Vertex positions of the generalized orthocenter and a related elliptic curve

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March 29, 2017

1 Introduction

In the third of our series of papers on cevian geometry [9], we have studied the properties of the generalized orthocenter H of a point P with respect to an ordinary triangle ABC in the extended Euclidean plane, using synthetic techniques from projective geometry. This generalized orthocenter is defined as follows. Letting K denote the complement map with respect to ABC and ι the isotomic map (see [1], [6]), the point $Q = K \circ \iota(P)$ is called the isotomcomplement of P. Further, let D, E, F denote the traces of P on the sides of ABC. The generalized orthocenter H is defined to be the unique point H for which the lines HA, HB, HC are parallel to QD, QE, QF, respectively. We showed (synthetically) in [7] that H is given by the formula

$$H = K^{-1} \circ T_{P'}^{-1} \circ K(Q),$$

where $T_{P'}$ is the unique affine map taking ABC to the cevian triangle $D_3E_3F_3$ of the isotomic conjugate $P' = \iota(P)$ of P. The related point

$$O = K(H) = T_{P'}^{-1} \circ K(Q)$$

is the generalized circumcenter (for P) and is the center of the circumconic $\tilde{\mathcal{C}}_O = T_{P'}^{-1}(\mathcal{N}_{P'})$, where $\mathcal{N}_{P'}$ is the nine-point conic for the quadrangle $ABCP'$ (see [9], Theorems 2.2 and 2.4; and [2], p. 84).

We also showed in [9], Theorem 3.4, that if T_P is the unique affine map taking ABC to the cevian triangle DEF of P, then the affine map $\mathsf{M} = T_P \circ K^{-1} \circ T_{P'}$ is a homothety or translation which maps the circumconic $\tilde{\mathcal{C}}_O$ to the inconic $\mathcal{I}$, defined to be the conic with center Q which is tangent to the sides of ABC at the points D, E, F. In the classical case, when $P = Ge$ is the Gergonne point of triangle ABC, the points O and H are the usual circumcenter and orthocenter, and the conics $\tilde{\mathcal{C}}_O$ and $\mathcal{I}$ are the circumcircle and incircle, respectively. In that case the map M taking $\tilde{\mathcal{C}}_O$ to $\mathcal{I}$ is a homothety, and its center is the insimilicenter S. In general, if G is the centroid of ABC, and $Q' = K(P)$, then the center of the map M is the point

$$S = OQ \cdot GV = OQ \cdot O'Q', \quad \text{where } V = PQ \cdot P'Q',$$

This is the author's manuscript of the article published in final edited form as:
Minevich, I., & Morton, P. (2017). Vertex Positions of the Generalized Orthocenter and a Related Elliptic Curve. Journal for Geometry and Graphics, 21(1), 7-27.

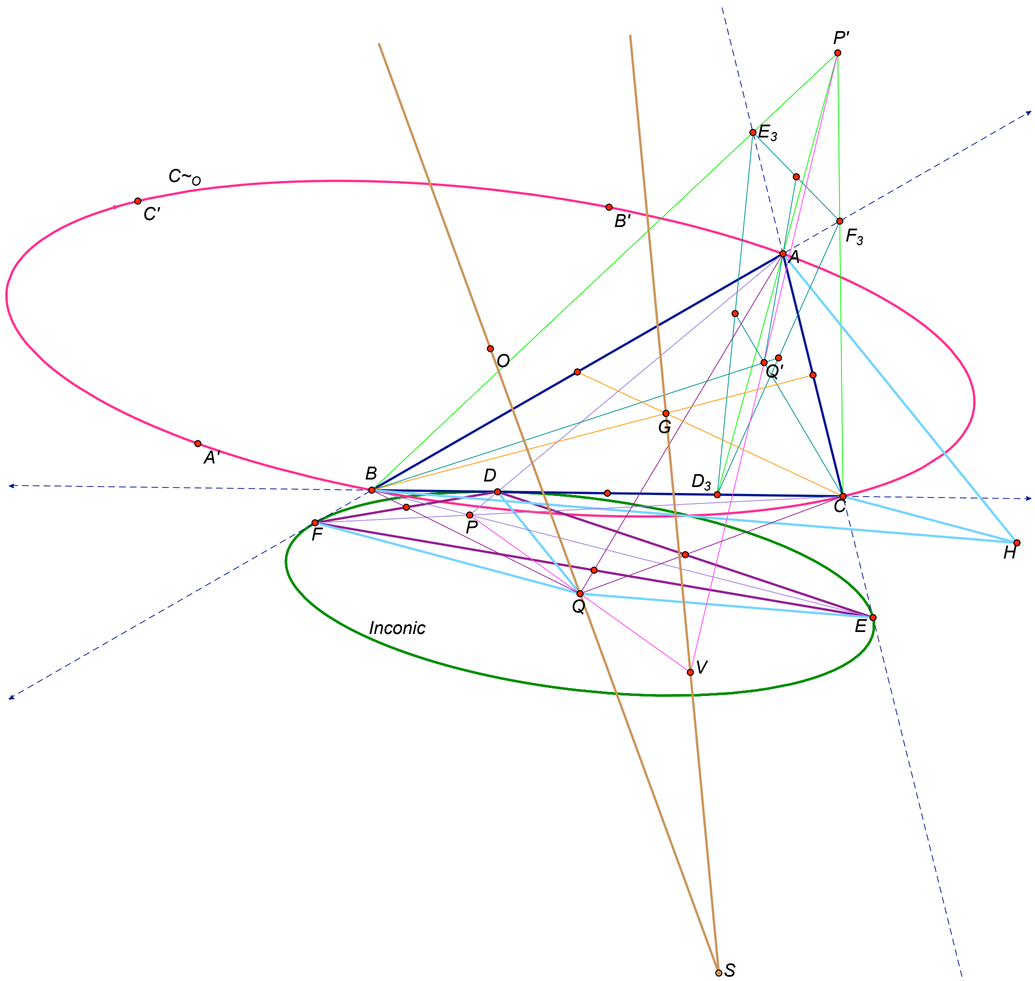

Figure 1: The conics $\tilde{\mathcal{C}}_O$ (strawberry) and $\mathcal{I}$ (green).

and $O' = T_P^{-1} \circ K(Q')$ is the generalized circumcenter for the point P'. (See Figure 1.)

In this paper we first determine synthetically the locus of points P for which the generalized orthocenter is a vertex of ABC. This turns out to be the union of three conics minus six points. Excepting the points A, B, C, these three conics lie inside the Steiner circumellipse $\iota(l_\infty)$ (l_∞ is the line at infinity), and each of these conics is tangent to $\iota(l_\infty)$ at two of the vertices. (See Figure 2.) We also consider a special case in which H is a vertex of ABC and the map M is a translation, so that the circumconic $\tilde{\mathcal{C}}_O$ and the inconic are congruent. (See Figures 3 and 4 in Section 2.) In Section 3, we synthetically determine the locus of all points P for which M is a translation, which is the set of P for which $S \in l_\infty$. We determine necessary and sufficient conditions for this to occur in Theorem 3.1; for example, we show M is a translation if and only if the point P lies on the conic $\tilde{\mathcal{C}}_O$. (This situation does not occur in the classical situation, when P is the Gergonne point, since this point always lies *inside* the circumcircle.) Using barycentric coordinates we show that this locus is an elliptic curve minus 6 points. (See Figure 5.) We also show that there are infinitely many points P in this locus which can be defined over the quadratic field $\mathbb{Q}(\sqrt{2})$, i.e., whose barycentric coordinates can be taken to lie in this field. In particular, given two points in this locus, a third point can be constructed using the addition on the elliptic curve. In Section 4 we show how this elliptic curve, minus a set of 12 torsion points, may be constructed as the locus of points $P = \mathsf{A}(P_1)$, where A runs over the affine mappings taking inscribed triangles (with a fixed centroid) on a subset $\mathscr{A}$ of a hyperbola $\mathcal{C}$ (consisting of six open arcs; see equation (5)) to a fixed triangle ABC, and where P_1 is a fixed point on the hyperbola (an endpoint of one of the arcs making up $\mathscr{A}$). In another paper [11] we will show that the locus of points P, for which the map M is a half-turn, is also an elliptic curve, which can be synthetically constructed in a similar way using the geometry of the triangle.

We adhere to the notation of [6]-[10]: P is always a point not on the extended sides of the ordinary triangles ABC and $K^{-1}(ABC)$; $D_0E_0F_0 = K(ABC)$ is the medial triangle of ABC, with D_0 on BC, E_0 on CA, F_0 on AB (and the same for further points D_i, E_i, F_i); DEF is the cevian triangle associated to P; $D_2E_2F_2$ the cevian triangle for $Q = K \circ \iota(P) = K(P')$; $D_3E_3F_3$ the cevian triangle for $P' = \iota(P)$. As above, T_P and $T_{P'}$ are the unique affine maps taking triangle ABC to DEF and $D_3E_3F_3$, respectively, and $\lambda = T_{P'} \circ T_P^{-1}$. See [6] and [8] for the properties of these maps.

We also refer to the papers [6], [8], [9], and [10] as I, II, III, and IV respectively. See [1], [2], [3] for results and definitions in triangle geometry and projective geometry.

2 The special case $H = A, O = D_0$.

We now consider the set of all points P such that $H = A$ and $O = K(H) = K(A) = D_0$. We start with a lemma.

Lemma 2.1. *Provided the generalized orthocenter H of P is defined, the following are equivalent:*

(a) $H = A$.

(b) $QE = AF$ *and* $QF = AE$.

(c) F_3 *is collinear with* Q, E_0, *and* $K(E_3)$.

(d) E_3 *is collinear with* Q, F_0, *and* $K(F_3)$.

Proof. (See Figure 2.) We use the fact that $K(E_3)$ is the midpoint of segment BE and $K(F_3)$ is the midpoint of segment CF from I, Corollary 2.2. Statement (a) holds iff $QE \parallel AB$ and $QF \parallel AC$, i.e. iff $AFQE$ is a parallelogram, which is equivalent to (b). Suppose (b) holds. Let $X = BE \cdot QF_3$. Then triangles BXF_3 and EXQ are congruent since $QE \parallel BF_3 = AB$ and $QE = AF = BF_3$. Therefore, $BX = EX$, i.e. X is the midpoint $K(E_3)$ of BE, so Q, F_3, and $X = K(E_3)$ are collinear. The fact that E_0 is also collinear with these points follows from $K(BP'E_3) = E_0QK(E_3)$ and the collinearity of B, P', E_3. Similarly, Q, E_3, F_0, and $K(F_3)$ are collinear. This shows (b) $\Rightarrow$ (c), (d).

Next, we show (c) and (d) are equivalent. Suppose (c) holds. The line $F_3E_0 = E_0K(E_3) = K(BE_3)$ is the complement of the line BE_3, hence the two lines are parallel and

$$\frac{AF_3}{F_3B} = \frac{AE_0}{E_0E_3}. \tag{1}$$

Conversely, if this equality holds, then the lines are parallel and F_3 lies on the line through $K(E_3)$ parallel to $P'E_3$, i.e. the line $K(P'E_3) = QK(E_3)$, so (c) holds. Similarly, (d) holds if and only if

$$\frac{AE_3}{E_3C} = \frac{AF_0}{F_0F_3}. \tag{2}$$

A little algebra shows that (1) holds if and only if (2) holds. Using signed distances, and setting $AE_0/E_0E_3 = x$, we have $AE_3/E_3C = (x+1)/(x-1)$. Similarly, if $AF_0/F_0F_3 = y$, then $AF_3/F_3B = (y+1)/(y-1)$. Now (1) is equivalent to $x = (y+1)/(y-1)$, which is equivalent to $y = (x+1)/(x-1)$, hence also to (2). Thus, (c) is equivalent to (d). Note that this part of the lemma does not use that H is defined.

