Non-Conventional Stochastic Resonance

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ABSTRACT
It is argued, on the basis of linear response theory (LRT), that new types of stochastic resonance (SR) are to be anticipated in diverse systems, quite different from the one most commonly studied to date, which has a static double-well potential and is driven by a net force equal to the sum of periodic and stochastic terms. On this basis, three new non-conventional forms of SR are predicted, sought, found and investigated both theoretically and by analogue electronic experiment: (a) in monostable systems; (b) in bistable systems with periodically modulated noise; (c) in a system with coexisting periodic attractors. In each case, it is shown that LRT can provide a good quantitative description of the experimental results for sufficiently weak driving fields. It is concluded that SR is a much more general phenomenon than has hitherto been appreciated.

KEY WORDS: Analogue simulation; fluctuation phenomena; resonance; noise; spectral density; linear response; periodic attractors.

1. INTRODUCTION

The remarkable diversity[1,2] of the systems in which stochastic resonance (SR) has already been found, or is being sought - ice-ages, lasers, electronic circuits, electron spin resonance (ESR), superconducting quantum interference devices (SQUIDs), sensory neurons, and passive optical systems, for example - is in a sense slightly misleading because, at a fundamental level, the underlying phenomenon in all of these apparently disparate cases is exactly the same. It arises because of the noise-induced increase in the system's generalised susceptibility $\chi(\Omega)$ at some frequency Ω on the wing of the zero-frequency spectral peak corresponding to hopping between two (or more) static attractors[3,4]. For convenience, we shall refer to the noise-induced enhancement of a weak periodic signal in systems of this kind, where the net applied force is a sum of regular and stochastic terms, as *conventional SR*. The overwhelming majority of earlier work on SR[1,2] has related to conventional SR. We note that the description of conventional SR in terms of a susceptibility[3,4], i.e. within the scope of linear response theory, has not only proven to be correct[5], but is also simple and revealing.

The aim of the present paper is to return to the interesting question of whether there may be other, quite different, classes of systems also able to support SR phenomena: that is, to explore the possibility of *non-conventional SR*. We shall use the latter term to describe SR in systems that do not have static potentials of the usual bistable (or multistable) type, or for which the periodic and stochastic forces are not mutually additive: in other words, we describe as non-conventional those systems which cannot be mapped into conventional SR systems by a suitable change of variable.

In Section 2, we consider SR phenomena in thermal equilibrium systems with static attractors, and ask whether there may be new forms of SR not related to the zero-frequency spectral peaks associated with fluctuational transitions between the stable states of bistable systems. We show that LRT leads immediately to the prediction and observation of *high-frequency* SR in underdamped *monostable* systems. Section 3 addresses the interesting question of whether SR occurs in systems with static attractors when the stochastic and periodic forces are applied *multiplicatively*, in the sense that the former is modulated by the latter. It turns out that SR does manifest itself, but with a phenomenology different from that of conventional SR. In Section 4, we describe the first search for evidence of SR in a system with *periodic attractors*, and demonstrate that the phenomenon does indeed occur at a (tunable) high frequency close to that of the main periodic drive. The results are discussed, and general conclusions are drawn, in Section 5.

2. STOCHASTIC RESONANCE IN MONOSTABLE SYSTEMS

The advantages of treating conventional SR by LRT[3,4] have already been discussed in an earlier paper[6] in this volume. We now show that, in addition to the obvious advantages of simplicity, elegance, and wide applicability, the LRT approach also possesses strong predictive power. This is *a fortiori* the case when seeking SR in systems of the thermal equilibrium type where the stochastic force is white, Gaussian and additive, and the weak periodic signal is also additive. The fluctuation dissipation theorem[7] is then applicable so that the generalized susceptibility $\chi(\Omega)$ of a given system at frequency Ω may be written as

$$\mathrm{Re}\ \chi(\Omega) = \frac{2}{T}\mathrm{P}\int_0^\infty d\omega_1[\omega_1^2/(\omega_1^2-\Omega^2)]Q^{(0)}(\omega_1) \qquad (1)$$

$$\mathrm{Im}\ \chi(\Omega) = (\pi\Omega/T)Q^{(0)}(\Omega)$$

where $Q^{(0)}(\omega)$ is the spectral density of the fluctuations (SDF) of the system in the absence of the periodic signal, T is the temperature and P implies the Cauchy principal part. Eq (1) also holds for quasi-thermal noise-driven systems, for example those moving in a static potential with friction proportional to velocity under the influence of white Gaussian noise; in such cases, T characterises the intensity of the noise.

All that is needed for a weak trial force at frequency Ω to be enhanced by added noise, i.e. for SR to occur, is that $|\chi(\Omega)|^2$ should rise with increasing T. What this means in practice is that SR may reasonably be sought in any system for which $Q^{(0)}(\omega)$ exhibits a well-resolved narrow peak that is *strongly* dependent on T. If the value of $Q^{(0)}(\Omega)$ of the SDF for Ω within the range of the peak increases faster than linearly with T, then signal enhancement may be expected; for the signal/noise *ratio* to increase with noise intensity, it is necessary that $Q^{(0)}(\Omega)$ should rise faster than quadratically with T. Dramatic manifestations of SR are to be anticipated in those cases where $Q^{(0)}(\Omega)$ rises extremely rapidly (e.g. exponentially) with T, as it does in the case of conventional SR[6].

This perception of the origins of SR suggests that it is actually a very general phenomenon. In particular, there is no obvious reason why it should be confined to bistable (or multistable) systems. It is equally likely, for example, to manifest itself in single-well nonlinear oscillators under appropriate conditions, i.e. in *monostable* systems. We now consider two cases of non-conventional SR of this kind that we have found in the single-well Duffing oscillator, driven by Gaussian white noise of intensity T and a weak periodic force of amplitude A

$$\ddot{q} + 2\Gamma\dot{q} + \frac{dU(q)}{dq} = f(t) + A\cos\Omega t \qquad (2)$$

$$U(q) = \frac{1}{2}q^2 + \frac{1}{4}q^4 + Bq$$

$$\Gamma \ll 1, \quad \langle f(t)\rangle = 0$$

$$\langle f(t)f(t')\rangle = 4\Gamma T\delta(t-t')$$

We distinguish two cases, depending[8] on whether or not $|B| > 8/7^{\frac{3}{2}} \simeq 0.43$. In case (a), with $|B| \leq 0.43$, the variation of the oscillator's eigenfrequency with energy is monotonic, as sketched in Fig 1(a). In the absence of the periodic force ($A = 0$), for small noise intensity T, the SDF consists of a narrow Lorentzian peak of width $\sim \Gamma$ at frequency $\omega(0)$, where $\omega(E)$ is the frequency of eigenvibrations of a given energy E measured from the bottom of the potential well. As T is increased, the average energy $\overline{E}$ of the oscillator rises, and the peak broadens asymmetrically[8] towards higher frequencies. For an $[\Omega - \omega(0)] > \Gamma$ in the position shown by the dashed line in Fig 1(a), initially on the tail of the spectral peak the magnitude of $Q^{(0)}(\Omega)$ will therefore increase very rapidly (approximately exponentially) with T. The corresponding increase in the susceptibility $\chi(\Omega)$ of the system implied by (1) means that, when there is a weak periodic force on the right hand side of (2), it will be amplified by an increase of T, i.e. SR will occur. The SR maximum is to be expected when T has been "tuned" to adjust $\overline{E}$ such that $\omega(\overline{E}) \simeq \Omega$. The argument is

