A mean-variance analysis of the Global Minimum Variance Portfolio Constructed using the CARBS indices

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Abstract The purpose of this paper is to construct a global minimum variance portfolio (GMVP) using the log returns of the CARBS (Canada, Australia, Russia, Brazil, South Africa) indices. The weights obtained indicate that most of the portfolio should be invested in Canadian equity. The returns series of the CARBS and the GMVP seem to be consistent with the stylised facts of financial time series. Further empirical analysis shows that the CAPM relationship holds for Canada, South Africa, and the GMVP. The systematic risk (β) of the GMVP is the lowest, and the Russian equity index is the highest. However the R^2 of all the models indicate that the CAPM relationship is not a good fit for all the variables, and can therefore not be considered a reliable measure of risk.

1 Introduction

This paper derives an expression for the portfolio weights of a GMVP that consists of the CARBS equity indices. The method is based on constant means, variances, and covariances determined using historical data. The risk of the individual assets and the portfolio is then measured using classical methods, this gives an opportunity to compare the measured risk of the individual assets and the portfolio. And in later

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chapters compare the measured risk using classical methods to the risk obtained using volatility models.

Conventional wisdom among finance researchers and practitioners is that it is necessary to diversify. Diversification is necessary to reduce the level of idiosyncratic risk. However, another important aspect to consider is how to determine the weight of each asset in a diversified portfolio. Furthermore, by minimising risk, it is possible that the return might be insignificant. Therefore, a GMVP is constructed in this chapter. By viewing each asset simultaneously in mean variance space, it will in some cases be clear which asset is more attractive than another.

In mean-variance space, it is that an asset with a greater return, but less risk is more attractive than an asset with a lower return and the same level of risk or greater. This is referred to as mean-variance dominance (Bailey (2005) [1]). Therefore, the validity of the capital asset pricing model (CAPM) relationship is tested in this chapter. This gives an indication of the level of systematic risk, and the expected return. The method used gives an indication of the goodness of fit of the model, this determines the applicability.

In this chapter, the relevant literature on mean-variance analysis will be discussed, and the dataset of the CARBS indices will be analysed. Using the data, a GMVP will be constructed. In addition, the risk return trade-off of the CARBS indices and the GMVP will be illustrated, finally the CAPM relationship will be estimated for the CARBS indices and the GMVP. The empirical analysis of this chapter makes extensive use of the `PerformanceAnalytics` package (Peterson et al. (2014) [2]) of the `R` statistical programming language.

2 Literature review

Several existing papers make use of matrix algebra techniques to construct a GMVP, or a target return portfolio that minimises risk. Watson et al. (2014) [3] estimated a GMVP which included the equity indices of the BRICS (Brazil, Russia, India, China, and South Africa). The purpose of the analysis was to determine which indices are the most significant. The findings suggest that more than half of the GMVP should be invested in South African equity. Moreover, Brazilian and Russian equity are insignificant when constructing a GMVP for the BRICS equity indices.

Cardoso (2015) [4] constructed efficient portfolios using 15 equity shares on the S&P500 index, both the maximum likelihood and robust methods were used. In order to test the performance of the methods, Cardoso (2015) [4] performed two Monte Carlo simulations, one assuming a normal distribution, and another with contaminated non-normal samples. The results show that when a normal distribution is assumed, the maximum likelihood method outperforms the robust method. However, the robust method proved to be more efficient when non-normal samples were used.

Li et al. (2003) [5] investigated the benefit of diversification subject to portfolio weight restrictions, more specifically constraints on sort-selling. The portfolio was constructed by minimising the variance, subject to two constraints: the portfolio weights must sum to one, and weights cannot be negative. Using a dataset that consists the dollar denominated equity indices of seven developed countries, and eight emerging countries, the results indicate that short-sale constraint reduce, but do not eliminate the benefit of diversification.

There have also been many studies that have tested the validity of the CAPM relationship. Iqbal et al. (2007) [6] tested the linearity of the CAPM relationship of the Karachi equity index, and found that the relationship between risk and return is generally not linear, which is inconsistent with the relationship predicted by the CAPM.

In a similar, more recent study, Ali et al. (2010) [7] tested the applicability of the CAPM relationship for the Dhaka (Bangladesh) equity index. In terms of methodology, the Fama and Macbeth approach was used. The findings finally showed that the systematic risk coefficient is not sufficient to explain the risks for returns. Hence it is necessary to consider other factors in addition to the risk premium of the market portfolio.

