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Application of optimal control theory in finance and economy

Financial mathematics

Master’s Thesis (15 ECTS)

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Master's thesis

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Abstract. The aim of current master thesis is to give the appropriate knowledge for the full understanding of models used in the optimization of the economical processes. A comparison was made of whether the size of the company influences the order of the solution and its general look. Now it's known that both huge and tiny companies, as well as individuals, who are about to make some investment decision, and use optimal control theory for the optimization of their activity. The model of the optimal economic growth can easily find its use in real economic and experience various improvements and extensions. There might be derived the unified models for groups of typical cases, as we can say that all decisions to be made can be summed under one variable.

CERCS research specialization: P160 Statistics, operations research, programming, actuarial mathematics

Keywords: control, optimum, variation, functional, Hamiltonian

Optimaalse juhtimise rakendused majandusteaduses

Magistritöö

Olesia Kucheryk

Lühikokkuvõte. Käesoleva magistritöö eesmärk on tutvuda optimaalse juhtimise teooria põhialustega ja selle rakendusvõimalustega majandusteaduses ning finantsmatemaatikas. Töö esimesed kaks peatükki sisaldavad variatsioonarvutuse ja optimaalse juhtimise teooria põhiseoseid. Siin tuletatakse Euleri võrrandid ning transversaalsuse tingimused, samuti formuleeritakse maksimumprintsiip. Kolmandas peatükis vaadeldakse erinevaid rakendusi majandusteaduses, tuuakse kolm erinevat näidet optimaalse juhtimise teooria rakendamise kohta majanduses ja üks näide finantsmatemaatika alalt. Optimaalse juhtimise teooriat on võimalik kasutada nii suurte kui ka väikeste ettevõtete jaoks oma finantsotsuste langetamiseks optimaalse tulemuse saavutamiseks.

CERCS teaduseriala: P160 Statistika, operatsioonianalüüs, programmeerimine, finants- ja kindlustusmatemaatika.

Märksõnad: optimaalne juhtimine, variatsioon, funktsionaal, Hamiltoni funktsioon.

Table of contents

The introduction

In the modern society one can notice the tendency to optimize every possible thing that can be measured with any valuable unit. In the economy, the unlimited demand has to be satisfied with limited supply, what arises the problem of the most productive use of the given input. One can't sound the decision only on historical data, experience and objective mind, decision must have appropriate economic and mathematical methods otherwise it's just a winning-lose game. The previous economical background of the author was one of the reasons of choosing the topic as there could be found an intersection between mathematics and economy. The mathematical essentials of the optimal control are relatively new, so this theory still reaches improvements in different spheres.

The maximum principle was first introduced by Pontryagin and the team of scientists involving Boltyanski, Gamkrelidze and Mischenko [1] in 1961. Their work laid the foundation for the development of the optimal control theory. The book is written in pure scientific language what required the advance level of mathematics for understanding. So, few years after it was introduced, the first interpretations with some improvements were published. In finance, the application of the optimal control theory was studied with economists and scientists such as Sethi [2], Davis and Elzinga [3]. The earliest papers devoted to the economic interpretation of optimal control theory were made by Arrow and Shell [4], after that such scientists as Leban and Lesourne [5], Chiang [6], Hadley and M. C. Kemp [7]. The authors noted gave the appropriate and understandable interpretation of the theory that could be used. To perform perfect computation in the middle of 80'th the programming environment MATLAB was mostly used.

Optimal control theory was widely used after its introduction, but now the number of publications devoted to the topic is significantly smaller than it was before. For me there arises the question whether the optimal control theory is still relevant in finance and economy and how it can be used by different size firms. If one claims that the theory can be unified for the use in finance in economy, how will the standardized solution differ is various cases. In this thesis the answer to the question, whether some group of economic indexes can be grouped under one mathematical denotation and used, was studied.

To answer the questions above, the paper was structured so that the reader can understand the logic of the optimal control theory. The first chapter is theoretical, it is devoted to the necessary theoretical background of the optimal control theory. Here you can find the essentials of the calculus of variations, Lemma of Lagrange, Euler's equations and some necessary extensions that needed to be known to understand the information in the Chapter II. The material given is based on the result of researches made by mathematicians before Pontryagin has presented the maximum principle. It's is all unified under similar notations and logically contained. Second chapter gives explains the maximum principle and the meaning of the Hamiltonian, on the base of the theory presented in both chapters, there presented examples from finance and economy and personal understanding of the problem.

Chapter I: Foundations of the calculus of variations

1.1 Weak variation

Let's assume that a functional $J(\overrightarrow{x})$ is defined for $\vec{x} \in C_n^1[t_0, T]$. It is said that it reaches its minimum at $\overrightarrow{x_*}$, if $J(\vec{x}) \geq J(\overrightarrow{x_*})$ for each $\vec{x}$ satisfying the condition $\|\vec{x} - \overrightarrow{x_*}\|_1 < \varepsilon$, where ε is an arbitrary positive small number. Here

$$\|\vec{x}\|_1 = \max_{t \in [t_0, T]} \left(|\vec{x}(t)|, |\dot{\vec{x}}(t)| \right), \qquad (1.1.1)$$

the functional is weakly differentiable at $\vec{x}$, if there is a limit

$$\delta J = \lim_{\varepsilon \to 0} \frac{J(\vec{x} + \varepsilon \vec{h}) - J(\vec{x})}{\varepsilon}, \qquad (1.1.2)$$

where ε is a small number and $\vec{h}$ is a given function that belongs to the space $C_1[t_0, T]$. The quantity δJ in (1.1.2) is called a weak variation of the functional J. To calculate the weak variation, the equation (1.1.2) should be modified with the respect to the identity $\vec{x} = \left(\vec{x} + \varepsilon \vec{h}\right)_{\varepsilon = 0}$, and ε should be changed to $\Delta\varepsilon$. Then the equality (1.1.2) can be rewritten as

$$\delta J = \lim_{\Delta\varepsilon \to 0} \left. \frac{J\left[\vec{x} + (\varepsilon + \Delta\varepsilon)\vec{h}\right] - J\left(\vec{x} + \varepsilon \vec{h}\right)}{\Delta\varepsilon} \right|_{\varepsilon = 0}. \qquad (1.1.3)$$

Using (1.1.3) one can derive the equation that is more convenient for the calculation of weak variation

$$\delta J = \left. \frac{\partial}{\partial \varepsilon} J\left(\vec{x} + \varepsilon \vec{h}\right) \right|_{\varepsilon = 0}. \qquad (1.1.4)$$

Let us denote $\overrightarrow{x_\varepsilon} = \vec{x} + \varepsilon \vec{h}$ and $J_\varepsilon = J(\overrightarrow{x_\varepsilon})$. We assume that the functional (1.1.1) has an extremum at $\overrightarrow{x_*}$. The right-hand side of (1.1.4) can be considered as a function of ε. It is known that if $\varepsilon = 0$, then the function has the extremum [8]. Consequently, according to the necessary condition of the extremum of the function with one variable

$$\left. \frac{\partial J_\varepsilon}{\partial \varepsilon} \right|_{\varepsilon = 0} = 0. \qquad (1.1.5)$$

Comparing (1.1.5) with (1.1.4) it becomes obvious that at the point of extremum the first variation vanishes. Thus, the necessary condition of optimality is

$$\delta J(\overrightarrow{x_*}) = 0. \qquad (1.1.6)$$

1.2 Lemma of Lagrange

While deriving the necessary conditions of optimality, it is convenient to use the Lemma of Lagrange. If $\vec{\varphi} = \vec{\varphi}(t)$ is a vector function so that t belongs to the interval $[t_0, T]$, then consequently $\vec{\varphi} \in C_n^1[t_0, T]$. Let's assume that $\vec{h}(t)$ is continuously differentiable function on the interval $[t_0, T]$. Then the following lemma holds good.

If for all continuous and differentiable $\vec{h}(t)$, that satisfies the conditions $\vec{h}(t_0) = 0$ and $\vec{h}(T) = 0$

$$\int_{t_0}^{T} \sum_{j=0}^{n} \varphi_j(t) h_j(t) dt = 0 \tag{1.2.1}$$

then $\vec{\varphi}(t) = 0$.

1.3 The Euler equations

Let's derive Euler's equations. The easiest and the most fundamental issue of the calculus of variations is the minimization or maximization of a functional. This means finding the best way between A and B (Fig.1.1). Usually line connecting A and B is a smooth curve, so the aim is to find the smooth line between points A and B so that it is the best path that minimizes a cost criterion.