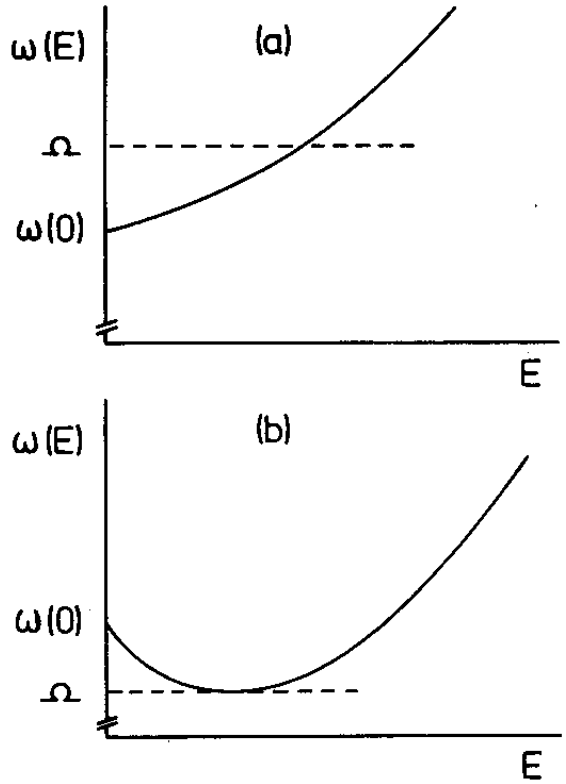

Figure 1: Sketches to show the dependence of the eigenfrequency $\omega(E)$ on energy E for the nonlinear oscillator (2): (a) for $|B| < 0.43$; (b) for $|B| > 0.43$. The frequencies Ω at which a weak periodic force will be amplified by SR are indicated.

obviously extremely general and can be applied, with minor variations where necessary, to any underdamped nonlinear oscillator.

In case (b), on the other hand, where $|B| > 0.43$, $\omega(E)$ is nonmonotonic, as sketched in Fig 1(b). In the absence of the periodic force ($A = 0$), the system is known[(8)] to exhibit noise-induced spectral narrowing of the main peak in the SDF; and, for sufficiently small values of the damping constant Γ, exceedingly sharp zero-dispersion peaks (ZDPs) of width $\propto \Gamma^{\frac{1}{2}}$ appear[(9,10)] in the SDF close to the frequency ω_m of the extremum where $d\omega(E)/dE = 0$. The magnitude of the ZDP rises exponentially fast with increasing T so that, just as in case (a), the correspondingly rapid increase of $\chi(\Omega)$ will imply a manifestation of SR for Ω close to ω_m. The extreme narrowness of the ZDP, and its very rapid rise with T, suggests that SR in case (b) will be a much more dramatic phenomenon than in case (a).

To test these predictions, we have sought evidence of SR in an electronic model of (2) designed, constructed and operated according to standard practice[(11)]. The parameter values used were: $\Gamma = 0.011$; $A = 0.020$; and $B = 0$ or 2.00 for cases (a) or (b) respectively. The model was driven with quasi-white noise from an external noise generator, and with a weak periodic force from an HP3325 frequency synthesizer. The resultant fluctuating $q(t)$ was digitized in 1024 point blocks and ensemble-averaged by a Nicolet LAB80 data-processor to yield $\langle q(t) \rangle$. The advantages of averaging in the time domain, rather than as more commonly[(1)] in the frequency domain, is that it enables measurements to be made

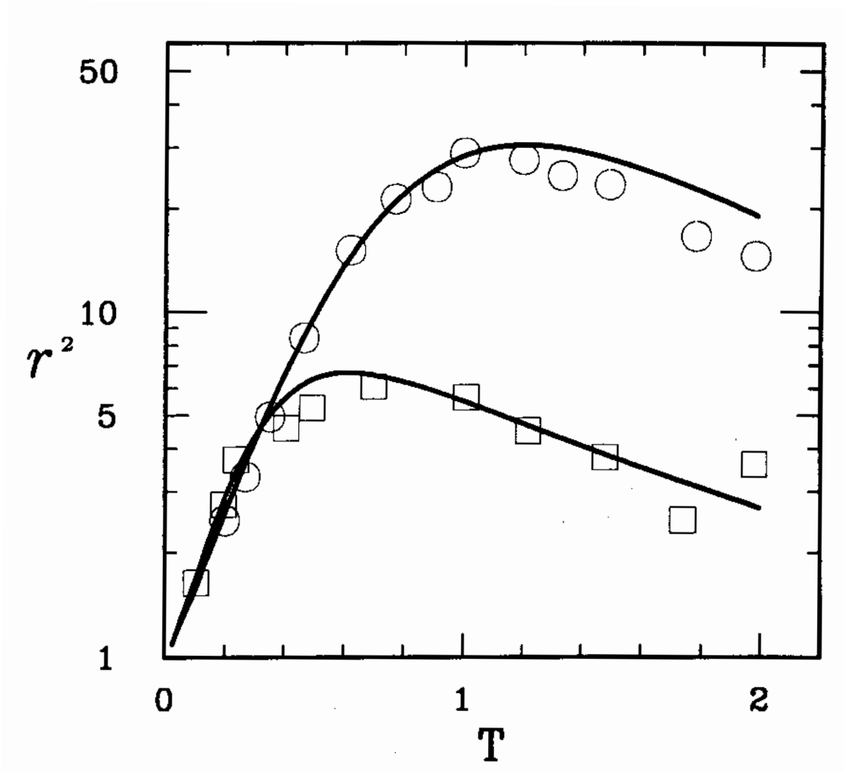

Figure 2: SR in monostable systems. The squared stochastic amplification factors, r^2, measured for case (a) with $B = 0$ (squares) and case (b) with $B = 2$ (circles) are plotted as functions of noise intensity T for the electronic circuit model of (2) with $A = 0.02$, $\Gamma = 0.011$. The full curves represent theoretical predictions derived using LRT and the fluctuation dissipation theorem.

of the phase shift ϕ between the drive and the response, as well as yielding the amplitude a of the response directly.

Some typical experimental results are shown by the data points in Fig 2, for case (a) (squares) and case (b) (circles) respectively. The measurements are expressed in terms of a *stochastic amplification factor* $r = a(T)/a(0)$, where $a(0)$ is the amplitude of the periodic response $\langle q(t) \rangle$ when $T = 0$; for more convenient comparison with earlier results in conventional SR[1], we have plotted r^2 rather than r. The form of the data is strikingly similar to that found in conventional SR[1], in that the variation of r^2 with T passes through a bell-shaped maximum, steeper on its low-T side. As inferred above, the maximum is indeed higher for case (b) than for case (a).

The measured phase lag $-\phi$ between the drive and the response is plotted for case (a) (squares) and case (b) (circles) in Fig 3. The forms of $-\phi(T)$ for the two cases are strikingly different, but they can readily be understood by analogy with an ordinary (deterministic) resonance. In case (a) with $T = 0$, the periodic driving force is being applied to the oscillator at a frequency well beyond its natural frequency, $\Omega > \omega(0)$ [see Fig 1(a)]. Consequently, $-\phi$ is close to 180° (just as it would be for a harmonic oscillator with $\Omega/\omega(0) \ll 1$). As T is increased, however, the natural frequency is effectively being tuned past the fixed driving frequency. Near resonance $-\phi$ passes through 90° and, in the high T limit where the natural frequency substantially exceeds Ω, $-\phi$ decreases towards 0° exactly as it would in a deterministic resonance.