3 Portfolio theory: matrix algebra

The objective of this section is to derive an expression for the weighting of each asset included in a GMVP, when five assets are included in the portfolio. The following derivation follows Zivot (2011) [8] closely. Let R_i[1] denote the return on asset i, and σ_{ij} the covariance between rates of return R_i and R_j (for $i, j = 1,2,..,5$). Furthermore, assume that R_i is identically and independently (normally) distributed, with constant mean and variance. More specifically,

$$R_i \sim \mathcal{N}\left(\mu_i, \sigma_i^2\right)$$
$$\operatorname{cov}(R_i, R_j) = \sigma_{ij}.$$

A portfolio can be defined as a vector of asset holdings with weighting, $\omega_1, \omega_2, ..., \omega_5$. Bailey (2005) [1] explains that the expected rate of return (μ_p) and variance of the rate of return (σ_p^2) of a portfolio that contains five risky assets take the form

$$\mu_p = \sum_{i=1}^{5} \omega_i \mu_i, \text{ and} \tag{1}$$

$$\sigma_p^2 = \sum_{i=1}^{5} \sum_{j=1}^{5} \omega_i \omega_j \sigma_{ij}. \tag{2}$$

[1] $R_{i,t} = \ln\left(\frac{S_{i,t+1}}{S_{i,t}}\right)$ where $S_{i,t}$ is the value of asset i at time t and $t \in \mathbb{N}$.

To find an expression for the weighting of each asset included in a GMVP, the following problem needs to be evaluated:

The problem is given by

$$\min_{\sum_{i=1}^{5}\omega_i=1}\left\{\sum_{i=1}^{5}\sum_{j=1}^{5}\omega_i\omega_j\sigma_{ij}\right\}.$$

This implies that we need to minimise the variance of the portfolio, subject to the constraint that the sum of the portfolio weightings should be equal to one.

The method of Lagrange multipliers is used to solve the problem above. The method of Lagrange multipliers is best explained by the following theorem from Wrede (2010) [9]

Theorem 1. *A method for obtaining the relative maximum or minimum values of a function $F(x,y,z)$ subject to the constraint $\phi(x,y,z)=0$ consists of the formation of the auxiliary function*

$$L(x,y,z,\lambda)=F(x,y,z)+\lambda\phi(x,y,z)$$

subject to the conditions

$$\frac{\partial L}{\partial x}=0,\qquad \frac{\partial L}{\partial y}=0,\qquad \frac{\partial L}{\partial z}=0,\qquad \frac{\partial L}{\partial \lambda}=0,$$

which are required conditions for a relative maximum or minimum.

The Lagrangian for minimisation problem 3 is given by

$$L(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5,\lambda)=\sum_{i=1}^{5}\sum_{j=1}^{5}\omega_i\omega_j\sigma_{ij}+\lambda\left(\sum_{i=1}^{5}\omega_i-1\right).$$

The first order conditions are given by

$$\begin{aligned}\frac{\partial L}{\partial \omega_i}&=\sum_{j=1}^{5}\omega_j\sigma_{ij}=0,\ \text{for } i=1,2,...,5,\\ \frac{\partial L}{\partial \lambda}&=\sum_{i=1}^{5}\omega_i-1=0\end{aligned}$$

The first order conditions give rise to the following system of linear equations

$$\begin{bmatrix}2\sigma_1^2 & 2\sigma_{12} & 2\sigma_{13} & 2\sigma_{14} & 2\sigma_{15} & 1\\ 2\sigma_{12} & 2\sigma_2^2 & 2\sigma_{23} & 2\sigma_{24} & 2\sigma_{25} & 1\\ 2\sigma_{13} & 2\sigma_{23} & 2\sigma_3^2 & 2\sigma_{34} & 2\sigma_{35} & 1\\ 2\sigma_{14} & 2\sigma_{24} & 2\sigma_{34} & 2\sigma_4^2 & 2\sigma_{45} & 1\\ 2\sigma_{15} & 2\sigma_{25} & 2\sigma_{35} & 2\sigma_{45} & 2\sigma_5^2 & 1\\ 1 & 1 & 1 & 1 & 1 & 0\end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\\ \omega_5\\ \lambda\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix}.$$

For any system of linear equations $Ax = b$, the solution is equal to $x = A^{-1}b$. Solving the above system of linear equations will provide the necessary portfolio weights $(\omega_1, ..., \omega_5)$ to construct a GMVP.

