ON SEPARATION OF A DEGENERATE DIFFERENTIAL OPERATOR IN HILBERT SPACE

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Abstract. A coercive estimate for a solution of a degenerate second order differential equation is installed, and its applications to spectral problems for the corresponding differential operator is demonstrated. The sufficient conditions for existence of the solutions of one class of the nonlinear second order differential equations on the real axis are obtained.

1. Introduction and main results

The concept of a separability was introduced in the fundamental paper [1]. The Sturm-Liouville's operator

$$Jy = -y'' + q(x)y, \quad x \in (a, +\infty),$$

is called separable [1] in space $L_2(a, +\infty)$, if $y, -y'' + qy \in L_2(a, +\infty)$ imply $-y'', qy \in L_2(a, +\infty)$. The separability of the operator J is equivalent to the following inequality

$$\|y''\|_{L_2(a,+\infty)} + \|qy\|_{L_2(a,+\infty)} \leq c\left(\|Jy\|_{L_2(a,+\infty)} + \|y\|_{L_2(a,+\infty)}\right), \quad y \in D(J). \quad (1.1)$$

In [1] (see also [2, 3]) for J some criteria of the separability depended on the behavior q and its derivatives are received, and an examples of not separable J with non-smooth potential q is shown. When q isn't necessarily differentiable function the sufficient separabilities conditions of J is obtained in [4, 5]. In [6,7] it was developed so-called "the localization principle" of proof of the separability of higher order binomial elliptic operators in Hilbert space. In [8,9] it was shown that the local integrability and the semi-boundedness from below of q are sufficient for separability of J in space $L_1(-\infty, +\infty)$. The valuation method of Green's functions [1-3,8,9] (see also [10]), a parametrix method [4,5], as well as a method of local estimates of the resolvents of some regular operators [6, 7] have been used in these works.

The sufficient conditions of the separability for the Sturm-Liouville's operator

$$y'' + Q(x)y$$

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are obtained in [11-15] where Q is an operator. There are a number of works where a separation of the general elliptic, hyperbolic, and mixed-type operators is discussed.

The separability estimate (1.1) is used in the spectral theory of J [15-18], and it allows us to prove an existence and a smoothness of solutions of one class of nonlinear differential equations in unbounded domains [11, 17-20]. Brown [21] has shown that certain properties of positive solutions of disconjugate second order differential expressions imply the separation. The connection of separation with the concrete physical problems is noted in [22].

The main aim of this paper is to study the separation and approximate properties for the differential operator

$$ly := -y'' + r(x)y' + q(x)y$$

in Hilbert space $L_2 := L_2(R),\ R = (-\infty, +\infty)$, as well as the existence problem for certain nonlinear differential equation in L_2. The operator l is said to be separable in space L_2, if the following estimate holds:

$$\|y''\|_2 + \|ry'\|_2 + \|qy\|_2 \le c\left(\|ly\|_2 + \|y\|_2\right), \quad y \in D(l), \tag{1.2}$$

where $\|\cdot\|_2$ is the norm in L_2.

We assume that the function r is positive and increases at infinity faster than $|q|$. The operator l occurs in the oscillatory processes in a medium with a resistance that depends on velocity [23] (page 111-116). The operator J same as the operator l when $r = 0$. Nevertheless, note that the sufficient conditions for the invertibility, respectively, of l and of J, are principally different from each other. The separability estimate for l can not be obtained by applying of results of the works [1-15].

We denote

$$\alpha_{g,h}(t) = \|g\|_{L_2(0,t)} \left\|h^{-1}\right\|_{L_2(t,+\infty)} (t > 0),\ \ \beta_{g,h}(\tau) = \|g\|_{L_2(\tau,0)} \left\|h^{-1}\right\|_{L_2(-\infty,\tau)} (\tau < 0)$$

$$\gamma_{g,h} = max\left(\sup_{t>0} \alpha_{g,h}(t), \sup_{\tau<0} \beta_{g,h}(\tau)\right),$$

where g and h are given functions. By $C_{loc}^{(1)}(R)$ we denote the set of functions f such that $\psi f \in C^{(1)}(R)$ for all $\psi \in C_0^{\infty}(R)$.