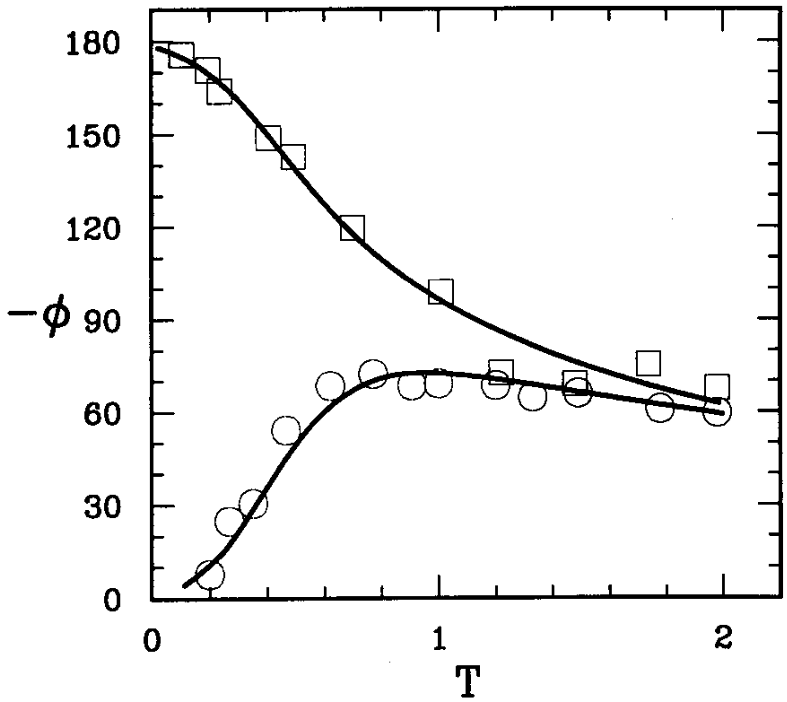

Figure 3: Phase shifts for SR in monostable systems. The phase differences $-\phi$ (in degrees) between the periodic driving force $A\cos\Omega t$ and the periodic response $\langle q(t)\rangle$ measured for case (a) with $B = 0$ (squares) and case (b) with $B = 2$ (circles) are plotted as functions of noise intensity T for the electronic circuit model of the monostable system (2) with $A = 0.02$. The full curves represent theoretical predictions derived using LRT and the fluctuation dissipation theorem.

In case (b) on the other hand, the natural frequency never falls below its minimum value, to which the periodic force $A\cos\Omega t$ has been tuned [see Fig 1(b)]. Consequently the phase lag $-\phi$ is always less than 90°, although it approaches 90° near the resonance maximum just as one might have expected from the analogy with the deterministic case. The resultant $-\phi(T)$ is therefore nonmonotonic, and similar in form to the phase shift observed previously[4,6] for conventional SR.

The application of LRT to provide a quantitative theoretical description of these interesting phenomena is relatively straightforward, because the SDF of (2) for $A = 0$, $Q^{(0)}(\omega)$, is already known[8], theoretically and experimentally, for both cases (a) and (b). Inserting the relevant expressions in the fluctuation dissipation relations (1) yields $\chi(\Omega)$ immediately, whence

$$r^2 = [a(T)/a(0)]^2 = |\chi(\Omega)|^2/\{[\omega(0)^2 - \Omega^2]^2 + 4\Gamma^2\Omega^2\} \qquad (3)$$

$$-\phi = \tan^{-1}[\mathrm{Im}\chi(\Omega)/\mathrm{Re}\chi(\Omega)] \qquad (4)$$

These two quantities, calculated for (2) with the parameters used in the circuit, are plotted (full curves) as functions of T in Figs 2 and 3 for comparison with the experimental measurements. Given that there are no adjustable parameters, the agreement between experiment and theory can be regarded as excellent.

The results of Figs 2 and 3, representing the first observation of SR in a monostable system, demonstrate that the bistability (or multistability) of a system is not in fact a necessary condition for SR to occur, as had previously been assumed[(1)]. We would emphasize that the case (a) variant of SR investigated in the present work is in no way confined to the particular system (2). Rather, it is a quite general phenomenon that is to be anticipated in *all* underdamped nonlinear oscillators (including, for example, those for which $|B| \geq 0.43$ in the model (2)). In all situations where, as in case (b) of (2), the eigenfrequency varies nonmonotonically with energy (or has singular points of higher order), a more pronounced manifestation of SR is to be anticipated near the frequency ω_m of the extremum; it might reasonably be called *zero-dispersion stochastic resonance* (ZDSR). There is an interesting distinction between the present results and those of conventional SR. In the latter case, stochastic amplification occurs in both overdamped and underdamped systems but is at its most pronounced when the damping is large; the nonconventional forms of SR studied above, on the other hand, are restricted to underdamped systems and are most pronounced when the damping is small.

3. STOCHASTIC RESONANCE FOR PERIODICALLY MODULATED NOISE INTENSITY

In this section we consider a system with a static bistable potential, as in conventional SR, but with an unconventional driving force. In conventional SR, the net driving force is a sum of periodic and stochastic terms; we now address the rather different situation that arises when the noise and the periodic force are introduced multiplicatively, so that the former is amplitude-modulated by the latter. Periodically modulated noise is not uncommon and arises, for example, at the output of any amplifier (e.g. in optics, or radio-astronomy) whose gain varies periodically with time. It is of obvious importance, therefore, to establish whether or not a modulated zero-mean noise can give rise to a periodic signal in the system it is driving. Such an effect would not, of course, occur in a linear system where the signal is directly proportional to the driving force so that they must both, on average, vanish. In a *nonlinear* system, on the other hand (e.g. a diode rectifier) there obviously can be a periodic signal in the output. We now show a form of SR can occur for the particular case where the system has a bistable potential.

To demonstrate the onset of this new form of SR, and to reveal its characteristic features, we treat the simplest nontrivial system: an overdamped Brownian particle, moving in an asymmetric bistable potential, with equation of motion

$$\dot{q} + \frac{dU(q)}{dq} = f(t) \equiv (\frac{1}{2}A\cos\Omega t + 1)\xi(t) \qquad (5)$$

$$U(q) = -\frac{1}{2}q^2 + \frac{1}{4}q^4 + Bq$$

Here again, B characterizes the asymmetry of the potential. For $-2/(3\sqrt{3}) < B < 2/(3\sqrt{3})$ the potential $U(q)$ has two minima, i.e. the system is bistable. The function $\xi(t)$ represents white Gaussian noise of intensity D, so that

$$\langle f(t)f(t')\rangle = 2D\delta(t-t')\left[1 + A\cos(\Omega t) + \frac{A^2}{8}(1+\cos(2\Omega t))\right] \qquad (6)$$

i.e. the intensity of the driving force $f(t)$ is periodic in time. In what follows, we assume the modulation to be weak, $A \ll 1$, and neglect the term $\sim A^2$ in (6).