4 The dataset

The purpose of this section is to illustrate the statistical properties of the dataset used in this study. In order to avoid loss of information, daily data from 11 January 2010 until 31 December 2015 was used for the purpose of most of the empirical analysis that follow.[2] Missing data points were interpolated using a cubic spline. Data from 2010 was used to exclude the GFC period which is considered a structural break in the data. When financial time series data is considered, Embrechts et al. (2005) [10] outlines the following stylised facts:

1. Return series show little autocorrelation.
2. Squared returns show evidence of significant autocorrelation.
3. The conditional expectation of returns is close to zero.
4. Volatility seems to fluctuate over time.
5. Return series show signs of leptokurtosis or fat tails.
6. Return series exhibit evidence of volatility clustering.

The following indices are used in this study: The Dow Jones will be used as a benchmark in the empirical analysis.

Table 1 Indices used in this study

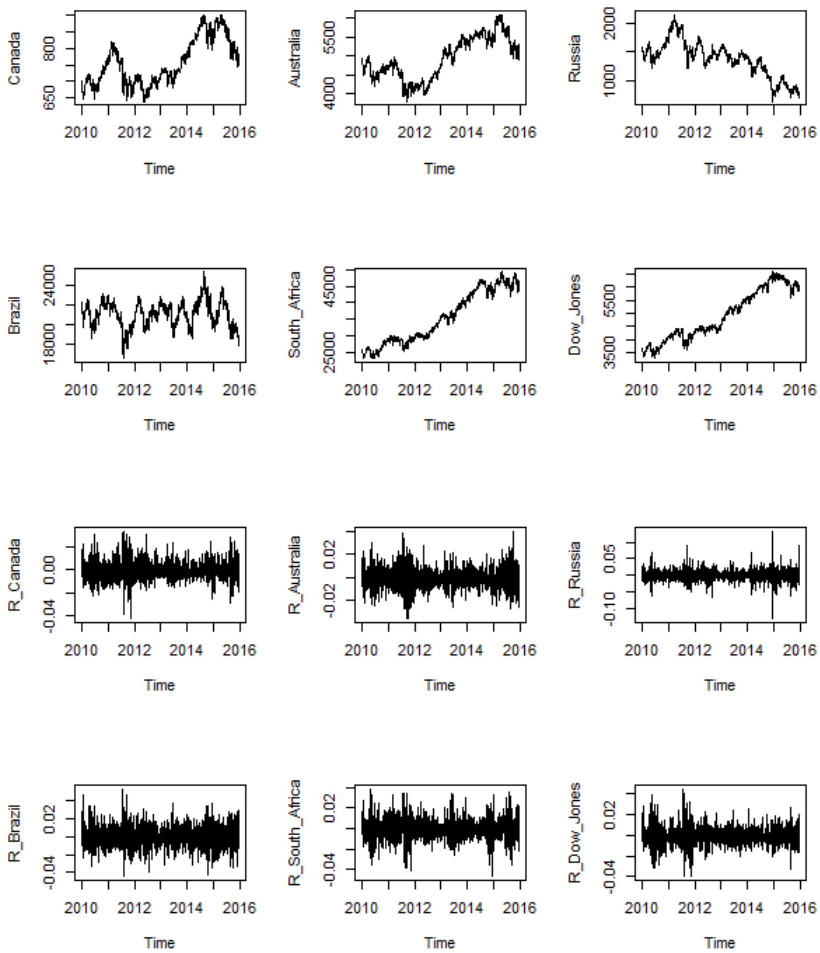
Country	Index
Canada	S&P/TSX 60
Australia	A&P/ASX All Australian 50
Russia	IRTS
Brazil	IBRX
South Africa	FTSE/JSE Top 40
USA	Dow Jones Composite

4.1 Graphical analysis

From the above, it is evident that the index values of all the indices are trended, except Russia, which seems to be mean reverting. Furthermore, the log returns of the indices all seem to be mean reverting and also tend to exhibit signs of volatility

[2] The data was obtained from the Thomson Reuters Eikon databank.