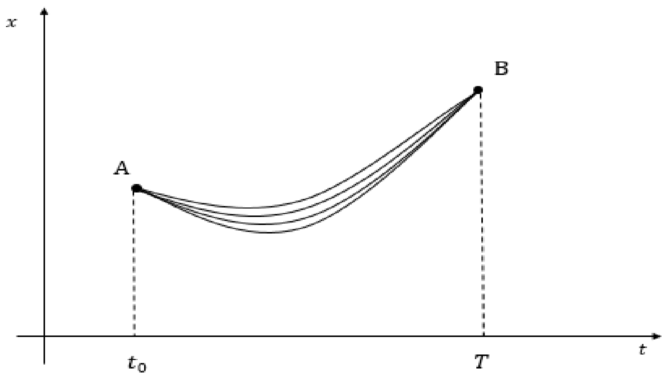

Figure 1.1. Smooth curves connecting points A and B. (see Lellep [9])

Let's define the cost function as

$$J = \int_{t_0}^{T} F(\vec{x}, \dot{\vec{x}}, t)dt. \tag{1.3.1}$$

Let's assume that integrand function F is continuous and twice differentiable. A smooth $\vec{x} = \vec{x}(t)$ that yields an extremum to J is called an extremal. Here $\dot{\vec{x}}$=$\dot{\vec{x}}(t)$ and $\ddot{\vec{x}} = \ddot{\vec{x}}(t)$ are continuous on the interval $[t_0, T]$. The aim of this section is to find the curve that minimizes the functional (1.3.1). For maximizing the same approach is used. So J reaches its extremum at point $\vec{x} = \vec{x}(t)$. When minimizing the cost function (1.3.1) it's assumed that

$$\vec{x}(t_0) = \overrightarrow{x_0},$$
$$\vec{x}(T) = \overrightarrow{x_T}. \tag{1.3.2}$$

According to (1.1.6) one can write that the weak variation of the function of J equals to zero:

$$\delta J = \int_{t_0}^{T} \sum_{j=1}^{n} \left(\frac{\partial F}{\partial x_j} h_j + \frac{\partial F}{\partial \dot{x}_j} \dot{h}_j \right) dt = 0, \tag{1.3.3}$$

where $\dot{h}_j = \frac{d}{dt} h_j$.

The integration by parts in (1.3.3) gives

$$\int_{t_0}^{T} \frac{\partial F}{\partial \dot{x}_j} \dot{h}_j dt = \frac{\partial F}{\partial \dot{x}_j} h_j \Bigg|_{t_0}^{T} - \int_{t_0}^{T} \frac{d}{dt} \frac{\partial F}{\partial \dot{x}_j} h_j dt. \tag{1.3.4}$$

The weak variation will take the form

$$\delta J = \sum_{j=1}^{n} \frac{\partial F}{\partial x_j} h_j \Bigg|_{t_0}^{T} + \int_{t_0}^{T} \sum_{j=1}^{n} \left(\frac{\partial F}{\partial x_j} h_j - \frac{d}{dt} \frac{\partial F}{\partial \dot{x}_j} h_j \right) dt = 0. \tag{1.3.5}$$

Under the terms of the task, all curves comparable to those which are sought among extremums undergo predetermined points $\overrightarrow{x_0}$ and $\overrightarrow{x_T}$. As it is shown in (1.1.4), if the extremum is achieved at

$$\vec{x} = \vec{x}(t)$$

then the appropriate curves are given by the equation

$$\vec{x} = \vec{x}(t) + \delta\vec{x}(t), \tag{1.3.6}$$

where $\delta\vec{x}(t) = \varepsilon\vec{h}(t)$. Here $\vec{h}(t)$ is a function of t where t belongs to the interval $[t_0, T]$ is a continuously differentiable function, ε is a small number. As $\vec{x}$ and $\vec{h}(t)$ are given functions, each value of ε will determine a particular value of J.

As $\vec{x}(t_0) = \overrightarrow{x_0}$ and $\vec{x}(T) = \overrightarrow{x_T}$, then

$$\overrightarrow{x_0} = \vec{x}(t_0) + \delta\vec{x}(t_0)$$

and

$$\overrightarrow{x_T} = \vec{x}(T) + \delta\vec{x}(T).$$

As all boundary conditions are met it can be said that

$$\delta\vec{x}(t_0) = 0 \tag{1.3.7}$$

and

$$\delta\vec{x}(T) = 0. \tag{1.3.8}$$

We have to use the equation (1.3.5) where $\delta\vec{x}(t_0) = \delta\vec{x}(T) = 0$. Thus, we obtain

$$\int_{t_0}^{T} \sum_{j=1}^{n} \left(\frac{\partial F}{\partial x_j} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}_j} \right) \frac{d}{dt} \delta x_j dt = 0.$$

Here δx_j $(j = 1,2 \dots n)$ is arbitrary. Thus, according to the lemma of Lagrange one obtains

$$\frac{\partial F}{\partial x_j} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}_j} = 0 \tag{1.3.9}$$

for all $t \in [t_0, T]$ and $j = 1, \dots, n$. This formula is known as Euler's equation and since now we can start applying it for particular mathematical issues. The equation (1.3.9) is a necessary condition for weak extremum, that means that the functional can reach extremum only on curves, that satisfy the Euler's equation. As weak extremum is meantime a strong extremum, then the necessary conditions of weak extremum are the necessary conditions for strong extremum, but not the opposite.

1.4 Extensions

1.4.1 Problems with functional constraints

Let's find the extremum of the functional (1.3.1) with additional constraints

$$g_j(\vec{x}, t) = 0, \tag{1.4.1}$$

for $j = 1, \dots, q$ and boundary conditions (1.3.2).

Let's assume F and g_j to be continuous up to the second order derivatives. Beside that the rank of $\frac{\partial g_j}{\partial x_i}$ is equal to q, for all $q < n$. To derive the necessary conditions for the extremum, one should use the method of Lagrange multipliers. The essence of this method is to use the extended functional $J_* = J + < \varphi , g >$ to find the conditional stationary for the functional J with $g = 0$. Here $< \varphi , g >$ is the scalar product, φ and g can be vectors or scalars. For the given issue

$$< \varphi , g >= \int_{t_0}^{T} \sum_{j=1}^{q} \varphi_j(t) g_j(\vec{x}, t) dt, \tag{1.4.2}$$

where $\vec{\varphi} = (\varphi_1, \dots \varphi_q)$ and $\vec{g} = (g_1, \dots, g_q)$, φ_j is an unknown Lagrange multipliers. Consequently, one can apply the condition $\delta J_* = 0$, where

$$J_* = J + \int_{t_0}^{T} \sum_{j=1}^{q} \varphi_j g_j dt. \tag{1.4.3}$$

Now one should calculate the weak variation and equalize it to zero

$$\int_{t_0}^{T} \sum_{i=1}^{n} \left(\frac{\partial F}{\partial x_i} - \frac{d}{dt} \frac{\partial F}{\partial \dot{x}_i} + \sum_{j=1}^{q} \varphi_j \frac{\partial g_j}{\partial x_i} \right) \delta x_i dt = 0. \tag{1.4.4}$$

Equation (1.4.4) allows to derive the modified Euler's equation

$$\frac{\partial F}{\partial x_i} + \sum_{j=1}^{q} \varphi_j \frac{\partial g_j}{\partial x_i} - \frac{d}{dt} \frac{\partial F}{\partial \dot{x}_i} = 0, \tag{1.4.5}$$

or

$$\frac{\partial F_*}{\partial x_i} - \frac{d}{dt} \frac{\partial F_*}{\partial \dot{x}_i} = 0, \tag{1.4.6}$$

for $i = 1, \dots, n$.

1.4.2. Problem with integral constraints

Let's solve the same problem that was described in section 1.4.1, so that instead of constraints (1.4.1) one has integral constraints

$$\int_{t_0}^{T} g_i(\vec{x}, t) dt = A_i, \tag{1.4.7}$$

for $i = 1, \dots, n$, where A_i is a set of given constants [8]. Here k can take the value of any natural number. The method of Lagrange multipliers is valid for the problems with integral constraints, so the modified Euler's equation is derived by the same way as in (1.4.1), but with constant $\varphi_j, j = 1, \ldots, k$.