For sufficiently weak noise, when D is much less than the depths $\Delta U_{1,2}$ of the potential wells,

$$D \ll \Delta U_1,\ \Delta U_2, \quad \Delta U_n \ = \ U(q_s) - U(q_n), \quad n = 1, 2$$

$$U'(q_{1,2}) \ = \ U'(q_s) \ = \ 0, \quad q_1 < q_s < q_2 \tag{7}$$

the motion of the system consists mostly of small intrawell fluctuations about the equilibrium positions at the potential minima at $q_{1,2}$. Occasionally, large fluctuations will occur, sufficient to cause interwell transitions across the potential maximum at q_s. Periodic modulation of the noise influences both types of fluctuation, and so there are two contributions to the signal $\langle q(t)\rangle$: one from the modulation of the intrawell fluctuations; and the other from the modulation of the populations $w_{1,2}(t)$ of the wells 1, 2

$$\langle q(t)\rangle \ \simeq \ \sum_{n=1,2} \langle q(t)\rangle_n w_n(t) \tag{8}$$

where, $\langle\rangle_n$ implies averaging over the n$^{\text{th}}$ well. The system (5), (6) is not of the thermal equilibrium type, and so cannot be described by the fluctuation dissipation relations (1). We can still apply LRT, however, and we assume that the periodic response to weak modulation can be described by a generalised susceptibility $\kappa(\Omega)$

$$\langle q(t)\rangle \ = \ \langle q\rangle^{(0)} \ + \ A\,\mathrm{Re}\,[\kappa(\Omega)\ \exp(-i\Omega t)] \tag{9}$$

where, as previously, the superscript (0) means that the corresponding quantity refers to the case $A = 0$.

We shall consider the response for the physically important case of low frequency modulation, $\Omega \ll U''(q_{1,2})$, where the adiabatic approximation holds. Both the intrawell fluctuations and the transition probabilities W_{12}, W_{21} are then the same as they would be for white noise of instantaneous intensity $D\,(1 + A\cos\Omega t)$. The well populations w_1, w_2 for periodically modulated noise depend on the relationship between Ω and the W_{nm}. To lowest order in the modulation amplitude A, the probability W_{nm} of an $n \to m$ transition is

$$W_{nm} \equiv W_{nm}(t) \simeq W_{nm}^{(0)}(1 + A\frac{\Delta U_n}{D}\cos\Omega t) \tag{10}$$

where $W_{nm}^{(0)} \propto \exp(-\Delta U_n/D)$ is the usual Kramers transition rate. The corresponding periodic modulation of the well populations $w_{1,2}$ is described by the balance equation $\dot{w}_1 = -W_{12}w_1 + W_{21}w_2$. The periodic redistribution over the wells gives a contribution $\kappa_{tr}(\Omega)$ to the susceptibility $\kappa(\Omega)$ of the form

$$\kappa_{tr}(\Omega) \ = \ -\frac{1}{D}(q_1 - q_2)(\Delta U_1 - \Delta U_2)w_1^{(0)}w_2^{(0)}\frac{W^{(0)}}{W^{(0)} - i\Omega}$$

$$W^{(0)} \ = \ W_{12}^{(0)} + W_{21}^{(0)} \tag{11}$$

$$w_1^{(0)} \ = \ W_{21}^{(0)}/W^{(0)}, \quad w_2^{(0)} \ = \ 1 - w_1^{(0)}$$

In obtaining (11) from (8)-(10), we have neglected the deviations of $\langle q_n\rangle$ from q_n in comparison with $|q_2 - q_1|$. According to (10), (11),

$$|\kappa_{tr}(\Omega)| \propto \zeta \exp(-\zeta), \quad \zeta \;=\; |\Delta U_1 - \Delta U_2|/D \tag{12}$$

i.e. the interwell transitions contribute to $\kappa(\Omega)$ provided that the potential is asymmetric. This is easily understood qualitatively. For a symmetric potential, the wells are equally populated irrespective of noise intensity and so the modulation of the latter does not influence the populations w_1, w_2. For asymmetric potentials, on the other hand, the ratio of the populations $w_1^{(0)}/w_2^{(0)} \propto \exp\,[(\Delta U_2 - \Delta U_1)/D]$ depends sharply on the noise intensity, and will be strongly influenced by the modulation of D. It is also evident that, for very large ζ, a weak modulation will not result in a substantial redistribution over the wells because the product $w_1 w_2 \propto \exp(-\zeta)$ will remain exponentially small: $|\kappa_{tr}(\Omega)|$ must therefore vary nonmonotonically with $\zeta \propto D^{-1}$, with a maximum at $\zeta = 1$, and increase rapidly with D in the range $\exp\,(\zeta) \gg 1$. This increase can in itself give rise to stochastic resonance, since the periodic signal is rising rapidly with increasing noise intensity.

However, the intrawell fluctuations are also to be considered. Their contribution to the susceptibility $\kappa(\Omega)$ is connected with the local asymmetry of the potential about each of its minima (just as for the zero frequency peaks in the power spectra of single-well underdamped systems[(12)]). The partial susceptibility for the n$^{\text{th}}$ well, $\kappa_n(\Omega)$, can be obtained for small D by expanding $U(q)$ in (1) to second order in $(q - q_n)$ and calculating $\langle q - q_n \rangle$ formally to second order in $f(t)$. For $\Omega \ll U''(q_n)$ one arrives at the expression

$$\kappa_n(\Omega) \;=\; -U'''(q_n)\,[U''(q_n)]^{-2}\, D/2 \tag{13}$$

The susceptibility $\kappa(\Omega)$ as a whole is then given by the sum of the above contributions

$$\kappa(\Omega) \;=\; \sum_{n=1,2} \kappa_n(\Omega) w_n^{(0)} \;+\; \kappa_{tr}(\Omega) \tag{14}$$

Eqs (9), (11), (13), (14) describe completely the periodic response of the system to periodically modulated noise. Following Ref 12, the influence of the noise intensity on the response can be characterised by a signal/noise ratio R equal to the ratio of the δ-like spike in the power spectral density of the fluctuations of the system

$$Q(\omega) \;=\; \frac{1}{4\pi\tau} \Big| \int_{-\tau}^{\tau} dt\; e^{i\omega t} q(t) \Big|^2, \quad \tau \to \infty \tag{15}$$

at the modulation frequency Ω to the value $Q^{(0)}(\Omega)$ of $Q(\Omega)$ in the absence of modulation. According to (9)

$$R \;=\; \frac{1}{4} A^2 |\kappa(\Omega)|^2 / Q^{(0)}(\Omega) \tag{16}$$

[Note that a similar equation was given in Ref 6 for the case of additive periodic forcing; we emphasize, however, that in contrast with Refs 3, 4, 6, the effective susceptibility $\kappa(\Omega)$ is not now given directly by the fluctuation dissipation theorem in terms of $Q^{(0)}(\omega)$].

The most interesting and important situation arises when the main contributions to both $\kappa(\Omega)$ and $Q^{(0)}(\Omega)$ are due to fluctuational interwell transitions. In this case, (16) simplifies and, allowing for the explicit form[(6)] of $Q^{(0)}(\Omega)$, one obtains

$$R \simeq R_{tr} \;=\; \frac{\pi}{4} A^2 \zeta^2 W_{12}^{(0)} W_{21}^{(0)} / (W_{12}^{(0)} \;+\; W_{21}^{(0)}) \tag{17}$$

It can be seen from the Kramers expression for the transition probability that $R_{tr} \propto \zeta^2$ $\exp\,(-\Delta U/D)$ where $\Delta U = \max\,(\Delta U_1, \Delta U_2)$ is the depth of the deeper potential well.