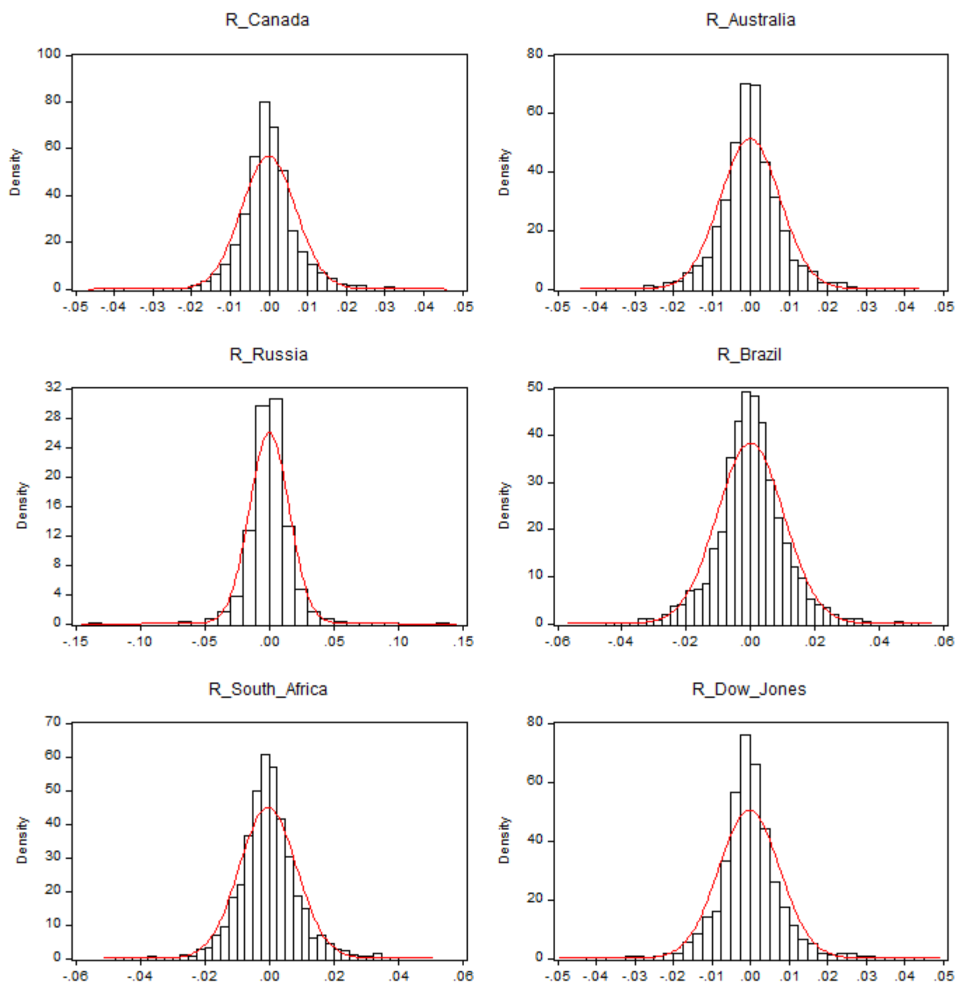
Fig. 1 Line graphs of the CARBS indices



clustering. Brooks (2014) [11] explains that volatility clustering is the tendency for volatility to occur in bunches, hence periods of high volatility are usually followed by periods of high volatility, and low volatility is expected to follow low volatility. This will be investigated in more detail in later chapters. The line graphs above are consistent with facts 4 and 6 listed above.

Figure 2 illustrates the histograms of the log returns of the indices, the normal density (in red) is also included for comparison purposes. The log return series do not seem normally distributed. The log return series are more highly peaked at the mean, and show signs of fat tails, this is referred to as leptokurtosis. In addition, the

Fig. 2 Histograms of the log returns



mean of the log return series seems to be approximately equal to zero in each case. The histograms provide evidence of facts 3 and 5 mentioned above.

Tsay (2005) [12] explains that the ACF is the correlation of a variable and lagged values of itself. Figure 3 shows that the log returns do not exhibit signs of high autocorrelation. When there is autocorrelation, it seems to die out immediately. However, when the squared returns are considered (figure 3), there is evidence of significant autocorrelation. This is consistent with the stylised facts of financial time series.

Another important aspect to consider in this section, is the statistical properties of the GMVP. By making use of the data of the CARBS indices, a GMVP is constructed by using the method outlined in section 3 The weights obtained are given by the table below:

Table 2 GMVP weights

Country	Weight
Canada	0.4760
Australia	0.3899
Russia	-0.0851
Brazil	0.0485
South Africa	0.1707

The portfolio weights above show that most of the GMVP should be invested in Canadian equity, which is close to half of the portfolio. Similar to the findings by Watson et al. (2014) [3], a small amount of the portfolio is invested in Brazilian equity. A negative amount (sell short) should be invested in Russian equity. Finally, a small positive amount should be invested in South African equity.