1.4.3 Problems with unknown terminal time

Transversality conditions are used when the terminal or initial point is variable, as the boundary condition is not fixed. In the current case one can also apply the method of Lagrange multipliers to derive the optimal path. Let's solve the case when the terminal time T is preliminary unknown. Now

$$J_\varepsilon = \int_{t_0}^{T_\varepsilon} F(\vec{x}, \dot{\vec{x}}, t)dt. \tag{1.4.8}$$

Although, the boundary conditions can be arbitrary. Let us assume for the simplicity that the boundary conditions (1.1.2) are satisfied by the optimal solution. Evidently the optimal curve satisfies the Euler's equations

To receive the needed conditions for the extremal the first step is using ε to generate a perturbing curve to be compared with extremal. Perturbing curve $\vec{h}(t)$ creates the neighboring paths that must pass through endpoints (Fig.1.2) A and $B2$. Let's assume that T is given optimal terminal time, then all T_ε are in the immediate neighborhood. The property may be written as

$$T_\varepsilon = T + \varepsilon \Delta T\ , \tag{1.4.9}$$

where T is given and ΔT is its small change. Evidently T is a function of ε, its derivative will take the form

$$\frac{dT_\varepsilon}{d\varepsilon} = \Delta T.$$

To find the neighboring paths of the extremal $\vec{x}(t)$ one can state that (see (1.3.6))

$$\overrightarrow{x_\varepsilon}(t) = \vec{x}(t) + \varepsilon\vec{h}(t), \tag{1.4.10}$$

where $\vec{h} = \vec{h}(t)$ is a smooth vector function on the interval $[t_0, T]$. Substituting (1.4.10) into the given functional (1.3.1) and taking into account that T is a function of ε, the following function is received

$$J(\varepsilon) = \int_{t_0}^{T_\varepsilon} F\left(\vec{x} + \varepsilon\vec{h}, \dot{\vec{x}} + \varepsilon\dot{\vec{h}}, t\right) dt. \tag{1.4.11}$$

1.4.4 Transversality conditions

The transversality conditions can be derived in three steps. Firstly, let us define

$$\frac{\partial J}{\partial \varepsilon} = \int_{t_0}^{T_\varepsilon} \frac{\partial F}{\partial \varepsilon} dt + F\left(\vec{x}(T), \dot{\vec{x}}(T), T\right) \frac{\partial T}{\partial \varepsilon}. \tag{1.4.12}$$

According to the Leibnitz's rule we can differentiate under the integral sign. Since we assumed that $\vec{h}(t_0) = 0$, but $\vec{h}(T) \neq 0$ one has

$$\int_{t_0}^{T_\varepsilon} \frac{\partial F}{\partial \varepsilon} dt = \int_{t_0}^{T} \sum_{j=1}^{n} h_j(t) \left(F_{xj} - \frac{d}{dt} F_{\dot{x}_j}\right) dt + \sum_{j=1}^{n} \left(F_{\dot{x}_j}\right)_{t=T} h_j(T) \tag{1.4.13}$$

and

$$F(x(T), \dot{x}(T), T) \frac{\partial T}{\partial \varepsilon} = (F)_{t=T} \Delta T.$$

After substituting these two formulas into (1.4.11) it takes the form

$$\sum_{j=1}^{n} \left(\int_{t_0}^{T} h_j(t) [F_{xj} - \frac{d}{dt} F_{\dot{x}j}] dt + \left[F_{\dot{x}j}\right]_{t=T} h_j(T) \right) + (F)_{t=T} \Delta T = 0 \tag{1.4.14}$$

Since $h_j(t)$, where $j = 1,2,\dots,n$ and ΔT are independent, the last term in (1.4.14) must be equal to zero. The last terms relate only to the terminal condition, while the first one equals zero, because of Euler's equations.

Second, we should get rid of $\vec{h}(t)$ transforming it into terms ΔT and $\Delta\overrightarrow{x_T}$, that denotes the change in T, and $\overrightarrow{x_T}$, that denotes the two principle variables and in the variable-terminal point issue. The easiest way is to do it is to build a graph (Fig.1.2) that will illustrate $\vec{x}(t)$ and $\vec{h}(t)$ meantime. In Fig. 1.2 the variations of the trajectory and of the terminal line are shown.

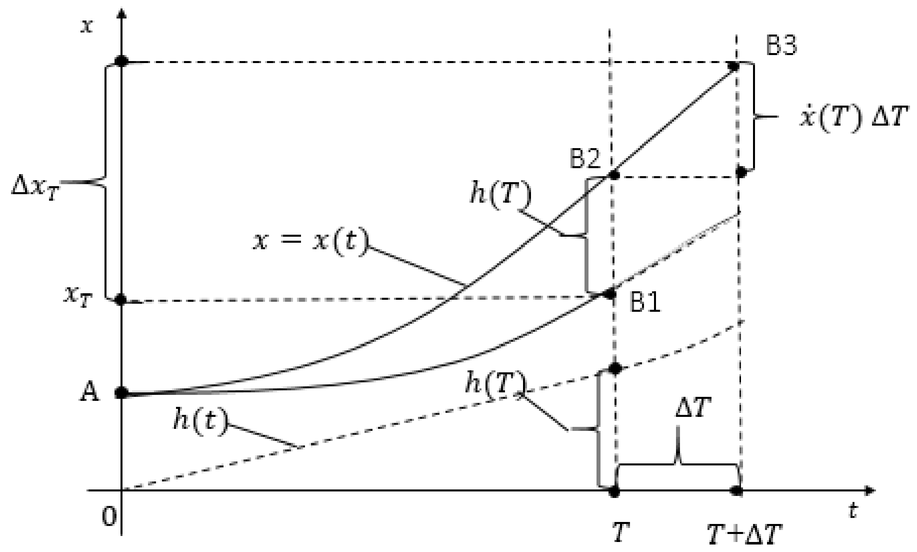

Figure 1.2. Total variation (see Chiang [6])

Basically, here curve $AB2$ is the neighboring path that starts with the same initial point. Curve $AB1$ was perturbed with $\varepsilon\vec{h}(t)$. $B1B2$ is the segment that characterize the change in $\vec{x}(t)$ caused by perturbation, and we can change T with $\varepsilon\Delta T$. As a consequence, $\vec{x}(t)$ has been pushed up further by segment $B2B3$. If we assume that ΔT is small, we can conclude that next change in $\vec{x}(t)$ is approximated by $\dot{\vec{x}}(T)\Delta T$. The entire change in $\vec{x}(t)$ between points $B2$ and $B3$ that is called total variation, can be written as

$$\Delta\overrightarrow{x_T} = \vec{h}(T) + \dot{\vec{x}}(T)\Delta T. \tag{1.4.15}$$

This approximation lets us derive a formula for the weak variation and the total variation coupling

$$\vec{h}(T) = \Delta\overrightarrow{x_T} - \dot{\vec{x}}(T)\Delta T. \tag{1.4.16}$$

Using the approximation (1.4.16) we can easily avoid $\vec{h}(T)$. So, to reach the general transversality condition (1.4.14) should be rewritten without the integral term as

$$\left(F - \sum_{j=1}^{n} \dot{x}_j F_{\dot{x}_j}\right)_{t=T} \Delta T + \sum_{j=1}^{n} \left(F_{\dot{x}_j}\right)_{t=T} \Delta x_{T_j} = 0. \tag{1.4.17}$$

The role of the derived condition is to replace the missing terminal point in the current problem and can be relevant to only one point of time T. It can be written in different forms depending on peculiarities of the terminal line.

For fixed T there won't be any change in T, what means that the terminal line will be vertical, as $\Delta T = 0$. The drop out the first term in (1.4.17) will be resulted. If $\Delta\vec{x}_T$ is arbitrary, then

$$\left(F_{\dot{x}_j}\right)_{t=T} = 0,$$

for $i = 1,2,\dots,n$. In case if the situation is opposite $\Delta\overrightarrow{x_T} = 0$, terminal line is horizontal and T is arbitrary, there will be a drop out of the second sum in (1.4.17). The only way to eliminate the term with ΔT is to make the entire expression in brackets equal to zero. In this case, the transversality conditions will take the form

$$\left(F - \sum_{j=1}^{n} \dot{x}_j F_{\dot{x}_j}\right)_{t=T} = 0. \tag{1.4.18}$$

Chapter II: Optimal control theory

2.1 The Hamiltonian

In the calculus of variations, the problem of optimization includes the variables $\vec{x} = \vec{x}(t)$. In the optimal control theory, we have to deal with one or more control variables. To reach the key insights of optimal control theory it's necessary first to determine the problem of finding optimal state variable $\vec{x}$, what means the finding of the optimal control variable $\vec{u}$ and the optimal state path $\vec{x}(t)$. The problem posed as a minimization problem since the maximization can be easily transformed into minimization one by just putting minus sign in front of the functional. The functional takes the form

$$J = \int_{t_0}^{T_\varepsilon} F(\vec{x}, \vec{u}, t)dt. \tag{2.1.1}$$

The functional (2.1.1) must be minimized among the solutions of the system

$$\dot{\vec{x}} = \vec{f}(\vec{x}, \vec{u}, t). \tag{2.1.2}$$

The boundary conditions can be different. For instance, in case of a problem with fixed initial point and free terminal point one has

$$\begin{gathered} \vec{x}(t_0) = \overrightarrow{x_0} \\ \vec{x}(T) = \text{free.} \end{gathered} \tag{2.1.3}$$

Similarly, in the case of a free initial point and fixed terminal point ones has

$$\vec{x}(T) = \overrightarrow{x_T},$$

$\vec{x}(t_0)$ being arbitrary.