Now assume (c) and (d) hold. We will show (b) holds in this case. By the reasoning in the previous paragraph, we have $F_3Q \parallel E_3P'$ and $E_3Q \parallel F_3P'$, so $F_3P'E_3Q$ is a parallelogram. Therefore, $F_3Q = P'E_3 = 2 \cdot QK(E_3)$, so $F_3K(E_3) = K(E_3)Q$. This implies the triangles $F_3K(E_3)B$ and $QK(E_3)E$ are congruent (SAS), so $AF = BF_3 = QE$. Similarly, $AE = CE_3 = QF$, so (b) holds. □

Theorem 2.2. *The locus $\mathscr{L}_A$ of points P such that $H = A$ is a subset of the conic $\overline{\mathcal{C}}_A$ through B, C, E_0, and F_0, whose tangent at B is $K^{-1}(AC)$ and whose tangent at C is $K^{-1}(AB)$. Namely, $\mathscr{L}_A = \overline{\mathcal{C}}_A \setminus \{B, C, E_0, F_0\}$.*

Proof. Given E on AC we define F_3 as $F_3 = E_0K(E_3) \cdot AB$, and F to be the reflection of F_3 in F_0. Then we have the following chain of projectivities (G is the centroid):

$$BE \;\overline{\wedge}\; E \;\overline{\wedge}\; E_3 \;\overset{G}{\overline{\overline{\wedge}}}\; K(E_3) \;\overset{E_0}{\overline{\overline{\wedge}}}\; F_3 \;\overline{\wedge}\; F \;\overline{\wedge}\; CF.$$

Then $P = BE \cdot CF$ varies on a line or a conic. From the lemma it follows that: (a) for a point P thus defined, $H = A$; and (b) if $H = A$ for some P, then P arises in this way, i.e. F_3 is on $E_0K(E_3)$.

Now we list four cases in the above projectivity for which H is undefined, namely when $P = B, C, E_0, F_0$. Let $A_\infty, B_\infty, C_\infty$ represent the points at infinity on the respective lines BC, AC, and AB.

1. For $E = B_\infty = E_3 = K(E_3)$, we have $E_0K(E_3) = AC$ so $F_3 = A, F = B$, and $P = BE \cdot CF = B$.

2. For $E = C$, we have $E_3 = A, K(E_3) = D_0, E_0K(E_3) = D_0E_0 \parallel AB, F = F_3 = C_\infty$, so $P = BE \cdot CF = C$.

3. For $E = E_0$, we have $E_3 = E_0$ and $K(E_0)$ is the midpoint of BE_0, so $F_3 = B, F = A$, and $P = BE \cdot CF = E_0$.

4. For $E = A$, we have $E_3 = C, K(E_3) = F_0, F_3 = F = F_0$, and $P = BE \cdot CF = F_0$.

Since the four points B, C, E_0, F_0 are not collinear, this shows that the locus of points $P = BE \cdot CF$ is a conic $\overline{\mathcal{C}}_A$ through B, C, E_0, F_0. Moreover, the locus $\mathscr{L}_A$ of points P such that $H = A$ is a subset of $\overline{\mathcal{C}}_A \setminus \{B, C, E_0, F_0\}$.

We claim that if E is any point on line AC other than A, C, E_0, or B_∞, then P is a point for which H is well-defined. First, E_3 is an ordinary point because $E \neq B_\infty$. Second, because $E \neq B_\infty$, the line $E_0K(E_3)$ is not a sideline of ABC. The line $E_0K(E_3)$ intersects AB in A if and only if $K(E_3)$ lies on AC, which is true only if $E_3 = B_\infty$. The line $E_0K(E_3)$ intersects AB in B iff $K(E_3)$ is on BE_0, which holds iff E_3 is on $K^{-1}(B)B = BE_0$, and this is the case exactly when $E = E_3 = E_0$. Since $K(E_3)$ lies on $K(AC) = D_0F_0$, the line $E_0K(E_3)$ is parallel to AB iff $K(E_3) = D_0$, giving $E_3 = A$ and $E = C$. Thus, the line $E_0K(E_3)$ intersects AB in an ordinary point which is not a vertex, so F_3 and F are not vertices and $P = BE \cdot CF$ is a point not on the sides of ABC.

It remains to show that P does not lie on the sides of the anticomplementary triangle of ABC. If P is on $K^{-1}(AB)$ then $F = F_3 = C_\infty$, which only happens in the excluded case $E = C$ (see Case 2 above). If P is on $K^{-1}(AC)$ then $E = B_\infty$, which is also excluded. If P is on $K^{-1}(BC)$ then P' is also on $K^{-1}(BC)$ so $Q = K(P')$ is on BC.

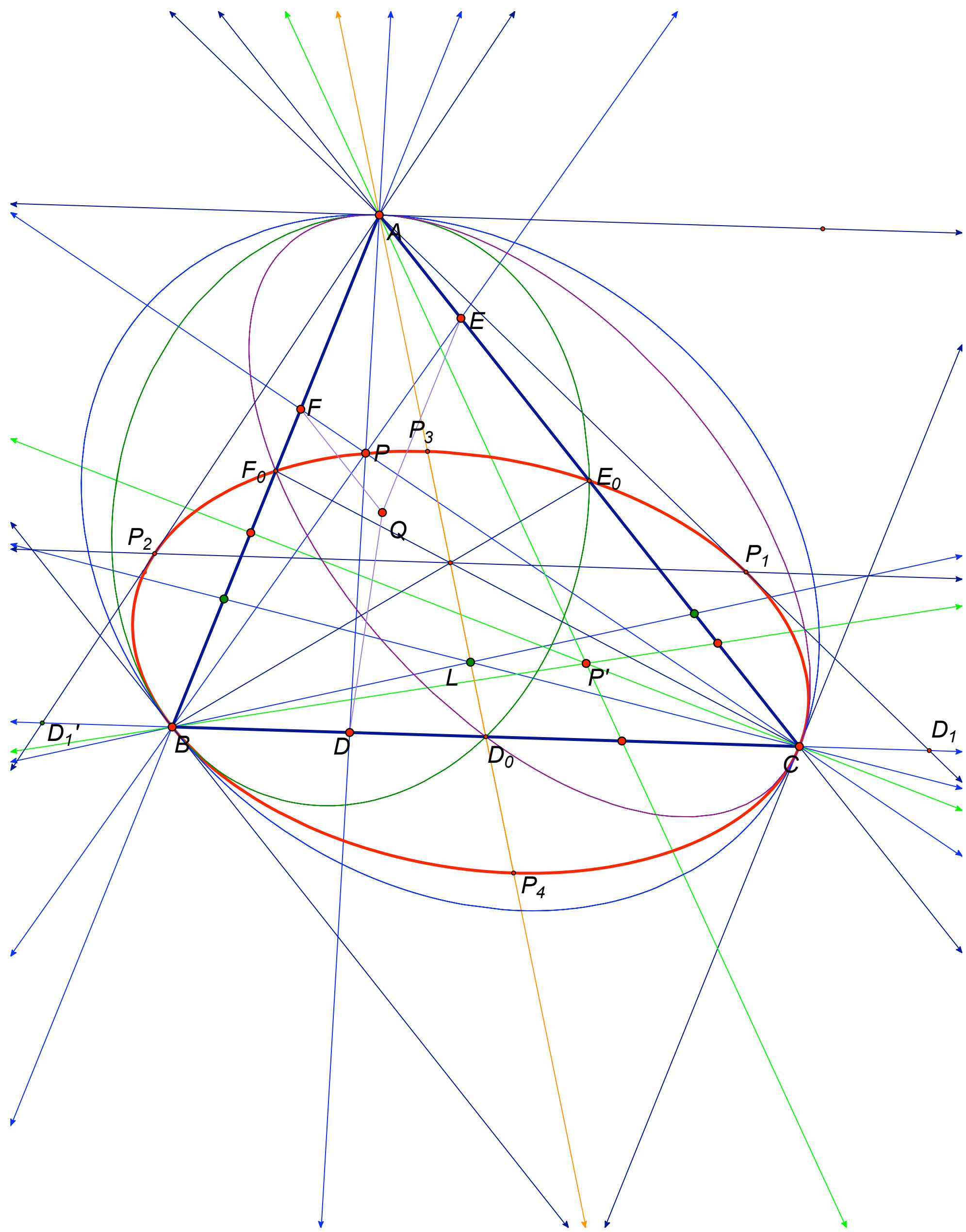

Figure 2: The conics $\overline{\mathcal{C}}_A$ (red), $\overline{\mathcal{C}}_B$ (purple), $\overline{\mathcal{C}}_C$ (green), and $\iota(l_\infty)$ (blue).

To handle the last case, suppose Q is on the same side of D_0 as C. Then P' is on the opposite side of line AD_0 from C, so it is clear that CP' intersects AB in the point F_3 between A and B. If Q is between D_0 and C, then F_3 is between A and F_0 (since F_0, C, and G are collinear), and it is clear that F_3E_0 can only intersect BC in a point outside of the segment D_0C, on the opposite side of C from Q. But this is a contradiction, since by construction F_3, E_0, and $K(E_3)$ are collinear, and $Q = K(P')$ lies on $K(BE_3) = E_0K(E_3)$. On the other hand, if the betweenness relation $D_0 * C * Q$ holds, then F_3 is between B and F_0, and it is clear that F_3E_0 can only intersect BC on the opposite side of B from C. This also applies when $P' = Q$ is a point on the line at infinity, since then $F_3 = B$, and B, E_0 and $Q = A_\infty$ (the point at infinity on BC) are not collinear, contradicting part (c) of Lemma 2.1. A symmetric argument applies if Q is on the same side of D_0 as B, using the fact that parts (c) and (d) of Lemma 2.1 are equivalent. Thus, no point P in $\overline{\mathcal{C}}_A \setminus \{B, C, E_0, F_0\}$ lies on a side of ABC or its anticomplementary triangle, and the point H is well-defined; further, $H = A$ for all of these points.

Finally, by the above argument, there is only one point P on $\overline{\mathcal{C}}_A$ that is on the line $K^{-1}(AB)$, namely C, and there is only one point P on $\overline{\mathcal{C}}_A$ that is on the line $K^{-1}(AC)$, namely B, so these two lines are tangents to $\overline{\mathcal{C}}_A$. □

This theorem shows that the locus of points P, for which the generalized orthocenter H is a vertex of ABC, is the union of the conics $\overline{\mathcal{C}}_A \cup \overline{\mathcal{C}}_B \cup \overline{\mathcal{C}}_C$ minus the vertices and midpoints of the sides. The Steiner circumellipse is tangent to the sides of the anticomplementary triangle $K^{-1}(ABC)$, so the conic $\overline{\mathcal{C}}_A$, for instance, has the double points B, C in common with $\iota(l_\infty)$. Since the conic $\overline{\mathcal{C}}_A$ lies on the midpoints E_0 and F_0, which lie inside $\iota(l_\infty)$, it follows from Bezout's theorem that the set $\overline{\mathcal{C}}_A - \{B, C\}$ lies entirely in the interior of $\iota(l_\infty)$, with similar statements for $\overline{\mathcal{C}}_B$ and $\overline{\mathcal{C}}_C$.

In the next proposition and its corollary, we consider the special case in which $H = A$ and D_3 is the midpoint of AP'. We will show that, in this case, the map M is a translation. (See Figure 4.) We first show that this situation occurs.