As mentioned, the return of the portfolio is given by equation 1. The line graph below illustrates the returns of the GMVP: It is evident that the log returns of the portfolio also show signs of volatility clustering. Furthermore, the time series seems to be mean reverting, the mean is approximately equal to zero. The histogram also indicates that the mean of the distribution is close to zero. In addition, the histogram shows signs of leptokurtosis which is consistent with the stylised facts of financial time series.
Finally, figure 6 shows that there does not seem to be significant autocorrelation when the log returns are considered. However, the squared log returns show signs of profound autocorrelation. It is important to note that visual methods are subjective and not always reliable. Therefore, the descriptive statistics are considered in the following subsection.

4.2 Descriptive statistics

The descriptive statistics in the table above confirm our expectations. The mean is approximately equal to zero for each index. The Jarque-Bera probability shows that all the return series are not normally distributed. The skewness and kurtosis show that the return series are slightly positively skewed and all show signs of leptokurtosis (kurtosis>3).

It is evident that the standard deviation of the GMVP is less than the individual assets (CARBS indices), this is consistent with expectations. The standard deviation of the GMVP is also slightly less than the standard deviation of the Dow Jones

Table 3 Descriptive statistics of the log returns

	R_Can	R_Aus	R_Rus	R_Bra	R_SA	R_DJ	R_GMVP
Mean	0	0	0.0004	0.0001	-0.0003	-0.0002	-0.0001
Median	-0.0003	-0.0001	0.0002	0	-0.0006	-0.0004	-0.0003
Maximum	0.0334	0.0401	0.1325	0.052	0.0384	0.0448	0.0267
Minimum	-0.0424	-0.0361	-0.1325	-0.0445	-0.0468	-0.0402	-0.0279
Std. Dev.	0.0070	0.0078	0.0154	0.0104	0.0089	0.0079	0.0056
Skewness	0.2039	0.1241	0.2105	0.056	0.1452	0.3683	0.1640
Kurtosis	6.2056	5.1704	10.9406	4.7523	5.3709	7.1022	6.0977
Jarque-Bera	948	433	5743	280	518	1577	881
Probability	0	0	0	0	0	0	0
Sum	-0.0865	-0.0698	0.8089	0.2213	-0.5777	-0.4921	-0.2251

index. Moreover, based on the standard deviation, the Russian equity index is the most volatile

5 Empirical results

In this section, the risk vs return tradeoff of the different indices included in this study will be analysed. This will provide an indication of whether it would be more beneficial to invest in individual portfolios rather than the GMVP. Furthermore, the systematic risk of the different indices will be estimated in the CAPM framework. The Dow Jones index will be used as a benchmark.

For the purpose of this empirical analysis, monthly return data was used. According to Bodie et al. (2013) [13], it is common to assume that treasury bills are risk-free. Because treasury bills are relatively short term investments, hence their prices are usually insensitive to changes in interest rates. Monthly data for the US treasury bill rate was obtained from the IMF International Financial Statistics databank.

5.1 Mean-variance approach

Figure 7 above illustrates the risk return trade-off of the CARBS indices and the GMVP, the dotted lines give an indication of the Sharpe ratio which is given by

$$\frac{R_i - R_f}{\sigma_i}.$$

Consider the following definition of mean-variance dominance from Bailey (2005) [1]

Portfolio A mean-variance dominates portfolio B if either of the following conditions are satisfied

1. $R_A \geq R_B$ and $\sigma_A < \sigma_B$
2. $R_A > R_B$ and $\sigma_A \leq \sigma_B$.

Hence, it is clear that the GMVP contains the least risk, and mean-variance dominates the individual assets, the indices of Canada, Australia, Russia, and Brazil. However, none of the 2 conditions above are met when the GMVP is compared to the South African index. This is considered a shortcoming of the mean-variance approach, it is unclear which portfolio is dominated by the other.

5.2 Risk measurement in the CAPM framework

As shown above, the mean-variance model can only be used in certain cases. The mean variance model does not give an indication of the expected return of the asset. Bodie et al. (2013) [13] explains that the CAPM provides a prediction of the relationship between the expected return of an asset and its risk. Therefore, the model provides a reliable estimate of the rate of return, which is invaluable when evaluating possible investments.