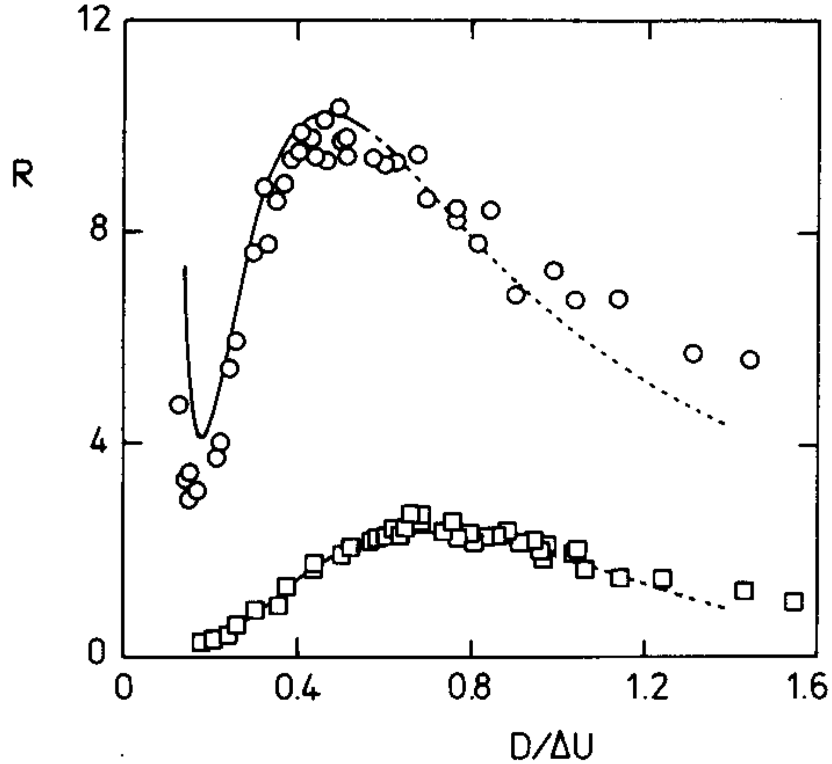

Figure 4: SR for periodically modulated noise. Measurements (square data points) of the signal/noise ratio R ($\times$ 15) for the system (5) are compared with theory (lower full curve), plotted as a function of reduced noise intensity $D/\Delta U$ with $B = 0.2$, $A = 0.14$, $\Omega = 0.029$. The circle data points represent measurements on the same electronic circuit with *additive* periodic forcing (conventional SR) under similar conditions; the upper full curve represents the theoretical prediction of Ref 3. The dashed regions of each curve lie beyond the parameter range where the theory is strictly valid.

For non-equal well depths, it is obvious that R_{tr} increases sharply with increasing D, i.e. stochastic resonance occurs. We emphasize that (17) holds for ζ not too large: this is because the contributions to $\kappa(\Omega), Q^{(0)}(\Omega)$ from the interwell transitions are proportional to exp $(-\zeta)$ and, for large ζ, they become small compared to the intrawell contributions.

The theory has been tested by means of an electronic analog experiment, using a circuit of conventional design[11] to simulate (5). Measurements of the signal/noise ratio R are shown by the square data points in Fig 4. We note immediately that the existence of stochastic resonance for the case of periodically modulated noise is confirmed by the data. We stress here that the rate of increase of R is faster than D, so that it does not represent merely the proportionality of the modulation to D in Eq (6). The lower solid line in Fig 4 represents a fit of Eqs (13-17) to the experimental data, demonstrating the universal character of the shape of the SR.

It is interesting to compare SR with periodically modulated noise with conventional SR for which $f(t)$ in (5) is replaced by

$$\tilde{f}(t) = \xi(t) + A\cos\Omega t \tag{18}$$

Experiment and the LRT prediction for signal/noise in conventional SR are also shown in

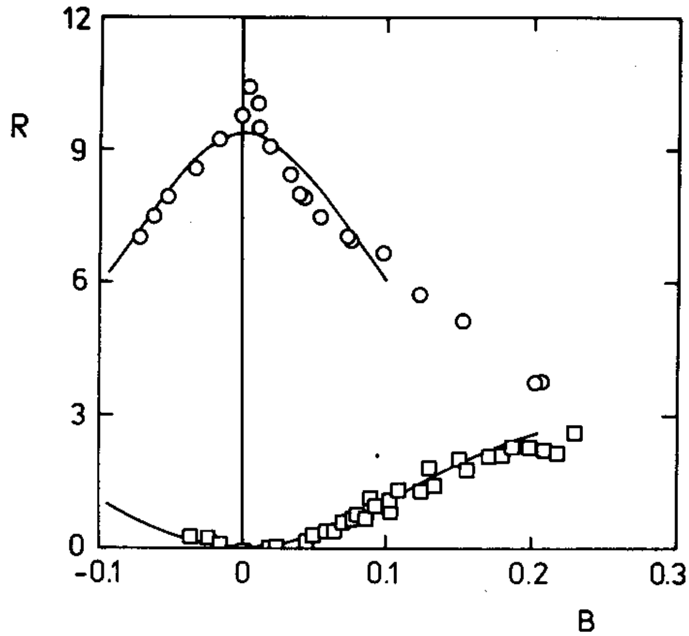

Figure 5: Measurements (square data points) of the signal/noise ratio R ($\times$ 15) for the system (5) with periodically modulated noise, compared with theory (lower curve), plotted as a function of the asymmetry parameter B with $A = 0.15$, $(D/\Delta U)_{B=0} = 0.303$, $\Omega = 0.029$. The circle data represent measurements on the same electronic circuit with *additive* periodic forcing (conventional SR) under similar conditions; the upper curve represents the theoretical prediction of Ref 3.

Fig 4, by the circle data points and upper full curve respectively. The behaviour is seen to be remarkably similar in the two cases, although the size of the SR effect for periodically modulated noise tends to be much smaller. The most striking difference between these two forms of SR relates, however, to the variation of R with the asymmetry parameter, B, shown in Fig 5. For periodically modulated noise, the signal seems to disappear for equal well depths ($B = 0$), and steadily increases as the difference in well depths with increasing B. For conventional SR, on the other hand, the situation is reversed: it is most pronounced for equally populated stable states.

These ideas are readily set on a more quantitative basis. The asymmetry of our model (5) is controlled by B, and when B is small we can write

$$\zeta \equiv |\Delta U_1 - \Delta U_2|/D \simeq 2|B|/D \tag{19}$$

Consequently, one would expect from (17), (19) that $R \propto B^2$ for small B. This is to be compared with conventional SR, where R_{tr} decreases[(6)] with increasing B. For large B, however, the rise in R_{tr} may be expected to saturate because the depth of the deeper well increases until, eventually, the interwell transitions get frozen out. We note that, for periodic forcing of the system described by (5), (18), R should be larger than for periodic modulation of the noise for the same dimensionless amplitude A, just because of the additional asymmetry factor ζ^2 in (17). It can be seen from Fig 5 that the above theory (full curves) is in good agreement with the experiment.