Theorem 1. *Let the function r satisfy the conditions*

$$r \in C_{loc}^{(1)}(R), \quad r \ge \delta > 0, \quad \gamma_{1,r} < \infty, \tag{1.3}$$

$$c^{-1} \le \frac{r(x)}{r(\eta)} \le c \ \ at \ \ |x - \eta| \le 1, \quad c > 1, \tag{1.4}$$

and the function q such that

$$\gamma_{q,r} < +\infty. \tag{1.5}$$

Then for $y \in D(l)$ the estimate

$$\|y''\|_2 + \|ry'\|_2 + \|qy\|_2 \leq c_l \, \|ly\|_2 \tag{1.6}$$

holds, in particular, the operator l is separable in L_2.

The following Theorems 2-4 are applications of Theorem 1.

Theorem 2. *Let functions q, r satisfy the conditions (1.3)-(1.5) and the equalities $\lim\limits_{t \to +\infty} \alpha_{q,r}(t) = 0$, $\lim\limits_{\tau \to -\infty} \beta_{q,r}(\tau) = 0$ hold. Then an inverse operator l^{-1} is completely continuous in L_2.*

We assume that the conditions of Theorem 2 hold, and consider a set

$$M = \left\{ y \in L_2 : \|ly\|_2 \leq 1 \right\}.$$

Let

$$d_k = \inf_{\Sigma_k \subset \{\Sigma_k\}} \sup_{y \in M} \inf_{w \in \Sigma_k} \|y - w\|_2 \, (k = 0, 1, 2, ...)$$

be the Kolmogorov's widths of the set M in L_2. Here $\{\Sigma_k\}$ is a set of all subspaces Σ_k of L_2 whose dimensions are not more than k. Through $N_2(\lambda)$ denote the number of widths d_k which are not smaller than a given positive number λ. Estimates of the width's distribution function $N_2(\lambda)$ are important in the approximating problem of solutions of the equation $ly = f$. The following statement holds.

Theorem 3. *Let the conditions of Theorem 2 be fulfilled. Then the following estimates hold:*

$$c_1 \lambda^{-2} \mu \left\{ x : |q(x)| \leq c_2^{-1} \lambda^{-1} \right\} \leq N_2(\lambda) \leq c_3 \lambda^{-2} \mu \left\{ x : |q(x)| \leq c_2 \lambda^{-1} \right\}.$$

Example. Let $q = -x^\alpha \ (\alpha \geq 0)$, $\ r = (1+x^2)^\beta \ (\beta > 0)$. Then the conditions of Theorem 1 are satisfied if $\beta \geq \frac{1+\alpha}{2}$. If $\beta > \frac{1+\alpha}{2}$, then the conditions of Theorem 3 are satisfied and for some $\epsilon > 0$ the following estimates hold:

$$c_0 \lambda^{\frac{-7-2\beta+\epsilon}{4}} \leq N_2(\lambda) \leq c_1 \lambda^{\frac{-7-2\beta+\epsilon}{4}}.$$

Consider the following nonlinear equation

$$Ly = -y'' + [r(x, y)]y' = f(x), \tag{1.7}$$

where $x \in R$, r is real-valued function and $f \in L_2$.

Definition 1. *A function $y \in L_2$ is called a solution of (1.7), if there is a sequence of twice continuously differentiable functions $\{y_n\}_{n=1}^{\infty}$ such that $\|\theta(y_n - y)\|_2 \to 0$, $\|\theta(Ly_n - f)\|_2 \to 0$ as $n \to \infty$ for any $\theta \in C_0^\infty(R)$.*

Theorem 4. *Let the function r be continuously differentiable with respect to both arguments and satisfies the following conditions*

$$r \geq \delta_0(1 + x^2) \quad (\delta_0 > 0), \quad \sup_{|x-y| \leq 1} \ \sup_{|C_1| \leq A, |C_2| \leq A, |C_1 - C_2| \leq A} \frac{r(x, C_1)}{r(\eta, C_2)} < \infty. \tag{1.8}$$

Then there is a solution y of the equation (1.7), and

$$\|y''\|_2 + \|[r(\cdot, y)]y'\|_2 < \infty. \tag{1.9}$$

2. Auxiliary statements

The next statement is a corollary of the well known Muckenhoupt's inequality [25].