Bailey (2015) [1] outlines the assumptions of the CAPM as follows:

- Asset markets are in equilibrium,
- investors are rational in the sense that they behave according to mean-variance portfolio criterion, and
- investors have the same beliefs, which implies that all investors use the same probability distributions when it comes to asset returns.

The CAPM model is specified as follows,

$$R_i - R_f = \beta_i \left(R_M - R_f\right) \text{ for } i = 1, ..., n. \tag{3}$$

where R_i is the return on asset i, R_f is the risk-free rate, R_M is the rate of return on the market portfolio, n is the number of assets, and β is an estimate of the systematic risk. Clearly, the β_i of the market portfolio is equal to one.

Using a similar approach to Zivot (2013) [14], the CAPM relationship is tested by adding an intercept to equation 3. The following model is estimated

$$R_i - R_f = \alpha_i + \beta_i \left(R_M - R_f\right) \text{ for } i = 1, ..., n.$$

where α_i is the excess return on asset i. For the purpose of the empirical analysis, R_f is set equal to the US treasury bill rate, and R_M is set equal to the monthly return on the Dow Jones index. The following results were obtained: The CAPM parameters are consistent with the findings of the mean-variance model. The GMVP does have the lowest systematic risk. The variance of Brazil was the highest when

Table 4 CAPM parameters

	Canada	Australia	Russia	Brazil	SA	GMVP
α	-0.0035	-0.0054*	-0.0195**	-0.0084*	0.0019	-0.0009
β	0.6435***	0.7667***	1.2707***	0.7712***	0.7687***	0.5522***
R^2	0.4762	0.4705	0.2306	0.2605	0.4488	0.4439

*(**) [***]: Statistically significant at a 10(5)[1] % level

implementing the mean variance model, when estimating the CAPM model it has the highest value of β. Furthermore, the CAPM relationship holds for Canada, South Africa and the GMVP. The intercept (α) is statistically significant when Australia, Russia, and Brazil are considered.

Asteriou et al. (2015) [15] explain that the R^2 of a simple regression model indicates the amount of variation in the dependent variable that is explained by variation in the independent variable. The R^2 of the estimated regression models are relatively low in each case. The models explain roughly 40% of the variation in the dependent variables, and can therefore not be considered an accurate measurement of risk.

The CAPM does give an indication of the systematic risk of an asset and gives an idea of a relationship between the risk and return of an asset. However, neither the mean-variance model nor the CAPM give an indication of how the risk of an asset varies over time. This notion, and different ways of modelling time-varying volatility is discussed in the next chapter.

6 Conclusion

In this paper, using a similar approach to Zivot (2011) [8] an expression for the weights of a GMVP is derived for the CARBS indices. The weights are computed using the historical means, variances, and covariances of the indices. The GFC can be considered a structural break in the data, and is therefore excluded from the dataset. The weights obtained indicate that most of the portfolio should be invested in Canadian equity. Furthermore, a small negative amount should be invested in Russian equity and a small positive amount be invested Brazilian equity, this is consistent with the findings by Watson et al. (2015) [3]. Finally, a positive amount should be invested in South African equity.

In addition to the GMVP weights, the statistical properties of the CARBS indices and the GMVP are explored. The statistical properties of the indices and the GMVP seem to be consistent with the stylised facts of financial time series, as outlined by Embrecths (2005) [10]. The return series seem to exhibit signs of volatility clustering. However, this is investigated using a formal test in the next chapter. Another important aspect of the historical return series data is that it is not normally distributed, there are signs of leptokurtosis. This presents a challenge when it comes to the quantitative measurement of risk.

Classical methods were used to measure risk of the CARBS indices and the GMVP. A risk vs return scatter plot, and risk measurement in the CAPM framework were considered. The risk vs return scatter diagram indicates that the GMVP mean-variance dominates all the indices, except South Africa. The GMVP is the least risky according to the risk vs return scatter diagram, which is consistent with expectations. Furthermore, the Russian equity index is the most risky. In addition, the CAPM relationship holds for Canada, South Africa, and the GMVP. The systematic risk (β) of the GMVP is the lowest, and the Russian equity index is the highest. However the R^2 of all the models indicate that the CAPM relationship is not a good fit for all the variables, and can therefore not be considered a reliable measure of risk.

Another important aspect to consider, is the strong assumptions made by the mean-variance and CAPM frameworks. It is unreasonable to assume that investors have the same beliefs, which makes the model less realistic. In addition, a constant volatility is assumed, in reality volatility is very volatile as shown by figures **??** and 4. Therefore, different time-varying volatility models are considered in the next chapter in order to determine the best fitting model for the CARBS indices and the GMVP.