Lemma 2.3. *If the equilateral triangle ABC has sides of length 2, then there is a point P with $AP \cdot BC = D$ and $d(D_0, D) = \sqrt{2}$, such that D_3 is the midpoint of the segment AP' and $H = A$.*

Proof. (See Figure 3.) We will construct P' such that D_3 is the midpoint of AP' and $H = A$, and then show that P satisfies the hypothesis of the lemma. The midpoint D_0 of BC satisfies $D_0B = D_0C = 1$ and $AD_0 = \sqrt{3}$. Let the triangle be positioned as in Figure 3. Let $\tilde{A}$ be the reflection of A in D_0, and let D be a point on BC to the right of C such that $D_0D = \sqrt{2}$. In order to ensure that the reflection D_3 of D in D_0 is the midpoint of AP', take P' on $l = K^{-2}(BC)$ with $P'\tilde{A} = 2\sqrt{2}$ and P' to the left of $\tilde{A}$. Then $Q = K(P')$ is on

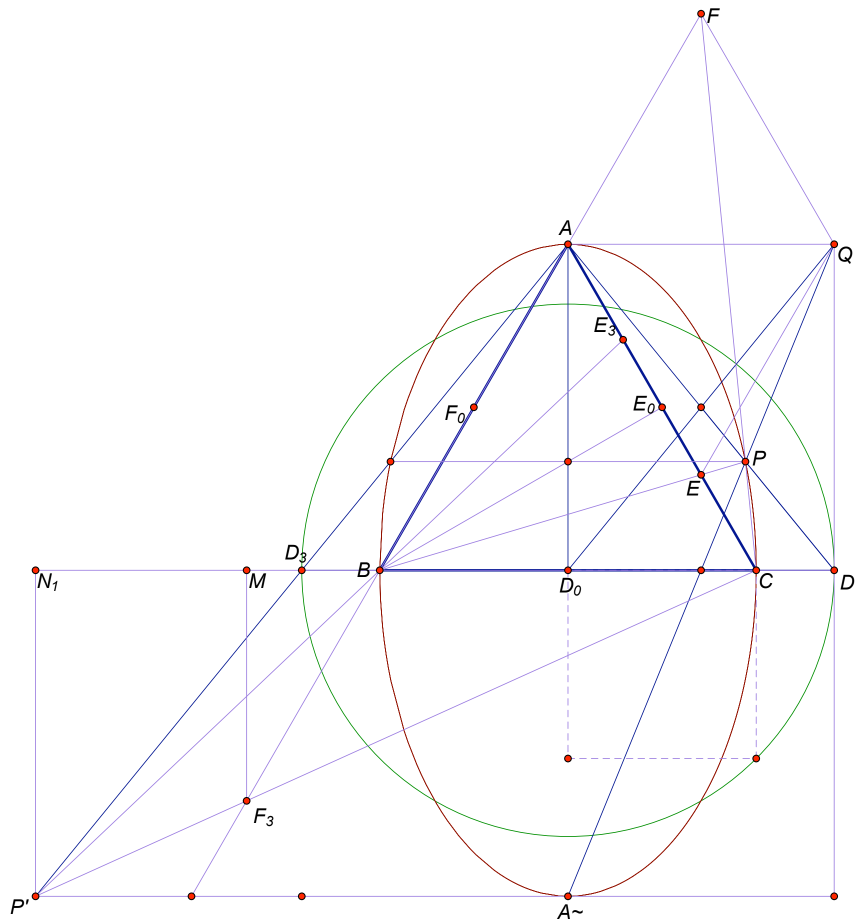

Figure 3: Proof of Lemma 2.3

$K^{-1}(BC)$, to the right of A, and $AQ = \sqrt{2}$. Let E_3 and F_3 be the traces of P' on AC and BC, respectively.

We claim $BF_3 = \sqrt{2}$. Let M be the intersection of BC and the line through F_3 parallel to AD_0. Then triangles BMF_3 and BD_0A are similar, so $F_3M = \sqrt{3} \cdot MB$. Let N_1 be the intersection of BC and the line through P' parallel to AD_0. Triangles $P'N_1C$ and F_3MC are similar, so

$$\frac{F_3M}{MC} = \frac{P'N_1}{N_1C} = \frac{AD_0}{P'\tilde{A}+1} = \frac{\sqrt{3}}{2\sqrt{2}+1}.$$

Therefore,

$$\frac{\sqrt{3}}{2\sqrt{2}+1} = \frac{F_3M}{MC} = \frac{\sqrt{3} \cdot MB}{MB+2}$$

which yields that $MB = 1/\sqrt{2}$. Then $BF_3 = \sqrt{2}$ is clear from similar triangles.

Now, let F be the reflection of F_3 in F_0 (the midpoint of AB). Then AQF is an equilateral triangle because $m(\angle FAQ) = 60^\circ$ and $AQ \cong BF_3 \cong AF$, so $\angle AQF \cong \angle AFQ$. Therefore, $QF \parallel AC$. It follows that the line through F_0 parallel to QF is parallel to AC, hence is a midline of triangle ABC and goes through D_0. Hence, the point O, which is the intersection of the lines through D_0, E_0, F_0, parallel to QD, QE, QF, respectively, must be D_0, giving that $H = K^{-1}(O) = A$. Clearly, $P = AD \cdot CF$ is a point outside the triangle ABC, not lying on an extended side of ABC or its anticomplementary triangle, which satisfies the conditions of the lemma. □

The next proposition deals with the general case, and shows that the point P we constructed in the lemma lies on a line through the centroid G parallel to BC. In this proposition and in the rest of the paper, we will use various facts about the center Z of the cevian conic $\mathcal{C}_P = ABCPQ$, which we studied in detail in the papers [8] and [9]. Recall that Z lies on the nine-point conic $\mathcal{N}_H$. We also recall the definition of the affine reflection η from II, p. 27, which fixes the line GV, with $V = PQ \cdot P'Q'$, and moves points parallel to the line PP'.

Proposition 2.4. *Assume that $H = A, O = D_0$, and D_3 is the midpoint of AP'. Then the circumconic $\tilde{\mathcal{C}}_O = \iota(l)$, where $l = K^{-1}(AQ) = K^{-2}(BC)$ is the line through the reflection $\tilde{A}$ of A in O parallel to the side BC. The points O, O', P, P' are collinear, with $d(O,P') = 3d(O,P)$, and the map M taking $\tilde{\mathcal{C}}_O$ to the inconic $\mathcal{I}$ is a translation. In this situation, the point P is one of the two points in the intersection $l_G \cap \tilde{\mathcal{C}}_O$, where l_G is the line through the centroid G which is parallel to BC.*

Proof. (See Figure 4.) Since the midpoint R_1' of segment AP' is D_3, lying on BC, P' lies on the line l which is the reflection of $K^{-1}(BC)$ (lying on A) in the line BC. It is easy to see that this line is $l = K^{-2}(BC)$, and hence $Q = K(P')$ lies on $K^{-1}(BC)$. From I, Corollary 2.6 we know that the points $D_0, R_1' = D_3$, and $K(Q)$ are collinear. Since $K(Q)$ is the center of the conic $\mathcal{N}_{P'}$ (the nine-point conic of quadrilateral $ABCP'$; see III, Theorem 2.4), which lies on D_0

and D_3, $K(Q)$ is the midpoint of segment D_0D_3 on BC. Applying the map $T_{P'}^{-1}$ gives that $O = T_{P'}^{-1}(K(Q))$ is the midpoint of $T_{P'}^{-1}(D_3D_0) = AT_{P'}^{-1}(D_0)$. It follows that $T_{P'}^{-1}(D_0) = \tilde{A}$ is the reflection of A in O, so that $\tilde{A} \in \tilde{\mathcal{C}}_O$. Moreover, $K(A) = O$, so $\tilde{A} = K^{-1}(A)$ lies on $l = K^{-1}(AQ) \parallel BC$.

Next we show that $\tilde{\mathcal{C}}_O = \iota(l)$, where the image $\iota(l)$ of l under the isotomic map is a circumconic of ABC (see Lemma 3.4 in [10]). It is easy to see that $\iota(\tilde{A}) = \tilde{A}$, since $\tilde{A} \in AG$ and $AB\tilde{A}C$ is a parallelogram. Therefore, both conics $\tilde{\mathcal{C}}_O$ and $\iota(l)$ lie on the 4 points $A, B, C, \tilde{A}$. To show they are the same conic, we show they are both tangent to the line l at the point $\tilde{A}$. From III, Corollary 3.5 the tangent to $\tilde{\mathcal{C}}_O$ at $\tilde{A} = T_{P'}^{-1}(D_0)$ is parallel to BC, and must therefore be the line l. To show that l is tangent to $\iota(l)$, let L be a point on $l \cap \iota(l)$. Then $\iota(L) \in l \cap \iota(l)$. If $\iota(L) \neq L$, this would give three distinct points, $L, \iota(L)$, and $\tilde{A}$, lying on the intersection $l \cap \iota(l)$, which is impossible. Hence, $\iota(L) = L$, giving that L lies on AG and therefore $L = \tilde{A}$. Hence, $\tilde{A}$ is the only point on $l \cap \iota(l)$, and l is the tangent line. This shows that $\tilde{\mathcal{C}}_O$ and $\iota(l)$ share 4 points and the tangent line at $\tilde{A}$, proving that they are indeed the same conic.

From this we conclude that $P = \iota(P')$ lies on $\tilde{\mathcal{C}}_O$. Hence, P is the fourth point of intersection of the conics $\tilde{\mathcal{C}}_O$ and $\mathcal{C}_P = ABCPQ$. From III, Theorem 3.14 we deduce that $P = \tilde{Z} = \mathsf{R}_O K^{-1}(Z)$, where R_O is the half-turn about O; and we showed in the proof of that theorem that $\tilde{Z}$ is a point on the line OP'. Hence, P, O, P' are collinear, and applying the affine reflection η gives that $O' = \eta(O)$ lies on the line PP', as well (see III, Theorem 2.4). Now, Z is the midpoint of $HP = AP$, since $\mathsf{H} = K \circ \mathsf{R}_O$ is a homothety with center $H = A$ and similarity factor $1/2$. Since Z lies on GV, where $V = PQ \cdot P'Q'$ (II, Prop. 2.3), it is clear that P and Q are on the opposite side of the line GV from P', Q', and A. The relation $K(\tilde{A}) = A$ means that $\tilde{A}$ and also O are on the opposite side of GV from A and O'. Also, $J = K^{-1}(Z) = \mathsf{R}_O(\tilde{Z}) = \mathsf{R}_O(P)$ lies on the line GV and on the conic $\tilde{\mathcal{C}}_O$. This implies that O lies between J and P, and applying η shows that O' lies between J and P'. Hence, OO' is a subsegment of PP', whose midpoint is exactly $J = K^{-1}(Z)$, since this is the point on GV collinear with O and O'. Now the map η preserves distances along lines parallel to PP' (see II, p. 27), so $JO' \cong JO \cong OP \cong O'P'$, implying that OO' is half the length of PP'. Furthermore, segment $QQ' = K(PP')$ is parallel to PP' and half as long. Hence, $OO' \cong QQ'$, which implies that $OQQ'O'$ is a parallelogram. Consequently, $OQ \parallel O'Q'$, and III, Theorems 3.4 and 3.9 show that M is a translation. Thus, the circumconic $\tilde{\mathcal{C}}_O$ and the inconic $\mathcal{I}$ are congruent in this situation. This argument implies the distance relation $d(O, P') = 3d(O, P)$.

The relation $O'Q' \parallel OQ$ implies, finally, that $T_P(O'Q') \parallel T_P(OQ)$, or $K(Q')P \parallel A_0Q = AQ$, since $O' = T_P^{-1}K(Q')$ from [7], Theorem 6; $T_P(Q') = P$ from I, Theorem 3.7; $T_P(O) = T_P(D_0) = A_0$; and A_0 is collinear with A and the fixed point Q of T_P by I, Theorem 2.4. Hence, $PG = PQ' = PK(Q')$ is parallel to AQ and BC. $\square$

There are many interesting relationships in the diagram of Figure 4. We

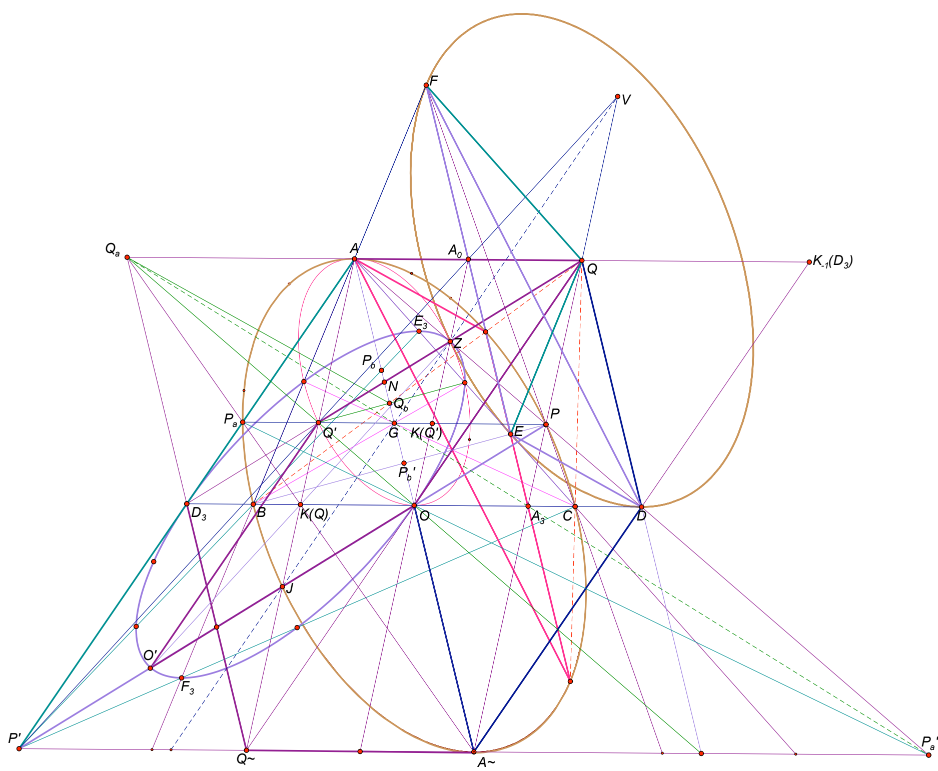

Figure 4: The case $H = A, O = D_0$, and midpoint of $AP' = D_3$.

point out several of these relationships in the following corollary.