Lemma 2.1. *Let the functions g, h such that $\gamma_{g,h} < \infty$. Then for $y \in C_0^\infty(R)$ the following inequality holds:*

$$\int_{-\infty}^{\infty} |g(x)y(x)|^2 dx \le C \int_{-\infty}^{\infty} |h(x)y'(x)|^2 dx. \tag{2.1}$$

Moreover, if C is a smallest constant for which the estimate (2.1) holds, then $\gamma_{g,h} \le C \le 2\gamma_{g,h}$.

The following lemma is a special case of Theorem 2.2 [26].

Lemma 2.2. *Let the given function r satisfies conditions*

$$\lim_{x\to+\infty} \sqrt{x} \left\|r^{-1}\right\|_{L_2(x,+\infty)} = \lim_{x\to+\infty} \sqrt{x} \left(\int_x^\infty r^{-2}(t)dt \right)^{\frac{1}{2}} = 0,$$

$$\lim_{x\to-\infty} \sqrt{|x|} \left\|r^{-1}\right\|_{L_2(-\infty,x)} = \lim_{x\to-\infty} \sqrt{|x|} \left(\int_{-\infty}^x r^{-2}(t)dt \right)^{\frac{1}{2}} = 0. \tag{2.2}$$

Then the set

$$F_k = \left\{ y : y \in C_0^\infty(R), \int_{-\infty}^{+\infty} |r(t)y'(t)|^2 dt \le K \right\}, \quad K > 0,$$

is a relatively compact in $L_2(R)$.

Denote by $\mathscr{L}$ a closure in L_2-norm of the differential expression

$$\mathscr{L}_0 z = -z' + rz \tag{2.3}$$

defined on the set $C_0^\infty(R)$.

Lemma 2.3. *Let the function r satisfies conditions (1.3) and (1.4). Then the operator $\mathscr{L}$ is boundedly invertible and separable in L_2. Moreover, for $z \in D(\mathscr{L})$ the following estimate holds:*

$$\|z'\|_2 + \|rz\|_2 \le c\, \|\mathscr{L}z\|_2 . \tag{2.4}$$

Proof. Let $\mathscr{L}_\lambda = \mathscr{L} + \lambda E, \ \ \lambda \geq 0$. Note that the operators $\mathscr{L}$ and $\mathscr{L}_\lambda = \mathscr{L} + \lambda E$ are separated to one and the same time. If z is a continuously differentiable function with the compact support, then

$$(\mathscr{L}_\lambda z, z) = -\int\limits_R z'\bar{z}dx + \int\limits_R [r(x) + \lambda]|z|^2 dx. \tag{2.5}$$

But

$$T := -\int\limits_R z'\bar{z}dx = \int\limits_R z\bar{z}'dx = -\bar{T}.$$

Therefore $ReT = 0$ and it follows from (2.5)

$$Re(\mathscr{L}_\lambda z, z) = \int\limits_R [r(x) + \lambda]|z|^2 dx. \tag{2.6}$$

We estimate the left-hand side of the equality (2.6) by using the Holder's inequality. Then we have

$$\left\| \sqrt{r(\cdot) + \lambda} z \right\|_2 \leq \left\| \frac{1}{\sqrt{r(\cdot) + \lambda}} \mathscr{L}_\lambda z \right\|_2 . \tag{2.7}$$

It is easy to show that (2.7) holds for any solution of (2.3).