It's assumed that for each $t \in [t_0, T]$ the control function satisfies the requirement

$$\vec{u} \in U, \tag{2.1.4}$$

where U is a convex closed set of admissible controls. Function $F(\vec{x}, \vec{u}, t)$ no longer contains the derivative $\dot{\vec{x}}$, but it depends on states variables x_j $(j = 1, \dots, n)$ and controls u_i $(i = 1, \dots r)$. The connection between them can only be seen through the first order differential equations

$$\frac{dx_j}{dt} = f_j(\vec{x}, \vec{u}, t),$$

for $j = 1, \dots, n$. Assume that at time $t = t_0$ $\vec{x}(t_0) = \overrightarrow{x_0}$. Depending on different $\vec{u}(t)_1, \dots \vec{u}(t)_r$ the derivative will have different values, what will predetermine a special direction of movement for $\vec{x}$. The first order necessary condition is known as the maximum principle and it's the most important result of optimal control theory [6]. In order to formulate H, let us introduce the Hamiltonian. In this case, a new adjoint variable $\vec{\varphi}$ is added to the existing variables $t, \vec{x}$ and $\vec{u}$. Later it will be shown that $\vec{\varphi}$ is closely related to Lagrange multipliers and it evaluates the shadow price of the problem. It depends on time as well as $\vec{x}(t)$ and $\vec{u}(t)$. The Hamiltonian function, that is more often called *the Hamiltonian*, is the tool to be used to reach the optimal control problem. It contains of integrand function F and the product of the adjoint variable with function $\vec{f}$, and takes the form

$$H = \varphi_0 F(\vec{x}, \vec{u}, t) + \sum_{j=1}^{n} \varphi_j f_j(\overrightarrow{x}, \overrightarrow{u}, t). \tag{2.1.5}$$

We assume that $\varphi_0 = -1$, as it can't be equal to zero and should be a negative constant, that can be normalized to unity.

The application of maximum principle applied to the Hamiltonian involves pair of first-order partial derivatives $\frac{d\vec{x}}{dt}$ and $\frac{d\vec{\varphi}}{dt}$. The maximum principle states that the Hamiltonian must be maximized with respect to $\vec{u}$ at every time instant. The maximization of $H(\vec{x}, \vec{u}, \vec{\varphi}, t)$ is to be done in accordance with the relations

$$\frac{dx_J}{dt} = \frac{\partial H}{\partial \varphi_j}$$

and

$$\frac{d\varphi_j}{dt} = -\frac{\partial H}{\partial x_j} \tag{2.1.6}$$

for $j = 1, \dots, n$.

2.2 The variational problem

The transversality condition corresponding to (2.1.3) will take the form $\vec{\varphi}(T) = 0$. Let us consider now the variational problem (1.3.1) - (1.3.3) with the integrand F. This problem can be treated as a particular problem of the optimal control.

In the case of problems of the calculus of variations it can be stated that

$$\dot{x}_j = u_j \tag{2.2.1}$$

$f$or $j = 1, \dots, n$. The equation (2.2.1) can be treated as the state equations for the variational problem (1.3.1), (1.3.2). Let's write the Hamiltonian function in the following way

$$H = -F(\vec{x}, \vec{u}, t) + \sum_{j=1}^{n} \varphi_j u_j . \tag{2.2.2}$$

According to the principle of maximum the function $H = H(\vec{x}, \vec{u}, \vec{\varphi}, t)$ attains maximum with respect to $\vec{u}$, we receive the following system that will give us the value. Since the maximum is a local maximum, the condition $\frac{\partial H}{\partial u_j} = 0$ must be satisfied. Evidently,

$$\frac{\partial H}{\partial u_j} = -\frac{\partial F}{\partial u_j} + \varphi_j = 0. \tag{2.2.3}$$

From (2.2.3) one obtains

$$\varphi_j = \frac{\partial F}{\partial u_j}. \tag{2.2.4}$$

It can be seen from (2.2.2) that

$$\frac{\partial H}{\partial \varphi_j} = u_j. \tag{2.2.5}$$

According to (2.1.6)

$$\dot{\varphi}_J = -\frac{\partial H}{\partial x_j}. \tag{2.2.6}$$

The transversality condition yields in the case of free terminal state

$$\varphi_j(T) = 0. \tag{2.2.7}$$

According to (2.2.2)

$$\frac{\partial H}{\partial x_j} = -\frac{\partial F}{\partial x_j}. \tag{2.2.8}$$

Thus, according to (2.2.6) one has

$$\dot{\varphi}_J = \frac{\partial F}{\partial x_j}. \qquad (2.2.9)$$

Combining (2.2.4) and (2.2.9) gives

$$\frac{\partial F}{\partial x_j} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}_J} = 0\,, \qquad (2.2.10)$$

for $j = 1, \dots, n$. It is seen that this system of equations is identical to Euler's equations, that was derived previously. So, there is an obvious connection between the Hamiltonian and Euler's equation. Since in the case of the maximum of H, its second derivative must be negative. Thus, $\frac{\partial^2 H}{\partial u_j \partial u_i}$ is negatively assigned.

Taking into account the transversality condition, that is obtained in the present case, it can be written

$$\left(\frac{\partial F}{\partial \dot{x}_j}\right)_{t=T} = 0. \qquad (2.2.11)$$

It was also obtained in the calculus of variations (see Troickij [11], Lellep [6]).

If we have an issue with horizontal terminal n the transversality condition will take the form

$$\left(F - \sum_{j=1}^{n} \varphi_j u_j\right)_{t=T} = 0. \qquad (2.2.12)$$

The last equation can be rewritten as

$$\left(F - \sum_{j=1}^{n} \frac{\partial F}{\partial \dot{x}_J}\dot{x}_J\right)_{t=T} = 0. \qquad (2.2.13)$$

So, all the issues in the calculus of variations can be derived from the maximum principle and the Hamiltonian function.

2.3 The principle of maximum

In the calculus of variations, the Hamiltonian is assumed to be differentiable with respect to the control variable $\vec{u}$, and the equalities $\frac{\partial H}{\partial u_i} = 0, i = 1, \dots, n$ replace the condition of maximum of the Hamiltonian. However, in the theory of optimal control these conditions are not satisfied in general.

Consider now the control problem, which consists in the minimization of the functional

$$J = \int_{t_0}^{T} F(\vec{x}, \vec{u}, t)\, dt, \tag{2.3.1}$$

subjected to differential constraints

$$\dot{\vec{x}} = \vec{f}(\vec{x}, \vec{u}, t).$$

To apply the maximum principle, the equation of the motion should be incorporated into the objective functional, rewritten in terms of Hamiltonian. The equation $\dot{\vec{x}} = \vec{f}(\vec{x}, \vec{u}, t)$ must be satisfied for all t on the interval $[t_0, T]$. In this case, evidently,

$$\int_{t_0}^{T} \sum_{j=1}^{n} \varphi_j\,(t)\big(f_j(\vec{x}, \vec{u}, t) - \dot{x}_j\big) dt = 0 \tag{2.3.2}$$

We can add (2.3.2) to the functional J without changing its value [6]. Thus, the extended functional can be presented as

$$J^* = J + \int_{t_0}^{T} \sum_{j=1}^{n} \varphi_j\,(t)\big(-f_j(\vec{x}, \vec{u}, t) + \dot{x}_j\big) dt$$

$$= \int_{t_0}^{T} \left(\mathrm{F}(\vec{x}, \vec{u}, t) + \sum_{j=1}^{n} \varphi_j\,(t)\big(-f_j(\vec{x}, \vec{u}, t) + \dot{x}_j\big) \right) dt. \tag{2.3.3}$$

Evidently, $J = J^*$, but the difference between them is that the derivative of J^* will be different. Inserting the Hamiltonian defined by (2.1.5) into (2.3.3) the following equality is obtained

$$J^* = -\int_{t_0}^{T} \left(H(\vec{x}, \vec{u}, \vec{\varphi}, t) - \sum_{j=1}^{n} \varphi_j\,(t) \dot{x}_j \right) dt =$$

$$= \int_{t_0}^{T} -H\,(\vec{x}, \vec{u}, \vec{\varphi}, t) dt + \int_{t_0}^{T} \sum_{j=1}^{n} \varphi_j\,(t) \dot{x}_j dt. \tag{2.3.4}$$

Here $\vec{x}$ and $\vec{u}$ are optimal trajectory and optimal control, respectively. The non-optimal control can be presented in the form $\vec{u} + \Delta\vec{u} \in U$, so that $\Delta\vec{u}(t) = 0$ if $t \in [\tau, \tau + \varepsilon]$. Evidently, moving from non-optimal trajectory to the optimal one, the increase of the functional J_* is non-negative [8].