Corollary 2.5. *Assume the hypotheses of Proposition 2.4.*

a) If Q_a is the vertex of the anticevian triangle of Q (with respect to ABC) opposite the point A, then the corresponding point P_a is the second point of intersection of the line PG with $\tilde{\mathcal{C}}_O$.

b) The point $A_3 = T_P(D_3)$ is the midpoint of segment OD and P is the centroid of triangle ODQ.

c) The ratio $\frac{OD}{OC} = \sqrt{2}$.

Proof. The anticevian triangle of Q with respect to ABC is $T_{P'}^{-1}(ABC) = Q_aQ_bQ_c$. (See I, Cor. 3.11 and III, Section 2.) Since D_3 is the midpoint of AP', this gives that $T_{P'}^{-1}(D_3) = A$ is the midpoint of $T_{P'}^{-1}(AP') = Q_aQ$. Therefore, Q_a lies on the line $AQ = K^{-1}(BC)$, so $P'_a = K^{-1}(Q_a)$ lies on the line l and is the reflection of P' in the point $\tilde{A}$. Thus, the picture for the point P_a is obtained from the picture for P by performing an affine reflection about the line $AG = A\tilde{A}$ in the direction of the line BC. This shows that P_a also lies on the line $PG \parallel BC$. The conic $\tilde{\mathcal{C}}_O$ only depends on O, so this reflection takes $\tilde{\mathcal{C}}_O$ to itself. This proves a).

To prove b) we first show that P lies on the line $Q\tilde{A}$. Note that the segment $K(P'\tilde{A}) = AQ$ is half the length of $P'\tilde{A}$, so $P'\tilde{A} \cong Q_aQ$. Hence, $Q_aQ\tilde{A}P'$ is a parallelogram, so $Q\tilde{A} \cong Q_aP'$. Suppose that $Q\tilde{A}$ intersects line PP' in a point X. From the fact that $K(Q)$ is the midpoint of D_3D_0 we know that Q is the midpoint of $K^{-1}(D_3)A$. Also, D_3Q' lies on the point $\lambda(A) = \lambda(H) = Q$, by II, Theorem 3.4(b) and III, Theorem 2.7. It follows that $K^{-1}(D_3), P = K^{-1}(Q'), P' = K^{-1}(Q)$ are collinear and $K^{-1}(D_3)QX \sim P'\tilde{A}X$, with similarity ratio $1/2$, since $K^{-1}(D_3)Q$ has half the length of $P'\tilde{A}$. Hence $d(X, K^{-1}(D_3)) = \frac{1}{2}d(X, P')$. On the other hand, $d(O, P) = \frac{1}{3}d(O, P')$, whence it follows, since O is halfway between P' and $K^{-1}(D_3)$ on line BC, that $d(P, K^{-1}(D_3)) = \frac{1}{2}d(P, P')$. Therefore, $X = P$ and P lies on $Q\tilde{A}$.

Now, $\mathsf{P} = AD_3OQ$ is a parallelogram, since $K(AP') = OQ$, so opposite sides in AD_3OQ are parallel. Hence, $T_P(\mathsf{P}) = DA_3A_0Q$ is a parallelogram, whose side $A_3A_0 = T_P(D_3D_0)$ lies on the line $EF = T_P(BC)$. Applying the dilatation $\mathsf{H} = K\mathsf{R}_O$ (with center $H = A$) to the collinear points $Q, P, \tilde{A}$ shows that $\mathsf{H}(Q), Z$, and O are collinear. On the other hand, $O = D_0, Z$, and A_0 are collinear by [7], Corollary 5 (since $Z = R$ is the midpoint of AP), and A_0 lies on AQ by I, Theorem 2.4. This implies that $A_0 = \mathsf{H}(Q) = AQ \cdot OZ$ is the midpoint of segment AQ, and therefore A_3 is the midpoint of segment OD. Since P lies on the line PG, $2/3$ of the way from the vertex Q of ODQ to the opposite side OD, and lies on the median QA_3, it must be the centroid of ODQ. This proves b).

To prove c), we apply an affine map taking ABC to an equilateral triangle. It is clear that such a map preserves all the relationships in Figure 4. Thus we may assume ABC is an equilateral triangle whose sidelengths are 2. By

Lemma 2.3 there is a point P for which $AP \cdot BC = D$ with $D_0D = \sqrt{2}, O = D_0$, and D_3 the midpoint of AP'. Now Proposition 2.4 implies the result, since the equilateral diagram has to map back to one of the two possible diagrams (Figure 4) for the original triangle. □

By Proposition 2.4 and III, Theorem 2.5 we know that the conic $\overline{\mathcal{C}}_A$ lies on the points P_1, P_2, P_3, P_4, where P_1 and $P_2 = (P_1)_a$ are the points in the intersection $\tilde{\mathcal{C}}_O \cap l_G$ described in Corollary 2.5, and $P_3 = (P_1)_b, P_4 = (P_1)_c$ are the anti-isotomcomplements of the points $(Q_1)_b, (Q_1)_c$, since these points all have the same generalized orthocenter $H = A$. (See Figure 2.) It can be shown that the equation of the conic $\overline{\mathcal{C}}_A$ in terms of the barycentric coordinates of the point $P = (x, y, z)$ is $xy + xz + yz = x^2$ (see [7]). Furthermore, the center of $\bar{\mathcal{C}}_A$ lies on the median AG, 6/7-ths of the way from A to D_0.

Remarks. 1. The polar of A with respect to the conic $\bar{\mathcal{C}}_A$ is the line l_G through G parallel to BC. This holds because the quadrangle BCE_0F_0 is inscribed in $\bar{\mathcal{C}}_A$, so its diagonal triangle, whose vertices are A, G, and $BC \cdot l_\infty$, is self-polar. Thus, the polar of A is the line l_G.

2. The two points P in the intersection $\bar{\mathcal{C}}_A \cap l_G$ have tangents which go through A. This follows from the first remark, since these points lie on the polar $a = l_G$ of A with respect to $\bar{\mathcal{C}}_A$. As a result, the points D on BC, for which there is a point P on AD satisfying $H = A$, have the property that the ratio of unsigned lengths $DD_0/D_0C \le \sqrt{2}$. This follows from the fact that $\bar{\mathcal{C}}_A$ is an ellipse: since it is an ellipse for the equilateral triangle, it must be an ellipse for any triangle. Then the maximal ratio DD_0/D_0C occurs at the tangents to $\bar{\mathcal{C}}_A$ from A; and we showed above that for these two points P, $D = AP \cdot BC$ satisfies $DD_0/D_0C = \sqrt{2}$.

3 The locus of points P for which M is a translation.

We can characterize the points P, for which M is a translation, as follows. We will have occasion to use the fact that $\mathsf{M} = T_P \circ K^{-1} \circ T_{P'}$ is symmetric in the points P and P', since $T_P \circ K^{-1} \circ T_{P'} = T_{P'} \circ K^{-1} \circ T_P$. This follows easily from the fact that the maps $T_P \circ K^{-1}$ and $T_{P'} \circ K^{-1}$ commute with each other. See III, Proposition 3.12 and IV, Lemma 5.2.

Theorem 3.1. *Let P and P' be ordinary points not on the sides or medians of ABC or $K^{-1}(ABC)$. Then the map $\mathsf{M} = T_P \circ K^{-1} \circ T_{P'}$ is a translation if and only if any one of the following statements holds.*

1. *$OQQ'O'$ is a parallelogram;*

2. *P is on the circumconic $\tilde{\mathcal{C}}_O$;*

3. *O and O' lie on PP';*

4. Z lies on QQ';

5. The signed ratio $\frac{GZ}{ZV} = \frac{1}{3}$;

6. $U = K^{-1}(Z) = K(V)$.

Proof. It is easy to see that M is a translation if and only if $OQQ'O'$ is a parallelogram, since $\mathsf{M}(O) = Q$ and $\mathsf{M}(O') = Q'$ and the center of M is the point $S = OQ \cdot GV = OQ \cdot O'Q'$. (See III, Theorem 3.4 and the proof of III, Theorem 3.9.) Thus, we will prove that the statements (2)-(6) are equivalent to (1).

First we prove that (1) $\Rightarrow$ (4). If $OQQ'O'$ is a parallelogram, then $OQ \parallel O'Q'$. By IV, Propsition 3.10, $q = OQ$ is the tangent to the conic $\mathcal{C}_P = ABCPQ$ at Q and $q' = O'Q'$ is the tangent to $\mathcal{C}_P$ at Q'. It follows that $q \cdot q'$ is on l_∞, which is the polar of the center Z of $\mathcal{C}_P$. Therefore, QQ' lies on Z.

Conversely, assume (4). Then $Z \in QQ'$ implies that $q \cdot q'$ lies on l_∞, so $OQ \parallel O'Q'$, giving that $S \in l_\infty$ and M is a translation. Hence, (4) $\Rightarrow$ (1). Furthermore, (1) $\Rightarrow$ (3), as follows. M is a translation so $Q\mathsf{M}(Q) \cong O\mathsf{M}(O) = OQ$, i.e. Q is the midpoint of $O\mathsf{M}(Q)$, where $\mathsf{M}(Q) = T_{P'} \circ K^{-1} \circ T_P(Q) = T_{P'}(P')$. But Q is also the midpoint of PV, so triangles PQO and $VQ\mathsf{M}(Q)$ are congruent, giving $\mathsf{M}(Q)V \parallel OP$. We know $\mathsf{M}(Q)V = K^{-1}(PP')$ by II, Proposition 2.3(f) and IV, Theorem 3.11(7.). Since the line through P parallel to $K^{-1}(PP')$ is PP', O lies on PP'. Hence, $\eta(O) = O'$ also lies on PP', giving (3).

Now (4) holds if and only if Z is the midpoint of QQ' (Z lies on GV, the fixed line of η, and $\eta(Q) = Q'$). The point V is the midpoint of segment $K^{-1}(PP')$ (II, Proposition 2.3), so $K(V)$ is the midpoint of segment PP' and $K^2(V)$ is the midpoint of $K(PP') = QQ'$. Hence, (4) holds if and only if $K^2(V) = Z$, which holds if and only if $K^{-1}(Z) = K(V)$. Thus, (4) $\iff$ (6). This allows us to show (4) $\Rightarrow$ (2), as follows. Since (4) also implies (1) and (3), we have that $OO' = QQ' = \frac{1}{2}PP'$. Also, $K(V)$ on GV is the midpoint of PP' and OO'. Since $OQ \parallel GV$ and QP' intersects GV at G, it is clear that O and P lie on the same side of line GV. Hence, O must be the midpoint of $PK(V)$ (the dilation with center $U = K(V)$ takes OO' to PP'). Now $K(V) = K^{-1}(Z) \in \tilde{\mathcal{C}}_O$, since $Z \in \mathcal{N}_H = K(\tilde{\mathcal{C}}_O)$; so $P = \mathsf{R}_O(K^{-1}(Z))$ lies on $\tilde{\mathcal{C}}_O$, hence (2). Thus (4) $\Rightarrow$ (2).