Let $\Delta_j = (j-1, j+1) \ \ (j \in Z), \ \ \{\varphi_j\}_{j=-\infty}^{+\infty}$ be a sequence of such functions from $C_0^\infty(\Delta_j)$, that

$$0 \leq \varphi_j \leq 1, \ \ \sum_{j=-\infty}^{+\infty} \varphi_j^2(x) = 1.$$

We continue $r(x)$ from Δ_j to R so that its continuation $r_j(x)$ was a bounded and periodic function with period 2. Denote by $\mathscr{L}_{\lambda,j}$ the closure in $L_2(R)$ of the differential operator $-z' + [r_j(x) + \lambda]z$ defined on the set $C_0^\infty(R)$. Similarly to the derivation of (2.7) one can proof the inequality

$$\left\| (r_j + \lambda)^{\frac{1}{2}} z \right\|_2 \leq \left\| (r_j + \lambda)^{-\frac{1}{2}} \mathscr{L}_{\lambda,j} z \right\|_2 , \ \ z \in D(\mathscr{L}_{\lambda,j}). \tag{2.8}$$

It follows from the estimates (2.7), (2.8) and from general theory of linear differential equations that the operators $\mathscr{L}_\lambda, \mathscr{L}_{\lambda,j} \ \ (j \in Z)$, are invertible, and their inverses $\mathscr{L}_\lambda^{-1}$ and $\mathscr{L}_{\lambda,j}^{-1}$ are defined in all L_2. From the estimate (2.8) by (1.4) follows

$$\|\mathscr{L}_{\lambda,j} z\|_2 \geq c \sup_{x \in \Delta_j} [r_j(x) + \lambda] \|z\|_2 , \ \ z \in D(\mathscr{L}_{\lambda,j}). \tag{2.9}$$

Let us introduce the operators $B_\lambda, \ \ M_\lambda$:

$$B_\lambda f = \sum_{j=-\infty}^{+\infty} \varphi_j'(x) \mathscr{L}_{\lambda,j}^{-1} \varphi_j f, \quad M_\lambda f = \sum_{j=-\infty}^{+\infty} \varphi_j(x) \mathscr{L}_{\lambda,j}^{-1} \varphi_j f.$$

At any point $x \in R$ the sums of the right-hand side in these terms contain no more than two summands, so B_λ and M_λ is defined on all L_2. It is easy to show that

$$\mathscr{L}_\lambda M_\lambda = E + B_\lambda. \tag{2.10}$$

Using (2.9) and properties of the functions φ_j $(j \in Z)$ we find that $\lim\limits_{\lambda\to+\infty} \|B_\lambda\| = 0$, hence there exists a number λ_0, such that $\|B_\lambda\| \le \frac{1}{2}$ for all $\lambda \ge \lambda_0$. Then it follows from (2.10)

$$\mathscr{L}_\lambda^{-1} = M_\lambda (E + B_\lambda)^{-1}, \quad \lambda \ge \lambda_0. \tag{2.11}$$

By (2.11) and properties of the functions φ_j $(j \in Z)$ again, we have

$$\left\|(r+\lambda)\mathscr{L}_\lambda^{-1} f\right\|_2 \le c_1 \sup_{j\in Z} \left\|(r+\lambda)\mathscr{L}_{\lambda,j}^{-1}\right\|_{L_2(\Delta_j)} \|f\|_2 . \tag{2.12}$$

From (2.9) by conditions (1.4) follows

$$\sup_{j\in Z} \left\|(r+\lambda)\mathscr{L}_{\lambda,j}^{-1} F\right\|_{L_2(\Delta_j)} \le \frac{\sup\limits_{x\in\Delta_j} [r(x)+\lambda]}{\inf\limits_{x\in\Delta_j} [r(x)+\lambda]} \|F\|_{L_2(\Delta_j)} \le$$

$$\le \sup_{|x-z|\le 2} \frac{r(x)+\lambda}{r(z)+\lambda} \|F\|_{L_2(\Delta_j)} \le c_2 \|F\|_{L_2(\Delta_j)} .$$

From the last inequalities and (2.12) we obtain $\|(r+\lambda)z\|_2 \le c_3 \|\mathscr{L}_\lambda z\|_2 , \quad z \in D(\mathscr{L}_\lambda)$, therefore

$$\|z'\|_2 + \|(r+\lambda)z\|_2 \le (1+2c_3) \|\mathscr{L}_\lambda z\|_2 .$$

From this taking into account (2.7) we have the estimate (2.4). The lemma is proved. $\square$

Denote by L a closure in the L_2 -norm of the differential expression

$$L_0 y = -y'' + r(x) y'$$

defined on the set $C_0^\infty(R)$.