Let's write the Hamiltonian

$$H = -F(\vec{x}, \vec{u}, t) + \sum_{j=1}^{n} \varphi_j(t)\, f_j(\vec{x}, \vec{u}, t). \tag{2.3.5}$$

The Hamiltonian (2.3.5) must be maximized among the solutions of the state equations and adjoint equations

$$\dot{\varphi}_J = -\frac{\partial H}{\partial x_i}. \tag{2.3.6}$$

The difference of the values of the functional J_* can be written as

$$\Delta J = \int_{t_0}^{T} \Bigg(\quad (F(\vec{x} + \Delta\vec{x}, \vec{u} + \Delta\vec{u}, t) - F(\vec{x}, \vec{u}, t) +$$

$$+ \sum_{j=1}^{n} \varphi_j(t)\big(\Delta\dot{x}_J - f_j(\vec{x} + \Delta\vec{x}, \vec{u} + \Delta\vec{u}, t) + f_j(\vec{x}, \vec{u}, t)\big) \Bigg) dt. \tag{2.3.7}$$

Simplifying and rewriting the equation (2.3.7) in case of the Hamiltonian (2.3.5), we have

$$\Delta J = \int_{t_0}^{T} \left(-\sum_{j=1}^{n} \dot{\varphi}_j(t)\Delta x_j(t) - H(\vec{x} + \Delta\vec{x}, \vec{u} + \Delta\vec{u}, \vec{\varphi}, t) + H(\vec{x}, \vec{u}, \vec{\varphi}, t) \right) dt. \tag{2.3.8}$$

We know that the change of state variable $\vec{x}$ and control variable $\vec{u}$ equals to zero and $t \leq \tau$ if $t > \tau + \varepsilon$ [6]. The integral in (2.3.8) can be divided into parts. Taking into the consideration (2.3.6) we have

$$\Delta J = \int_{\tau}^{\tau+\varepsilon} \left(\sum_{j=1}^{n} \frac{\partial H}{\partial x_j}\Delta x_j(t) - H(\vec{x} + \Delta\vec{x}, \vec{u} + \Delta\vec{u}, \vec{\varphi}, t) + H(\vec{x}, \vec{u}, \vec{\varphi}, t) \right) dt +$$

$$+ \int_{\tau+\varepsilon}^{\tau} \left(\sum_{j=1}^{n} \frac{\partial H}{\partial x_j}\Delta x_j(t) - H(\vec{x} + \Delta\vec{x}, \vec{u}, \vec{\varphi}, t) - H(\vec{x}, \vec{u}, \vec{\varphi}, t) \right) dt. \tag{2.3.9}$$

Those two integrals from the equation (2.3.9) can be rewritten as

$$\Delta J = -\int_{\tau}^{\tau+\varepsilon} \big(H(\vec{x}, \vec{u} + \Delta\vec{u}, \vec{\varphi}, t) - H(\vec{x}, \vec{u}, \vec{\varphi}, t)\big) dt + \eta. \tag{2.3.10}$$

In equation (2.3.9) η is a small value of ε^2 order (see Lellep [6]). So, as the value of the ΔJ is nonnegative and ε is a small value, one can write an inequality

$$H(\vec{x}, \vec{u}, \vec{\varphi}, t)|_{t=\tau} \geq H(\vec{x}, \vec{u} + \Delta\vec{u}, \vec{\varphi}, t)|_{t=\tau}. \tag{2.3.11}$$

As $\tau + \varepsilon \in [t_0, T]$, the optimal control satisfies the maximum principle

$$H(\vec{x}(t), \vec{u}(t), \vec{\varphi}(t), t) = \max_{\dot{\vec{x}} \in U} H(\vec{x}(t), \dot{\vec{x}}(t), \vec{\varphi}(t), t). \qquad (2.3.12)$$

So, if the control $\vec{u}$ and trajectory $\vec{x}$ give the minimum to the functional (2.3.1), so that the boundary conditions (2.3.2) are met and the control belongs to the closed set U, then there exists a continuous vector function $\vec{\varphi} = (\varphi_1, \ldots, \varphi_n)$ that satisfies (2.3.6) so that for each $t \in [t_0, T]$, the Hamiltonian (2.3.5) attains its maximum over the set U.

It should be noted the maximum principle (2.3.12) is just a necessary condition of optimality. In various situations, non-optimal conditions may satisfy the maximum principle as well, because the principle of maximum presents a necessary condition of optimality.

Chapter III: The application of optimal control theory

3.1 Simple economic interpretation of optimal control theory

In theory, everything looks good but when coming to the real issues the mathematics described in the first before has to be converted into applicable terms and be economically interpreted so that it's easy to use. Each mathematical term that was used has its intuitive meaning in economy, this statement is well described in the article published by Robert Dorfman [10] in that is still relevant, but need to be amended according to changes that have happened during the last years. To connect reality with the theory the first step is to match meanings in the maximum principle. In his work Dorfman assumed that there is a company that wants to maximize its total profit over some period of time $[0, T]$, here we assume that $t_0 = 0$.

Let's assume k to be the value of the capital, at any time t company has to make some business decisions, like price of output, supplies cost, rate of output and others. Let us denote these factors by the vector $\vec{u} = (u_1, \dots, u_r)$ As capital and decision making process are interdependent one can introduce the profit function as $\varrho = \varrho(k(t), \vec{u}(t), t)$. It's known that there is a dependency between $\vec{u}(t)$ and $k(t)$ as decisions are made upon rate at which capital changes.

According to this, the total profit earned over time period T is the solution of the optimal control problem, that consists in the maximization (see Chiang [6])

$$J = \int_0^T \varrho(k, \vec{u}, t) dt. \tag{3.1.1}$$

The rate of change of the capital stock k at any moment is a function of the current standing, time and the decision made. Thus, one can state that

$$\dot{k} = f(k, \vec{u}, t). \tag{3.1.2}$$

These two formulas above (2.1.1) and (2.1.2) describe a problem of optimal control. Strictly speaking, the main problem is to find $\vec{u}$ so that the total profit J is as big as possible under the condition, that the rate of the capital satisfies (3.1.2).

The problem (2.1.1) and (2.1.2) will be treated as a particular problem of the optimal control (2.1.1), (2.1.2). Here the state variable k and the control variable is $\vec{u}$. Thus, $n = 1$ and $F = \rho$.

In the present case. Therefore, the Hamiltonian function is

$$H = -\varrho(k, \vec{u}, t) + \varphi(t) f(k, \vec{u}, t). \tag{3.1.3}$$

After applying (2.2.8) to the existing problem and putting in all the values for all variables

$$J^* = \int_0^T -(H(k, \vec{u}, \varphi, t) + f(k, \vec{u}, t)\varphi(t))dt - \varphi(T)k(T) + \varphi(0)k_0 , \tag{3.1.4}$$

where $k_0 = k(0)$. In equation (3.1.4) $\varphi(t)$ measures the shadow price of capital at each time moment. The transversality conditions take the form

$$\varphi(0) = \frac{\partial J^*}{\partial k_0},$$
$$\tag{3.1.5}$$
$$\varphi(T) = -\frac{\partial J^*}{\partial k(T)}.$$

The first equation in (3.1.5) shows the interdependency between the functional J^* and initial capital stock, while the other equation in (3.1.5) displays the negative rate of change of J^* with respect to the terminal capital stock.

The functional (3.1.1) can be rewritten as Hamiltonian that basically represents overall profit prospect of different meanings of decision-making function $u(t)$ with immediate and future effects taken into account.

So, one can say that the first term of (3.1.4) can be called as current-profit effect as it's a is the profit function dependent on time t, and the second term – as future-profit effect of $\vec{u}$, is a monetary value, that consists of shadow value multiplied with rate of change of capital. Now a very controversial situation arises, some optimal decision $\vec{u}$ at some time t influences the current profit, it will naturally require a sacrifice in the future profit.

As we need to make overall profit represented by (3.1.1) the greatest possible, we should apply the maximum principle to (2.3.7). In the present case, we have a local maximum. Therefore, one must compute its partial derivatives with respect to $\vec{u}$ and equate partial derivative to zero

$$\frac{\partial H}{\partial u_j} = \frac{\partial \varrho}{\partial u_j} + \varphi(t)\frac{\partial f}{\partial u_j} = 0.$$

It can be rewritten as

$$\frac{\partial \varrho}{\partial u_j} = -\varphi(t)\frac{\partial f}{\partial u_j}, \quad (3.1.6)$$

where $j = 1, \dots, r$. Taking into account (3.1.6) it becomes obvious, that the optimal choice in $\vec{u}$ should cause an increase in current profit and avoid the drop down in the future profit meantime.

