, Compositio Math. **24** (1972), no. 2, 129–174 (English). MR 50:4592

[Mum72b] David Mumford, *An analytic construction of degenerating abelian varieties over complete rings*, Compositio Math. **24** (1972), no. 3, 239–272 (English). MR 50:4593

[Mus78] G. A. Mustafin, *Nonarchimedean uniformization.*, Math. USSR, Sb. **34** (1978), no. 2, 187–214 (English. Russian original), English translation of: Mat. Sb. (N.Ser) 105(147) (1978), no. 2, 207–237 (1978).

[Par00] Anne Parreau, *Dégénérescences de sous-groupes discrets de groupes de lie semisimples et actions de groupes sur des immeubles affines.*, Ph.D. thesis, Université de Paris-Sud U.F.R. scientifique d'Orsay, January 2000.

[Ras05] Wayne Raskind, *A generalized Hodge-Tate conjecture for algebraic varieties with totally degenerate reduction over p-adic fields*, Algebra and number theory. Proceedings of the silver jubilee conference, Hyderabad, India, December 11–16, 2003 (Delhi) (Rajat Tandon, ed.), Hindustan Book Agency, 2005, pp. 99–115 (English). MR 2006k:14037

[RX07a] Wayne Raskind and Xavier Xarles, *On the étale cohomology of algebraic varieties with totally degenerate reduction over p-adic fields*, J. Math. Sci. Univ. Tokyo **14** (2007), no. 2, 261–291 (English). MR 2009h:14034

[RX07b] ______, *On p-adic intermediate Jacobians*, Trans. Amer. Math. Soc. **359** (2007), no. 12, 6057–6077 (electronic) (English). MR 2008i:14071

[RTW10] Bertrand Rémy, Amaury Thuillier, and Annette Werner, *Bruhat-Tits theory from Berkovich's point of view. I. Realizations and compactifications of buildings*, Ann. Sci. Éc. Norm. Supér. (4) **43** (2010), no. 3, 461–554 (English). MR 2011j:20075

[RTW12] ______, *Bruhat-Tits theory from Berkovich's point of view. II Satake compactifications of buildings*, J. Inst. Math. Jussieu **11** (2012), no. 2, 421–465 (English). MR 2905310

[RTW15] ______, *Bruhat-tits buildings and analytic geometry*, Berkovich Spaces and Applications (Antoine Ducros, Charles Favre, and Johannes Nicaise, eds.), Lecture Notes in Mathematics, vol. 2119, Springer International Publishing, 2015, pp. 141–202 (English). MR 3330766

[Rou09] Guy Rousseau, *Euclidean buildings.*, Géométries à courbure négative ou nulle, groupes discrets et rigidités, Séminaires et Congrès, vol. 18, Paris: Société Mathématique de France (SMF), 2009, pp. 77–116 (English). MR 2655310

[SX16] Dani Samaniego and Xavier Xarles, *Schottky groups over valuation rings.*

[SS91] P. Schneider and U. Stuhler, *The cohomology of p-adic symmetric spaces*, Invent. Math. **105** (1991), no. 1, 47–122 (English). MR 92k:11057

[Ser80] Jean-Pierre Serre, *Trees*, Springer-Verlag, Berlin-New York, 1980 (English), Translated from the French by John Stillwell. MR 82c:20083

[Tem15] Michael Temkin, *Introduction to Berkovich analytic spaces*, Berkovich spaces and applications. Based on a workshop, Santiago de Chile, Chile, January 2008 and a summer school, Paris, France, June 2010, Lecture Notes in Math., vol. 2119, Springer, Cham, 2015, pp. 3–66 (English). MR 3330762

[Thu05] Amaury Thuillier, *Potential theory on curves in non-archimedean geometry. applications to arakelov theory.*, Ph.D. thesis, Université Rennes 1, October 2005, Laurent Moret-Bailly (Président) Jean-Benoît Bost (Rapporteur) Robert Rumely (Rapporteur) Antoine Chambert-Loir (Directeur de thèse) Antoine Ducros (Examinateur) Charles Favre (Examinateur).

[Val03] Frank Vallentin, *Sphere coverings, lattices and tilings (in low dimensions)*, Ph.D. thesis, Technische Universität München, 2003.

[vdP92] Marius van der Put, *Discrete groups, Mumford curves and theta functions*, Ann. Fac. Sci. Toulouse Math. (6) **1** (1992), no. 3, 399–438 (English). MR 94h:14021

[Var98] Yakov Varshavsky, *p-adic uniformization of unitary Shimura varieties*, Inst. Hautes Études Sci. Publ. Math. (1998), no. 87, 57–119 (English). MR 99m:14045

[Wer01] Annette Werner, *Compactification of the Bruhat-Tits building of PGL by lattices of smaller rank*, Doc. Math. **6** (2001), 315–341 (English). MR 2002j:20060

[Wer04] ———, *Compactification of the Bruhat-Tits building of PGL by seminorms*, Math. Z. **248** (2004), no. 3, 511–526 (English). MR 2005k:20075

[Wil11] Rolf Stefan Wilke, *Line bundles on totally degenerated formal schemes*, Int. Math. Res. Not. IMRN (2011), no. 22, 5019–5044. MR 2854721