Lemma 2.4. *Assume that the function r satisfies the condition (1.3). Then for $y \in D(L)$ the estimate*

$$\left\|\sqrt{r} y'\right\|_2 + \|y\|_2 \le c \|Ly\|_2 \tag{2.13}$$

holds.

Proof. Let $y \in C_0^\infty(R)$. Integrating by parts, we have

$$(Ly, y') = -\int_R y'' \overline{y}' dx + \int_R r(x) |y'|^2 dx. \tag{2.14}$$

Since

$$A := -\int_R y'' \overline{y}' dx = \int_R y' \overline{y}'' dx = -\bar{A},$$

we see $ReA = 0$.
Therefore, it follows from (2.14)

$$Re\,(Ly, y') = \int_R r(x)|y'|^2 dx.$$

Hence, applying the Holder's inequality and using the condition (1.3) we obtain the following estimate

$$c_0 \left\|\sqrt{r}y'\right\|_2 \leq \|Ly\|_2 . \tag{2.15}$$

The inequality (2.15) and Lemma 2.1 imply the estimate (2.13) for $y \in C_0^\infty(R)$. If y is an arbitrary element of $D(L)$, then there is a sequence of functions $\{y_n\}_{n=1}^\infty \subset C_0^\infty(R)$ such that $\|y_n - y\|_2 \to 0$, $\|Ly_n - Ly\|_2 \to 0$ as $n \to \infty$. For y_n the estimate (2.13) holds. From (2.13) taking the limit as $n \to \infty$ we obtain the same estimate for y. The lemma is proved. □

Remark 2.1. The statement of Lemma 2.1 is valid, if $r(x)$ is a complex-valued function, and instead of (1.3) the conditions

$$Re\ r \geq \delta > 0, \quad \gamma_{1,Re\ r} < \infty, \tag{2.16}$$

hold. It follows from Lemma 2.1 that the conditions related to the function r in Lemma 2.4 are natural.

We consider the equation

$$Ly \equiv -y'' + r(x)y' = f, \quad f \in L_2. \tag{2.17}$$

By a solution of (2.17) we mean a function $y \in L_2$ for which there exists a sequence $\{y_n\}_{n=1}^\infty \subset C_0^\infty(R)$ such that $\|y_n - y\|_2 \to 0$, $\|Ly_n - f\|_2 \to 0$, $n \to \infty$.

Lemma 2.5. *If the function r satisfies the condition (1.3), then the equation (2.17) has a unique solution. If, in addition, the function r satisfies the condition (1.4), then for a solution y of the equation (2.17) the following estimate*

$$\|y''\|_2 + \|ry'\|_2 \leq c_L \|Ly\|_2$$

holds i.e. the operator L is separated in the space L_2.

Proof. It follows from the estimate (2.13) that a solution y of the equation (2.17) is unique and belongs to $W_2^1(R)$. Let us prove that the equation (2.17) is solved. Assume the contrary. Then $R(L) \neq L_2$, and there exists a non-zero element $z_0 \in L_2$ such that $z_0 \perp R(L)$. According to operator's theory z_0 is a generalized solution of the equation

$$L^* y \equiv -y' + [r(x)y]' = 0,$$

where L^* is an adjoint operator. Then

$$-z_0' + r(x)z_0 = C.$$

Without loss of generality, we set $C = 1$. Then

$$z_0 = c_0 exp\left[-\int_a^x r(t)dt\right] + \int_a^x exp\left[-\int_a^t r(\tau)d\tau\right] dt := z_1 + z_2. \qquad (2.18)$$

In (2.18) if $c_0 > 0$, then $z_0 \geq c_0$ when $x > a$. If in (2.18) $c_0 \leq 0$, then $z_1 \to 0$ when $x \to -\infty$, and $|z_2(x)| \geq c_1 exp[-\delta_0 x]$ $(0 < \delta_0 < \delta)$ when $x << a$. So $z_0 \notin L_2$. We obtained a contradiction, which shows that the solution of the equation (2.17) exists.