The results in Fig 6 demonstrate that the signal/noise ratio saturates with increasing

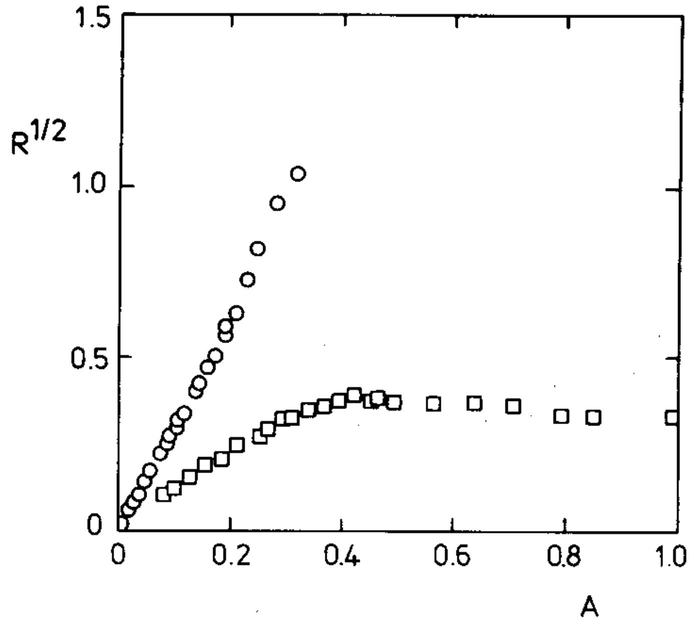

Figure 6: Comparison of (signal/noise)$^{\frac{1}{2}}$ $= R^{\frac{1}{2}}$ measured as a function of signal amplitude A for conventional SR (circles) and ($\times$ 5) for SR with periodically modulated noise (squares) for the same values of B and D. Note that the dependence is linear for small signals in both cases, and that the rise in R with A saturates within the parameter range investigated for the case of periodically modulated noise.

amplitude of the periodic modulation. The effect is easily understood, because the amplitude of the signal due to interwell transitions in effectively limited to one half of the distance between the attractors. It is more striking than the corresponding saturation effect in conventional SR, for which the additive periodic force also distorts the shape of the potential (cf Ref 6 where nonlinear effects for large amplitude modulation in conventional SR are considered).

4. STOCHASTIC RESONANCE FOR PERIODIC ATTRACTORS

The third form of non-conventional SR that we treat relates to an entirely different form of bistability - one where the coexisting attractors are not static, but *periodic*. We shall consider the case where the period of vibration for each of the two attractors is the same, and we will assume that they correspond to two different stable states of forced vibration induced by an external periodic field driving the system. The underdamped nonlinear oscillator to be considered provides a well-known simple, but nontrival, example(14) of a system that behaves in just this way; its bistability under periodic, nearly resonant, driving has recently been investigated in the context of nonlinear optics(15) and in experiments on a confined relativistic electron excited by cyclotron resonant radiation(16).

The particular model we treat, the nearly-resonantly-driven, underdamped, single-well Duffing oscillator(14) with additive noise, which serves as an archetype for the study of fluctuation phenomena associated with coexisting periodic attractors(17), is described by

$$\ddot{q} + 2\Gamma\dot{q} + \omega_0^2 q + \gamma q^3 = F\cos(\omega_F t) + f(t) \qquad (20)$$

$$\Gamma, |\delta\omega| \ll \omega_F, \qquad \gamma\delta\omega > 0, \qquad \delta\omega = \omega_F - \omega_0$$

$$\langle f(t)\rangle = 0, \qquad \langle f(t)f(t')\rangle = 4\Gamma T\delta(t-t')$$

Note that, in contrast to (2), there is a *strong* periodic force on the right hand side of (20) and it is this that can give rise[14] to the pair of periodic attractors. Weak Gaussian noise of intensity T causes transitions to take place between the attractors; a statistical distribution over the attractors is formed as a result, with the populations of the attractors, w_1 and w_2, differing exponentially strongly from each other except in the close neighbourhood of the kinetic phase transition[17–21] (KPT), where they become equal. It is in this parameter range (the KPT range) that a supernarrow spectral peak[17,20] arises in the SDF. It broadens dramatically with increasing noise intensity T, just like the zero-frequency spectral peak[22] that is responsible[3,4,6] for conventional SR[1,2] in systems with static attractors. The value of $Q^{(0)}(\Omega)$ for ω close to ω_F therefore rises very rapidly (approximately exponentially) with increasing T. Correspondingly, by analogy with thermal equilibrium systems (see above, Section 2), the generalised susceptibility $\overline{\chi}(\Omega)$ of the system and hence the response to a trial force at frequency Ω may also be expected to rise rapidly with T, i.e. SR is to be anticipated. Note that large supernarrow peaks in the imaginary part of $\overline{\chi}(\Omega)$ were predicted theoretically[17] to arise for the case of (20) within the KPT range and with Ω near ω_F. We therefore consider the effect of an additional extremely weak trial force A cos $(\Omega t + \psi)$, of frequency Ω very close to ω_F, on the right-hand side of (20).

The trial force beats with the main periodic force and thus gives rise to vibrations, not only at Ω, but also at the combination frequencies $|\Omega \pm \omega_F|, |\Omega \pm 2\omega_F| \ldots$, the response being strongest at Ω and the nearest resonant combination $|\Omega - 2\omega_F|$. The amplitudes of vibrations at the latter two frequencies can be described by generalised susceptibilities $\chi(\Omega)$, $X(\Omega)$, so that the trial-force-induced modification of the coordinate q, averaged over noise, is of the form

$$\delta\langle q(t)\rangle \simeq A\mathrm{Re}\left[\overline{\chi}(\Omega)\exp(-i\Omega t - i\psi) + \overline{X}(\Omega)\exp[i(2\omega_F - \Omega)t - i\psi\right]] \tag{21}$$

To obtain the susceptibilities, it is convenient to transform to slow variables u, u^* in the frame rotating at ω_F and to introduce the usual dimensionless parameters[17–20] η, β and α which characterise, respectively, the frequency detuning, the strength of the main periodic field and the noise intensity

$$\eta = \Gamma/|\delta\omega|, \qquad \beta = \frac{3|\gamma|F^2}{32\omega_F^3(|\delta\omega|)^3}, \qquad \alpha = 3|\gamma|T/8\omega_F^3\Gamma \tag{22}$$

In what follows, we will assume $\delta\omega, \gamma > 0$. For small α (weak noise intensity) the trial force has two main effects. First, it causes small periodic fluctuations about the stable states. Secondly, by modulating the probability of fluctuational transitions between the stable states, it causes a periodic modulation of their populations. As a result, each of the susceptibilities has a structure similar to that found in conventional SR,

$$\overline{\chi}(\Omega) = \sum_j w_j \overline{\chi}_j(\Omega) + \overline{\chi}_{tr}(\Omega) \tag{23}$$

$$\overline{X}(\Omega) = \sum_j w_j \overline{X}_j(\Omega) + \overline{X}_{tr}(\Omega)$$