We next show that (2) $\Rightarrow$ (3). Assume that P lies on $\tilde{\mathcal{C}}_O$. Then P is the fourth point of intersection of the conics $\tilde{\mathcal{C}}_O$ and $\mathcal{C}_P = ABCPQ$. From III, Theorem 3.14 we deduce that $P = \tilde{Z} = \mathsf{R}_O \circ K^{-1}(Z)$. Furthermore, $\tilde{Z}$ lies on OP'. Hence, P, O, P' are collinear, and applying the affine reflection η gives that O' lies on the line PP', as well.

Now suppose that (3) holds, so that O lies on PP'. From II, Corollary 2.2 we know that $T_P(P')$ lies on PP', so that IV, Theorem 3.11 implies that $O = OQ \cdot PP' = T_P(P')$. Let $\tilde{H} = T_P^{-1}(H) = T_{P'}^{-1}(Q)$, as in III, Theorem 2.10. Note that

$$\mathsf{M}(\tilde{H}) = T_P \circ K^{-1} \circ T_{P'}(\tilde{H}) = T_P \circ K^{-1}(Q) = T_P(P') = O.$$

Hence, $\mathsf{M}(\tilde{H}O) = OQ$. Part III, Lemma 3.8 says that O is the midpoint of $\tilde{H}Q$, so $\mathsf{M}(\tilde{H}O) \cong \tilde{H}O$. (Note that $\tilde{H} = T_{P'}^{-1}(Q) \neq T_{P'}^{-1} \circ K(Q) = O$.) The result of III, Theorem 3.4 says that M is a homothety or translation. A homothety expands or contracts all segments on lines by the same factor k, so if M were not a translation, the factor $k = \pm 1$. But $k \neq 1$ since M is not the identity map and $k \neq -1$ since it preserves the orientation of the segment $\tilde{H}O$ on the line OQ. Hence, M must be a translation. This proves (3) $\Rightarrow$ (1), and therefore (1) $\Rightarrow$ (4) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

Furthermore, (5) $\iff$ (6). If $U = K^{-1}(Z) = K(V)$, then taking signed distances yields $GZ = GK(U) = \frac{1}{2}UG = \frac{1}{4}GV$, which gives that $\frac{GZ}{ZV} = \frac{1}{3}$. Conversely, if $\frac{GZ}{ZV} = \frac{1}{3}$, then $\frac{GZ}{GV} = \frac{1}{4}$, which implies $Z = K^2(V)$, since $K^2(V) = K(K(V))$ is the unique point X on GV for which the signed ratio $\frac{GX}{GV} = \frac{1}{4}$. Thus, (5) $\iff$ (6) $\iff$ (4) (from above).

This completes the proof that (1)-(6) are equivalent. □

Corollary 3.2. *Under the hypotheses of Theorem 3.1, if M is a translation:*

1. *$HK^{-1}(Z)PV$ is a parallelogram;*
2. *$T_P(P)$ is the midpoint of segment HV;*
3. *$T_P(P') = O$;*
4. *The points $P', O', U = K^{-1}(Z), O, P$ are equally spaced on line PP';*
5. *OH is tangent to the conic $\mathcal{C}_P = ABCPQ$ at H.*

Proof. (See Figure 5.) Statements (3) and (4) were proved in the course of proving (3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (2) above. With $U = K^{-1}(Z) = K(V)$, (4) gives that that $UO = \frac{1}{2}UP$, so $UO \cong K(UP) = ZQ'$. This shows that $UO \parallel ZQ'$ and $OZQ'U$ is a parallelogram. Thus, $K^{-1}(OZQ'U) = HUPV$ is also a parallelogram. In particular, $QQ' \parallel PP' \parallel HV$. Since $\mathsf{M}(U) = \mathsf{M} \circ K^{-1}(Z) = Z$ (by the Generalized Feuerbach Theorem in III) and Z is the midpoint of segment UV, the translation M maps parallelogram $OZQ'U$ to $QV\mathsf{M}(Q')Z$, where $\mathsf{M}(Q') = T_P \circ K^{-1} \circ T_{P'}(Q') = T_P(P)$. As Z is the center of parallelogram $HUPV$ and O is the midpoint of the side UP, while Q is the midpoint of side PV, it follows that Q' is the midpoint of side HU and $\mathsf{M}(Q') = T_P(P) = OZ \cdot HV$ is the midpoint of HV, proving statement (2). Finally, let $O^* = PP' \cdot QH$. Triangles PO^*Q and UO^*H are similar ($UH \parallel PV = PQ$) and $PQ = \frac{1}{2}UH$, so PQ is the midline of triangle UO^*H and $PO^* = UP = \frac{1}{2}PP'$. This implies $\frac{PO^*}{O^*P'} = -\frac{1}{3} = -\frac{PO}{OP'}$, and therefore O^* is the harmonic conjugate of O with respect to P and P'. Thus, O^* is conjugate to O with respect to the polarity induced by $\mathcal{C}_P$. As in the proof of the theorem, $q = OQ$ is tangent to C_P at Q, so Q is also conjugate to O. Thus, the polar of O is $o = QO^* = QH$ and since H and Q are on $\mathcal{C}_P$ (III, Theorem 2.8), this implies that OH is the tangent to $\mathcal{C}_P$ at H. This proves (5). □

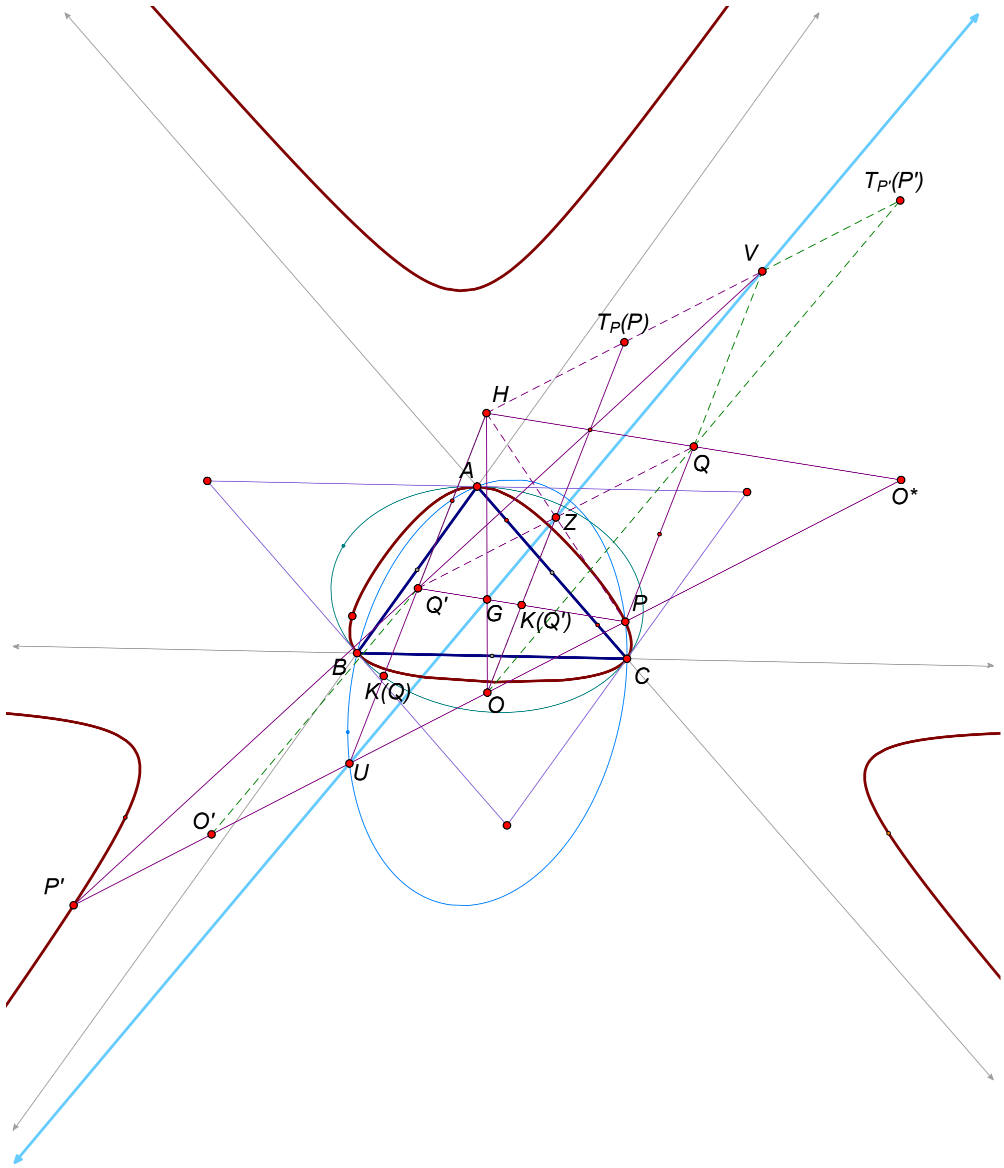

Figure 5: Locus of M a translation (curve $\mathcal{E}_S$ in brown) with Steiner circumellipse (teal) and $\tilde{\mathcal{C}}_O$ (blue).

Remark. The statements in Corollary 3.2 are actually all equivalent to the map M being a translation. For the sake of brevity, we leave this verification to the interested reader.

The set of points $P = (x, y, z)$, for which M is a translation, can be determined using barycentric coordinates. This is just the set of points for which $S \in l_\infty$. It can be shown (see [5], eq. (8.1)) that homogeneous barycentric coordinates of the point S are

$$S = (x(y+z)^2, y(x+z)^2, z(x+y)^2),$$

where $P = (x, y, z)$. Thus, the the locus in question has the projective equation

$$\mathcal{E}_{\mathcal{S}}: \quad x(y+z)^2 + y(x+z)^2 + z(x+y)^2 = 0.$$

Setting $z = 1 - x - y$, so that (x, y, z) are absolute barycentric coordinates, the set of ordinary points for which M is a translation has the affine equation

$$(3x+1)y^2 + (3x+1)(x-1)y + x^2 - x = 0. \tag{3}$$

Since the discriminant of this equation, with respect to y, is $D = (x-1)(3x+1)(3x^2 - 6x - 1)$, this curve is birationally equivalent to

$$Y^2 = (X-1)(3X+1)(3X^2 - 6X - 1).$$

Using $X = \frac{u-1}{u+3}, Y = \frac{8v}{(u+3)^2}$, this equation can be written in the form

$$\mathcal{E}'_S: \ v^2 = u(u^2 + 6u - 3),$$

which is an elliptic curve with j-invariant $j = 54000 = 2^4 3^3 5^3$ and infinitely many points defined over real quadratic fields. Thus, we see that there are infinitely many points for which M is a translation. Note that $\mathcal{E}'_S$ is isomorphic to the curve (36A2) in Cremona's tables [4] (via the substitution $u = x-2, v = y$). Consequently, $\mathcal{E}'_S$ has the torsion points $T = \{\tilde{O}, (0,0), (1, \pm 2), (-3, \pm 6)\}$ ($\tilde{O}$ is the base point) and rank $r = 0$ over $\mathbb{Q}$. These 6 points correspond to the vertices of triangle ABC and the infinite points on its sides; the latter points are $(0, 1, -1), (1, 0, -1), (1, -1, 0)$.