Further, it follows from Lemma 2.3 that the operator $\mathscr{L}$ is separated in L_2. Then by construction the operator L is also separated in L_2. The proof is complete. $\square$

Lemma 2.6. *Let the function r satisfies conditions (1.3), (1.4), $\gamma_{1,r} < \infty$ and*

$$\lim_{t\to+\infty} \sqrt{t}\left\|r^{-1}\right\|_{L_2(t,+\infty)} = 0, \quad \lim_{t\to-\infty} \sqrt{|t|}\left\|r^{-1}\right\|_{L_2(-\infty,t)} = 0. \qquad (2.19)$$

Then the inverse operator L^{-1} is completely continuous in L_2.

Proof. From Lemma 2.5 follows that the operator L^{-1} exists and translates L_2 into space $W^2_{2,r}(R)$ with the norm $\|y''\|_2 + \|ry'\|_2 + \|y\|_2$. By Lemma 2.2 and (2.19) space $W^2_{2,r}(R)$ is compactly embedded into L_2. The proof is complete. $\square$

3. Proofs of Theorems 1-4

Proof of Theorem 1. It follows from Lemma 2.5 that the operator $Ly \equiv -y'' + r(x)y'$ is separated in L_2. From (1.5) and (2.1) we get the estimates

$$\|qy\|_2 \leq 2\gamma_{q,r}\|ry'\|_2 \leq \frac{2}{\sqrt{\delta}}\gamma_{q,r} c \|Ly\|_2, \quad y \in D(L).$$

This means that the operator $l = L + qE$ is also separated in L_2. The theorem is proved. $\square$

Theorem 2 is a consequence of Lemma 2.2, Lemma 2.5 and Theorem 1.

Statement of Theorem 3 follows from Theorem 2 and Theorem 1 [27].

Proof of Theorem 4. Let ϵ and A be positive numbers. We denote

$$S_A = \left\{z \in W^1_2(R) : \|z\|_{W^1_2(R)} \leq A\right\}.$$

Let ν be an arbitrary element of S_A. Consider the following linear "perturbed" equation

$$l_{0,\nu,\epsilon} y \equiv -y'' + \left[r(x,\nu(x)) + \epsilon(1+x^2)^2\right] y' = f(x). \qquad (3.1)$$

Denote by $l_{\nu,\epsilon}$ the minimal closed in L_2 operator generated by expression $l_{0,\nu,\epsilon}y$. Since

$$r_\epsilon(x) := r(x,\nu(x)) + \epsilon(1+x^2)^2 \geq 1 + \epsilon(1+x^2)^2,$$

the function $r_\epsilon(x)$ satisfies the condition (1.3). Further, when $|x-\eta| \leq 1$ for $\nu \in S_A$ we have

$$|\nu(x) - \nu(\eta)| \leq |x-\eta| \, \|\nu'\|_p \leq |x-\eta| \, \|\nu\|_{W_2^1} \leq A. \tag{3.2}$$

It is easy to verify that

$$\sup_{|x-\eta|\leq 1} \frac{(1+x^2)^2}{(1+\eta^2)^2} \leq 3.$$

Then, assuming $\nu(x) = C_1, \quad \nu(\eta) = C_2$, by (1.8) and the inequality (3.2) we obtain

$$\sup_{|x-\eta|\leq 1} \frac{r_\epsilon(x)}{r_\epsilon(\eta)} \leq \sup_{|x-\eta|\leq 1} \sup_{|C_1|\leq A, |C_2|\leq A, |C_1-C_2|\leq A} \frac{r(x, C_1)}{r(\eta, C_2)} + 3 < \infty.$$