The variable of motion k specifies only the effect of the policy decision on the rate of change of capital. Assume that the shadow price is constant. Now we have

$$\frac{\partial H}{\partial k} = \frac{\partial \varrho}{\partial k} + \varphi(t)\frac{\partial f}{\partial k} + \varphi(t) = 0.$$

The rewritten equation will take the form

$$-\varphi(t) = \frac{\partial \varrho}{\partial k} + \varphi(t)\frac{\partial f}{\partial k}. \quad (3.1.7)$$

The basic ideas of the usage of the maximum principle was explained. In terms of the Hamiltonian equations (3.1.6), (3.1.7) and $\dot{k} = f(k, \vec{u}, t)$, we can be rewritten them into the following system

$$\begin{aligned} \frac{\partial H}{\partial \varphi} &= k, \\ \frac{\partial H}{\partial u} &= 0, \\ \frac{\partial H}{\partial k} &= -\varphi. \end{aligned} \quad (3.1.8)$$

The system of equations (3.1.8) determines the optimal paths for all variables starting from given initial point, so that the problem reduces to the issue of finding the optimal initial value of the capital.

To use the transversality condition properly, it’s essential to understand the given boundary conditions. It’s seen that the starting values are already given and they determine the terminal values. The task is to find starting values that will lead to wanted terminal values to find the path, that will satisfy conditions of optimality.

In case of free terminal state $k(T)$ with fixed terminal time T, the shadow price of capital should be equal zero, because value of the capital appears from its ability to bring profit in the future

$$\varphi(T) = 0.$$

To see how terminal line can, differ, let’s review some special cases. One can make a conclusion, that for the company it makes sense to use the initial capital by the time T to receive higher income, as there is no reason to accumulate capital closer to the end of the period. If the company wants to persuade consistent and continuous growth it should assign definite minimum acceptable level of terminal capital, then the terminal line will be truncated and transversality condition is

$$(k^*(T) - k_{\min})\varphi(T) = 0$$

for all $\varphi(t) \geq 0$. When we have definite terminal capital, we can assign the time T, at which the company wants to reach some level of income, the transversality condition here basically means that at some T the sum of current and future income should be equal to zero. Thus,

$$(H)_{t=T} = 0.$$

3.2 Problems in small business

Now let's for comparison try to use the optimal control theory for the problem of small businesses.

Let's assume that there is a start-up company that works with long time and short time projects. At any time t company has to make business decisions on how much to invest into full time workers. Assume that k is the value of budget planned for operational costs, it should be noted that the salaries are the only costs the company has, x and y will denote the amount of work done by full-time workers and part-time workers respectively. Let w_1 and w_2 be labour utility on long and short time projects, respectively. Let c be the amount paid to full time workers during their vacations. According to the current problem the objective is to maximize the income of the company with limited budget. We can present the income function as $p = p(k(t), u(t), t)$ as budget and the decision-making function doesn't depend on each other. So, we need to maximize the functional on the interval $[t_0, T]$,

$$J = x(T) + y(T). \tag{3.2.1}$$

The variables x and y are state variables. The state equations will take the form (see Sethi [2])

$$\dot{x} = w_1 x - c + u \tag{3.2.2}$$

and

$$\dot{y} = w_2 y - u. \tag{3.2.3}$$

The aim is to determine u, that will let the company to receive the highest possible income, and the budget rate should satisfy the conditions given by (3.2.2) and (3.2.3). Let's assume that $x(0) = x_0$ and $y(0) = y_0$. One can write the Hamiltonian as

$$H = \varphi_1(w_1 x - c + u) + \varphi_2(w_2 y - u). \tag{3.2.4}$$

Let's describe some economic meanings of all the variables and their combination given above. Let $w_1 x$ and $w_2 y$ give the values of income based on the utility of the labour that is involved. The adjoint variables φ_1 and φ_2 characterize the value of one money unit, that is invested in long-time and short-time project, respectively. The future value of these adjoint variables should satisfy the equations

$$\dot{\varphi}_1 = -\varphi_1(t) w_1$$

and (3.2.5)

$$\dot{\varphi}_2 = -\varphi_2(t) w_2.$$

The necessary condition for an optimum is that the first derivatives of state variables are equal to zero. Then we can assume that the transversality conditions equals to one, so the most accurate equations to give

$$\varphi_1(t) = e^{-\int_t^T w_1(\tau) d\tau} \tag{3.2.6}$$

and

$$\varphi_2(t) = e^{-\int_t^T w_2(\tau) d\tau}. \tag{3.2.7}$$

To receive the proper solution for the current problem one just need to define the boundary conditions and substitute all given values into the equations of the adjoint variables (3.2.6), (3.2.7). As current problem deals with real problem there can't be any negative value of x or y. Let us consider the previous problem once more in the case when additional constraints are $x \geq 0$ and $y \geq 0$. The extended Hamiltonian will take the form

$$\begin{aligned} H_{**} = H + \theta_1 \dot{x} + \theta_2 \dot{y} = \varphi_1(w_1 x - c + u) + \varphi_2(w_2 y - u) + \\ + \theta_1(w_1 x - c + u) + \theta_2(w_2 y - u). \end{aligned} \tag{3.2.8}$$

Here we can derive the equations for the adjoint variables in the form

$$\dot{\varphi}_1 = -\frac{\partial H_{**}}{\partial x} = -(\varphi_1 + \theta_1) w_1 \tag{3.2.9}$$

and

$$\dot{\varphi}_2 = -\frac{\partial H_{**}}{\partial y} = -(\varphi_2 + \theta_2) w_2. \tag{3.2.9}$$

In present case

$$\frac{\partial H_{**}}{\partial u} = 0. \qquad (3.2.10)$$

The transversality conditions of the current problem are (see Sethi[])

$$(\varphi_1(T) - 1)x(T) = 0$$

and

$$(\varphi_2(T) - 1)y(T) = 0. \qquad (3.2.11)$$

The slack variables θ_1 and θ_2 must be positive or equal to zero and the complimentary conditions to be satisfied are

$$\theta_1(t)x(t) = 0 \qquad (3.2.12)$$

and

$$\theta_1(t)\big(w_1x(t) - c + u(t)\big) = 0. \qquad (3.2.13)$$

Similarly, θ_2 must satisfy the equations

$$\theta_2(t)y(t) = 0 \qquad (3.2.14)$$

and

$$\theta_2(t)\big(w_2x(t) + u(t)\big) = 0. \qquad (3.2.15)$$

So, looking at these two examples above, it's easy to say that the optimal control theory is applicable for the problems of different size of the company as long as there is some variable upon which the decision should be made. In case of a big corporation this control variable was a vector of meanings, let's say without proof that it was the rate of output. In case of small start-up company, the control variable is the fraction of the budget to be spend on the employees' wages. Both problems have different economic meanings but they have the same aim – to maximize the profit of the company. It makes the optimal control theory very useful for the small companies. Of course, real cases are way more complicated and need more input to be correctly solved, but there is no difference between the production or service sector on which the company is oriented; there are always decisions to be made. Of course, optimal control can deal only with that ones that have the real measures.

3.3 Models of optimal economic growth

Many economic problems have a very complicated structure of decision making function, it may be the product of more than two components that influence the final result. In this case, the application of optimal control theory requires finding optimal growth path for all variables to provide the best possible final result.

Consider the company that wants to increase its profit over some period of time $[0, T]$. Assume that the labour amount is denoted by $l(t)$. As its growth is exponential with rate g , at time t the value of $\vec{l}(t)$ will follow the equation

$$l(t) = l(0)e^{gt}.$$

Let's assume the stock of capital $k(t)$ and labor $l(t)$ to be the only production factors, then $F(k, l)$ is the production function that gives the output rate of the company. At time $t = 0$ the output rate will equal to zero as well, but at any following time moment $[0, T]$ it will be more than zero. The first and second order derivatives should satisfy the conditions for all k more than zero

$$\dot{F}(k, l) > 0,$$

and

$$\ddot{F}(k, l) < 0.$$

The output of the company can be sold or reinvested for the future enrichment of capital stock. Lets' define $K = \frac{k}{l}$, then the function $f(K)$ defines production per capita

$$f(K) = \frac{F(k, l)}{l} = F(K, 1). \tag{3.3.1}$$

Let $c(t)$ be the output allocated to sale, and $C = \frac{c}{l}$ be the consumption per capita then the investment amount is $I(t) = F[k, l] - c(t)$. Let the δ be the constant rate of the depreciation of the capital and $\gamma = \delta + g$. The capital stock equation will take the form

$$\dot{K} = f(K) - c - \gamma K, \tag{3.3.2}$$

for $K(0) = K_0$. It will also determine the first boundary condition. The utility of consumption is a function of the output denoted by $U(C)$ [13]. We assume that $\dot{U}(C) = \infty$. The company management will face the following maximization problem on the time interval $[0, T]$

$$J = \int_0^T e^{-\varrho t} U(C) dt. \tag{3.3.3}$$

Where ϱ denotes the social discount rate. One of the boundary conditions will take the form

$$K(T) = K_T. \tag{3.3.4}$$

In the equation (2.3.4) K is a constant vector of predetermined values. Substituting the given values into (2.3.3) one can write the Hamiltonian

$$H = U(C) + \varphi(f(K) - C - \gamma k). \tag{3.3.5}$$

The first term in (3.3.5) determines the utility of current consumption and the second term is the value of the net investment that is measured by the adjoint variable φ.