It is not hard to calculate, using the equation (3) and the equation $xy + xz + yz = x^2$ for $\overline{\mathcal{C}}_A$ that the intersection $\mathcal{E}_S \cap \overline{\mathcal{C}}_A$ consists of the points $B = (0, 1, 0)$ and $C = (0, 0, 1)$, with intersection multiplicity 2 at both points, together with the points

$$P = \left(\frac{1}{3}, \frac{1+\sqrt{2}}{3}, \frac{1-\sqrt{2}}{3}\right) \quad \text{and} \quad P_a = \left(\frac{1}{3}, \frac{1-\sqrt{2}}{3}, \frac{1+\sqrt{2}}{3}\right), \tag{4}$$

where P and P_a are the points pictured in Figure 4. (These points are labeled P_1 and P_2 in Figure 2.) That these are the correct points follows from the fact that

$$P - G = P - \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{\sqrt{2}}{3}(0, 1, -1),$$

and therefore $PG \parallel BC$. The affine coordinates of P on (3) are $(x,y) = \left(\frac{1}{3}, \frac{1+\sqrt{2}}{3}\right)$, which corresponds to the point $\tilde{P} = (u,v) = (3, 6\sqrt{2})$ on $\mathcal{E}_S'$. The double of the latter point is $[2]\tilde{P} = \left(\frac{1}{2}, \frac{\sqrt{2}}{4}\right)$, and $[4]\tilde{P} = \left(\frac{169}{8}, -\frac{2483\sqrt{2}}{32}\right)$. Using [12], Theorem VII.3.4 (p. 193) with $p = 2$ over the local field $K = \mathbb{Q}_2(\sqrt{2})$, the coordinates of the last point show that $\tilde{P}$ is a point of infinite order on $\mathcal{E}_S'$, and therefore P is a point of infinite order on (3). Hence, there are infinitely many points on $\mathcal{E}_S$ which have coordinates in the field $\mathbb{Q}(\sqrt{2})$.

It follows from this calculation that the only points, other than the vertices of ABC, in the intersection $\mathcal{E}_S \cap \mathscr{L}$ of $\mathcal{E}_S$ and the locus $\mathscr{L} = \mathscr{L}_A \cup \mathscr{L}_B \cup \mathscr{L}_C$ of Section 2 are the 6 points obtained by permuting the coordinates of the point P in (4). There are, however, 6 more important points on the curve $\mathcal{E}_S$. These are the intersections of $\mathcal{E}_S$ with the medians of triangle ABC, which are found by setting two variables equal to each other in the equation for $\mathcal{E}_S$. This yields the following six points on $\mathcal{E}_S$, paired with their isotomic conjugates:

$$P_1 = (1, -2+\sqrt{3}, -2+\sqrt{3}), \quad P_1' = (1, -2-\sqrt{3}, -2-\sqrt{3})$$
$$P_2 = (-2+\sqrt{3}, 1, -2+\sqrt{3}), \quad P_2' = (-2-\sqrt{3}, 1, -2-\sqrt{3})$$
$$P_3 = (-2+\sqrt{3}, -2+\sqrt{3}, 1), \quad P_3' = (-2-\sqrt{3}, -2-\sqrt{3}, 1).$$

These correspond to the six points

$$(x,y) = \left(1 \pm \frac{2}{3}\sqrt{3}, \mp\frac{\sqrt{3}}{3}\right), \ \left(\pm\frac{\sqrt{3}}{3}, 1 \mp \frac{2}{3}\sqrt{3}\right), \ \left(\pm\frac{\sqrt{3}}{3}, \pm\frac{\sqrt{3}}{3}\right)$$

on (3); and to the six points

$$(u,v) = (-3 \pm 2\sqrt{3}, 0), \ (3 \pm 2\sqrt{3}, 12 \pm 6\sqrt{3}), \ (3 \pm 2\sqrt{3}, -12 \mp 6\sqrt{3})$$

on $\mathcal{E}_S'$. Together with the points in T, these points form a torsion group T_{12} of order 12 defined over $\mathbb{Q}(\sqrt{3})$, with $T_{12} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$. The points in T_{12} are the points which are excluded in Theorem 3.1 and Corollary 3.2. In particular, there are only two excluded points on each median, for which M is a translation.

The equation (3) is a special case of the equation

$$E_a: \quad (ax+1)y^2 + (ax+1)(x-1)y + x^2 - x = 0,$$

which we call the *geometric normal form* of an elliptic curve. It can be shown that for real values of $a \notin \{3, 0, -1, 9\}$, the set of points (not a vertex or an infinite point on the sides of ABC) on this elliptic curve is the locus of points P for which the map M is a homothety with ratio $k = \frac{4}{a+1}$; and every elliptic curve defined over $\mathbb{R}$ is isomorphic to a curve in this form.

4 Constructing the elliptic curve.

In this section we will use the results of the previous section to give a geometric construction of the elliptic curve $\mathcal{E}_S$. We start with the following lemma.

Lemma 4.1. *Assume that P is a point for which the map M is a translation. Then the line $GZ = GV$ does not intersect the conic $\mathcal{C}_P$, which is a hyperbola.*

Proof. We will use the characterization of $\mathcal{C}_P$ as the set of points Y for which P, Y, and $T_P(Y)$ are collinear (II, Corollary 2.2).

Let Y be a point on GV and $Y' = PT_P(Y) \cdot GV$ the projection of $Y_P = T_P(Y)$ onto GV from P. The mapping $Y \rightarrow Y_P$ is projective, since T_P is an affine map, so the mapping $\pi : \ Y \barwedge Y'$ is a projectivity from GV to itself. We will show that π has no invariant points. This will imply the lemma, since if $Y \in \mathcal{C}_P$, then Y lies on $PT_P(Y)$, implying that $Y = Y'$.

We will show that the projectivity π has order 3 by showing that π coincides with the projectivity $UZV \barwedge ZVU$ on GV. First, $\pi(U) = Z$, because $T_P(U) = T_P(K^{-1}(Z)) = Z$ is already on GV. Also, since Z is the midpoint of QQ', $T_P(Z)$ is the midpoint of $T_P(QQ') = QP$. This implies that $\pi(Z) = QP \cdot GV = V$. Now, $T_P(V)$ is the intersection of $T_P(PQ) = QT_P(P)$ and $T_P(P'Q') = OP = PP'$ by Corollary 3.2. Hence, $\pi(V) = PT_P(V) \cdot GV = PP' \cdot GV = U$.

Since π has order 3, it cannot have any invariant points. See [2], p. 43 or [3], p.35, Exercises. Finally, since GV lies on the center Z of $\mathcal{C}_P$, but does not intersect $\mathcal{C}_P$, the conic must be a hyperbola. This completes the proof. □

Thus, the line GV is an exterior line of $\mathcal{C}_P$ ([2], p. 72), so its pole V_∞ is an interior point, which implies that the line $GV_\infty \parallel PP'$ is a secant for the conic $\mathcal{C}_P$, and therefore meets $\mathcal{C}_P$ in two points E and F. (These are different points from the similarly named points in Figure 4.) Hence, as η fixes the line EF and maps the conic $\mathcal{C}_P$ to itself (II, p. 27), we have $\eta(E) = F$ and G on GV is the midpoint of segment EF. But $EF = GV_\infty$ is the polar of V with respect to $\mathcal{C}_P$, so VE and VF are tangent to $\mathcal{C}_P$ at E and F, respectively. We choose notation so that E is the intersection of GV_∞ with the branch of the hyperbola through P' and Q', which exists since P' and Q' are on the same side of the line GV.

Proposition 4.2. *Assume P is a point for which M is a translation. If E' and F' are the midpoints of segments EG and GF, then the lines ZE' and ZF' are the asymptotes of $\mathcal{C}_P$.*

Proof. (See Figure 6.) We know $Z = K(U)$ is the midpoint of segment UV and the center of $\mathcal{C}_P$. If we rotate the tangents EV and FV by a half-turn about Z, we obtain two lines $E''U$ and $F''U$ through U which are also tangent at points E'' and F'' respectively. In particular, Z is the midpoint of segments EE'', FF'', and UV. This implies $F''V \parallel UF$ and Z is the center of the parallelogram $VF''UF$. Now *define* E' to be the midpoint of UF''. The line ZE' is halfway between $F''V$ and UF, hence $ZE' \parallel F''V$.

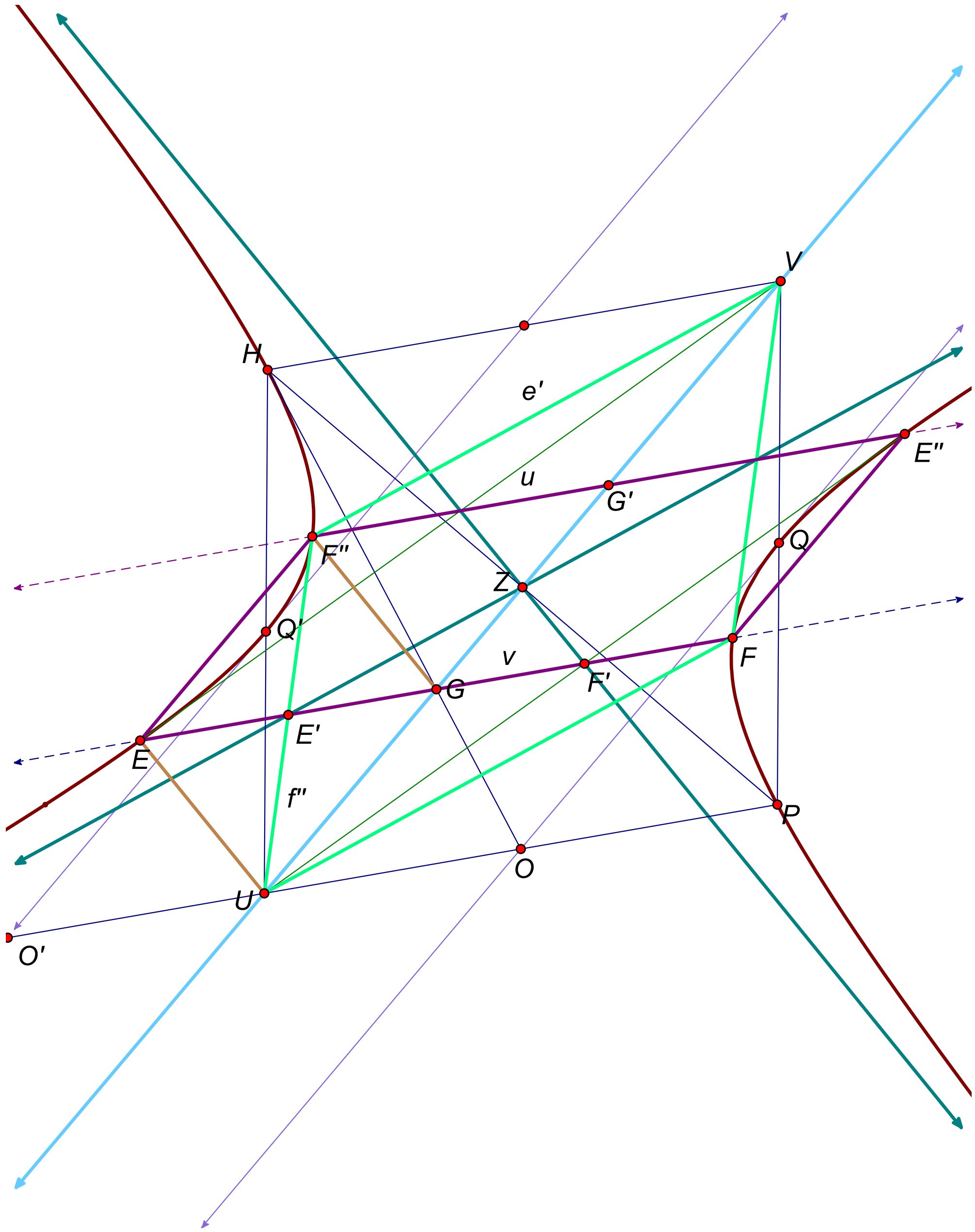

Figure 6: The parallelogram $HVPU$ and conic $\mathcal{C}_P$.

Next we show that $EF''G'G$ is a parallelogram, where $G' = \mathsf{R}_Z(G)$. Now, Z is the midpoint of EE'' and FF'', so $EF''E''F$ is a parallelogram. Since G is the midpoint of EF, G' is the midpoint of $E''F''$. This implies $F''G' \cong EG$, which proves $EF''G'G$ is a parallelogram. Also, using part (6.) of Theorem 3.1 it is easy to see that G is the midpoint of UG' so $UG \cong GG' \cong EF''$ and $EF''GU$ is a parallelogram, with center E'. This verifies that E' is the midpoint of segment EG.