Thus the coefficient $r_\epsilon(x)$ in (3.1) satisfies the conditions of Lemma 2.5. Therefore, the equation (3.1) has unique solution y and for y the estimate

$$\|y''\|_2 + \left\| [r(\cdot, \nu(\cdot)) + \epsilon(1+x^2)^2] y' \right\|_2 \leq C_3 \, \|f\|_2 \tag{3.3}$$

holds (an operator $l_{\nu,\epsilon}$ is separated). By (1.8) and (2.1)

$$\|y\|_2 \leq C_0 \, \|ry'\|_2 \, , \quad \left\|(1+x^2)y\right\|_2 \leq C_4 \left\|(1+x^2)^2 y'\right\|_2 . \tag{3.4}$$

Taking them into account from (3.3) we have

$$\|y''\|_2 + \frac{1}{2} \left\|(1+x^2)y'\right\|_2 + \frac{1}{2C_0} \|y\|_2 + \frac{\epsilon}{C_4} \left\|(1+x^2)y\right\|_2 \leq C_3 \, \|f\|_2 .$$

Then for some $C_5 > 0$ the following estimate

$$\|y\|_W := \|y''\|_2 + \left\|(1+x^2)y'\right\|_2 + \left\|[1+\epsilon(1+x^2)]y\right\|_2 \leq C_5 \, \|f\|_2 \tag{3.5}$$

holds. We choose $A = C_5 \, \|f\|_2$, and denote $P(\nu, \epsilon) := L^{-1}_{\nu,\epsilon} f$. From the estimate (3.5) follows that the operator $P(\nu, \epsilon)$ translates the ball $S_A \subset W_2^1(R)$ to itself. Moreover, the operator $P(\nu, \epsilon)$ translates the ball S_A into a set

$$Q_A = \left\{ y : \|y''\|_2 + \left\|(1+x^2)y'\right\|_2 + \left\|[1+\epsilon(1+x^2)]y'\right\|_2 \leq C_5 \, \|f\|_2 \right\} .$$

The set Q_A is the compact in Sobolev's space $W_2^1(R)$. Indeed, if $y \in Q_A, \quad h \neq 0$ and $N > 0$, then the following relations (3.6), (3.7) hold:

$$\|y(\cdot + h) - y(\cdot)\|^2_{W_2^1(R)} = \int_{-\infty}^{+\infty} \left[|y'(t+h) - y'(t)|^2 + |y(t+h) - y(t)|^2 \right] dt =$$

$$= \int_{-\infty}^{+\infty} \left[\left| \int_t^{t+h} y''(\eta) d\eta \right|^2 + \left| \int_t^{t+h} y'(\eta) d\eta \right|^2 \right] dt \leq$$

$$\leq |h| \int_{-\infty}^{+\infty} \left[\left| \int_t^{t+h} y''(\eta) d\eta \right| + \left| \int_t^{t+h} y'(\eta) d\eta \right| \right] dt =$$

$$= |h|^2 \int\limits_{-\infty}^{+\infty} \left[|y''(\eta)|^2 + |y'(\eta)|^2\right] d\eta \le C_5 \|f\|_2 |h|^2, \tag{3.6}$$

$$\|y\|^2_{W_2^1(R\setminus[-N,N])} = \int\limits_{|\eta|\ge N} \left[|y'(\eta)|^2 + |y(\eta)|^2\right] d\eta \le$$

$$\le \int\limits_{|\eta|\ge N} (1+\eta^2)^{-2} \left[|y''(\eta)|^2 + (1+\eta^2)^2|y'(\eta)|^2 + (1+\eta^2)^2|y(\eta)|^2\right] d\eta \le$$

$$\le C_5^2 \|f\|_2^2 (1+N^2)^{-2}. \tag{3.7}$$

The expressions in the right-hand side of (3.6) and (3.7), respectively, tend to zero as $h \to 0$ and as $N \to +\infty$. Then by Kolmogorov-Frechet's criterion the set Q_A is compact in space $W_2^1(R)$. Hence $P(\nu, \epsilon)$ is a compact operator.