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Fig. 3 Autocorrelation function (ACF) graphs.

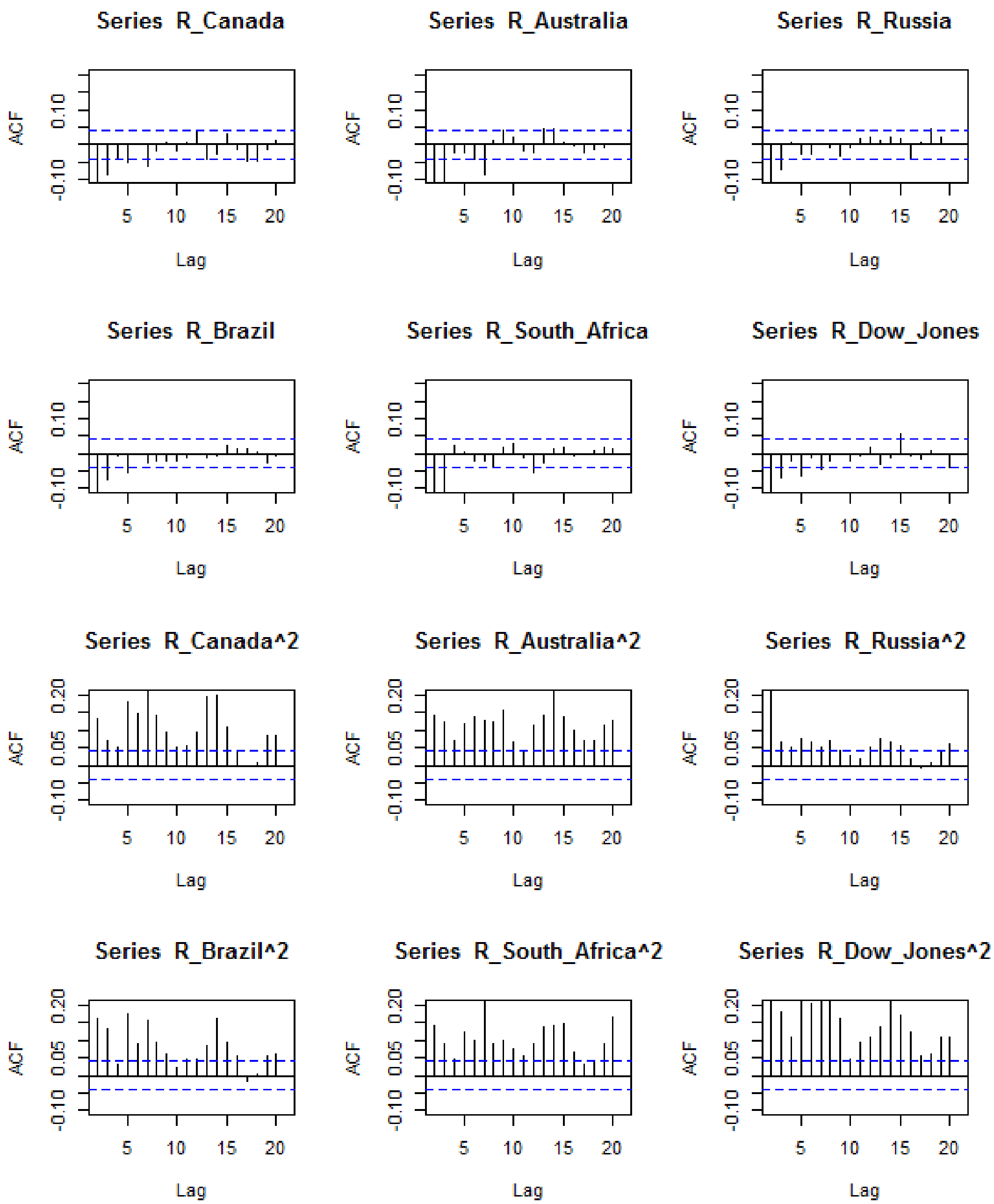

Fig. 4 Log returns of the GMVP

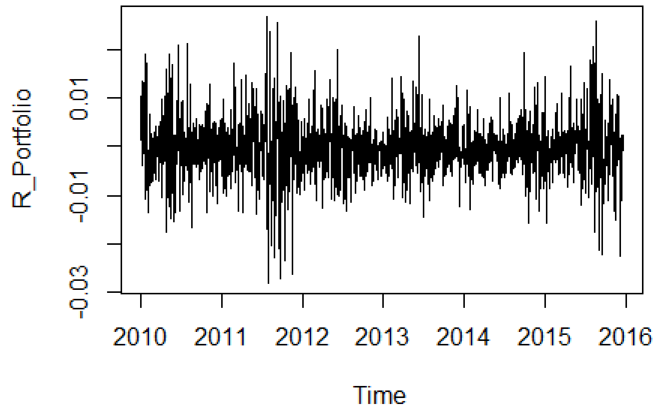