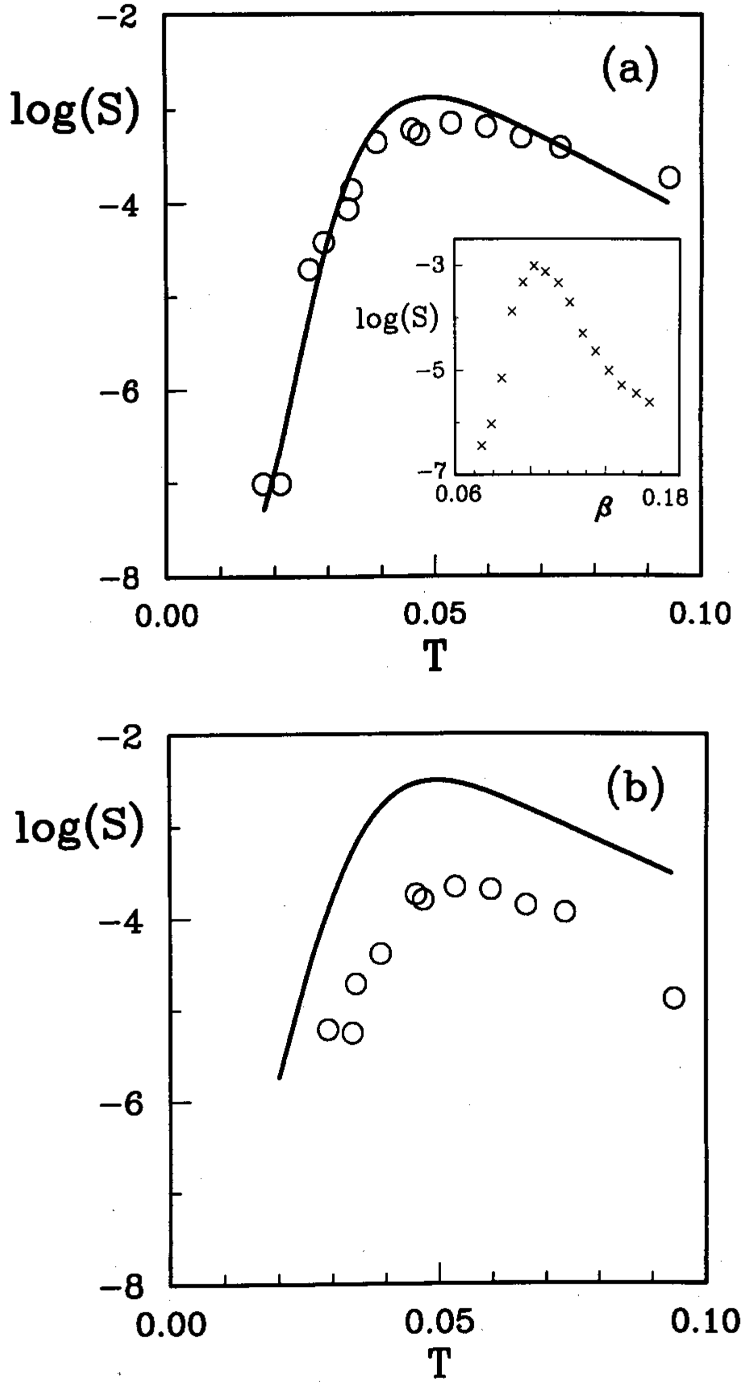

Figure 7: SR in a system (20) with coexisting periodic attractors. The natural log of its response S to a weak trial force at frequency Ω is plotted as a function of noise intensity T: (a) at the trial frequency Ω; (b) at the mirror-reflected frequency $(2\omega_F - \Omega)$. The data points are experimental results from the electronic model. The curves represent the theory, incorporating measured values of the activation energies R_j. Inset in (a): variation of the response S, measured at the trial force frequency Ω with distance from the kinetic phase transition, indicated by β, for fixed noise intensity $T = 0.05$. It takes the form of a cusp (note the log scale), rounded by noise.

where $\overline{\chi}_j$, $\overline{X}_j$ are the partial susceptibilities corresponding to vibrations about the stable states, $\overline{\chi}_{tr}$, $\overline{X}_{tr}$ correspond to transitions between them, and their populations in the absence of the trial force

$$w_1 = \frac{W_{21}}{W_{12} + W_{21}}, \qquad w_2 = \frac{W_{12}}{W_{12} + W_{21}} \tag{24}$$

are expressed in terms of the transition probabilities W_{ij}, which are known[17,20] to be of the activation type $\propto \exp(-R_j/\alpha)$. By linearizing the equations of motion near the stable states, the intra-attractor susceptibilities are readily shown to be of the form

$$\overline{\chi}_j(\Omega) = \frac{i}{2\omega_F} \frac{\Gamma - i(\Omega - \omega_F) - i(2|u_j|^2 - 1)(\omega_F - \omega_0)}{\Gamma^2 \nu_j^2 - 2i\Gamma(\Omega - \omega_F) - (\Omega - \omega_F)^2} \tag{25}$$

$$\overline{X}_j(\Omega) = \frac{-1}{2\omega_F} \frac{u_j^2(\omega_F - \omega_0)}{\Gamma^2 \nu_j^2 - 2i\Gamma(\Omega - \omega_F) - (\Omega - \omega_F)^2}$$

where

$$\nu_j^2 = 1 + \eta^{-2}(3|u_j|^2 - 1)(|u_j|^2 - 1)$$

and the u_j are the values[17–20] of the slow variables (complex dimensionless envelopes) corresponding to each of the stable states.

For present purposes, however, the most significant effect of the trial field is that it smoothly raises and lowers the effective barriers for transitions from each of the stable states, with a period $2\pi/|\Omega - \omega_F|$. Consequently, the activation energies R_1, R_2 vary periodically in time; so also do the transition probabilities W_{ij} and, through the balance equations, the populations of the stable states. The final expressions for the redistribution-induced additions to the generalised susceptibilities are

$$\overline{chi}_{tr}(\Omega) = \frac{w_1 w_2}{2\omega_F(\omega_F - \omega_0)}(u_1^* - u_2^*)\frac{\mu_1 - \mu_2}{\alpha}\left[1 - \frac{i(\Omega - \omega_F)}{W_{12} + W_{21}}\right]^{-1} \tag{26}$$

$$\overline{X}_{tr}(\Omega) = \frac{u_1 - u_2}{u_1^* - u_2^*}\chi_{tr}(\Omega), \qquad \mu_j = \sqrt{\beta}\left(\frac{\partial R_j}{\partial \beta}\right)$$

The strengths (integrated powers) of the periodic signals at Ω and $(2\omega_F - \Omega)$ are then given by

$$S(\Omega) = \frac{1}{4}A^2|\chi(\Omega)|^2 \tag{27}$$

$$S(2\omega_F - \Omega) = \frac{1}{4}A^2|X(\Omega)|^2$$

Given the very rapid dependences of W_{12}, W_{21} on noise intensity near the kinetic phase transition[20], (26) suggests immediately that there will be a range of noise intensity near the kinetic phase transition, $w_1 \simeq w_2$, in which $S(\Omega)$ and $S(2\omega_F - \Omega)$ should each increase very rapidly with T, i.e. the system should indeed exhibit SR.

An electronic model has been used to test this theoretical prediction. Details of both the model and the data-processing techniques will be given elsewhere[23] but, briefly, the arrangements were as follows. The circuit model of (20), designed according to standard practice[11], was driven by Gaussian pseudo-white noise from a feedback shift-register noise generator and additionally by strong and weak periodic forces from a pair of HP3325 frequency synthesizers. The fluctuating voltage representing $q(t)$ in the circuit was digitized (12-bit precision) and the SDF of the fluctuations was computed by means of a standard FFT routine using a Nicolet 1280 data-processor. In terms of scaled units, the circuit parameters were: $2\Gamma = 0.0397$; $\omega_0 = 1.00$; $\gamma = 0.1$; $\omega_F = 1.07200$; $\Omega = 1.07097$; $F = 0.068$; $A = 0.006$. The acquisition process was then repeated, averaging the SDFs until the statistical quality of the result was considered acceptable (typically including 500 realizations). Because of the (necessarily) very close values of ω_F and Ω, a relatively large block size (8K) was used in order to provide the necessary frequency resolution.