Let us show that the operator $P(\nu, \epsilon)$ is continuous with respect to ν in S_A. Let $\{\nu_n\} \subset S_A$ be a sequence such that $\|\nu_n - \nu\|_{W_2^1} \to 0$ as $n \to \infty$, and y_n and y such that $L^{-1}_{\nu,\epsilon} y = f, \;\; L^{-1}_{\nu_n,\epsilon} y_n = f$. Then it is sufficient to show that the sequence $\{y_n\}$ converges to y in $W_2^1(R)$ - norm as $n \to \infty$. We have

$$P(\nu_n, \epsilon) - P(\nu, \epsilon) = y_n - y = L^{-1}_{\nu_n,\epsilon}[r(x, \nu_n(x)) - r(x, \nu(x))]y'_n.$$

The functions $\nu(x)$ and $\nu_n(x)$ $(n = 1, 2, ...)$ are continuous, then by conditions of the theorem the difference $r(x, \nu_n(x)) - r(x, \nu(x))$ is also continuous with respect to x, so that for each finite interval $[-a, a], \;\; a > 0$, we have

$$\|y_n - y\|_{W_2^1(-a,a)} \le c \max_{x\in[-a,a]} |r(x, \nu_n(x)) - r(x, \nu)| \cdot \|y'_n\|_{L_2(-a,a)} \to 0 \tag{3.8}$$

as $n \to \infty$. On the other hand, it follows from Lemma 2.4 that $\{y_n\} \in Q_A$, $\|y_n\|_W \le A, \;\; y \in Q_A, \;\; \|y\|_W \le A$. Since the set Q_A is compact in $W_2^1(R)$, then $\{y_n\}$ converges in the norm of $W_2^1(R)$. Let z be a limit. By properties of $W_2^1(R)$

$$\lim_{|x|\to\infty} y(x) = 0, \quad \lim_{|x|\to\infty} z(x) = 0. \tag{3.9}$$

Since $L^{-1}_{\nu,\epsilon}$ is a closed operator, from (3.8) and (3.9) we obtain $y = z$. So $\|P(\nu_n, \epsilon) - P(\nu, \epsilon)\|_{W_2^1(R)} \to 0, \;\; n \to \infty$.

Hence $P(\nu, \epsilon)$ is the completely continuous operator in space $W_2^1(R)$ and translates the ball S_A to itself. Then, by Schauder's theorem the operator $P(\nu, \epsilon)$ has in S_A a fixed point y $(P(y, \epsilon) = y)$, and y is a solution of the equation

$$L_\epsilon y := -y'' + \left[r(x, y) + \epsilon(1 + x^2)^2\right] y' = f(x).$$

By (3.3) for y the estimate

$$\|y''\|_2 + \left\|\left[r(\cdot, y) + \epsilon(1 + x^2)^2\right] y'\right\|_2 \le C_3 \|f\|_2$$

holds.

Now, suppose that $\{\epsilon_j\}_{j=1}^{\infty}$ is a sequence of the positive numbers converged to zero. The fixed point $y_j \in S_A$ of the operator $P(\nu, \epsilon_j)$ is a solution of the equation

$$L_{\epsilon_j} y_j := -y_j'' + \left[r(x, y_j) + \epsilon_j(1 + x^2)^2\right] y_j' = f(x).$$

For y_j the estimate

$$\left\|y_j''\right\|_2 + \left\|\left[r(\cdot, y_j(\cdot)) + \epsilon(1 + x^2)^2\right] y_j'\right\|_2 \leq C_3 \left\|f\right\|_2 \tag{3.10}$$

holds.

Suppose (a, b) is an arbitrary finite interval. By (3.10) from the sequence $\{y_j\}_{j=1}^{\infty} \subset W_2^2(a, b)$ one can select a subsequence $\left\{y_{\epsilon_j}\right\}_{j=1}^{\infty}$ such that $\left\|y_{\epsilon_j} - y\right\|_{L_2[a,b]}$ $\to 0$ as $j \to \infty$. A direct verification shows that y is a solution of the equation (1.7). In (3.10) passing to the limit as $j \to \infty$ we obtain (1.9). The theorem is proved. $\square$

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