The adjoint equation for the problem (3.3.3), (3.3.4) is

$$\dot{\varphi} = \varrho\varphi - \frac{\partial H}{\partial K} = (\varrho - \gamma)\varphi - \varphi\dot{f}(K), \tag{3.3.6}$$

where $\varphi(T) = \alpha$. The latter can be considered as a boundary condition, α is simply a predetermined constant.

To receive the optimal solution of the current problem one has to find the local maximum of H. Thus, the condition $\frac{\partial H}{\partial C} = 0$ must be applied. Therefore,

$$\frac{dU}{dC} - \varphi = 0, \tag{3.3.7}$$

where $\dot{U}(0) = \infty$. From the equation (3.3.7) the last boundary condition to be determined for the current problem. There are two more conditions to be satisfied in the optimal run of the company. The dynamic efficiency condition that is described in (3.3.6) causes the change in the price φ of the capital over definite time period $[0, T]$. Multiplying (3.3.6) to dt one has

$$d\varphi + \frac{\partial H}{\partial K}dt = \varrho\varphi dt. \tag{3.3.8}$$

Summarizing (3.3.2) and (3.3.6) yields

$$\dot{K} = f(K) - h(\varphi) - \gamma K \tag{3.3.9}$$

and

$$\dot{\varphi} = (\varrho - \gamma)\varphi - \dot{f}(K)\varphi, \tag{3.3.10}$$

where $c = h$. The point of intersection of right-hand side of (3.3.9) and (3.3.10) is denoted by A in Fig.2.1. This point represents the long-run stationary equilibrium. After defining A, one should figure out whether there is an optimal path that satisfies the equilibrium.

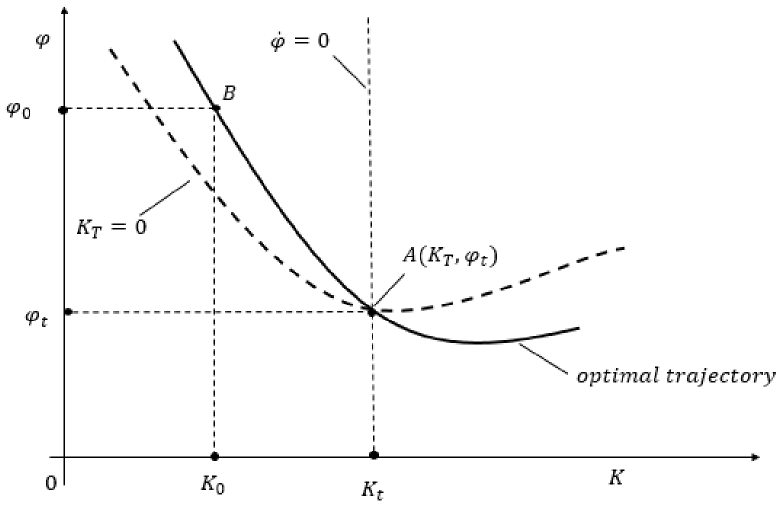

Figure 2.1. Phase diagram of the optimal control model [2]

3.4 Financial interpretation of the optimal control theory

The problem of implementation of the optimal control theory in finance mostly refers to investment and dividend policies, as these require decision making issues, earnings distribution, equity issuing, lending money, investment package and so on. Right now, we will give the solution in a form that is similar to the problem in the section 3.2 but contains complications and a little bit more explanations.

Consider that there is a company that wants to control the cash demand over some period of time T, in order not to lose possible income from shares and bonds that could've been bought for that money. Let's assume c to be a cash balance and s to be a security balance at any time t. The company has to make a decision on how much cash to hold and let u denoted the cost of chancery to be bought, d is assigned to be the rate of sales of securities, r_1 and r_2 are interest rates earned on the cash balance and security balance respectively, α some broker's commission. It's known that there is a dependency between $c(t)$ and $s(t)$ as decisions are made upon the cash balance. According to the given input the problem is to minimize the functional (see Sethi [2])

$$J = -s(T) - c(T). \tag{3.4.1}$$

The variables c and s are the state variables. The state equations will take the form

$$\dot{c} = r_1 c - d + u - \alpha |u|, \tag{3.4.2}$$

and

$$s = r_2 s - u. \quad (3.4.3)$$

Here $c(0) = c_0$ and $s(0) = s_0$. The aim of the problem is to maximize the sum of $c(T)$ and $s(T)$. The Hamiltonian has the form

$$H = \varphi_1\,(r_1 c - d + u - \alpha|u|) + \varphi_2(\,r_2 s - u). \quad (3.4.4)$$

The adjoint variables φ_1 and φ_2 relate to the Lagrange multipliers and represent the future value of one money unit being invested into cash or securities. They should satisfy the equations [2]

$$\dot{\varphi}_1 = -\varphi_1(t) r_1$$

and (3.4.5)

$$\dot{\varphi}_2 = -\varphi_2(t) r_2.$$

The transversality conditions φ_1 and φ_2 equals to one. The best equations to give

$$\varphi_1(t) = e^{\int_0^T r_1(\tau)d\tau}$$

and (3.4.6)

$$\varphi_2(t) = e^{\int_0^T r_2(\tau)d\tau}.$$

It appears that the control function must be rewritten as a difference of two nonnegative variables

$$u = u_1 - u_2. \quad (3.4.7)$$

Note that the value of the control variable u lies between nonnegative constants U_1 and U_2. To avoid any negative sign in the equation let's assume that $u_1 u_2 = 0$, so, at least one of it doesn't take the value zero. As our problem includes the broker's commission being paid out on every transaction, it doesn't make sense to buy and sell securities simultaneously, so the equation (3.4.7) can be rewritten as

$$|u| = u_1 - u_2. \quad (3.4.8)$$

Let's rewrite the Hamiltonian substituting (3.4.8) into (3.4.4)

$$H^* = u_1\big((1-\alpha)\varphi_1 - \varphi_2\big) - u_2\big((1+\alpha)\varphi_1 - \varphi_2\big). \quad (3.4.10)$$

Control variable u_1 of the rate of securities sale, it's function is to determine whether to sell or not sell the securities [12]. If the future value of the money unit φ_1 minus broker's commission is greater than the future value of the securities that can be bought for one money unit, then the securities should be sold on maximum possible rate. If the situation is opposite – the function of the control variable is to prevent the sale, if both values are equal, then the optimal policy is underdetermined. The same rule works with control variable u_2, that denotes the purchase of the securities, but here the purchase is recommended if the sun of the future value of the one money unit and the commission is less than the future value of the securities that can be bought for one money unit, purchase is not done when the situation is the opposite and if the values are equal, then the decision is underdetermined.

So, to receive the solution one just need to assign the boundary conditions and put values inside the equations of the adjoint variables (3.4.6) and the control variable (3.4.7).

To avoid overdrafts and short-sales in the cash balance problem few more additional constraints have to be added:

$$c(t) \geq 0 \tag{3.4.10}$$

and

$$s(t) \geq 0. \tag{3.4.11}$$

In order to fulfill the constraints (3.4.10) and (3.4.11) and to use the maximum principle, so, we the Hamiltonian for the extended functional in the form

$$\begin{aligned} H_{**} = H + \theta_1 c + \theta_2 s = \varphi_1(r_1 c - d + u - \alpha|u|) + \varphi_2(\, r_2 s - u) \\ +\theta_1(r_1 c - d + u - \alpha|u|) + \theta_2(\, r_2 s - u). \end{aligned} \tag{3.4.12}$$

Now let’s write the adjoint equations for this problem as

$$\dot{\varphi_1} = -\frac{\partial H_{**}}{\partial c} = -(\varphi_1 + \theta_1)r_1, \tag{3.4.13}$$

and

$$\dot{\varphi_2} = -\frac{\partial H_{**}}{\partial s} = -(\varphi_2 + \theta_2)r_2. \tag{3.4.14}$$

In the equations (3.4.13) and (3.4.14) $\varphi_1(T), \varphi_2(T) \geq 1$, $(\varphi_1(T) - 1)c(T) = 0$ and $(\varphi_2(T) - 1)s(T) = 0$.

The adjoint variables θ_1 and θ_2 should be more or equal to zero and the optimality conditions is $\frac{\partial H_{**}}{\partial u} = 0$. The complimentary conditions to be satisfied are

$$\theta_1(t)c(t) = 0 \tag{3.4.15}$$

and

$$\theta_1(t)(r_1 c(t) - d(t) + u(t) - \alpha|u(t)|) = 0. \tag{3.4.16}$$

Simply θ_2 should satisfy

$$\theta_2(t)s(t) = 0 \tag{3.4.17}$$

and

$$\theta_2(t)(r_2 s(t) - u(t)) = 0. \quad (3.4.18)$$

The problem described herein can be solved analytically or by the use of the computer program.