But E' on $EG = v$ implies V lies on its polar e'. Also, E' is on $UF'' = f''$, so F'' lies on e'. Together, this implies $e' = F''V$, so from the first paragraph of the proof, $e' \parallel ZE'$. Hence, e', ZE', and l_∞ are concurrent. The dual of this statement says that $E', l_\infty \cdot e'$, and Z are collinear. Thus, the infinite point $l_\infty \cdot e'$ lies on ZE', which its own polar! Hence, $l_\infty \cdot e'$ must lie on the conic and ZE' must be an asymptote. Applying the map η shows that $ZF' = \eta(ZE')$ is also an asymptote. □

We now consider a fixed configuration of points, as in Figure 6, consisting of the parallelogram $HUPV$, its center Z, the point O which is the midpoint of side UP, the point $G = UV \cdot HO$, the midpoints Q, Q' of opposite sides HU and PV, and the points O', P' which are the affine reflections of the points O, P through the line $UV = GZ$ in the direction of the line UP, together with the conic $\mathcal{C} = PQHQ'P'$. By Theorem 3.1 and Corollary 3.2 this configuration arises from a triangle ABC and the point P (not on the sides of ABC or $K^{-1}(ABC)$), for which the map M is a translation, and such a configuration certainly exists because it can be taken to be the image under an affine map of the configuration constructed in Lemma 2.3 and Proposition 2.4. For this configuration the conclusions of Lemma 4.1 and Proposition 4.2 hold, so that $\mathcal{C}$ is a hyperbola. Our focus now is on finding all triangles $A_1B_1C_1$ inscribed in the conic $\mathcal{C} = \mathcal{C}_P$ for which the map M_P corresponding to $A_1B_1C_1$ is a translation. This will lead us to a synthetic construction of the elliptic curve $\mathscr{E}_S$ discussed in Section 3.

Let A_1 be any point on the conic $\mathcal{C} = PQHQ'P'$, and define $D_0 = K(A_1)$, where K is the dilation about G with signed ratio $-1/2$. Further, let $\mathcal{C}(A_1)$ be the reflection of the conic $\mathcal{C}$ in the point D_0. If the conics $\mathcal{C}(A_1)$ and $\mathcal{C}$ intersect in two points B_1, C_1, then $A_1B_1C_1$ is the unique triangle with vertex A_1 and centroid G which is inscribed in $\mathcal{C}$. This is because D_0 must be the midpoint of side B_1C_1 in any such triangle, and lying on $\mathcal{C}$, B_1 and C_1 must both lie on $\mathcal{C}(A_1)$. Since $\mathcal{C}(A_1)$ is the reflection of $\mathcal{C}$ in D_0, its asymptotes $c = \mathsf{R}_{D_0}(a)$ and $d = \mathsf{R}_{D_0}(b)$ are parallel to the respective asymptotes a and b of $\mathcal{C}$. It follows that $\mathcal{C} \cap \mathcal{C}(A_1)$ can consist of at most two points other than the infinite points on the asymptotes.

Lemma 4.3. *The conics $\mathcal{C}(A_1)$ and $\mathcal{C} = PQHQ'P'$ intersect in two ordinary points if and only if A_1 does not lie on either of the closed arcs of $\mathcal{C}$ between the lines $EF = GV_\infty$ and $P'P$.*

Proof. It suffices to prove the lemma for the configuration pictured in Figures 7 and 8, since any two configurations for which M is a translation are related by an affine map. In particular, Corollary 3.2 shows that one configuration can be mapped to any other by an affine map taking the parallelogram $HUPV$ for the one configuration to the corresponding parallelogram for the other.

Let $c = \mathsf{R}_{D_0}(a)$ and $d = \mathsf{R}_{D_0}(b)$ be the asymptotes of $\mathcal{C}(A_1)$, where $a = ZF'$ and $b = ZE'$ are the asymptotes of $\mathcal{C}$. When $A_1 = E$, then $D_0 = K(E) = F'$ lying on $a = ZF'$, so the lines a, c coincide. Then $\mathcal{C}$ and $\mathcal{C}(A_1)$ have the common tangent $a = c$, so they intersect with multiplicity at least 2 at $a \cdot l_\infty$. They also intersect with multiplicity 1 at $b \cdot l_\infty$, since they have different tangents at that point ($D_0 = F'$ is on a but not b, so $b \neq d$). Hence, they can have at most one ordinary point in common. However, reflecting in D_0 (lying on $a = c$ and therefore in the exterior of $\mathcal{C}$), any ordinary intersection of $\mathcal{C}$ and $\mathcal{C}(A_1)$ would yield a second intersection, so the two conics can't have any ordinary points in common. On the other hand, if $A_1 = P'$ on arc EP', then $D_0 = K(P') = Q$ lies on $\mathcal{C} \cap \mathcal{C}(A_1)$ and also on the tangent OQ to $\mathcal{C}$, so that $\mathcal{C}$ and $\mathcal{C}(A_1)$ touch at $D_0 = Q$. Thus, they don't intersect in any other ordinary point.

We also claim that there are no ordinary points on $\mathcal{C} \cap \mathcal{C}(A_1)$ when A_1 lies between E and P' on the open arc $\mathscr{E} = EP'$. In this case $D_0 = K(A_1)$ lies on the open arc of the conic $K(\mathcal{C})$ between F' and Q. For any A_1 on the left branch of $\mathcal{C}$, let A_1' be the reflection of the point A_1 in D_0. Using $D_0 = K(A_1)$ it is easy to see that $A_1' = K^{-1}(A_1)$, so the tangent ℓ to $\mathcal{C}$ at A_1 is mapped to a parallel tangent $\ell' = K^{-1}(\ell)$ to $\mathcal{C}' = K^{-1}(\mathcal{C})$ at A_1'. On the other hand, ℓ is mapped by reflection in D_0 to a line through A_1' parallel to ℓ, so it must also be mapped to ℓ'. Therefore, ℓ' is tangent to both conics $\mathcal{C}'$ and $\mathcal{C}(A_1) = \mathsf{R}_{D_0}(\mathcal{C})$ at A_1'. Hence, these conics intersect with multiplicity 2 at A_1', and since their asymptotes are parallel, this is the only ordinary point where they can intersect. This holds for any point A_1 on the left branch of $\mathcal{C}$, and therefore the right branch of the conic $\mathcal{C}'$ is an *envelope* for the right branches of the conics $\mathcal{C}(A_1)$. For all A_1 on the left branch of $\mathcal{C}$, D_0 lies below the asymptote $K(b)$ of $K(\mathcal{C})$, which is parallel to and lies halfway between b and the asymptote $b' = K^{-1}(b) = UF$ of $\mathcal{C}'$; hence, the asymptote d of $\mathcal{C}(A_1)$ lies below b', implying that the right branch of $\mathcal{C}(A_1)$ lies in the interior of the right branch of $\mathcal{C}'$. Since $\mathcal{C}'$ intersects $\mathcal{C}$ at $K^{-1}(Q') = P$ and $K^{-1}(Q) = P'$, it crosses $\mathcal{C}$ at P, and points on $\mathcal{C}'$ to the right of P lie in the interior of $\mathcal{C}$. For $A_1 = E$, the right branches of $\mathcal{C}$ and $\mathcal{C}(A_1)$ are also asymptotic in the direction of line $c = a$. Therefore, as $A_1 \in \mathscr{E}$ moves from E to P', the right branch of $\mathcal{C}(A_1)$ moves to the right, separated from the right branch of $\mathcal{C}$ by the asymptote c and the conic $\mathcal{C}'$ and remaining in the interior of $\mathcal{C}$. It follows that the right branch of $\mathcal{C}(A_1)$ contains no ordinary points of $\mathcal{C}$ for $A_1 \in \mathscr{E}$. Since D_0 is in the exterior of $\mathcal{C}$ for these points A_1, any intersection of the left branch of $\mathcal{C}(A_1)$ and the left branch of $\mathcal{C}$ would reflect through D_0 to an intersection of the right branches. The left branch of $\mathcal{C}(A_1)$ also does not intersect the right branch of $\mathcal{C}$ since it is the reflection of that branch in D_0. This proves the claim.

Now assume A_1 lies outside the closed arc $\overline{\mathscr{E}}$ on the left branch of $\mathcal{C}$. First, if A_1 lies above the point E, then D_0 lies below the point F' on $K(\mathcal{C})$, and since c lies on the same side of the line a as D_0, c lies to the left of the line a in Figure

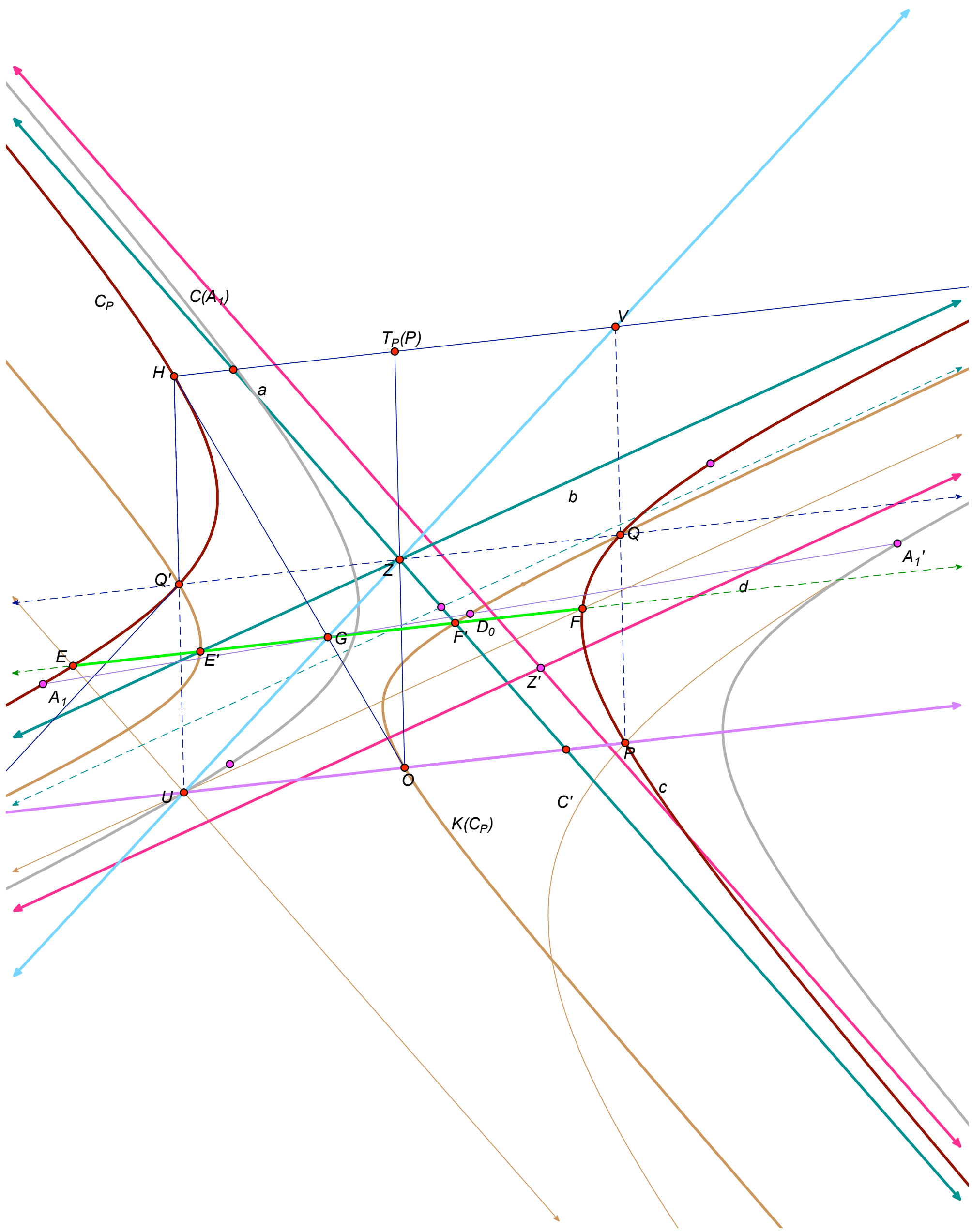

Figure 7: Conics $\mathcal{C}$ (brown), $K(\mathcal{C}_P)$ (tan), $\mathcal{C}(A_1)$ (grey), $\mathcal{C}'$ (light tan), with A_1 on the arc $\mathscr{E}$.