The signal strengths were determined from measurements of the magnitudes of the delta spikes corresponding to Ω and $(2\omega_F - \Omega)$, and are plotted (data points) as functions of noise intensity T in Fig 7(a) and (b). The predicted SR effect - strikingly similar in form to that seen in conventional SR and other forms of non-conventional SR (see above) - is clearly evident in each case as a rapid rise, followed by a slower fall, in S with increasing T. A quantitative comparison of these data with the theory is not entirely straightforward, however, because the activation energies R_j have not yet been computed within the relevant part of the phase diagram[17,20]. However, the values of R_j have been determined experimentally from transition rates measured as a function of T for the same model. Use of these values in conjunction with (23)-(27) yields the full curves of Fig 7. Given the large systematic errors inherent in the measurements - arising e.g. from $\delta\omega$ (20), a small difference between large quantities which, in β (22), is then raised to its third power - the agreement between theory and experiment can be considered excellent. The signal/noise *ratios* have also been measured: they each increase with T by a factor of about 25 between their minimum and maximum values.

The magnitude of the fluctuation-induced signal at Ω has been measured as a function of distance, expressed[17] in terms of β, from the kinetic phase transition. The result, shown in the inset of Fig 7(a), exhibits a fast cusp-like (note the log scale) decrease of S as β moves away from its critical value, demonstrating that, like the onset of the supernarrow peak itself[20], HFSR for periodic attractors has the character of a critical phenomenon. We note that conventional SR is also a KPT phenomenon[5,6] (cf Fig 5), and it is clear that SR in bistable systems of any kind is quite generally always of this nature except in those special cases (e.g. Sec 3) where the SR is suppressed by symmetry arguments for equally populated attractors.

An interesting feature of SR for periodic attractors is that it occurs at a frequency that is both high and tunable. Rather than being constrained to lie close to zero frequency on the wing of the zero-frequency spectral peak (as in conventional SR), or close to the characteristic high frequency of intrawell vibrations (as in SR for monostable systems), the frequency of the trial force in the present case is constrained to lie close to that of the strong periodic force in (20); and the latter can, of course, be adjusted within quite wide limits while still keeping the system within the regime of bistability[17,20].

5. CONCLUSIONS

The prediction and successful demonstration of quite new forms of SR in completely different classes of systems from that which supports conventional SR shows, first, that SR is actually a very general phenomenon. In other words, there are many physical situations where noise can be used to *increase* the response of a system to periodic driving;

the effect is not confined to systems with coexisting static stable states, as was thought. Correspondingly, SR may be more widespread in nature, and potentially of wider relevance in science and technology, than has hitherto been appreciated. Secondly, these results can be taken as a vindication of our contention[6] that LRT provides an approach to the SR problem that is not only valid[3,4], but is also of extremely wide applicability. The results that are discussed above demonstrate that the LRT treatment of SR possesses strong predictive power and can therefore provide a useful basis on which to search for yet more new variants of SR in systems that are far removed from the static double-well potential in which this remarkable phenomenon was originally discovered.

ACKNOWLEDGEMENTS
It is a pleasure to acknowledge valuable discussions with Paolo Grigolini and Slava Soskin. The work was supported by the Science and Engineering Research Council (UK), by the Royal Society of London, by the European Community under SC1-CT91-0697 (TSTS), and by the Ukrainian Academy of Sciences.

REFERENCES

1. For a review, see: F Moss, "Stochastic resonance: from the ice ages to the monkey's ear", in *Some Problems in Statistical Physics*, ed G H Weiss (SIAM, Philadelphia, 1992), in press.

2. M I Dykman, G P Golubev, D G Luchinsky, A L Velikovich and S V Tsuprikov, *Sov. Phys. JETP Letters* **53**: 193 (1991).

3. M I Dykman, R Mannella, P V E McClintock and N G Stocks, *Phys. Rev. Lett.* **65**: 2606 (1990).

4. M I Dykman, R Mannella, P V E McClintock and N G Stocks, *Sov. Phys. JETP Letters* **52**: 141 (1990).

5. M I Dykman, R Mannella, P V E McClintock and N G Stocks, *Phys. Rev. Lett* **68**: 2985 (1992).

6. M I Dykman, D G Luchinsky, R Mannella, P V E McClintock, N D Stein and N G Stocks, *J Stat Phys* [this volume].

7. L D Landau and E M Lifshitz, *Statistical Physics*, 3rd edn (Pergamon, New York, 1980).

8. M I Dykman, R Mannella, P V E McClintock, S M Soskin and N G Stocks, *Phys Rev A* **42**: 7041 (1990).

9. S M Soskin, *Physica A* **155**: 401 (1989); *Physica A* **180**: 386 (1992).

10. N G Stocks, P V E McClintock and S M Soskin, "Observation of zero-dispersion peaks in the fluctuation spectrum of an underdamped single-well oscillator", to be published.

11. L Fronzoni in *Noise in Nonlinear Dynamical Systems*, ed F Moss and P V E McClintock (CUP, Cambridge, 1989), vol 3, p 222; P V E McClintock and F Moss in *loc cit*, p 243; N G Stocks, *Experiments in Stochastic Nonlinear Dynamics* (PhD thesis, Lancaster, 1990).

12. M I Dykman, R Mannella, P V E McClintock, S M Soskin and N G Stocks, *Phys. Rev. A* **43**: 1701 (1991).

13. B McNamara, K Wiesenfeld and R Roy, *Phys Rev Lett* **60**: 2626 (1988).

14. L D Landau and E M Lifshitz, *Mechanics* (Pergamon, London, 1976).

15. H M Gibbs, *Optical Bistability: Controlling Light with Light* (Academic Press, New York, 1985); P D Drummond and D F Walls, *J Phys A* **13**: 725 (1980); Chr Flytzanis and C L Tang, *Phys Rev Lett* **45**: 441 (1980); J A Goldstone and E Garmire, *Phys Rev Lett* **53**: 910 (1984).

16. G Gabrielse, H Dehmelt and W Kells, *Phys Rev Lett* **54**: 537 (1985).

17. M I Dykman and M A Krivoglaz, *Sov. Phys. JETP* **50**: 30 (1979); M I Dykman and M A Krivoglaz, *Soviet Physics Reviews*, ed I M Khalatnikov (Harwood, New York, 1984) vol 5, p 265.

18. M I Dykman, M A Krivoglaz and S M Soskin in *Noise in Nonlinear Dynamical Systems*, ed F Moss and P V E McClintock (CUP, Cambridge, 1989), vol 2, p 347.

19. M I Dykman and V N Smelyanski, *Phys Rev A* **41**: 3090 (1990).

20. M I Dykman, R Mannella, P V E McClintock and N G Stocks, *Phys Rev Lett* **65**: 48 (1990).

21. R Bonifacio and L A Lugiato, *Phys Rev Lett* **40**: 1023 (1978); L A Lugiato in *Progress in Optics*, ed E Wolf (North-Holland, Amsterdam, 1984), vol 21, p 69.

22. M I Dykman, R Mannella, P V E McClintock, F Moss and S M Soskin, *Phys Rev A* **37**, 1303 (1988).

23. M I Dykman, R Mannella, P V E McClintock, N D Stein and N G Stocks, "High frequency stochastic resonance for periodic attractors", to be published.