One can notice that the problem discussed in the current section takes the same look as the problem described in the 3.2. It shows that the same approach can be used to different problems. The only condition to be satisfied is the existence of the factors on which the decisions can be done. Now it's obvious, that the simplest explanation of the control function u, is that it represents one or the combination of various key performance indicators, that means the value of u is essential for the income as it directly influences it. Simple examples of key performance indicators are labour cost, raw materials cost, the cost of one production unit, the value of rent, administrative cost, income rate, return on capital, capacity utilization and many others. So basically, one can derive the optimal control path storing all or some of the key performance indicators in u, that will basically be the product of all factors upon which the decision has to be made.

3.5 Application of the optimal control theory to monopolistic firm

Now we will apply optimization problem to a simple economic unit such as classic monopolistic firm. This example is believed to be one of the first economic interpretation of the variational calculus. Let's consider that there exists a monopolistic firm, that is a manufacturer of a single commodity and its cost function takes the form of quadratic equation

$$C = \alpha Q^2 + \beta Q + \gamma \quad (3.5.1)$$

$Q(t)$ denotes both the output and the quantity demanded, as there is no inventory and we can equalize them. We take into the account that the quantity demanded depends not only on price $P(t)$ but additionally on the rate of change of that price $\dot{P}(t)$. We have the equation for quantity

$$Q = a - bP(t) + h\dot{P}(t) \quad (3.5.2)$$

The equation of profit is a function of P and $\dot{P}$

$$I = PQ - C = P\left(a - bP + h\dot{P}\right) - \alpha\left(a - bP + h\dot{P}\right)^2 - \beta\left(a - bP + h\dot{P}\right) - \gamma \quad (3.5.3)$$

Having done simple manipulations such as multiplying out and collecting terms we can write an expression of the dynamic profit function

$$I(P,\dot{P}) = -b(1+\alpha b)P^2 + (a+2\alpha ab-\beta b)P - \alpha h^2\dot{P}^2 - h(2\alpha a+\beta)\dot{P}$$
$$+ h(1+2\alpha b)P\dot{P} - (\alpha a^2+\beta a+\gamma) \quad (3.5.4)$$

The company should find an optimal paths for P that will maximize the profit over the time *[0,T]*. We will not take into account the discount factor, as over the given period of time we will have fixed demand and cost functions. The aim is to maximize the functional

$$J = \int_0^T I(P,\dot{P})dt \quad (3.5.5)$$

In this case it's obvious that the easiest way is to use the classic Euler's equation *(1.3.9)* and to do it, we have to calculate partial derivatives based on the profit function

$$\frac{\partial I}{\partial P} = -2b(1+\alpha b)P + (a+1\alpha qb+\beta b)\dot{P} \quad (3.5.6)$$

$$\frac{\partial I}{\partial \dot{P}} = -2\alpha h^2\dot{P} - h(2\alpha a+\beta) + h(1+2\alpha b)P \quad (3.5.7)$$

And

$$\frac{\partial^2 I}{\partial \dot{P}^2} = -2\alpha h^2 \quad (3.5.8)$$

$$\frac{\partial^2 I}{\partial P\partial \dot{P}} = h(1+2\alpha b) \quad (3.5.9)$$

$$\frac{\partial^2 I}{\partial t\partial \dot{P}} = 0 \quad (3.5.10)$$

After substituting *(3.5.6)*, *(3.5.7)*, *(3.5.8)*, *(3.5.9)* and *(3.5.10)* into the Euler's equation *(1.3.9)* and normalizing it we receive a second order differential equation with constant coefficients and constant term

$$\ddot{P} - \frac{b(1+\alpha b)}{\alpha h^2}P = -\frac{a+2\alpha ab+\beta b}{2\alpha h^2} \quad (3.5.11)$$

There is a well-known general solution for it, that was described by A. Chiang [6]. A_1 and A_2 are two arbitrary constraints and in our case the solution will take the form

$$P^*(t) = A_1e^{r_1t} + A_2e^{r_2t} + \bar{P} \quad (3.5.12)$$

where characteristic roots

$$r_{1,}r_2 = \pm\sqrt{\frac{b(1+\alpha b)}{\alpha h^2}} \quad (3.5.13)$$

and particular integral

$$\bar{P} = \frac{a + 2\alpha ab + \beta b}{2b(1 + \alpha b)} \quad (3.5.14)$$

Taking into account that two characteristic roots are the exact negatives of each other we can denote *r* as absolute value of both roots. The rewritten solution

$$P^*(t) = A_1 e^{rt} + A_2 e^{-rt} + \bar{P} \quad (3.5.15)$$

We can define boundary conditions *P(0)=*P_0 and $P(T) = P_t$ from A_1 and A_2. Here we set *t=0* and *t=T*

$$P_0 = A_1 + A_2 + \bar{P} \quad (3.5.16)$$

and

$$P_T = A_1 e^{rt} + A_2 e^{-rt} + \bar{P} \quad (3.5.17)$$

The solution value for A_1 and A_2

$$A_1 = \frac{P_0 - \bar{P} - (P_T - \bar{P})e^{rT}}{1 - e^{2rT}} \quad (3.5.18)$$

$$A_2 = \frac{P_0 - \bar{P} - (P_T - \bar{P})e^{-rT}}{1 - e^{-2rT}} \quad (3.5.19)$$

It completes the solution of the problem as now all the parameters were taken into account except of *h*, but it doesn't seem to be an issue as this parameter enters the solution path only through *r* and as a squared term, so it's sign can't affect the result, but its value will.

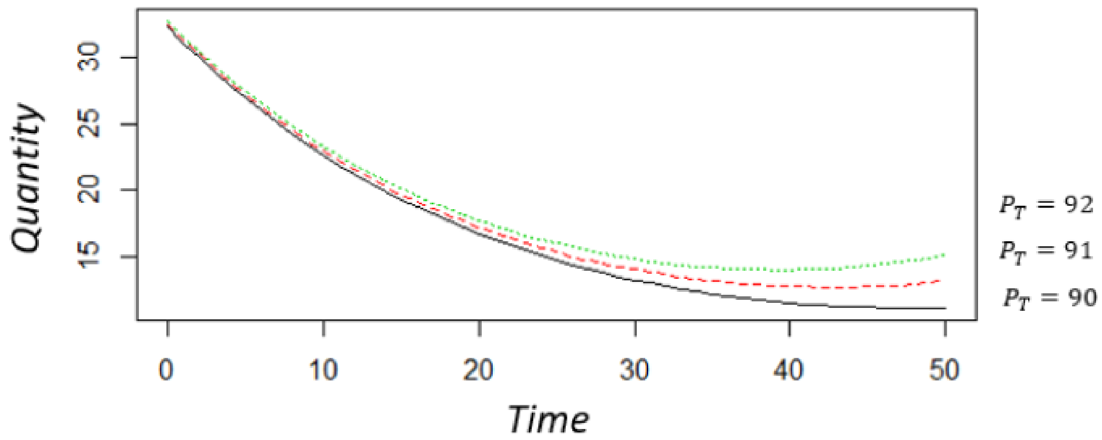

Figure 3.1. Production quantity over the period of time with different terminal prices

Figure 3.1 illustrates the change in quantity produced based on the price and its change. Obviously, the volume of the terminal price creates a change in the trajectory of each curve.

Summary

Optimal control theory is the extension of the calculus of the variations; it is relatively a new discipline. In this thesis, we showed the logical way of connecting them together. It appeared that the principle of maximum is actually the necessary part of it. To use the optimal control theory, it should be interpreted using all the mathematical laws and values. It can be applied to any problem that requires finding optimal decision connected with some value that will bring some positive input in the future. One can’t apply the theory to the very abstract meanings, just to those ones that can be measured with some measurable units.

The first two chapters are aimed to give the appropriate knowledge for the full understanding of the interpretation process. In the third chapter, we introduced three economic and one financial case of applying the theory to the real problems. There was made a comparison of whether the size of the company influences the order of the solution and its general look. Now it’s known that both huge and tiny companies, as well as individuals, who are about to make some investment decision, and use optimal control theory for the optimization of their activity. It is shown that control variable can be at some point a synonym to the mathematical meaning of the control variable. The model of the optimal economic growth can easily find its use in real economic and experience various of improvements and extensions. There might be derived the unified models for groups of typical cases, as we can say that all decisions to be made can be summed under one variable.

Optimal control theory is easy to be applied as it gives enough information to understand the mathematical reasons of the decision-making process in the real world.

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