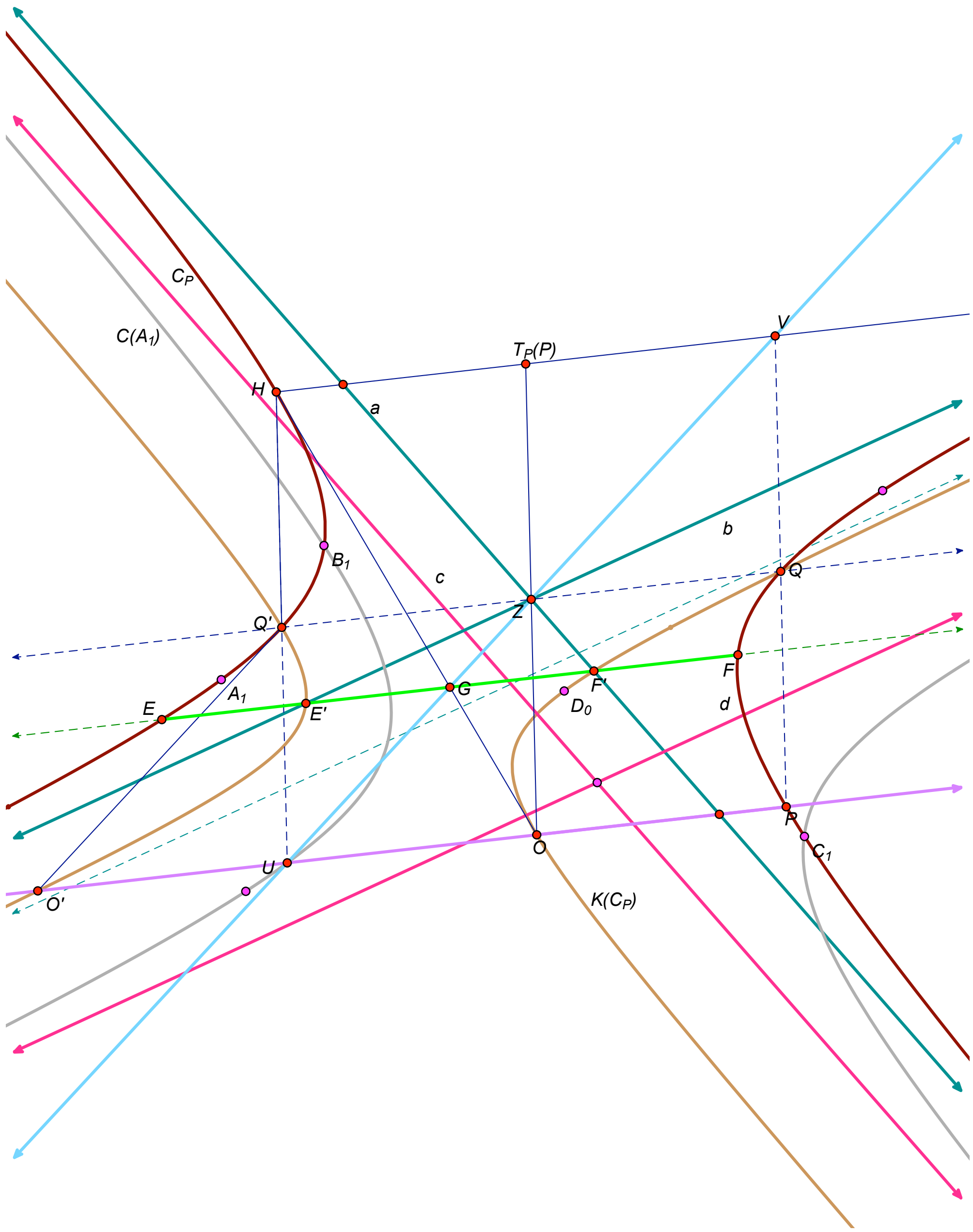

Figure 8: Conics $\mathcal{C}$ (brown), $K(\mathcal{C}_P)$ (tan), $\mathcal{C}(A_1)$ (grey), with A_1 above E.

7. Since c lies on $a \cdot l_\infty$ and is not the tangent a at that point, it must intersect the conic $\mathcal{C}$ in a second, ordinary, point. It cannot intersect the right branch of $\mathcal{C}$ because that branch is on the other side of the line a. Hence, c intersects the left branch of $\mathcal{C}$, whence it follows that the left branch of $\mathcal{C}(A_1)$ intersects $\mathcal{C}$ as well (because this branch of $\mathcal{C}(A_1)$ is asymptotic to an exterior ray of line d in one direction and to c in the other direction). At the same time, this shows that the asymptote a of $\mathcal{C}$ intersects the right branch of $\mathcal{C}(A_1)$, so the right branch of $\mathcal{C}$ intersects the latter. It is easy to see that these two intersection points are reflections of each other in the point D_0.

On the other hand, if A_1 lies below the point P' on the left branch of $\mathcal{C}$, then D_0 lies above Q on $K(\mathcal{C})$. Since $Q \in \mathcal{C} \cap K(\mathcal{C})$, points to the right of Q on $K(\mathcal{C})$ are in the interior of $\mathcal{C}$, so the reflection Q_0 of the point Q in D_0 lies on the left branch of $\mathcal{C}(A_1)$ in the interior of $\mathcal{C}$. It follows that the left branch of $\mathcal{C}(A_1)$ must intersect the right branch of $\mathcal{C}$ in two points. The same arguments apply to points A_1 on the right branch of $\mathcal{C}$, and this completes the proof. □

Lemma 4.4. *The points $A_1 = Q, Q'$ are the only points on $\mathcal{C} = PQHQ'P'$ for which A_1 lies on $\mathcal{C}(A_1)$.*

Proof. Certainly $Q' \in \mathcal{C}(Q')$ because $D_0 = K(Q')$ is the midpoint of segment $Q'P$, so $Q' = \mathsf{R}_{D_0}(P)$ lies on $\mathcal{C}(Q')$, the reflection of $\mathcal{C}$ in D_0. The same argument holds for Q. If A_1 is any point lying on $\mathcal{C}(A_1)$, then A_1 and its reflection A_1' in $D_0 = K(A_1)$ both lie on the conic $\mathcal{C}$ and are collinear with the point G. Since $A_1' = K^{-1}(A_1)$, the locus of points A_1' coincides with the conic $\mathcal{C}' = K^{-1}(\mathcal{C})$, whose asymptotes are parallel to the asymptotes of $\mathcal{C}$. Hence, $\mathcal{C}'$ intersects $\mathcal{C}$ in only the two points $P = K^{-1}(Q')$ and $P' = K^{-1}(Q)$. This proves the lemma. □

We now fix a parallelogram $H_1U_1P_1V_1$ with center Z_1, distinguished point $G_1 = U_1V_1 \cdot H_1O_1$, and its corresponding conic $\mathcal{C} = P_1Q_1H_1Q_1'P_1'$, as in Figure 6. We will call this configuration the P_1 configuration, and consider it fixed for the following discussion.

Let ABC be a given triangle. For any point P, not on a median of ABC, for which the map M, defined relative to ABC and P, is a translation, there is an affine map A^{-1} taking the parallelogram $HUPV$ for ABC to the parallelogram $H_1U_1P_1V_1$. (We avoid points on the medians of ABC, because for these points, the conic $\mathcal{C}_P = ABCPQ = AP \cup BC$ and parallelogram $HUPV$ are degenerate.) Since ABC is inscribed in the cevian conic $\mathcal{C}_P = ABCPQ = PQHQ'P'$ for P, and the points P', Q, Q' are defined by simple affine relationships in terms of the parallelogram $HUPV$, the image triangle $\mathsf{A}^{-1}(ABC) = A_1B_1C_1$ under the map A^{-1} is a triangle inscribed in the conic $\mathcal{C} = P_1Q_1H_1Q_1'P_1'$. By Theorem 3.1 and Corollary 3.2 all the same relationships hold for the two configurations. Hence, the centroid G maps to the centroid G_1 in the P_1 configuration. It follows from Lemma 4.3 and Lemma 4.4 that the image A_1 of the point A must lie in the complement of the union of closed arcs $\overline{\mathscr{E}}$ (from E to P_1') and $\overline{\mathscr{F}}$ (from F to P_1)

on $\mathcal{C}$, and that A_1 is also distinct from the points Q_1, Q_1' (as there is no triangle $A_1B_1C_1$ for these two points). Thus,

$$A_1 \in \mathscr{A} = \mathcal{C} - (\overline{\mathscr{E}} \cup \overline{\mathscr{F}} \cup \{Q_1, Q_1', A_\infty, B_\infty\}), \tag{5}$$

where $A_\infty = a \cdot l_\infty$ and $B_\infty = b \cdot l_\infty$ are the infinite points on the asymptotes. The set $\mathscr{A}$ is a union of 6 open arcs on $\mathcal{C}$.

Conversely, let $A_1 \in \mathscr{A}$ and let $A_1B_1C_1$ be the corresponding triangle inscribed in $\mathcal{C}_{P_1}$. Then the centroid of $A_1B_1C_1$ is G_1, and the cevian conic for $A_1B_1C_1$ and P_1 is $A_1B_1C_1P_1Q_1' = \mathcal{C}_{P_1}$, coinciding with the conic $\mathcal{C} = P_1Q_1H_1Q_1'P_1'$. Moreover, the point P_1 does not lie on a median of $A_1B_1C_1$; otherwise one of the vertices of the triangle would be collinear with P_1 and G_1, implying that this vertex would have to coincide with Q_1 or Q_1'. This conic has center Z_1, and the pole of G_1Z_1 is the point $V_\infty = P_1P_1' \cdot l_\infty$. Now we use the characterization of the isotomic conjugate $\iota(P_1)$ (with respect to $A_1B_1C_1$) as the unique point different from P_1 lying in the intersection $\mathcal{C}_{P_1} \cap P_1V_\infty = \mathcal{C} \cap P_1V_\infty$ to deduce that $\iota(P_1) = P_1'$. (See II, p. 26.) Theorem 3.1 shows that $\mathsf{M}_1 = \mathsf{M}_{P_1}$ for the triangle $A_1B_1C_1$ must be a translation. If A is an affine map taking $A_1B_1C_1$ to ABC, then Theorem 3.1 shows once again that the map $\mathsf{M} = \mathsf{A}\mathsf{M}_1\mathsf{A}^{-1}$ is a translation for the point $P = \mathsf{A}(P_1)$. Hence, P lies on the elliptic curve $\mathcal{E}_S$ of Section 3. The argument of the previous paragraph shows that every point P on $\mathcal{E}_S$, other than the 12 points in its torsion group T_{12}, is the image $P = \mathsf{A}(P_1)$ for some triangle $A_1B_1C_1$ inscribed in $\mathcal{C}$ and an affine map A for which $\mathsf{A}(A_1B_1C_1) = ABC$. This proves the following theorem.

Theorem 4.5. *Fix a parallelogram $H_1U_1P_1V_1$ and the corresponding hyperbola $\mathcal{C} = P_1Q_1H_1Q_1'P_1'$, as in Figure 6. The elliptic curve $\mathcal{E}_S$, minus the torsion subgroup T_{12}, corresponding to the vertices of ABC, the infinite points on its sides, and the points lying on the medians of ABC, is the locus of images $\mathsf{A}(P_1)$, where A_1 is a point in the set $\mathscr{A} \subset \mathcal{C}$ (a union of six open arcs on the hyperbola $\mathcal{C}$), B_1 and C_1 are the unique points on $\mathcal{C}$ for which triangle $A_1B_1C_1$ has centroid G_1, and A is one of the two affine maps for which $\mathsf{A}_1(A_1B_1C_1) = ABC$ or $\mathsf{A}_2(A_1C_1B_1) = ABC$.*

By virtue of the above discussion, we have taken the situation of Figure 8, where P_1 is fixed and the triangle $A_1B_1C_1$ varies, and transformed it, via the locus of maps A corresponding to $A_1 \in \mathscr{A}$, to the fixed triangle ABC and varying point $\mathsf{A}(P_1) = P$ lying on the elliptic curve $\mathcal{E}_S$.

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