Fig. 5 Histogram of the log returns of the GMVP

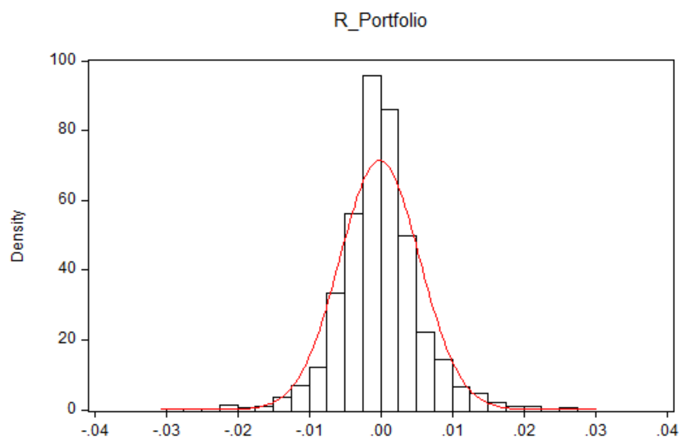

Fig. 6 Autocorrelation function (ACF) graphs of the GMVP

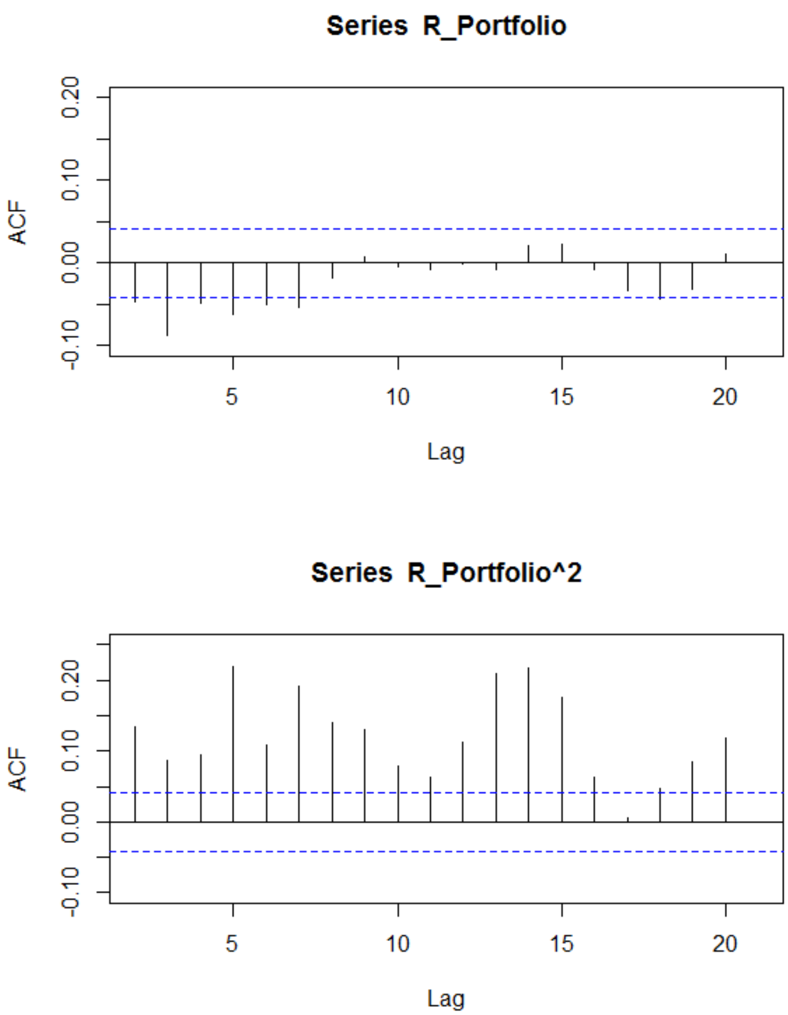

Fig. 7 Risk vs return scatter diagram